

DEFINABLE TRANSVERSALS OF ANALYTIC EQUIVALENCE RELATIONS

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ABSTRACT. We examine the circumstances under which certain analytic equivalence relations on Polish spaces have definable transversals.

1. INTRODUCTION

Suppose that X is a Polish space and E is a Σ_1^1 equivalence relation on X . A *transversal* of E is a set $B \subseteq X$ which intersects every equivalence class of E in exactly one point. Our goal here is to provide some insight into the circumstances under which E admits a definable transversal.

A *reduction* of E to another equivalence relation F on a Polish space Y is a function $\pi : X \rightarrow Y$ such that

$$\forall x_1, x_2 \in X (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2)).$$

An *embedding* is an injective reduction. Given a pointclass Γ of subsets of Polish spaces, we write $E \leq_\Gamma F$ to indicate the existence of a Γ -measurable reduction of E to F , and we write $E \sqsubseteq_\Gamma F$ to indicate the existence of a Γ -measurable embedding of E into F . Following standard convention, we use the subscripts c and B as shorthand for the classes of open and Borel sets in Polish spaces, respectively.

The *diagonal* on a Polish space Y is given by

$$\Delta(Y) = \{(y_1, y_2) \in Y \times Y : y_1 = y_2\}.$$

We say that E is Γ -*smooth* if $E \leq_\Gamma \Delta(\mathbb{N}^{\mathbb{N}})$. We say that E is *smooth* if it is Borel smooth. From the descriptive set-theoretic point of view, these are the simplest equivalence relations. Let E_0 denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N} \forall m \geq n (\alpha(m) = \beta(m)).$$

Harrington-Kechris-Louveau [3] have shown that E_0 is the \leq_B -minimal non-smooth Borel equivalence relation, and Hjorth-Kechris [5] have shown a similar theorem for Σ_1^1 equivalence relations, in the presence of mild large cardinals.

Following standard terminology, we say that E is *countable* if all of its equivalence classes are countable, and we say that E is Γ -*treeable* if Γ contains an acyclic graph \mathcal{G} such that $E = E_{\mathcal{G}}$, where $E_{\mathcal{G}}$ denotes the equivalence relation whose classes are the connected components of \mathcal{G} . We say that E is *treeable* if it is Borel-treeable. Our main result is the following:

Theorem. *Assume that $\forall x \in \mathbb{R}$ (x^\sharp exists). Suppose that X is a Polish space, E is a Σ_1^1 equivalence relation on X , and at least one of the following holds:*

- (a) E is Borel;
- (b) E is countable;
- (c) E is Σ_1^1 -treeable.

Then exactly one of the following holds:

- 1. E admits a Π_1^1 transversal;
- 2. $E_0 \sqsubseteq_c E$.

Moreover, if (a) holds and either (b) or (c) holds, then (1) is equivalent to the existence of a Borel transversal of E .

Remark 1.1. It seems worth noting that condition (c) can be weakened to:

- (c') E is the union of countably many Σ_1^1 -treeable equivalence relations.

This is a consequence of a straightforward generalization of our arguments here.

Case (a) follows easily from Theorem 1 of Harrington-Kechris-Louveau [3]. Case (a) + (b) is well known, and case (b) follows from case (a), case (a) + (b), and a result of Lecomte-Miller [9] regarding Σ_1^1 graphs of uncountable Borel chromatic number. We give the proofs of these cases in §2. Case (a) + (c) follows from Theorem 1 of Harrington-Kechris-Louveau [3] and a recent result of Hjorth [4]. In §3, we prove a technical modification of the main dichotomy theorem of Kechris-Solecki-Todorćević [8] regarding Σ_1^1 graphs of uncountable Borel chromatic number, which we use in §4 to prove case (c) and give a new proof of case (a) + (c).

2. TRANSVERSALS IN THE BOREL AND COUNTABLE CASES

We begin with the case of Borel equivalence relations, which has already been essentially dealt with by Harrington-Kechris-Louveau [3]:

Theorem 2.1. *Suppose that X is a Polish space and E is a Borel equivalence relation on X . Then at least one of the following holds:*

- 1. E admits a Π_1^1 transversal;
- 2. $E_0 \sqsubseteq_c E$.

Proof. If $E_0 \not\sqsubseteq_c E$, then Theorem 1 of Harrington-Kechris-Louveau [3] ensures that there is a finer Polish topology τ on X , compatible with its underlying Borel structure, such that E is τ -closed. Fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f : C \rightarrow X$ (see, for example, Exercise 15.3 of Kechris [7]), and define

$$\alpha F \beta \Leftrightarrow f(\alpha) E f(\beta),$$

for $\alpha, \beta \in C$. It is clear that F is closed, and it follows that the set

$$B = \{\alpha \in C : \forall \beta \in C (\alpha F \beta \Rightarrow \alpha \leq_{\text{lex}} \beta)\}$$

is a Π_1^1 transversal of F , thus $f[B]$ is a Π_1^1 transversal of E . □

It is not hard to see that under ZFC + All Δ_2^1 sets of reals are Baire measurable, if E admits a Π_1^1 transversal, then E is Δ_2^1 -smooth. As $\forall x \in \mathbb{R}$ ($x^\#$ exists) implies Σ_1^1 determinacy, which in turn implies that all Σ_1^1 sets of reals are Baire measurable, it follows from the well-known fact that E_0 is not BP -smooth that conditions (1) and (2) are mutually exclusive. This completes the proof of case (a) of our main theorem.

We next turn our attention to the countable case. We say that a set $B \subseteq X$ is a *partial transversal* of E if it intersects every equivalence class of E in at most one point. The following consequence of the Glimm-Effros dichotomy is well known:

Theorem 2.2. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X . Then exactly one of the following holds:*

1. X is the union of countably many Borel partial transversals of E ;
2. $E_0 \sqsubseteq_c E$.

Case (a) + (b) of our main theorem is therefore a consequence of the following:

Proposition 2.3. *Suppose that X is a Polish space and E is a countable Σ_1^1 equivalence relation on X . Then the following are equivalent:*

1. X is the union of countably many Borel partial transversals of E ;
2. E admits a Π_1^1 transversal.

Moreover, if E is Borel, then (1) is equivalent to the existence of a Borel transversal.

Proof. To see (1) \Rightarrow (2), suppose that there are countably many Borel partial transversals $B_n \subseteq X$ which cover X . The E -saturation of a set $B \subseteq X$ is given by

$$[B]_E = \{x \in X : \exists y \in B (xEy)\}.$$

For each $n \in \mathbb{N}$, define $C_n = B_n \setminus \bigcup_{m < n} [B_m]_E$, and observe that the set $\bigcup_{n \in \mathbb{N}} C_n$ is a Π_1^1 transversal of E . Moreover, if E is Borel, then so too is C .

To see (2) \Rightarrow (1), suppose that $B \subseteq X$ is a Π_1^1 transversal of E . As the property of having countable sections is Π_1^1 on Σ_1^1 , it follows from the first reflection theorem (see, for example, Theorem 35.10 of Kechris [7]) that E is contained in a Borel set $R \subseteq X \times X$ with countable sections. The Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of Kechris [7]) implies that the transitive closure F of the symmetrization of R is a countable Borel equivalence relation, and Theorem 1 of Feldman-Moore [1] implies that there are Borel automorphisms $f_n : X \rightarrow X$ such that $F = \bigcup_{n \in \mathbb{N}} \text{graph}(f_n)$. For each $n \in \mathbb{N}$, define $A_n \subseteq X$ by

$$A_n = \{x \in X : xE f_n(x)\}.$$

Given a graph \mathcal{G} on X , we say that a set $A \subseteq X$ is \mathcal{G} -discrete if $\mathcal{G} \cap (A \times A) = \emptyset$. A ($\mathbb{N}^{\mathbb{N}}$ -valued) coloring of \mathcal{G} is a function $c : X \rightarrow \mathbb{N}^{\mathbb{N}}$ such that each of the sets $c^{-1}(y)$ is \mathcal{G} -discrete. The Γ -measurable chromatic number of \mathcal{G} , or $\chi_\Gamma(\mathcal{G})$, is the least cardinal of the form $|c[X]|$, where c is a Γ -measurable coloring of \mathcal{G} .

Let $\mathcal{G} = E \setminus \Delta(X)$, noting that a set is a partial transversal of E if and only if it is \mathcal{G} -discrete, thus the disjointification of the sets of the form $f_n[A_n \cap B]$ witness that $\chi_{\sigma(\Sigma_1^1)}(E \setminus \Delta(X)) \leq \aleph_0$. Theorem 6.3 of Kechris-Solecki-Todorćević [8] then implies that $\chi_B(E \setminus \Delta(X)) \leq \aleph_0$, thus X is the union of countably many Borel partial transversals of E . \square

In order to obtain case (b) of our main theorem, it is now sufficient to prove the following generalization of Theorem 2.2:

Theorem 2.4. *Suppose that X is a Polish space and E is a countable Σ_1^1 equivalence relation on X . Then exactly one of the following holds:*

1. X is the union of countably many Borel partial transversals of E ;
2. $E_0 \sqsubseteq_c E$.

Proof. To see that (1) and (2) are mutually exclusive, fix a cover of X by countably many Borel partial transversals $B_n \subseteq X$, and suppose that $\pi : 2^{\mathbb{N}} \rightarrow X$ is a continuous embedding of E_0 into E . Fix $n \in \mathbb{N}$ such that $\pi^{-1}(B_n)$ is non-meager, and observe that $\pi^{-1}(B_n)$ is a partial transversal of E_0 , which contradicts the well known fact that E_0 does not admit a non-meager Baire measurable transversal.

It remains to show $\neg(1) \Rightarrow (2)$. Let $\mathcal{G} = E \setminus \Delta(X)$. If (1) is false, then $\chi_B(\mathcal{G}) > \aleph_0$, and Proposition 3.12 of Lecomte-Miller [9] ensures that there is a Borel set $B \subseteq X$ such that $\mathcal{G}|B$ is Borel and $\chi_B(\mathcal{G}|B) > \aleph_0$. The Lusin-Novikov uniformization theorem then implies that $E|B$ is Borel. Fix a finer Polish topology τ on X , compatible with its underlying Borel structure, such that B is τ -clopen (see, for example, Theorem 13.1 of Kechris [7]), and observe that Theorem 2.2 implies that $E_0 \sqsubseteq_c E|B \sqsubseteq_c E$. \square

3. BOREL CHROMATIC NUMBERS OF SEQUENCES OF GRAPHS

Suppose that $\mathcal{G} = \langle \mathcal{G}^i \rangle_{i \in I}$ is a sequence of graphs on a set X . We say that a set $A \subseteq X$ is \mathcal{G} -discrete if $\mathcal{G}^i \cap (A \times A) = \emptyset$, for some $i \in I$. A ($\mathbb{N}^{\mathbb{N}}$ -valued) coloring of \mathcal{G} is a function $c : X \rightarrow \mathbb{N}^{\mathbb{N}}$ such that each of the sets $c^{-1}(y)$ is \mathcal{G} -discrete. The Γ -measurable chromatic number of \mathcal{G} , or $\chi_{\Gamma}(\mathcal{G})$, is the least cardinal of the form $|c[X]|$, where c is a Γ -measurable coloring of \mathcal{G} .

Fix sequences $s_n \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N} (s \subseteq s_n)$, set

$$\mathcal{G}_0^n = \bigcup_{m \geq n} \{(s_m i \alpha, s_m \bar{i} \alpha) : i \in \{0, 1\} \text{ and } \alpha \in 2^{\mathbb{N}}\},$$

and define $\mathcal{G}_0 = \langle \mathcal{G}_0^n \rangle_{n \in \mathbb{N}}$.

Proposition 3.1. *If $A \subseteq 2^{\mathbb{N}}$ is a non-meager Borel set, then $\chi_{\text{BP}}(\mathcal{G}_0|A) > \aleph_0$.*

Proof. Fix $s \in 2^{<\mathbb{N}}$ such that A is comeager in \mathcal{N}_s . It is enough to show that no Baire measurable subset of A is both non-meager in \mathcal{N}_s and \mathcal{G}_0 -discrete. Towards this end, suppose that $B \subseteq A$ is Baire measurable and non-meager in \mathcal{N}_s , and fix $t \supseteq s$ such that B is comeager in \mathcal{N}_t . Given $m \in \mathbb{N}$, fix $u \supseteq t$ such that $m \leq |u|$, and fix $n \in \mathbb{N}$ such that $u \subseteq s_n$, noting that $m \leq n$. Then there are comeagerly many $\alpha \in 2^{\mathbb{N}}$ such that $(s_n 0 \alpha, s_n 1 \alpha) \in \mathcal{G}_0^m \cap (B \times B)$, so B is not \mathcal{G}_0^m -discrete, thus B is not \mathcal{G}_0 -discrete. \square

Suppose that $\mathcal{G} = \langle \mathcal{G}^n \rangle_{n \in \mathbb{N}}$ and $\mathcal{H} = \langle \mathcal{H}^n \rangle_{n \in \mathbb{N}}$ are decreasing vanishing sequences of graphs on X and Y , respectively, and fix a strictly increasing sequence $k \in \mathbb{N}^{\mathbb{N}}$. A k -homomorphism from \mathcal{G} to \mathcal{H} is a function $\pi : X \rightarrow Y$ such that

$$\forall n \in \mathbb{N} \forall x_1, x_2 \in X ((x_1, x_2) \in \mathcal{G}^n \setminus \mathcal{G}^{n+1} \Rightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}^{k_{2n}} \setminus \mathcal{H}^{k_{2n+1}}).$$

Given $f : \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}$, we say that k is *evenly f -dominating* if

$$\forall n \in \mathbb{N} (k_{2n} > f(k_0, k_1, \dots, k_{2n-1})).$$

We write $\mathcal{G} \preceq_c^f \mathcal{H}$ if there is a continuous k -homomorphism from \mathcal{G} to \mathcal{H} , for some strictly increasing, evenly f -dominating sequence k .

Theorem 3.2. *Suppose that X is a Polish space, $\mathcal{G} = \langle \mathcal{G}^n \rangle_{n \in \mathbb{N}}$ is a decreasing vanishing sequence of Σ_1^1 graphs on X , and $f : \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}$. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. $\mathcal{G}_0 \preceq_c^f \mathcal{G}$.

Proof. As Proposition 3.1 implies that (1) and (2) are mutually exclusive, it is enough to prove $\neg(1) \Rightarrow (2)$. In fact, it is enough to prove the special case of $\neg(1) \Rightarrow (2)$ in which $X = \mathbb{N}^{\mathbb{N}}$, as can be easily seen by appealing to the fact that every Polish space is the continuous injective image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. From this point forward, we will therefore assume that $X = \mathbb{N}^{\mathbb{N}}$ and $\chi_B(\mathcal{G}) > \aleph_0$.

Lemma 3.3. *Every \mathcal{G} -discrete Σ_1^1 set is contained in a \mathcal{G} -discrete Borel set.*

Proof. Simply note that a set $A \subseteq X$ is \mathcal{G} -discrete if and only if

$$\exists n \in \mathbb{N} \forall x, y \in X (x \notin A \text{ or } y \notin A \text{ or } (x, y) \notin \mathcal{G}^n).$$

As this is Π_1^1 on Σ_1^1 , the lemma follows from the first reflection theorem. \square

By the first reflection theorem, there is a decreasing vanishing sequence $\mathcal{H} = \langle \mathcal{H}^n \rangle_{n \in \mathbb{N}}$ of Borel graphs such that $\forall n \in \mathbb{N} (\mathcal{G}^n \subseteq \mathcal{H}^n)$. For each pair of natural numbers $k < \ell$, fix a tree $T^{[k, \ell]}$ on $\mathbb{N} \times (\mathbb{N} \times \mathbb{N})$ such that $p[T^{[k, \ell]}] = \mathcal{G}^k \setminus \mathcal{H}^\ell$.

For each pair of natural numbers $m < n$, set

$$J_{m,n} = \{(s_m 0t, s_m 1t) \in 2^n \times 2^n : t \in 2^{n-m-1}\},$$

as well as $J_n = \bigcup_{m < n} J_{m,n}$ and $J_{\leq n} = \bigcup_{k \leq n} J_k$. For $i, n \in \mathbb{N}$ and $s \in 2^i$, set

$$\mathcal{G}_s^n = \{(\langle x_t \rangle, \langle y_t \rangle) \in X^{2^i} \times X^{2^i} : (x_s, y_s) \in \mathcal{G}^n\},$$

and define $\mathcal{G}^i = \langle \mathcal{G}_s^n \rangle_{(s,n) \in 2^i \times \mathbb{N}}$.

We will recursively find $k_0, \dots, k_{2n} \in \mathbb{N}$, $\langle u_s \rangle \in (\mathbb{N}^n)^{2^n}$, and $\langle v_{s,t} \rangle \in (\mathbb{N}^n)^{J_n}$, from which we obtain the Σ_1^1 set X_n of sequences $\langle x_s \rangle_{s \in 2^n}$ such that:

- (a) $\forall s \in 2^n (x_s \in \mathcal{N}_{u_s})$;
- (b) $\forall m < n \forall (s, t) \in J_{m,n} ((x_s, x_t) \in p[T_{(v_{s,t}, (u_s, u_t))}^{[k_{2m}, k_{2m+1}]}])$.

We begin by fixing $k_0 > f(\emptyset)$ and setting $u_\emptyset = \emptyset$. Suppose now that we have found $\langle k_i \rangle_{i \leq 2n}$, $\langle u_s \rangle_{s \in 2^{\leq n}}$, and $\langle v_{s,t} \rangle_{(s,t) \in J_{\leq n}}$ which satisfy the following conditions:

1. $\forall i \leq n (\chi_B(\mathcal{G}^i | X_i) > \aleph_0)$;

2. $\forall i \leq n$ ($k_{2i-1} > k_{2i-2}$ and $k_{2i} > \max(k_{2i-1}, f(k_0, \dots, k_{2i-1}))$);
3. $\forall i < n \forall s \in 2^i$ ($u_s \subseteq u_{s0}, u_{s1}$);
4. $\forall i < n \forall (s, t) \in J_i$ ($v_{s,t} \subseteq v_{s0,t0}, v_{s1,t1}$);
5. $\forall i < n \forall m < i \forall (s, t) \in J_{m,i}$ ($(v_{s,t}, (u_s, u_t)) \in T^{[k_{2m}, k_{2m+1}]}$).

Let \mathbb{P} denote the set of triples

$$p = (k^p, \langle u_s^p \rangle, \langle v_{s,t}^p \rangle)$$

in $\mathbb{N} \times (\mathbb{N}^{n+1})^{2^{n+1}} \times (\mathbb{N}^{n+1})^{J_{n+1}}$ which satisfy the following conditions:

- 2'. $k^p > k_{2n}$;
- 3'. $\forall s \in 2^n$ ($u_s \subseteq u_{s0}^p, u_{s1}^p$);
- 4'. $\forall (s, t) \in J_n$ ($v_{s,t} \subseteq v_{s0,t0}^p, v_{s1,t1}^p$);
- 5'. $\forall m < n \forall (s, t) \in J_{m,n+1}$ ($(v_{s,t}^p, (u_s^p, u_t^p)) \in T^{[k_{2m}, k_{2m+1}]}$);
- 6'. $\forall (s, t) \in J_{n,n+1}$ ($(v_{s,t}^p, (u_s^p, u_t^p)) \in T^{[k_{2n}, k^p]}$).

For each $p \in \mathbb{P}$, let Y_p denote the set of sequences $\langle y_s \rangle_{s \in 2^{n+1}}$ such that:

- (a') $\forall s \in 2^{n+1}$ ($y_s \in \mathcal{N}_{u_s^p}$);
- (b') $\forall m < n \forall (s, t) \in J_{m,n+1}$ ($(y_s, y_t) \in p[T_{(v_{s,t}^p, (u_s^p, u_t^p))}^{[k_{2m}, k_{2m+1}]}]$);
- (c') $\forall (s, t) \in J_{n,n+1}$ ($(y_s, y_t) \in p[T_{(v_{s,t}^p, (u_s^p, u_t^p))}^{[k_{2n}, k^p]}]$).

Lemma 3.4. *There exists $p \in \mathbb{P}$ such that $\chi_B(\mathcal{G}^{n+1}|Y_p) > \aleph_0$.*

Proof. Suppose, towards a contradiction, that $\forall p \in \mathbb{P}$ ($\chi_B(\mathcal{G}^{n+1}|Y_p) \leq \aleph_0$). Then for each $p \in \mathbb{P}$, there are \mathcal{G}^{n+1} -discrete Borel sets $A_{p,m} \subseteq X^{2^{n+1}}$ such that $Y_p \subseteq \bigcup_{m \in \mathbb{N}} A_{p,m}$. For each $p \in \mathbb{P}$ and $m \in \mathbb{N}$, fix $i \in \{0, 1\}$ such that the set

$$\text{proj}_i[A_{p,m}] = \{\langle x_{si} \rangle_{s \in 2^n} : \langle x_s \rangle \in A_{p,m}\}$$

is \mathcal{G}^n -discrete, and set $i_{p,m} = i$. By Lemma 3.3, there are \mathcal{G}^n -discrete Borel sets $B_{p,m} \supseteq \text{proj}_{i_{p,m}}[A_{p,m}]$. Then the set

$$A = X_n \setminus \bigcup_{p \in \mathbb{P}, m \in \mathbb{N}} B_{p,m}$$

is Σ_1^1 and $\chi_B(\mathcal{G}^n|A) > \aleph_0$. In particular, it follows that A is not \mathcal{G}^n -discrete.

Fix $(\langle y_{s0} \rangle, \langle y_{s1} \rangle) \in \mathcal{G}_{s_n}^{k_{2n}}|A$. Then there exists $k^p > k_{2n}$ such that $(y_{s_n 0}, y_{s_n 1}) \notin \mathcal{H}^{k^p}$. For each $s \in 2^{n+1}$, set $u_s^p = y_s|(n+1)$. For each $m < n$, $(s, t) \in J_{m,n}$, and $i \in \{0, 1\}$, fix $v_{si,ti}^p \supseteq v_{s,t}$ in \mathbb{N}^{n+1} such that

$$(y_{si}, y_{ti}) \in p[T_{(v_{si,ti}^p, (u_{si}^p, u_{ti}^p))}^{[k_{2m}, k_{2m+1}]}].$$

As $(y_{s_n 0}, y_{s_n 1}) \in \mathcal{G}^{k_n} \setminus \mathcal{H}^{k^p}$, there exists $v_{s_n 0, s_n 1}^p \in \mathbb{N}^{n+1}$ such that

$$(y_{s_n 0}, y_{s_n 1}) \in p[T_{(v_{s_n 0, s_n 1}^p, (u_{s_n 0}^p, u_{s_n 1}^p))}^{[k_{2n}, k^p]}].$$

It is clear that $p = (k^p, \langle u_s^p \rangle, \langle v_{s,t}^p \rangle)$ is in \mathbb{P} , and since $\langle y_s \rangle \in Y_p$, there exists $m \in \mathbb{N}$ such that $\langle y_s \rangle \in A_{p,m}$, thus $\langle y_{s_i p, m} \rangle \in B_{p,k}$, which contradicts the definition of A , and therefore completes the proof of the lemma. \square

Fix $p \in \mathbb{P}$ as in Lemma 3.4. Set $k_{2n+1} = k^p$, $\langle u_s \rangle_{s \in 2^{n+1}} = \langle u_s^p \rangle_{s \in 2^{n+1}}$, and $\langle v_{s,t} \rangle_{(s,t) \in J_{n+1}} = \langle v_{s,t}^p \rangle_{(s,t) \in J_{n+1}}$. Fix $k_{2n+2} > \max(k_{2n+1}, f(k_0, \dots, k_{2n+1}))$. Then $X_{n+1} = Y_p$, so condition (1) holds, and conditions (2) – (7) follow from the definitions of \mathbb{P} and Y_p . This completes the recursive construction.

Condition (3) ensures that the function $\pi : 2^{\mathbb{N}} \rightarrow X$ given by

$$\pi(\alpha) = \lim_{n \rightarrow \infty} u_{\alpha|n}$$

is well-defined and continuous. Condition (2) ensures that the sequence $k = \langle k_n \rangle$ is strictly increasing and evenly f -dominating, so it only remains to check that π is a homomorphism from $\mathcal{G}_0^m \setminus \mathcal{G}_0^{m+1}$ to $\mathcal{G}^{k_{2m}} \setminus \mathcal{G}^{k_{2m+1}}$, for all $m \in \mathbb{N}$. Given $\alpha \in 2^{\mathbb{N}}$, fix $n \geq m$, set $k = n - m$, and observe that the pair $(s_m 0(\alpha|k), s_m 1(\alpha|k))$ is in $J_{m, n+1}$. Condition (5) then ensures that

$$(v_{s_m 0(\alpha|k), s_m 1(\alpha|k)}, (u_{s_m 0(\alpha|k)}, u_{s_m 1(\alpha|k)})) \in T^{[k_{2m}, k_{2m+1}]},$$

thus condition (4) implies that

$$(\pi(s_m 0\alpha), \pi(s_m 1\alpha)) \in \mathcal{G}^{k_{2m}} \setminus \mathcal{H}^{k_{2m+1}} \subseteq \mathcal{G}^{k_{2m}} \setminus \mathcal{G}^{k_{2m+1}},$$

which completes the proof of the theorem. \square

It is not difficult to modify the above proof so as to obtain new proofs of Theorems 6.3 and 6.6 of Kechris-Solecki-Todorćević [8] which do not require the use of effective descriptive set theory.

We say that a set $A \subseteq X$ is *globally Baire measurable* if for every Polish space Y and continuous function $\pi : Y \rightarrow X$, the set $\pi^{-1}(A)$ is Baire measurable. Let GB denote the class of all such sets. We will use Theorem 3.2 to study maps $C \mapsto d_C$ which assign a metric d_C on C to each equivalence class C of a GB-smooth Σ_1^1 equivalence relation E . We say that such a map is Σ_1^1 if the graph of the corresponding function $d : E \rightarrow \mathbb{R}$ given by $d(x, y) = d_{[x]_E}(x, y) = d_{[y]_E}(x, y)$ is Σ_1^1 . We say that a set $A \subseteq X$ is *d-bounded* if there exists $n \in \mathbb{N}$ such that

$$\forall x, y \in A \ (xEy \Rightarrow d(x, y) \leq n).$$

Theorem 3.5. *Suppose that X is a Polish space, E is a GB-smooth Σ_1^1 equivalence relation on X , and $C \mapsto d_C$ is a Σ_1^1 assignment of metrics to the equivalence classes of E . Then X is the union of countably many d -bounded Borel sets.*

Proof. For each $n \in \mathbb{N}$, let \mathcal{G}^n denote the graph on X given by

$$\mathcal{G}^n = \{(x, y) \in E : d(x, y) > n\},$$

and set $\mathcal{G} = \langle \mathcal{G}^n \rangle_{n \in \mathbb{N}}$. Note that a set is d -bounded if and only if it is \mathcal{G} -discrete, so it is enough to show that $\chi_B(\mathcal{G}) \leq \aleph_0$. Suppose, towards a contradiction, that $\chi_B(\mathcal{G}) > \aleph_0$, and fix $f : \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}$ such that

$$\forall n \in \mathbb{N} (f(k_0, \dots, k_{2n-1}) = 2^n k_1 + 2^{n-1} k_3 + \dots + 2k_{2n-1}).$$

By Theorem 3.2, there is a strictly increasing, evenly f -dominating sequence k and a continuous k -homomorphism $\pi : 2^{\mathbb{N}} \rightarrow X$ from \mathcal{G}_0 to \mathcal{G} .

Fix a globally Baire measurable reduction $\varphi : X \rightarrow \mathbb{N}^{\mathbb{N}}$ of E to $\Delta(\mathbb{N}^{\mathbb{N}})$. Then the composition $\varphi \circ \pi$ is Baire measurable, so there is a dense G_δ set $C \subseteq 2^{\mathbb{N}}$ such that $\varphi \circ \pi|_C$ is continuous. Then the set

$$R = \{(\varphi \circ \pi(\alpha), \pi(\alpha)) \in \mathbb{N}^{\mathbb{N}} \times X : \alpha \in C\}$$

is Σ_1^1 , so the Jankov-von Neumann uniformization theorem (see, for example, Theorem 18.1 of Kechris [7]) ensures the existence of a $\sigma(\Sigma_1^1)$ -measurable function $\psi : \varphi \circ \pi[C] \rightarrow \pi[C]$ such that

$$\forall \alpha \in C (\pi(\alpha) E \psi \circ \varphi \circ \pi(\alpha)).$$

As $\psi \circ \varphi \circ \pi|_C$ is $\sigma(\Sigma_1^1)$ -measurable, there is a comeager Borel set $D \subseteq C$ such that $\psi \circ \varphi \circ \pi|_D$ is Borel. Then for each $n \in \mathbb{N}$, the set

$$D_n = \{\alpha \in D : d(\pi(\alpha), \psi \circ \varphi \circ \pi(\alpha)) \leq n\}$$

is $\mathcal{G}_0^{2^n}$ -discrete, thus $\chi_{\sigma(\Sigma_1^1)}(\mathcal{G}_0|_D) \leq \aleph_0$, which contradicts Proposition 3.1. \square

4. TRANSVERSALS IN THE TREEABLE CASE

Suppose that \mathcal{G} is an acyclic Σ_1^1 treeing of E . Associated with \mathcal{G} is the corresponding graph metric $d_{\mathcal{G}}$, as well as the sequence $\mathcal{G} = \langle \mathcal{G}^n \rangle_{n \in \mathbb{N}}$, where $\mathcal{G}^n = \{(x, y) \in E : d_{\mathcal{G}}(x, y) > n\}$. The \mathcal{G} -convex closure of a set $A \subseteq X$ is the set $[A]_{\mathcal{G}}$ of points which lie along an injective \mathcal{G} -path from one point of A to another. We say that a set is \mathcal{G} -convex if it is equal to its convex closure.

Proposition 4.1. *Suppose that X is a Polish space, E is an equivalence relation on X , \mathcal{G} is a Σ_1^1 treeing of E , and $A \subseteq X$ is a \mathcal{G}^n -discrete Σ_1^1 set. Then there is a \mathcal{G} -convex \mathcal{G}^n -discrete Borel set $B \supseteq A$.*

Proof. Set $A_0 = A$, and given a \mathcal{G}^n -discrete Σ_1^1 set $A_n \subseteq X$, appeal to Lemma 3.3 to obtain a \mathcal{G}^n -discrete Borel set $B_n \supseteq A_n$. Then the set $A_{n+1} = [B_n]_{\mathcal{G}}$ is Σ_1^1 . It follows that the set $B = \bigcup_{n \in \mathbb{N}} B_n$ is Borel and \mathcal{G}^n -discrete, and since $B = \bigcup_{n \in \mathbb{N}} A_n$, it is also \mathcal{G} -convex. \square

The \mathcal{G} -interior of a set $A \subseteq X$ is the set of all $x \in A$ for which there is an injective \mathcal{G} -path x_0, x_1, \dots, x_n such that $x_0, x_n \in A$ and $x \in \{x_1, \dots, x_{n-1}\}$. A *partial quasi-transversal* of E is a set which intersects every equivalence class of E in at most two points.

Proposition 4.2. *Suppose that X is a Polish space, E is an equivalence relation on X , \mathcal{G} is a Σ_1^1 treeing of E , and $A \subseteq X$ is a \mathcal{G}^{2^n} -discrete Σ_1^1 set. Then there are \mathcal{G} -convex Borel sets $B_0, B_1, \dots, B_n \subseteq X$ such that the corresponding sets*

$$C_i = B_i \setminus [B_{i+1} \cup \dots \cup B_n]_E$$

are partial quasi-transversals and $A \subseteq [C_0 \cup \dots \cup C_n]_E$.

Proof. Set $A_0 = A$. Given $i < n$ and a $\mathcal{G}^{2^{(n-i)}}$ -discrete Σ_1^1 set $A_i \subseteq X$, appeal to Proposition 4.1 to obtain a \mathcal{G} -convex $\mathcal{G}^{2^{(n-i)}}$ -discrete Borel set $B_i \supseteq A_i$, and observe that the Σ_1^1 set $A_{i+1} = B_i'$ is $\mathcal{G}^{2^{(n-(i+1))}}$ -discrete.

To see that the corresponding sets C_i are as desired, note first that $C_i \subseteq B_i \setminus [B_i']_E$. As the latter set is \mathcal{G}^1 -discrete, so too is the former. As \mathcal{G}^1 -discrete sets are necessarily partial quasi-transversals of E , it follows that C_i is a partial quasi-transversal of E . It only remains to observe that if $x \in A$, then there is a maximal $i \leq n$ such that $x \in [B_i]_E$ is non-empty, from which it follows that $x \in [C_i]_E$. \square

We say that a set $B \subseteq X$ is an E -complete section if it intersects every E -class.

Theorem 4.3. *Suppose that X is a Polish space, E is a GB -smooth equivalence relation on X , and \mathcal{G} is a Σ_1^1 treeing of E . Then there are \mathcal{G} -convex Borel sets $B_n \subseteq X$ such that:*

1. $\bigcup_{n \in \mathbb{N}} B_n$ is an E -complete section;
2. $\forall n \in \mathbb{N} (B_n \setminus \bigcup_{m < n} [B_m]_E \text{ is a partial quasi-transversal of } E)$.

Proof. By Theorem 3.5, there are countably many $d_{\mathcal{G}}$ -bounded Borel sets $B_n \subseteq X$ which cover X . For each $n \in \mathbb{N}$, fix $k_n \in \mathbb{N}$ such that B_n is $\mathcal{G}^{2^{k_n}}$ -bounded, and fix Borel sets $B_{n0}, B_{n1}, \dots, B_{nk_n}$ as in Proposition 4.2. Then the sets $B_{0k_0}, \dots, B_{00}, B_{1k_1}, \dots, B_{10}, \dots$ are as desired. \square

We can now give a new proof of case (a) + (c) of our main theorem:

Theorem 4.4 (Hjorth). *Suppose that E is a treeable equivalence relation on a Polish space X . Then exactly one of the following holds:*

1. E admits a Borel transversal;
2. $E_0 \sqsubseteq_c E$.

Proof. To see (1) \Rightarrow \neg (2), fix a Borel transversal B of E , and define $\pi : X \rightarrow B$ by

$$\pi(x) = y \Leftrightarrow (xEy \text{ and } y \in B).$$

Then π is Borel. It follows that if $\varphi : 2^{\mathbb{N}} \rightarrow X$ is a continuous embedding of E_0 into E , then $\pi \circ \varphi$ is a Borel reduction of E_0 to $\Delta(Y)$, a contradiction.

It remains to show \neg (2) \Rightarrow (1). By Theorem 1 of Harrington-Kechris-Louveau [3], if $E_0 \not\sqsubseteq_c E$, then E is smooth. By Theorem 4.3, there are \mathcal{G} -convex Borel sets $B_n \subseteq X$ such that:

1. $\bigcup_{n \in \mathbb{N}} B_n$ is an E -complete section;

2. $\forall n \in \mathbb{N}$ ($B_n \setminus \bigcup_{m < n} [B_m]_E$ is a partial quasi-transversal of E).

As E -saturations of \mathcal{G} -convex Borel sets are Borel, it follows that the set

$$B = \bigcup_{n \in \mathbb{N}} B_n \setminus \bigcup_{m < n} [B_m]_E$$

is Borel. Fix a linear ordering \leq of X , and observe that

$$C = \{x \in B : \forall y \in B (xEy \Rightarrow x \leq y)\}$$

is a transversal of E . As the Lusin-Novikov uniformization theorem implies that C is Borel, this completes the proof of the theorem. \square

Before going further, we will need the following corollary of Hjorth-Kechris [2]:

Theorem 4.5. *Assume that $\forall x \in \mathbb{R}$ (x^\sharp exists). Suppose that X is a Polish space and E is a Σ_1^1 equivalence relation on X . Then exactly one of the following holds:*

1. E is GB-smooth;
2. $E_0 \sqsubseteq_c E$.

Proof. Let $\Delta(2^{<\omega_1})$ denote the equivalence relation on $2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N}}$ given by

$$(x_1, x_2) \Delta(2^{<\omega_1})(y_1, y_2) \Leftrightarrow (x_1, y_1 \notin \text{WO or } (x_1, x_2) \cong (y_1, y_2)),$$

where $(x_1, x_2) \cong (y_1, y_2)$ indicates the existence of a bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$ ($x_1(m, n) = y_1(\pi(m), \pi(n))$ and $x_2(n) = y_2(\pi(n))$). As noted by Hjorth-Kechris [2], the equivalence relation $\Delta(2^{<\omega_1})$ is not “definably” smooth under appropriate determinacy hypotheses. However, we do have the following:

Lemma 4.6. $\Delta(2^{<\omega_1})$ is GB-smooth.

Proof. Let $\Delta(\omega_1)$ denote the equivalence relation on $2^{\mathbb{N} \times \mathbb{N}}$ given by

$$x \Delta(\omega_1) y \Leftrightarrow (x, y \notin \text{WO or } x \cong y),$$

where $x \cong y$ indicates the existence of a bijection $\pi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n \in \mathbb{N}$ ($x(n) = y(\pi(n))$). Fix a reduction $\varphi : 2^{\mathbb{N} \times \mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ of $\Delta(\omega_1)$ to $\Delta(\mathbb{N}^{\mathbb{N}})$. For each ordinal $\alpha < \omega_1$, let $C_\alpha = \{x \in 2^{\mathbb{N} \times \mathbb{N}} : x \cong \alpha\}$ (where $x \cong \alpha$ indicates that the order type of $x^{-1}(1)$ is α), fix $x_\alpha \in C_\alpha$, and define $\psi_{C_\alpha} : C_\alpha \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by

$$\psi_{C_\alpha}(x_1, x_2) = y \Leftrightarrow (x_1, x_2) \cong (x_\alpha, y).$$

For the remaining $\Delta(\omega_1)$ -class C , define $\psi_C : C \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $\psi_C(x_1, x_2) = 0^\infty$. For each equivalence class C of $\Delta(\omega_1)$, define $\pi_C : C \times 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$\pi_C(x_1, x_2) = (\varphi(x_1), \psi_C(x_1, x_2)),$$

and define $\pi : 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$\pi(x_1, x_2) = \pi_{[x_1]_{\Delta(\omega_1)}}(x_1, x_2).$$

It is clear that π is a reduction of $\Delta(2^{<\omega_1})$ to $\Delta(\mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}})$, so it only remains to show that π is globally Baire measurable. That is, we must show that for every open set $U \subseteq \mathbb{N}^{\mathbb{N}} \times 2^{\mathbb{N}}$, every Polish space X , and every continuous function $f : X \rightarrow 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N}}$, the set $f^{-1}(\pi^{-1}(U))$ is Baire measurable. Towards this end, fix a Borel set $B \subseteq f^{-1}(\text{WO} \times 2^{\mathbb{N}})$ such that $f^{-1}(\text{WO} \times 2^{\mathbb{N}}) \setminus B$ is meager, and set

$$A = \{x \in X : \exists y \in B (f(x) = f(y))\}.$$

It is enough to show that both $f^{-1}(\pi^{-1}(U)) \setminus f^{-1}(\text{WO} \times 2^{\mathbb{N}})$ and $f^{-1}(\pi^{-1}(U)) \cap A$ are Σ_1^1 . The former set is clearly Σ_1^1 , since it is either empty or $X \setminus f^{-1}(\text{WO} \times 2^{\mathbb{N}})$. To see that the latter set is Σ_1^1 , note that the Kunen-Martin theorem (see, for example, Theorem 31.1 of Kechris [7]) ensures the existence of an ordinal $\alpha < \omega_1$ such that $\forall x \in A (\text{proj}_{2^{\mathbb{N} \times \mathbb{N}}}(f(x)) \leq \alpha)$, thus

$$\begin{aligned} f^{-1}(\pi^{-1}(U)) \cap A &= f^{-1}(\pi^{-1}(U) \cap f[A]) \\ &= \bigcup_{\beta \leq \alpha} f^{-1}(\pi_{C_\beta}^{-1}(U) \cap f[A]), \end{aligned}$$

which is clearly Σ_1^1 . \square

As it is well known that E_0 is not BP-smooth, it is enough to show $\neg(2) \Rightarrow (1)$. Theorem 1 of Hjorth-Kechris [2] ensures that if $E_0 \not\sqsubseteq_c E$, then there is a Δ_2^1 measurable reduction φ of E to $\Delta(2^{<\omega_1})$. Our assumption that $\forall x \in \mathbb{R} (x^\sharp \text{ exists})$ ensures that Σ_1^1 determinacy holds, so all Σ_2^1 sets are Baire measurable, thus φ is globally Baire measurable. By Lemma 4.6, there is a globally Baire measurable reduction ψ of $\Delta(2^{<\omega_1})$ to $\Delta(\mathbb{N}^{\mathbb{N}})$, and since the composition of globally Baire measurable functions is globally Baire measurable, the theorem follows. \square

We can now establish case (c) of our main theorem:

Theorem 4.7. *Assume that $\forall x \in \mathbb{R} (x^\sharp \text{ exists})$. Suppose that E is a Σ_1^1 treeable equivalence relation on a Polish space X . Then exactly one of the following holds:*

1. E admits a Π_1^1 transversal;
2. $E_0 \sqsubseteq_c E$.

Proof. To see $(1) \Rightarrow \neg(2)$, fix a Π_1^1 transversal C of E , and define $\pi : X \rightarrow C$ by

$$\pi(x) = y \Leftrightarrow (xEy \text{ and } y \in C).$$

Then π is Δ_2^1 measurable, thus our assumption that $\forall x \in \mathbb{R} (x^\sharp \text{ exists})$ implies that π is globally Baire measurable, and Theorem 4.5 implies that condition (2) fails.

It only remains to show $\neg(2) \Rightarrow (1)$. Theorem 4.5 implies that if $E_0 \not\sqsubseteq_c E$, then E is GB-smooth. By Theorem 4.3, there are \mathcal{G} -convex Borel sets $B_n \subseteq X$ such that:

1. $\bigcup_{n \in \mathbb{N}} B_n$ is an E -complete section;
2. $\forall n \in \mathbb{N} (B_n \setminus \bigcup_{m < n} [B_m]_E \text{ is a partial quasi-transversal of } E)$.

Fix a linear ordering \leq of X , and for each $n \in \mathbb{N}$, set

$$C_n = \{x \in B_n : \forall y \in B_n (xEy \Rightarrow x \leq y)\}.$$

Then the set $C = \bigcup_{n \in \mathbb{N}} C_n \setminus \bigcup_{m < n} [B_m]_E$ is the desired Π_1^1 transversal of E . \square

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