On the existence of invariant probability measures for Borel actions of countable semigroups

Benjamin D. Miller

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Abstract

We consider the problem of characterizing the circumstances under which a Borel action of a countable semigroup on a Polish space admits an invariant probability measure, and we prove that aperiodic Borel actions of countable semigroups generically lack invariant probability measures.

1 Introduction

Suppose that X is a Polish space, G is a countable semigroup of Borel functions on X, and μ is a (Borel) probability measure on X. For each $g \in G$, we use $g_* \mu$ to denote the measure on X given by $g_* \mu(B) = \mu(g^{-1}(B))$, and we say that μ is *G-invariant* if $g_* \mu = \mu$, for all $g \in G$.

The *orbits* of the action of G are the sets of the form $[x]_G = \{g \cdot x : g \in G\}$. A *compression* of the action of G is a partition $\langle B_g \rangle_{g \in G}$ of X for which the function $\pi = \bigcup_{g \in G} g | B_g$ is an injection such that $X \setminus \pi[X]$ intersects every G-orbit. (We break slightly with tradition here, as it is usually the map π itself which is called a *compression*.) We say that the action of G is *compressible* if it admits a Borel compression. Nadkarni has asked whether the following remarkable theorem can be generalized to semigroup actions of $\mathbb{N}$:

Theorem 1 (Nadkarni). *Suppose that X is a Polish space and G is a countable group of Borel automorphisms of X. Then exactly one of the following holds:*

1. *The action of G admits an invariant probability measure;*
2. *The action of G is compressible.*

It is not difficult to see that the notion of compressibility itself has little to do with the inexistence of invariant probability measures for semigroup actions. The proper interpretation of Nadkarni's question then is whether there is a property of semigroup actions which syntactically resembles compressibility, agrees with compressibility on group actions, easily rules out the existence of invariant probability measures for semigroup actions and, in fact, characterizes the inexistence of invariant probability measures. Here we suggest a notion which could conceivably satisfy these criteria. In the process, we discuss some of the obstacles that any such notion must overcome, we prove a theorem of independent interest regarding the generic inexistence of invariant probability measures, and we pose a new question about compressibility of Borel automorphisms.

A *redundant cover* of X is a sequence $\langle B_i \rangle_{i \in I}$ of Borel subsets of X such that, for every $x \in X$, there exist infinitely many $i \in I$ for which $x \in B_i$. A *spreading* of the action of G is a sequence $\langle B_g \rangle_{g \in G}$ of pairwise disjoint subsets of X such that $\langle g^{-1}(B_g) \rangle_{g \in G}$ is a redundant cover of X. We say that the action of G is *spreadable* if it admits a Borel spreading. It is straightforward to check that universally measurable spreadings rule out the existence of invariant probability measures.

In §2, we consider the question of whether Baire category can be used to distinguish spreadability from the inexistence of invariant probability measures. We say that an action of G is *aperiodic* if all of its orbits are infinite. We say that a property P *holds generically* of the action of G if there is a comeager G-invariant Borel set $C \subseteq X$ such that property P holds of the action of G on C. By an unpublished result of Kechris (which extends a result of Wright [2]), every aperiodic Borel action of a countable group is generically compressible. It follows that any notion which characterizes the inexistence of invariant probability measures must hold generically of every aperiodic Borel action of a countable group. The following fact both generalizes Kechris's theorem and shows that spreadability passes this test:

Theorem 2. *Every aperiodic Borel action of a countable semigroup on a Polish space is generically spreadable.*

In addition to a Kuratowski-Ulam argument in the style of Kechris's original argument, our proof of Theorem 2 uses an elementary generalization of the "marker lemma" from aperiodic countable Borel equivalence relations to transitive Borel subsets of the plane with countably infinite vertical sections.

In §3, we restrict our attention to actions of semigroups by Borel automorphisms. We note first the following fact:

Theorem 3. *Suppose that X is a Polish space and G is a semigroup of Borel automorphisms of X. Then the following are equivalent:*

1. *The action of G is compressible;*
2. *The action of G is spreadable.*

This implies that spreadability characterizes the inexistence of invariant probability measures for group actions. Unfortunately, the corresponding fact for semigroup actions by automorphisms remains open. In the special case that $G = \mathbb{N}$, however, this leads to an interesting question about compressibility.

Suppose that $T : X \to X$ is a Borel automorphism. We say that T is *compressible* if the corresponding action of $\mathbb{Z}$ is compressible, and that T is *forward compressible* if the corresponding action of $\mathbb{N}$ is compressible. By Theorems 1 and 3, the question of whether spreadability characterizes the inexistence of invariant probability measures for actions of $\mathbb{N}$ by Borel automorphisms is equivalent to the following:

Question 4. *Suppose that X is a Polish space and $T : X \to X$ is a Borel automorphism. Is T compressible iff T is forward compressible?*

Through an elementary argument, we reduce this to the following:

Question 5. *Suppose that X is a Polish space and $T : X \to X$ is a Borel automorphism. Is T forward compressible iff T^{-1} is forward compressible?*

By Theorem 2, there is not a generic negative answer to Question 5. The following fact implies that there is not a measure-theoretic negative answer:

Theorem 6 (MA). *Suppose that X is a Polish space and $T : X \to X$ is a Borel automorphism. Then exactly one of the following holds:*

1. *There is a T-invariant probability measure;*
2. *There is a universally measurable forward compression of T.*

Taken together, these results seem to provide fairly strong evidence that spreadability should characterize the inexistence of invariant probability measures, at least for semigroup actions by Borel automorphisms. It should be noted, however, that even the following basic question remains open:

Question 7. *Is the action of $\mathbb{N}$ induced by the shift on $[\mathbb{N}]^{\mathbb{N}}$ spreadable?*

2 Generic spreadability

Here we prove that aperiodic Borel actions of semigroups are generically spreadable. Before getting to our main result, we prove a generalization of the "marker lemma" for equivalence relations. The *vertical sections* of $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y : (x, y) \in R\}$, and we say that a set $B \subseteq Y$ is an *R-complete section* if it intersects every vertical section of R.

Proposition 8. *Suppose that X is a Polish space and $R \subseteq X \times X$ is a transitive Borel set all of whose vertical sections are countably infinite. Then there are Borel R-complete sections $B_0 \supseteq B_1 \supseteq \cdots$ such that $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$.*

It will be convenient to prove instead a slight rephrasing of Proposition 8. We use "$\exists^\infty n$" as shorthand for "there exist infinitely many n."

Proposition 9. *Suppose that X is a Polish space and $R \subseteq X \times X$ is a transitive Borel set all of whose vertical sections are countably infinite. Then there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of pairwise disjoint Borel sets such that*

$$\forall x \in X \, \exists^\infty n \in \mathbb{N} \; (R_x \cap B_n \neq \emptyset).$$

Proof. Fix a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of X such that

$$\forall x, y \in X \, \exists^\infty n \in \mathbb{N} \, (x \in U_n \text{ and } y \notin U_n).$$

Set $B_0 = \emptyset$. Given $B_n \subseteq X$, set $X_n = X \setminus \bigcup_{i \leq n} B_i$ and

$$V_n = \{x \in X : |R_x \cap (X_n \setminus U_n)| = \aleph_0\},$$

and define $B_{n+1} = X_n \setminus (U_n \Delta V_n)$.

Lemma 10. *For all $x \in X$ and $n \in \mathbb{N}$, the set $R_x \cap X_n$ is infinite.*

Proof. We proceed by induction. The case $n = 0$ is a triviality, so suppose that we have shown that each $R_x \cap X_n$ is infinite. Note that for all $y \in X$,

$$\begin{aligned} R_y \cap X_{n+1} &= (R_y \cap X_n) \setminus B_{n+1} \\ &= (R_y \cap X_n) \setminus (X_n \setminus (U_n \Delta V_n)) \\ &= (R_y \cap X_n) \cap (U_n \Delta V_n) \\ &= (R_y \cap X_n) \cap ((U_n \setminus V_n) \cup (V_n \setminus U_n)). \end{aligned}$$

There are now two cases.

If $R_x \nsubseteq V_n$, then fix $y \in R_x \setminus V_n$, note that $R_y \cap V_n = \emptyset$, and observe that

$$\begin{aligned} R_x \cap X_{n+1} &\supseteq R_y \cap X_{n+1} \\ &= (R_y \cap X_n) \cap ((U_n \setminus V_n) \cup (V_n \setminus U_n)) \\ &= (R_y \cap X_n) \cap (U_n \setminus V_n) \\ &= R_y \cap X_n \cap U_n. \end{aligned}$$

As our assumption that $y \in R_x \setminus V_n$ implies that $R_y \cap X_n \cap U_n$ is infinite, it follows that $R_x \cap X_{n+1}$ is infinite.

If $R_x \subseteq V_n$, then

$$\begin{aligned} R_x \cap X_{n+1} &= (R_x \cap X_n) \cap ((U_n \setminus V_n) \cup (V_n \setminus U_n)) \\ &= (R_x \cap X_n) \cap (V_n \setminus U_n) \\ &= R_x \cap (X_n \setminus U_n). \end{aligned}$$

As R_x is non-empty, R is transitive, and $R_x \subseteq V_n$, it follows that $x \in V_n$, so $R_x \cap (X_n \setminus U_n)$ is infinite, thus $R_x \cap X_{n+1}$ is infinite. □

Now suppose, towards a contradiction, that there exists $m \in \mathbb{N}$ such that $R_x \cap B_n = \emptyset$, for all $n > m$. By Lemma 10, we can fix $y \in R_x \cap X_m$ and $z \in R_y \cap X_m$, as well as $n > m$ such that $y \in U_n$ and $z \notin U_n$. As $y, z \in X_n$, our assumption that $y \notin B_{n+1}$ then ensures that $y \notin V_n$, so $z \notin V_n$, thus $z \in B_{n+1}$, the desired contradiction. □

We are now ready for the main result of this section:

Theorem 11. *Suppose that X is a Polish space and G is a countable semigroup of Borel endomorphisms of X which acts aperiodically. Then there is a G-invariant comeager Borel set on which the action of G is spreadable.*

Proof. Set $R = \bigcup_{g \in G} \mathrm{graph}(g)$, and note that by Proposition 9 there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of pairwise disjoint Borel subsets of X such that

$$\forall x \in X \, \exists^\infty n \in \mathbb{N} \; (R_x \cap B_n \neq \emptyset).$$

Let $\mathbb{P}$ denote the Polish space of all injections of G into $\mathbb{N}$. For each $p \in \mathbb{P}$ and $g \in G$, set $B^p_g = B_{p(g)}$ and define a G-invariant Borel set $C_p \subseteq X$ by setting

$$C_p = \{x \in X : \forall g \in G \; \exists^\infty h \in G \; (g \cdot x \in h^{-1}(B^p_h))\}.$$

Lemma 12. *There exists $p \in \mathbb{P}$ such that C_p is comeager.*

Proof. For each $g \in G$, $n \in \mathbb{N}$, and $x \in X$, let $U_{x,g,n}$ denote the set of $p \in \mathbb{P}$ for which there is a sequence $\langle h_i \rangle_{i<n}$ of pairwise distinct elements of G such that $g \cdot x \in h_i^{-1}(B^p_{h_i})$, for all $i < n$. As $U_{x,g,n}$ is clearly open dense, it follows that the set $V_x = \bigcap_{g \in G, n \in \mathbb{N}} U_{x,g,n}$ is comeager, for all $x \in X$. We use the shorthand "$\forall^* x$" to denote "for comeagerly many x." As

$$\forall x \in X \, \forall^* p \in \mathbb{P} \, \forall g \in G \, \exists^\infty h \in G \; (g \cdot x \in h^{-1}(B_h)),$$

the Kuratowski-Ulam Theorem (see, for example, §8 of Kechris [1]) implies that

$$\forall^* p \in \mathbb{P} \, \forall^* x \in X \, \forall g \in G \, \exists^\infty h \in G \; (g \cdot x \in h^{-1}(B_h)),$$

or equivalently, $\forall^* p \in \mathbb{P}$ (C_p is comeager). ☐

Fix $p \in \mathbb{P}$ such that C_p is comeager, and observe that $\langle B^p_g \rangle_{g \in G}$ is a spreading of the action of G on C_p. ☐

3 Spreadability of semigroups of automorphisms

In this section, we examine the spreadability of actions of semigroups of Borel automorphisms. We begin with the following fact:

Proposition 13. *Suppose that X is a Polish space and G is a countable semigroup of Borel automorphisms of X. Then the following are equivalent:*

1. *The action of G is spreadable;*
2. *The action of G is compressible.*

Proof. To see (1) ⇒ (2), suppose that $\langle A_g \rangle_{g \in G}$ is a spreading of the action of G, and fix a Borel function $\varphi : X \to G$ such that $\varphi(x) \cdot x \in A_{\varphi(x)}$, for all $x \in X$. Set $B_g = \varphi^{-1}(g)$, and observe that $\langle B_g \rangle_{g \in G}$ is a compression of the action of G.

To see (2) ⇒ (1), we note first a pair of lemmas:

Lemma 14. *Suppose that there is a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of elements of G and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of pairwise disjoint Borel subsets of X such that $\langle g_n^{-1}(A_n) \rangle_{n \in \mathbb{N}}$ is a redundant cover of X. Then the action of G is spreadable.*

Proof. For each $g \in G$, define $B_g \subseteq X$ by

$$B_g = \bigcup \{A_n : n \in \mathbb{N} \text{ and } g_n = g\}.$$

It is clear that these sets are pairwise disjoint. For each $x \in X$, there are infinitely many natural numbers $n_0, n_1, \ldots$ such that $x \in g_{n_i}^{-1}(A_{n_i})$. As the A_{n_i} are pairwise disjoint, it follows that the g_{n_i} are pairwise distinct. As $x \in g_{n_i}^{-1}(B_{g_{n_i}})$, for each $i \in \mathbb{N}$, it follows that $\langle B_g \rangle_{g \in G}$ is a spreading of the action of G. $\square$

We say that $B \subseteq X$ is a *G-complete section* if it intersects every orbit of G.

Lemma 15. *Suppose that there is a Borel G-complete section $B \subseteq X$, a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of elements of G, and a sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ of pairwise disjoint Borel subsets of X such that $\langle g_n^{-1}(A_n) \rangle_{n \in \mathbb{N}}$ is a redundant cover of B. Then the action of G is spreadable.*

Proof. Fix an enumeration $\langle h_n \rangle_{n \in \mathbb{N}}$ of G, as well as a bijection $\langle \cdot, \cdot \rangle$ of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$. For each $i, j, k \in \mathbb{N}$, define $A'_{ijk} \subseteq X$ by

$$A'_{ijk} = \{x \in A_i : \langle j, k \rangle = |\{\ell \leq i : g_\ell g_i^{-1} \cdot x \in A_\ell\}|\},$$

and set $g'_{ijk} = g_i h_k$. By Lemma 14, to see that the action of G is spreadable, it only remains to check that the sequence $\langle (g'_{ijk})^{-1}(A_{ijk}) \rangle_{i,j,k \in \mathbb{N}}$ is a redundant cover of X. Towards this end, note that for each $x \in X$, there exists $k \in \mathbb{N}$ such that $h_k \cdot x \in B$. Then for each $j \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $g_i h_k \cdot x \in A'_{ijk}$, thus $x \in (g'_{ijk})^{-1}(A'_{ijk})$. $\square$

Suppose now that $\langle B_g \rangle_{g \in G}$ is a compression of the action of G, let $\pi = \bigcup_{g \in G} g | B_g$, set $A = X \setminus \pi[X]$, and fix an enumeration $\langle g_n \rangle_{n \in \mathbb{N}}$ of G. For each $i, j \in \mathbb{N}$, define $A_{ij} \subseteq X$ by

$$A_{ij} = \{\pi^i(x) : x \in A \text{ and } j \text{ is least such that } \pi^i(x) = g_j \cdot x\},$$

and set $g_{ij} = g_j$. As $\langle g_{ij}^{-1}(A_{ij}) \rangle_{i,j \in \mathbb{N}}$ is a redundant cover of A, it follows that the action of G is spreadable. $\square$

As a corollary, we immediately obtain:

Theorem 16. *Suppose that X is a Polish space and G is a countable group of Borel automorphisms of X. Then exactly one of the following holds:*

1. *The action of G admits an invariant probability measure;*

2. *The action of G is spreadable.*

Proof. This follows from Theorem 1 and Proposition 13. □

Unfortunately, the generalization of Theorem 16 to semigroups of automorphisms remains open, although there are some interesting things to be said about the case $G = \mathbb{N}$.

Proposition 17. *Suppose that X is a Polish space and $T : X \to X$ is a compressible Borel automorphism. Then there is a partition of X into T-invariant Borel sets $A, B \subseteq X$ such that both $T|A$ and $T^{-1}|B$ are forward compressible.*

Proof. Fix a compression $\langle C_n \rangle_{n \in \mathbb{Z}}$ of the action of $\mathbb{Z}$ induced by T, put $\pi = \bigcup_{n \in \mathbb{N}} T^n | C_n$, and set

$$A' = \{x \in X \setminus \pi[X] : \exists^\infty n \in \mathbb{N} \, \exists i \in \mathbb{Z}^+ \ (\pi^n(x) = T^i(x))\}.$$

Set $A = \bigcup_{n \in \mathbb{Z}} T^n[A']$, $B = X \setminus A$, and

$$B' = \{x \in B \setminus \pi[B] : \exists^\infty n \in \mathbb{N} \, \exists i \in \mathbb{Z}^+ \ (\pi^n(x) = T^{-i}(x))\}.$$

For each $x \in A'$, set $k_0(x) = 0$ and recursively define

$$k_{n+1}(x) = \min\{k \in \mathbb{N} : \exists i \in \mathbb{Z}^+ \ (\pi^k(x) = T^i \circ \pi^{k_n(x)}(x))\}.$$

For each $x \in B'$, set $\ell_0(x) = 0$ and recursively define

$$\ell_{n+1}(x) = \min\{\ell \in \mathbb{N} : \exists i \in \mathbb{Z}^+ \ (\pi^\ell(x) = T^{-i} \circ \pi^{\ell_n(x)}(x))\}.$$

Finally, define $A_n, B_n \subseteq X$ by

$$A_n = \{x \in X : \exists y \in A' \, \exists i \in \mathbb{N} \ (x = \pi^{k_i(y)}(y) \text{ and } n = k_{i+1}(y) - k_i(y))\}$$

and

$$B_n = \{x \in X : \exists y \in B' \, \exists i \in \mathbb{N} \ (x = \pi^{\ell_i(y)}(y) \text{ and } n = \ell_{i+1}(y) - \ell_i(y))\}.$$

Then, off of a set where the orbit equivalence relation induced by T is smooth, the sequences $\langle A_n \rangle_{n \in \mathbb{N}}$ and $\langle B_n \rangle_{n \in \mathbb{N}}$ are forward compressions of $T|A$ and $T^{-1}|B$, respectively. □

This reduces the question of whether Theorem 16 generalizes to actions of $\mathbb{N}$ by Borel automorphisms to the following:

Question 18. *Suppose that X is a Polish space and $T : X \to X$ is a forward compressible Borel automorphism. Is T^{-1} forward compressible?*

In the measure-theoretic setting, this question has a positive answer. This is a consequence of the (proof of) the following fact:

Theorem 19 (`MA`). *Suppose that X is a Polish space and $T : X \to X$ is a Borel automorphism. Then exactly one of the following holds:*

1. *There is a T-invariant probability measure;*

2. *There is a universally measurable forward compression of T.*

Proof. To see (2) $\Rightarrow \neg$(1) suppose, towards a contradiction, that μ is a T-invariant probability measure and $\pi : X \to X$ is a μ-measurable forward compression of T. Set $B = X \setminus \pi[X]$, and observe that $\langle \pi^n[B] \rangle_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel sets of μ-measure $\mu(B)$, thus $\mu(B) = 0$. It then follows that $\mu(X) \leq \sum_{n \in \mathbb{Z}} \mu(T^n[B]) = 0$, the desired contradiction.

In order to show $\neg$(1) $\Rightarrow$ (2), assume that there is no T-invariant probability measure. We note first the following:

Lemma 20. *For every probability measure μ on X, there is a T-invariant, μ-conull Borel set $C \subseteq X$ such that $T|C$ is forward compressible.*

Proof. By Theorem 1.8 of Zakrzewski [3], there is a Borel set $A \subseteq X$ whose T-saturation $B = \bigcup_{n \in \mathbb{Z}} T^n[A]$ is of μ-positive measure, as well as an infinite set $S \subseteq \mathbb{N}$ such that $\langle T^n[A] \rangle_{n \in S}$ is a sequence of pairwise disjoint sets. For each $n \in S$, set $A_n = T^n[A]$ and $g_n = T^n$, and observe that $\langle g_n^{-1}(A_n) \rangle_{n \in S}$ is a redundant cover of A, thus Lemma 15 ensures that $T|B$ is forward compressible. By repeating this argument countably many times and taking the union of the resulting sets, we obtain the desired μ-conull Borel set. □

Now fix an enumeration $\langle \mu_\alpha \rangle_{\alpha < \mathfrak{c}}$ of the probability measures on X, and fix a sequence $\langle B_\alpha \rangle_{\alpha < \mathfrak{c}}$ of pairwise disjoint, T-invariant Borel sets such that each of the restrictions $T|B_\alpha$ is forward compressible and for each $\alpha < \mathfrak{c}$, the set $\bigcup_{\beta \leq \alpha} B_\beta$ is μ_α-conull. Fix forward compressions π_α of $T|B_\alpha$, and let π be any forward compression of T which agrees with π_α on B_α. It is now easily verified that π is the desired universally measurable forward compression of T. □

References

[1] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[2] J.D.M. Wright. Generic countable equidecomposability. *Quart. J. Math. Oxford Ser. (2)*, 42(165):125–128, 1991.

[3] Piotr Zakrzewski. Measures on algebraic-topological structures. In *Handbook of measure theory, Vol. I, II*, pages 1091–1130. North-Holland, Amsterdam, 2002.