A REDUCIBILITY LEMMA

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Abstract. At the request of Kechris, we prove a technical lemma involved with weak reducibility.

Lemma 1. *Suppose that E, F are countable Borel equivalence relations on uncountable Polish spaces X, Y and $E \leq_B F$. Then for every Borel subequivalence relation E' of E, there is a Borel subequivalence relation F' of F such that $E' \sim_B F'$.*

Proof. Fix a Borel reduction $\pi : X \to Y$ of E into F. By the Lusin-Novikov uniformization theorem, there is a partition of X into Borel sets $X_n \subseteq X$ on which π is injective. Define $Z \subseteq X$ by

$$Z = \{x \in X : \forall n \in \mathbb{N} \ (X_n \cap [x]_{E'} \text{ is empty or infinite})\}.$$

For each $n \in \mathbb{N}$, set $Z_n = X_n \cap Z$ and define F_n on $\pi(Z_n)$ by

$$\pi(z) F_n \pi(z') \Leftrightarrow z E' z'.$$

Then F_n is an aperiodic countable Borel equivalence relation on $\pi(Z_n)$, so Proposition 7.4 of Kechris-Miller [1] implies that there is a Borel subequivalence relation F_n' of F_n, all of whose classes are of cardinality 2^{n+1}. Fix a Borel linear ordering $\leq$ of Y, and let $\varphi_n : \pi(Z_n) \to \pi(Z_n)$ be the map which sends $y \in \pi(Z_n)$ to the $\leq$-minimal element of $[y]_{F_n'}$. Note that $\varphi_n \circ \pi(Z_n)$ is of measure at most $1/2^{n+1}$ with respect to every F-invariant probability measure on Y.

By repeatedly appealing to Proposition 7.4 of Kechris-Miller [1], we can find Borel sets $X = B_0 \supseteq B_1 \supseteq \cdots$ and fixed-point free Borel involutions $i_n : B_n \to B_n$ such that B_{n+1} consists of exactly one point from each i_n-orbit. Then the sets $i_0(B_1), i_1(B_2), \ldots$ are pairwise disjoint, and $i_n(B_{n+1})$ is of measure exactly $1/2^{n+1}$ with respect to every F-invariant probability measure on Y.

By the proof of Lemma 7.10 of Kechris-Miller [1], there is an F-invariant Borel set $C \subseteq Y$ on which F is compressible, off of which we can find Borel injections $\psi_n : \pi(Z_n) \setminus C \to i_n(B_{n+1}) \setminus C$ such that graph$(\psi_n) \subseteq F$. As $F|C$ is compressible, there are injections $\psi_n' : C \to C$ whose graphs are contained in F and whose ranges are pairwise disjoint.

Define now $\theta : Z \to Y$ by

$$\theta(z) = \begin{cases} \psi_n \circ \varphi_n \circ \pi(z) & \text{if } z \in Z_n \text{ and } \pi(z) \notin C, \\ \psi_n' \circ \pi(z) & \text{if } z \in Z_n \text{ and } \pi(z) \in C. \end{cases}$$

Since $\forall z, z' \in Z \ (\theta(z) = \theta(z') \Rightarrow z E' z')$, we can define F' on $\theta(Z)$ by

$$\theta(z) F' \theta(z') \Leftrightarrow z E' z'.$$

Then θ is a reduction of $E'|Z$ into F', thus $E'|Z \sim_B F'$. As $E'|(X \setminus Z)$ is smooth, it then follows that either: (1) $E'|Z$ is non-smooth, in which case $E' \sim_B E'|Z \sim_B F' \sim_B F' \cup \Delta(Y \setminus \theta(Z))$, or (2) $E'|Z$ is smooth, in which case E' is smooth, so the lemma trivializes. $\square$

References

[1] A. Kechris and B. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2004)