BASES FOR NOTIONS OF RECURRENCE

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Abstract. We investigate the existence of non-trivial bases for actions of locally-compact Polish groups satisfying a broad array of recurrence properties.

Introduction

Given an ordered family $(\mathcal{F}, \preceq)$ of mathematical structures and an upward-closed property Φ of structures in $\mathcal{F}$, a *basis* for the family $\mathcal{F}_\Phi = \{F \in \mathcal{F} \mid \Phi(F)\}$ under $\preceq$ is a set $\mathcal{B} \subseteq \mathcal{F}_\Phi$ with the property that $\forall F \in \mathcal{F}_\Phi \exists B \in \mathcal{B}\ B \preceq F$. Singleton bases are particularly useful, as their existence ensures that satisfying Φ is equivalent to containing a copy of a canonical structure. Even when there are no small bases, the existence of a basis consisting solely of particularly simple structures nevertheless yields substantial insight into the nature of Φ. Here we show that this is the case for myriad properties of actions of locally-compact Polish groups, including non-smoothness, the inexistence of suitably-large weakly-wandering Borel sets, and weak mixing.

In §1, we introduce the actions in our bases. In the special case of $\mathbb{Z}$-actions, these are made up of actions induced by transformations obtained via cutting and stacking with a sufficiently quickly growing number of insertions at each stage. In order to endow our actions with appropriate topologies and handle groups other than $\mathbb{Z}$, we use quotients associated with cocycles to generalize the cutting and stacking construction to produce continuous actions of non-compact locally-compact Polish groups G on locally-compact Polish spaces that are *minimal*, in the sense that their orbits are dense. We refer to these actions as being obtained through *expansive cutting and stacking*. More generally, we define *continuous disjoint unions* of such actions.

In §2, we consider a generalized notion of recurrence. Given $d \in \mathbb{Z}^+$ and a binary relation R on a set X, we say that a sequence $x \in X^{\{0,\ldots,d\}}$ is *R-discrete* if there do not exist distinct $i, j \leq d$ for which

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$x_i \mathrel{R} x_j$. The *orbit relation* on X associated with an action $G \curvearrowright X$ and a set $K \subseteq G$ is given by $x \mathrel{R_K^X} y \iff x \in Ky$. For all sets $R \subseteq \bigcup_{d \in \mathbb{Z}^+} X^{\{0,\ldots,d\}}$, define $\Delta_G^X(R) = \{g \in \bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}} \mid \exists x \in X \ \overline{g}x \in R\}$, where $\overline{g} \in G^{\{0,\ldots,d\}}$ is the extension of g given by $\overline{g}_0 = 1_G$. Given a family $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, we say that a set $Y \subseteq X$ is *expansively $\mathcal{S}$-transient* if there exist a compact set $K \subseteq G$ and $S \in \mathcal{S}$ for which $\Delta_G^X(\{y \in \bigcup_{d \in \mathbb{Z}^+} Y^{\{0,\ldots,d\}} \mid y \text{ is } R_K^X\text{-discrete}\}) \cap S = \emptyset$. We say that a G-action by homeomorphisms of a topological space is *expansively $\mathcal{S}$-recurrent* if no non-empty open set is expansively $\mathcal{S}$-transient, and a Borel G-action on a standard Borel space X is *σ-expansively $\mathcal{S}$-transient* if X is a union of countably-many expansively-$\mathcal{S}$-transient Borel sets. Every minimal continuous G-action on a Polish space is either expansively $\mathcal{S}$-recurrent or σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient. Given a family $\boldsymbol{\mathcal{S}}$ of subsets of $\mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, we say that a Borel G-action on a standard Borel space is *σ-expansively $\boldsymbol{\mathcal{S}}$-transient* if it is σ-expansively $\mathcal{S}$-transient for some $\mathcal{S} \in \boldsymbol{\mathcal{S}}$. A *homomorphism* from $G \curvearrowright X$ to $G \curvearrowright Y$ is a function $\phi \colon X \to Y$ with the property that $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$, a *stabilizer-preserving homomorphism* is a homomorphism whose restriction to each orbit is injective, and an *embedding* is an injective homomorphism.

Building on arguments of [Wei84], we show that if $\mathcal{S}$ is a non-empty countable family, then among all non-σ-expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient Borel G-actions on Polish spaces, those obtained via expansive cutting and stacking form a basis under continuous embeddability. Similarly, we show that if $\boldsymbol{\mathcal{S}}$ is a family of non-empty countable families, then among all non-σ-expansively-$\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient Borel G-actions on Polish spaces, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under continuous stabilizer-preserving homomorphism.

Building on arguments of [EHW98], we show that if $\mathcal{S}$ is a non-empty countable family and $G \curvearrowright X$ is a non-σ-expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient Borel action on a Polish space, then there is a family $\mathcal{F}$ of $2^{\aleph_0}$-many non-σ-expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient Borel actions on Polish spaces such that every action in $\mathcal{F}$ admits a continuous embedding into $G \curvearrowright X$, but every Borel G-action on a standard Borel space admitting a Borel stabilizer-preserving homomorphism to at least two actions in $\mathcal{F}$ is σ-expansively $\{G\}$-transient. Building on this, we show that if $\boldsymbol{\mathcal{S}}$ is a family of non-empty countable families and $G \curvearrowright X$ is a non-σ-expansively-$\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient

Borel action on a Polish space, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of non-σ-expansively-$\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}\}$-transient Borel G-actions on Polish spaces that admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

In §3, we turn our attention to actions that are particularly simple from the descriptive-set-theoretic point of view. The *orbit equivalence relation* associated with an action $G \curvearrowright X$ is given by $E_G^X = R_G^X$. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ such that $w \; E \; x \iff \pi(w) \; F \; \pi(x)$ for all $w, x \in X$, a Borel equivalence relation on a standard Borel space is *smooth* if it admits a Borel reduction to equality on a standard Borel space, and a Borel action $G \curvearrowright X$ on a standard Borel space is *smooth* if E_G^X is smooth. It is easy to see that the latter notion is equivalent to σ-expansive $\{G\}$-transience, from which it follows that the family of actions obtained via expansive cutting and stacking is a basis, under continuous embeddability, for the family of all non-smooth Borel G-actions on Polish spaces. This generalizes and strengthens the original Glimm-Effros dichotomy [Gli61, Eff65], as well as the subsequent results of [SW82, Wei84] (and strengthens the corresponding special case of [HKL90]). It also follows that if $G \curvearrowright X$ is a non-smooth Borel action on a Polish space, then there is no basis of cardinality strictly less than $2^{\aleph_0}$, under Borel stabilizer-preserving homomorphism, for the family of non-smooth Borel G-actions on Polish spaces that admit a continuous embedding into $G \curvearrowright X$. This negatively answers Louveau's question as to whether there is a singleton basis, under Borel embeddability, for the family of all non-smooth Borel $\mathbb{Z}$-actions on standard Borel spaces.

In an attempt to salvage the hope underlying Louveau's question, we also consider Borel free G-actions on standard Borel spaces that *contain a basis*, in the sense that their non-smooth G-invariant Borel restrictions form a basis, under Borel embeddability, for the family of all non-smooth Borel free G-actions on standard Borel spaces. We show that this notion is robust, in the sense that it remains unchanged if Borel embedding is replaced with Borel stabilizer-preserving homomorphism. Recalling that the *diagonal product* of $G \curvearrowright X$ and $G \curvearrowright Y$ is the action $G \curvearrowright X \times Y$ given by $g \cdot (x, y) = (g \cdot x, g \cdot y)$, we also show that a Borel free action $G \curvearrowright X$ on a standard Borel space contains a basis if and only if $G \curvearrowright X \times Y$ is non-smooth for every non-smooth Borel free action $G \curvearrowright Y$ on a standard Borel space. Examples of such actions include all continuous free G-actions on compact Polish spaces, as well as all Borel free G-actions on standard Borel spaces that are *invariant* with respect to some Borel probability measure μ on X, in the sense that $\mu = g_* \mu$ for all $g \in G$. Let $\mathbb{s}$ denote the *shift* on the class of $\mathbb{N}$-sequences given by $\mathbb{s}_n(\mathbf{g}) = \mathbf{g}_{n+1}$, and define $\mathrm{IP}(\mathbf{g}) = \{\mathbf{g}^s \mid s \in 2^{<\mathbb{N}}\}$ for all $\mathbf{g} \in G^{\mathbb{N}}$, where $\mathbf{g}^s = \prod_{n<|s|} \mathbf{g}_n^{s(n)}$ for all $s \in 2^{<\mathbb{N}}$. Letting $\boldsymbol{\mathcal{S}}_{cb}$ denote the family of sets of the form $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$, where $\mathbf{g} \in G^{\mathbb{N}}$ is an injective sequence for which $\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}$ is closed and discrete, we show that if G is abelian, then a Borel free G-action on a standard Borel space contains a basis if and only if it is not σ-expansively $\boldsymbol{\mathcal{S}}_{cb}$-transient. It follows that among all Borel free G-actions on Polish spaces that contain a basis, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under Borel stabilizer-preserving homomorphism. It also follows that if $G \curvearrowright X$ is a Borel free action on a Polish space containing a basis, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of Borel free G-actions on Polish spaces that contain a basis and admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

We also consider sets $Y \subseteq X$ that are *weakly wandering*, in the sense that there is an infinite set $S \subseteq G$ such that $g^{-1}Y \cap h^{-1}Y = \emptyset$ for all distinct $g, h \in S$. We say that a set $Y \subseteq X$ is *complete* if $X = GY$, and *σ-complete* if there is a countable set $H \subseteq G$ for which $X = HY$. When $G \curvearrowright X$ is continuous and Y is open, these notions are equivalent. Letting $\boldsymbol{\mathcal{S}}_{ww\sigma}$ denote the family of sets consisting of a single closed discrete infinite subset of G of the form SS^{-1}, and $\boldsymbol{\mathcal{S}}_{\sigma ww}$ denote the family of countable sets of closed discrete infinite subsets of G of the form SS^{-1}, we note that a Borel free G-action on a standard Borel space admits a weakly-wandering σ-complete Borel set if and only if it is σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{ww\sigma}\}$-transient, whereas the underlying space is a union of countably-many weakly-wandering Borel sets if and only if it is σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{\sigma ww}\}$-transient. These notions are the same for minimal continuous free actions, but the latter is strictly weaker outside of the minimal case. Strengthening the earlier measure-theoretic result, we show that the failure of either of these properties ensures that the action in question contains a basis. We also show that if G admits a compatible two-sided-invariant metric, then the failure of either of these properties is strictly stronger than containing a basis. It also follows that among all Borel free G-actions on Polish spaces that do not have one of these properties, those that are continuous disjoint unions of actions obtained via expansive cutting and stacking form a basis under Borel stabilizer-preserving homomorphism. In addition, we show that if $G \curvearrowright X$ is a Borel free action on a Polish space that does not have one of these properties, then there is no countable basis, under Borel stabilizer-preserving homomorphism, for the family of Borel G-actions on Polish spaces that do not have the property and admit a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$. This answers [EHN93, Question 1] concerning the circumstances under which a Borel $\mathbb{Z}$-action on a standard Borel space admits a weakly-wandering σ-complete Borel set.

The main result of [EHN93] is the existence of a Borel $\mathbb{Z}$-action on a standard Borel space that admits neither an invariant Borel probability measure nor a weakly-wandering σ-complete Borel set. Their example is a disjoint union of $2^{\aleph_0}$-many $\mathbb{Z}$-actions obtained via expansive cutting and stacking. We show that there is an example that is itself obtained via expansive cutting and stacking, and retains the advantages of the more recent examples appearing in [Mil04, Tse15, IM17], in that the same straightforward argument not only rules out weakly-wandering σ-complete Borel sets, but also σ-complete Borel sets satisfying still weaker wandering conditions, yielding a structurally simpler negative answer to [EHN93, Question 2].

In §4, we turn our attention towards mixing conditions. An action $G \curvearrowright X$ by homeomorphisms of a topological space is *topologically transitive* if $\Delta_G^X(U \times V) \neq \emptyset$ for all non-empty open sets $U, V \subseteq X$. More generally, such an action is *topologically d-transitive* if $G \curvearrowright X^d$ is topologically transitive. In the special case that $d = 2$, we also say that $G \curvearrowright X$ is *weakly mixing*. Fix a countable dense subset H of G. Setting $\mathcal{S}_{tdt} = H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \ g_{2i+1} = g_1 g_{2i}\}$, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is topologically d-transitive if and only if it is expansively $(\bigcup_{g \in G} g\mathcal{S}_{tdt}g^{-1})$-recurrent. It follows that among all topologically-d-transitive continuous G-actions on Polish spaces with no open orbits, those obtained via expansive cutting and stacking form a basis under continuous embeddability. It also follows that if $G \curvearrowright X$ is a topologically-d-transitive continuous G-action on a Polish space with no open orbits, then there is no basis of cardinality strictly less than $2^{\aleph_0}$, under Borel stabilizer-preserving homomorphism, for the family of topologically-d-transitive continuous G-actions on Polish spaces with no open orbits that admit a continuous embedding into $G \curvearrowright X$.

A Borel action $G \curvearrowright X$ on a standard Borel space is *ergodic* with respect to a Borel measure μ on X if every G-invariant Borel set is μ-conull or μ-null, and *weakly mixing* with respect to μ if $G \curvearrowright X \times X$ is $(\mu \times \mu)$-ergodic. In the spirit of [SW82, Wei84], we show that if G is abelian, then a Borel action $G \curvearrowright X$ on a standard Borel space is weakly mixing with respect to a Polish topology compatible with the Borel structure of X on a G-invariant closed set if and only if it is weakly mixing with respect to a G-invariant σ-finite Borel measure on X.

We also note that if G has a compatible two-sided-invariant metric and $G \curvearrowright X$ is a continuous action on a Polish space with no open orbits satisfying any mixing condition at least as strong as weak mixing, then there is no basis of cardinality strictly less than the additivity of the meager ideal on $\mathbb{R}$, under continuous stabilizer-preserving homomorphism, for the family of continuous G-actions on Polish spaces with no open orbits satisfying the mixing condition and admitting a continuous embedding into $G \curvearrowright X$.

We say that a continuous action $G \curvearrowright X$ on a Polish space with no open orbits is *mildly mixing* if $G \curvearrowright X \times Y$ is topologically transitive for every topologically-transitive continuous action $G \curvearrowright Y$ on a Polish space with no open orbits. Letting $\boldsymbol{\mathcal{S}}_{mm}$ denote the family of sets consisting of a single closed discrete subset of G of the form $g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}$, where $g \in G$ and $\mathbf{g} \in G^{\mathbb{N}}$ is injective, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is mildly mixing if and only if it is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient if and only if there is a non-σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient continuous disjoint union of G-actions obtained via expansive cutting and stacking that admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

A continuous action $G \curvearrowright X$ on a Polish space is *strongly mixing* if $\Delta_G^X(U \times V)$ is co-compact for all non-empty open sets $U, V \subseteq X$. Letting $\boldsymbol{\mathcal{S}}_{sm}$ denote the family of sets consisting of a single closed discrete infinite subset of G, we note that a topologically-transitive continuous G-action on a Polish space with no open orbits is strongly mixing if and only if it is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient if and only if there is a non-σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient continuous disjoint union of G-actions obtained via expansive cutting and stacking that admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.

1. A generalization of cutting and stacking

1.1. **Quotients.** Given a topological space X and an equivalence relation E on X, we endow X/E with the topology consisting of all sets $U \subseteq X/E$ for which $\bigcup U$ is an open subset of X. We begin by noting a sufficient condition under which such quotients are Polish spaces:

Proposition 1.1.1. *Suppose that X is a Polish space and E is an equivalence relation on X for which every E-class is closed, E-saturations of open sets are open, and there is a basis of open sets $U \subseteq X$ such that $\overline{[U]_E} \subseteq [\overline{U}]_E$. Then X/E is a Polish space.*

Proof. The fact that every E-class is closed ensures that X/E is T_1, and the fact that X is second countable implies that so too is X/E, for if $(U_n)_{n \in \mathbb{N}}$ is a basis for X, then $([U_n]_E/E)_{n \in \mathbb{N}}$ is a basis for X/E. To see that X/E is regular, note that if $V \subseteq X/E$ is an open neighborhood of $[x]_E$, then there is an open neighborhood $U \subseteq \bigcup V$ of x such that $\overline{U} \subseteq \bigcup V$ and $\overline{[U]_E} \subseteq [\overline{U}]_E$, in which case $\overline{[U]_E/E} \subseteq \overline{[U]_E}/E \subseteq [\overline{U}]_E/E \subseteq V$, where the first containment follows from the fact that $\overline{[U]_E}$ is E-invariant. The Urysohn metrization theorem (see, for example, [Kec95, Theorem 1.1]) therefore ensures that X/E is metrizable. As the surjection $\pi \colon X \to X/E$ given by $\pi(x) = [x]_E$ is continuous and open, it follows that X/E is Polish (see, for example, [Kec95, Theorem 8.19]). ⊠

In the special case that E is closed and X is locally compact, so too is the quotient:

Proposition 1.1.2. *Suppose that X is a locally-compact Polish space and E is a closed equivalence relation on X for which E-saturations of open sets are open. Then X/E is a locally-compact Polish space.*

Proof. To see that X/E is Hausdorff, note that if $[x]_E$ and $[y]_E$ are distinct elements of X/E, then there are open neighborhoods $U \subseteq X$ of x and $V \subseteq X$ of y whose product is disjoint from E, in which case $[U]_E/E$ and $[V]_E/E$ are disjoint open neighborhoods of $[x]_E$ and $[y]_E$. As the function $\pi \colon X \to X/E$ given by $\pi(x) = [x]_E$ is continuous, it follows that if $U \subseteq X$ is an open set with compact closure, then the set $\pi(\overline{U}) = [\overline{U}]_E/E$ is compact, so $[\overline{U}]_E$ is closed, thus $[U]_E/E$ is an open set with compact closure and $\overline{[U]_E} \subseteq [\overline{U}]_E$, hence X/E is locally compact, and Proposition 1.1.1 ensures that it is Polish. ⊠

Suppose that R and S are binary relations on X and Y. A *homomorphism* from R to S is a function $\phi \colon X \to Y$ for which $(\phi \times \phi)(R) \subseteq S$, a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, an *embedding* of R into S is an injective reduction of R to S, and an *isomorphism* of R with S is a surjective embedding of R into S. Note that if G is a group and $G \curvearrowright X$ is an action by homomorphisms from E to E, then it is an action by isomorphisms of E with E, and we obtain an action $G \curvearrowright X/E$ by setting $g \cdot [x]_E = [g \cdot x]_E$ for all $g \in G$ and $x \in X$.

Proposition 1.1.3. *Suppose that G is a topological group, X is a topological space, E is an equivalence relation on X for which the E-saturation of every open set is open, and $G \curvearrowright X$ is a continuous action by homomorphisms from E to E. Then $G \curvearrowright X/E$ is continuous.*

Proof. Suppose that $g \in G$, $x \in X$, and $W \subseteq X/E$ is an open neighborhood of $g \cdot [x]_E$. Then there are open neighborhoods $U \subseteq G$ of g and $V \subseteq X$ of x such that $UV \subseteq \bigcup W$, in which case U and $[V]_E/E$ are open neighborhoods of g and $[x]_E$ for which $U([V]_E/E) \subseteq W$. ⊠

Suppose that G is a group, X is a set, and E is an equivalence relation on X. A function $\rho \colon E \to G$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ for all $x \mathrel{E} y \mathrel{E} z$. This trivially implies that $\rho(x, x) = 1_G$ for all $x \in X$, thus $\rho(x, y) = \rho(y, x)^{-1}$ for all $x \mathrel{E} y$.

More generally, we say that a function $P \colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a *cocycle* if $P(x, z) = P(x, y)g$ for all $x \mathrel{E} y \mathrel{E} z$ and $g \in P(y, z)$. This trivially implies that $1_G \in P(x, x)$ for all $x \in X$, so $P(x, y) = P(y, x)^{-1}$ for all $x \mathrel{E} y$, thus $P(x, z) = gP(y, z)$ for all $x \mathrel{E} y \mathrel{E} z$ and $g \in P(x, y)$.

Let $\mathcal{S}(G)$ denote the set of all subgroups of G. We say that a function $\mathbf{G} \colon X \to \mathcal{S}(G)$ is *compatible* with a cocycle $\rho \colon E \to G$ if $\mathbf{G}_x\rho(x, y) = \rho(x, y)\mathbf{G}_y$ for all $x \mathrel{E} y$, in which case we define $P \colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ by setting $P(x, y) = \rho(x, y)\mathbf{G}_y$. Observe that if $x \mathrel{E} y \mathrel{E} z$ and $g \in P(y, z)$, then there exists $h \in \mathbf{G}_z$ for which $g = \rho(y, z)h$, and it follows that $P(x, z) = \rho(x, z)\mathbf{G}_z = \rho(x, y)\rho(y, z)\mathbf{G}_z h = \rho(x, y)\mathbf{G}_y\rho(y, z)h = P(x, y)g$, thus P is a cocycle.

The *orbit cocycle* on E^X_G associated with an action $G \curvearrowright X$ is given by $P^X_G(x, y) = \{g \in G \mid x = g \cdot y\}$. For each cocycle $P \colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$, define the equivalence relation $E_P \subseteq E$ by $x \mathrel{E_P} y \iff 1_G \in P(x, y)$. Suppose now that $E^X_G \subseteq E$ and $P^X_G(x, y) \subseteq P(x, y)$ for all $x \mathrel{E^X_G} y$. If $g \in G$ and $x \mathrel{E} y$, then the facts that $g \in P(g \cdot x, x)$ and $g^{-1} \in P(y, g \cdot y)$ ensure that $P(g \cdot x, g \cdot y) = gP(x, y)g^{-1}$, so $x \mathrel{E_P} y \Longrightarrow g \cdot x \mathrel{E_P} g \cdot y$, thus $G \curvearrowright X$ is an action by homomorphisms from E_P to E_P. The fact that $g^{-1} \in P(y, g \cdot y)$ also implies that $P(x, g \cdot y) = P(x, y)g^{-1}$, so $[x]_{E_P} = g \cdot [y]_{E_P} \iff 1_G \in P(x, g \cdot y) \iff g \in P(x, y)$, thus P factors over E_P to the orbit cocycle of $G \curvearrowright X/E_P$.

Let $G \curvearrowright G \times X$ denote the action given by $g \cdot (h, x) = (gh, x)$, set $I(G) = G \times G$, identify the product of equivalence relations E on X and F on Y with the equivalence relation on $X \times Y$ given by $(x_1, y_1) \mathrel{(E \times F)} (x_2, y_2) \iff (x_1 \mathrel{E} x_2 \text{ and } y_1 \mathrel{F} y_2)$, and let $\overline{P}$ denote the cocycle on $I(G) \times E$ given by $\overline{P}((g, x), (h, y)) = gP(x, y)h^{-1}$. Clearly $E^{G \times X}_G \subseteq I(G) \times E$. Moreover, if $g \in G$ and $(h, x) \in G \times X$, then $\overline{P}(g \cdot (h, x), (h, x)) = ghP(x, x)h^{-1}$, so $g \in \overline{P}(g \cdot (h, x), (h, x))$, thus $P_G^{G\times X}(g\cdot(h,x),(h,x))\subseteq \overline{P}(g\cdot(h,x),(h,x))$, hence $\overline{P}$ factors over $E_{\overline{P}}$ to the orbit cocycle of $G \curvearrowright (G\times X)/E_{\overline{P}}$.

An equivalence relation on a topological space is *minimal* if its equivalence classes are dense.

Proposition 1.1.4. *Suppose that G is a topological group, X is a topological space, E is a minimal equivalence relation on X for which the E-saturation of every open set is open, and $P\colon E\to \mathcal{P}(G)\setminus\{\emptyset\}$ is a cocycle. Then $G \curvearrowright (G\times X)/E_{\overline{P}}$ is minimal.*

Proof. Suppose that $W\subseteq (G\times X)/E_{\overline{P}}$ is a non-empty G-invariant open set. Then there are non-empty open sets $U\subseteq G$ and $V\subseteq X$ with the property that $U\times V\subseteq \bigcup W$. The fact that $\bigcup W$ is G-invariant then ensures that $G\times V\subseteq \bigcup W$. To see that $G\times X\subseteq \bigcup W$, suppose that $g\in G$ and $x\in X$, fix $y\in V$ such that $x\ E\ y$, fix $h\in gP(x,y)$, and observe that $1_G\in gP(x,y)h^{-1}=\overline{P}((g,x),(h,y))$, so the $E_{\overline{P}}$-invariance of $\bigcup W$ ensures that it contains (g,x). ⊠

When Y is a topological space, we use $\mathcal{F}(Y)$ to denote the family of all closed subsets of Y, equipped with the *Fell topology* generated by the sets of the form $\{F\mid F\cap K=\emptyset\}$ and $\{F\mid F\cap U\neq\emptyset\}$, where $K\subseteq Y$ is compact and $U\subseteq Y$ is open. We say that a function $\phi\colon X\to\mathcal{F}(Y)$ is *upper semi-continuous* if it is continuous with respect to the topology generated by the sets of the former type, and *lower semi-continuous* if it is continuous with respect to the topology generated by the sets of the latter type.

We say that a sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E is *exhaustive* if $E=\bigcup_{n\in\mathbb{N}} E_n$.

Proposition 1.1.5. *Suppose that G is a topological group, X is a topological space, E is an equivalence relation on X, and $P\colon E\to\mathcal{F}(G)$ is a cocycle for which there is an exhaustive increasing sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E such that E_n-saturations of open sets are open and $P\restriction E_n$ is lower semi-continuous for all $n\in\mathbb{N}$. Then $E_{\overline{P}}$-saturations of open sets are open.*

Proof. Suppose that $U\times V\subseteq G\times X$ is an open rectangle. Given $(g,x)\in[U\times V]_{E_{\overline{P}}}$, fix $(h,y)\in U\times V$ for which $(g,x)\ E_{\overline{P}}\ (h,y)$, as well as $n\in\mathbb{N}$ for which $x\ E_n\ y$, and open neighborhoods $U_g,U_h\subseteq G$ of g and h for which $U_g g^{-1}U_h\subseteq U$. As $g^{-1}h\in P(x,y)$, there is an open neighborhood $V_x\times V_y\subseteq X\times V$ of (x,y) with the property that $g^{-1}U_h\cap P(x',y')\neq\emptyset$ for all $(x',y')\in E_n\cap(V_x\times V_y)$. Define $V'_x=V_x\cap[V_y]_{E_n}$, and note that if $(g',x')\in U_g\times V'_x$, then there exists $y'\in V_y$ for which $x'\ E_n\ y'$, and since $g^{-1}U_h\cap P(x',y')\neq\emptyset$, there exists $h'\in g'P(x',y')\cap U$, so $(g',x')\ E_{\overline{P}}\ (h',y')$, thus $U_g\times V'_x\subseteq[U\times V]_{E_{\overline{P}}}$. ⊠

We say that an increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G is *exhaustive* if every compact subset of G is contained in some K_n.

Proposition 1.1.6. *Suppose that G is a locally-compact separable group. Then there is an exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G.*

Proof. Fix a countable dense set $D \subseteq G$ and a non-empty open set $U \subseteq G$ with compact closure. As $D^{-1}g$ is dense—and therefore intersects U—for all $g \in G$, it follows that $G = DU$. Fix an enumeration $(g_n)_{n \in \mathbb{N}}$ of D, set $F_n = \{g_m \mid m \leq n\}$ and $K_n = F_n\overline{U}$ for all $n \in \mathbb{N}$, and observe that if $K \subseteq G$ is compact, then the fact that $K \subseteq DU$ yields $n \in \mathbb{N}$ for which $K \subseteq F_nU \subseteq K_n$. ⊠

For each set $K \subseteq G$ and cocycle $P \colon E \to \mathcal{F}(G)$, define $R_K^X = P^{-1}(\{H \subseteq G \mid H \cap K \neq \emptyset\})$. Note that the relations R_K^X associated with an action and its orbit cocycle coincide. We say that P is $(E_n, K_n)_{n \in \mathbb{N}}$*-expansive* if $R_{K_n}^X \subseteq E_n$ for all $n \in \mathbb{N}$.

Proposition 1.1.7. *Suppose that G is a locally-compact group, X is a Polish space, E is an equivalence relation on X, $P \colon E \to \mathcal{F}(G)$ is a cocycle, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and there is an exhaustive increasing sequence $(E_n)_{n \in \mathbb{N}}$ of closed subequivalence relations of E such that P is $(E_n, K_n)_{n \in \mathbb{N}}$-expansive and $P \restriction E_n$ is upper semi-continuous for all $n \in \mathbb{N}$. Then $E_{\overline{P}}$ is closed.*

Proof. If $((g, x), (h, y)) \in \sim E_{\overline{P}}$, then $x \mathrel{E} y \implies g^{-1}h \notin P(x, y)$. The fact that every topological group is regular (see, for example, [HR79, Theorem 8.4]) yields an open neighborhood $U_g \times U_h \subseteq G \times G$ of (g, h) for which $\overline{U_g^{-1}U_h}$ is compact and $x \mathrel{E} y \implies P(x, y) \cap \overline{U_g^{-1}U_h} = \emptyset$. Fix $n \in \mathbb{N}$ sufficiently large that $U_g^{-1}U_h \subseteq K_n$, as well as an open neighborhood $V_x \times V_y \subseteq X \times Y$ of (x, y) with the property that $P(x', y') \cap \overline{U_g^{-1}U_h} = \emptyset$ for all $(x', y') \in E_n \cap (V_x \times V_y)$, and observe that $(U_g \times V_x) \times (U_h \times V_y)$ is disjoint from $E_{\overline{P}}$. ⊠

We say that an equivalence relation E on a metric space X is *locally generated by continuous actions of compact Polish groups* if X is the union of E-invariant open sets $U \subseteq X$ for which there are compact Polish groups G and continuous actions $G \curvearrowright U$ such that $E \restriction U = E_G^U$. Note that every such equivalence relation is necessarily closed, for if $(x, y) \in \overline{E}$, then it is the limit of a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ of elements of E, and if $U \subseteq X$ is an E-invariant open neighborhood of x for which there is a compact Polish group G and a continuous action $G \curvearrowright U$ such that $E \restriction U = E_G^U$, then by passing to a terminal subsequence, we can assume that $x_n \in U$ for all $n \in \mathbb{N}$, in which case there is a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $g_n \cdot x_n = y_n$ for all $n \in \mathbb{N}$, and by passing to infinite subsequences, we can assume that $(g_n)_{n\in\mathbb{N}}$ converges to some $g \in G$, so $g \cdot x = y$, thus $x \mathrel{E} y$.

Proposition 1.1.8. *Suppose that G is a locally-compact Polish group, X is a metric space, E is an equivalence relation on X, $P \colon E \to \mathcal{F}(G)$ is a cocycle, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and there is an exhaustive increasing sequence $(E_n)_{n\in\mathbb{N}}$ of subequivalence relations of E such that E_n is locally generated by continuous actions of compact Polish groups, P is $(E_n, K_n)_{n\in\mathbb{N}}$-expansive, and $P \restriction E_n$ is upper semi-continuous for all $n \in \mathbb{N}$. Then $\overline{[R]_{E_{\overline{P}}}} \subseteq [\overline{R}]_{E_{\overline{P}}}$ for all sets $R \subseteq G \times X$ with the property that $\overline{\mathrm{proj}_G(R)}$ is compact.*

Proof. Suppose that $(g, x) \in \overline{[R]_{E_{\overline{P}}}}$, and fix a sequence $(g_n, x_n)_{n\in\mathbb{N}}$ of elements of $[R]_{E_{\overline{P}}}$ for which $(g_n, x_n) \to (g, x)$, as well as a sequence $(h_n, y_n)_{n\in\mathbb{N}}$ of elements of R such that $(g_n, x_n) \mathrel{E_{\overline{P}}} (h_n, y_n)$ for all $n \in \mathbb{N}$. By passing to infinite subsequences, we can assume that $(h_n)_{n\in\mathbb{N}}$ converges to some $h \in G$. As the closure of $\{g_n \mid n \in \mathbb{N}\} \cup \{h_n \mid n \in \mathbb{N}\}$ is compact, so too is the closure of $\{g_n^{-1} h_n \mid n \in \mathbb{N}\}$. Fix $m \in \mathbb{N}$ for which the latter set is contained in K_m. As $g_n^{-1} h_n \in P(x_n, y_n)$ for all $n \in \mathbb{N}$, it follows that $x_n \mathrel{E_m} y_n$ for all $n \in \mathbb{N}$. Fix an E_m-invariant open neighborhood $V \subseteq X$ of x, a compact Polish group K, and a continuous action $K \curvearrowright V$ such that $E_K^V = E_m \restriction V$. By passing to terminal subsequences, we can assume that $x_n \in V$ for all $n \in \mathbb{N}$, so there is a sequence $(k_n)_{n\in\mathbb{N}}$ of elements of K such that $y_n = k_n \cdot x_n$ for all $n \in \mathbb{N}$. By passing to infinite subsequences, we can assume that $(k_n)_{n\in\mathbb{N}}$ converges to some $k \in K$, in which case $(y_n)_{n\in\mathbb{N}}$ converges to the point $y = k \cdot x$, so $x \mathrel{E_m} y$ and $(h_n, y_n) \to (h, y)$, thus $(h, y) \in \overline{R}$. To see that $(g, x) \mathrel{E_{\overline{P}}} (h, y)$, note that if $U \subseteq G$ is an open neighborhood of $g^{-1}h$, then $g_n^{-1} h_n \in U$—so $P(x_n, y_n) \cap U \neq \emptyset$—for all but finitely many $n \in \mathbb{N}$. The upper semi-continuity of $P \restriction E_m$ therefore ensures that if $\overline{U}$ is compact, then $P(x, y) \cap \overline{U} \neq \emptyset$, so the local compactness and regularity of G imply that $g^{-1}h \in P(x, y)$. ⊠

Suppose that $P \colon E \to \mathcal{P}(G)$ and $\Sigma \colon F \to \mathcal{P}(G)$. A *homomorphism* from P to Σ is a homomorphism ϕ from E to F such that $P(x, y) \subseteq \Sigma(\phi(x), \phi(y))$ for all $x \mathrel{E} y$, a *reduction* of P to Σ is a reduction ϕ of E to F such that $P(x, y) = \Sigma(\phi(x), \phi(y))$ for all $x \mathrel{E} y$, and an *embedding* of P into Σ is an injective reduction of P to Σ. Given an action $G \curvearrowright Y$ and a function $\phi \colon X \to Y$, define $\phi_G \colon G \times X \to Y$ by $\phi_G(g, x) = g \cdot \phi(x)$.

Proposition 1.1.9. *Suppose that G is a group, X and Y are sets, E is an equivalence relation on X, $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ is a cocycle, and $G \curvearrowright Y$ is an action.*

1. *If $\phi\colon X \to Y$ is a homomorphism from P to P_G^Y, then $\phi_G/E_{\overline{P}}$ is a homomorphism from $G \curvearrowright (G \times X)/E_{\overline{P}}$ to $G \curvearrowright Y$.*
2. *If $\phi\colon X \to Y$ is a reduction of P to P_G^Y, then $\phi_G/E_{\overline{P}}$ is an embedding of $G \curvearrowright (G \times X)/E_{\overline{P}}$ into $G \curvearrowright Y$.*

Proof. If $\phi\colon X \to Y$ is a homomorphism from P to P_G^Y, $g, h \in G$, and $w \mathrel{E} x$, then $\overline{P}((g,w),(h,x)) = gP(w,x)h^{-1} \subseteq gP_G^Y(\phi(w),\phi(x))h^{-1} = P_G^Y(\phi_G(g,w),\phi_G(h,x))$, so ϕ_G is a homomorphism from $\overline{P}$ to P_G^Y, and therefore factors over $E_{\overline{P}}$ to a homomorphism from $\overline{P}/E_{\overline{P}}$ to P_G^Y, thus to a homomorphism from $G \curvearrowright (G \times X)/E_{\overline{P}}$ to $G \curvearrowright Y$.

Similarly, if $\phi\colon X \to Y$ is a reduction of P to P_G^Y, $g, h \in G$, and $w \mathrel{E} x$, then $\overline{P}((g,w),(h,x)) = gP(w,x)h^{-1} = gP_G^Y(\phi(w),\phi(x))h^{-1} = P_G^Y(\phi_G(g,w),\phi_G(h,x))$, so ϕ_G is a reduction of $\overline{P}$ to P_G^Y, and therefore factors over $E_{\overline{P}}$ to an embedding of $\overline{P}/E_{\overline{P}}$ into P_G^Y, thus to an embedding of $G \curvearrowright (G \times X)/E_{\overline{P}}$ into $G \curvearrowright Y$. ⊠

1.2. **Cutting and stacking.** For all $n \in \mathbb{N}$, let $\mathbb{E}_{0,n}(\mathbb{N})$ denote the equivalence relation on $\mathbb{N}^{\mathbb{N}}$ given by $a \mathrel{\mathbb{E}_{0,n}(\mathbb{N})} b \iff \forall m \geq n\ a_m = b_m$, and define $\mathbb{E}_0(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \mathbb{E}_{0,n}(\mathbb{N})$. For all $s \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{<\mathbb{N}}$, define $X_s = \prod_{n<|s|}\{0,\ldots,|s_n|\}$, and for all $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, set $T_{\mathbf{g}} = \bigcup_{n \in \mathbb{N}} X_{\mathbf{g} \upharpoonright n}$ and $X_{\mathbf{g}} = \prod_{n \in \mathbb{N}}\{0,\ldots,|\mathbf{g}_n|\}$, and let $\wp_{\mathbf{g}}$ be the cocycle on $\mathbb{E}_0(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ given by $\wp_{\mathbf{g}}((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) = (\mathbf{g}_n)_k$ for all $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$, and $1 \leq k \leq |\mathbf{g}_n|$. We say that a function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is *compatible* with $\mathbf{g}$ if it is compatible with $\wp_{\mathbf{g}}$. For every such $\mathbf{G}$, define $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ on $\mathbb{E}_0(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ by $\mathbb{P}_{\mathbf{g},\mathbf{G}}(c,d) = \wp_{\mathbf{g}}(c,d)\mathbf{G}_d$, and set $E_{\mathbf{g},\mathbf{G}} = E_{\overline{\mathbb{P}_{\mathbf{g},\mathbf{G}}}}$ and $\mathbb{X}_{\mathbf{g},\mathbf{G}} = (G \times X_{\mathbf{g}})/E_{\mathbf{g},\mathbf{G}}$.

In the special case that $\mathbf{G}$ is the function $\mathbf{1_G}$ with constant value $\{1_G\}$, we use $\mathbb{P}_{\mathbf{g}}$, $E_{\mathbf{g}}$, and $\mathbb{X}_{\mathbf{g}}$ to denote $\mathbb{P}_{\mathbf{g},\mathbf{G}}$, $E_{\mathbf{g},\mathbf{G}}$, and $\mathbb{X}_{\mathbf{g},\mathbf{G}}$. When $G = \mathbb{Z}$ and $\forall n \in \mathbb{N} \forall k < |\mathbf{g}_n|\ (\mathbf{g}_n)_{k+1} > (\overline{\mathbf{g}_n})_k + \sum_{m<n}(\mathbf{g}_m)_{|\mathbf{g}_m|}$, it is not difficult to see that $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ is essentially generated by the automorphism obtained via cutting and stacking with stacks of height $|\mathbf{g}_n| + 1$ and $(\mathbf{g}_n)_{k+1} - 1 - (\overline{\mathbf{g}_n})_k - \sum_{m<n}(\mathbf{g}_m)_{|\mathbf{g}_m|}$ insertions between the k^{th} and $(k+1)^{\text{st}}$ levels of the n^{th} stack.

For all $s \in T_{\mathbf{g}}$, define $\mathbf{g}^s = \prod_{n<|s|}(\overline{\mathbf{g}_n})_{s(n)}$.

Proposition 1.2.1. *Suppose that G is a group, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^n(\mathbf{g})}$, and $s, t \in X_{\mathbf{g} \upharpoonright n}$. Then $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) = \mathbf{g}^s \mathbf{G}_{(0)^n \frown c}(\mathbf{g}^t)^{-1}$.*

Proof. As

$$\begin{aligned}\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) &= \wp_{\mathbf{g}}(s \frown c, t \frown c)\mathbf{G}_{t \frown c} \\ &= \wp_{\mathbf{g}}(s \frown c, (0)^n \frown c)\wp_{\mathbf{g}}((0)^n \frown c, t \frown c)\mathbf{G}_{t \frown c} \\ &= \wp_{\mathbf{g}}(s \frown c, (0)^n \frown c)\mathbf{G}_{(0)^n \frown c}\wp_{\mathbf{g}}((0)^n \frown c, t \frown c),\end{aligned}$$

it is sufficient to show that $\wp_{\mathbf{g}}(s \frown c, (0)^n \frown c) = \mathbf{g}^s$ for all $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^n(\mathbf{g})}$, and $s \in X_{\mathbf{g} \restriction n}$. But if this holds at n, and if $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$, $k \leq |\mathbf{g}_n|$, and $s \in X_{\mathbf{g} \restriction n}$, then

$$\begin{aligned}&\wp_{\mathbf{g}}(s \frown (k) \frown c, (0)^n \frown (0) \frown c) \\ &\quad = \wp_{\mathbf{g}}(s \frown (k) \frown c, (0)^n \frown (k) \frown c)\wp_{\mathbf{g}}((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) \\ &\quad = \mathbf{g}^s(\overline{\mathbf{g}_n})_k \\ &\quad = \mathbf{g}^{s \frown (k)},\end{aligned}$$

so it holds at $n+1$. ⊠

Given a binary relation R on X, we say that a sequence $(X_i)_{i \in I}$ of subsets of X is *R-discrete* if every element of $\prod_{i \in I} X_i$ is R-discrete. For all $n \in \mathbb{N}$, define $\mathrm{IP}(\mathbf{g} \restriction n) = \{\mathbf{g}^s \mid s \in X_{\mathbf{g} \restriction n}\}$. We say that $(\mathbf{g}, \mathbf{G})$ is *$(K_n)_{n \in \mathbb{N}}$-expansive* if $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1} \frown c}$ is $R^G_{\mathrm{IP}(\mathbf{g} \restriction n)^{-1} K_n \mathrm{IP}(\mathbf{g} \restriction n)}$-discrete for all $n \in \mathbb{N}$ and $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$. In the special case that $\mathbf{G} = \mathbf{1}_{\mathbf{G}}$, we say that $\mathbf{g}$ is *$(K_n)_{n \in \mathbb{N}}$-expansive.*

Proposition 1.2.2. *Suppose that G is a topological group, $(K_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact subsets of G, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive. Then $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $(\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}}, K_n)_{n \in \mathbb{N}}$-expansive.*

Proof. Simply observe that if $n \in \mathbb{N}$, $m \geq n$, $c \in X_{\mathfrak{s}^{m+1}(\mathbf{g})}$, $j, k \leq |\mathbf{g}_m|$ are distinct, and $s, t \in X_{\mathbf{g} \restriction m}$, then $(\overline{\mathbf{g}_m})_j \mathbf{G}_{(0)^{m+1} \frown c} \cap (\mathbf{g}^s)^{-1} K_m \mathbf{g}^t (\overline{\mathbf{g}_m})_k = \emptyset$, so $\mathbf{g}^s (\overline{\mathbf{g}_m})_j \mathbf{G}_{(0)^{m+1} \frown c} (\overline{\mathbf{g}_m})_k^{-1} (\mathbf{g}^t)^{-1} \cap K_m = \emptyset$, thus Proposition 1.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown (j) \frown c, t \frown (k) \frown c) \cap K_n = \emptyset$. ⊠

We say that an action of a locally compact Polish group is obtained via *expansive cutting and stacking* if it is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive for some exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G.

Proposition 1.2.3. *Suppose that G is a locally-compact Polish group and $G \curvearrowright X$ is obtained via expansive cutting and stacking. Then X is a locally-compact Polish space and $G \curvearrowright X$ is minimal and continuous.*

Proof. Fix an exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, as well as a continuous function $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ compatible with $\mathbf{g}$ for which $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive and $G \curvearrowright X$ is $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$. As $\mathbb{E}_0(\mathbb{N}) \restriction X_{\mathbf{g}}$ is minimal, Proposition 1.1.4 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is minimal. Proposition 1.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}} \restriction (\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}})$ is continuous for all $n \in \mathbb{N}$. As $(\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}})$-saturations of open sets are open for all $n \in \mathbb{N}$, Proposition 1.1.5 implies that $E_{\mathbf{g},\mathbf{G}}$-saturations of open sets are open, so Proposition 1.1.3 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is continuous. Proposition 1.2.2 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $(\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}}, K_n)_{n \in \mathbb{N}}$-expansive. As $\mathbb{E}_{0,n}(\mathbb{N}) \restriction X_{\mathbf{g}}$ is closed for all $n \in \mathbb{N}$, Proposition 1.1.7 implies that $E_{\mathbf{g},\mathbf{G}}$ is closed. As $G \times X_{\mathbf{g}}$ is a locally-compact Polish space, Proposition 1.1.2 ensures that so too is $\mathbb{X}_{\mathbf{g},\mathbf{G}}$. ⊠

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $RS = \{(x,z) \in X \times Z \mid \exists y \in Y \ x \mathrel{R} y \mathrel{S} z\}$.

Proposition 1.2.4. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a topological space, $K, L \subseteq G$ are compact, $R \subseteq X \times X$ is closed, and $(x,y) \in {\sim} R^X_{K^{-1}} R R^X_L$. Then there are open sets $U_K, U_L \subseteq G$ containing K and L and open neighborhoods $V_x, V_y \subseteq X$ of x and y such that $(V_x \times V_y) \cap R^X_{U_K^{-1}} R R^X_{U_L} = \emptyset$.*

Proof. The fact that R is closed ensures that for all $(g,h) \in K \times L$, there are open neighborhoods $W_{g,h,x}, W_{g,h,y} \subseteq X$ of $g \cdot x$ and $h \cdot y$ such that $R \cap (W_{g,h,x} \times W_{g,h,y}) = \emptyset$. As $G \curvearrowright X$ is continuous, there are open neighborhoods $U_{g,h,x}, U_{g,h,y} \subseteq G$ of g and h and $V_{g,h,x}, V_{g,h,y} \subseteq X$ of x and y such that $U_{g,h,x} V_{g,h,x} \subseteq W_{g,h,x}$ and $U_{g,h,y} V_{g,h,y} \subseteq W_{g,h,y}$. As $K \times L$ is compact, there is a finite set $F \subseteq K \times L$ such that $K \times L \subseteq \bigcup_{(g,h) \in F} U_{g,h,x} \times U_{g,h,y}$. Define $\mathcal{F}' = \{F' \subseteq F \mid L \subseteq \bigcup_{(g,h) \in F'} U_{g,h,y}\}$, $U_K = \bigcup_{F' \in \mathcal{F}'} \bigcap_{(g,h) \in F'} U_{g,h,x}$ and $U_L = \bigcap_{F' \in \mathcal{F}'} \bigcup_{(g,h) \in F'} U_{g,h,y}$, and $V_x = \bigcap_{(g,h) \in F} V_{g,h,x}$ and $V_y = \bigcap_{(g,h) \in F} V_{g,h,y}$. As the fact that $K \times L \subseteq \bigcup_{(g,h) \in F} U_{g,h,x} \times U_{g,h,y}$ implies that $K \subseteq U_K$, and the definition of $\mathcal{F}'$ ensures that $L \subseteq U_L$, it only remains to observe that if $g' \in U_K$, $h' \in U_L$, $x' \in V_x$, and $y' \in V_y$, then there exists $F' \in \mathcal{F}'$ such that $g' \in \bigcap_{(g,h) \in F'} U_{g,h,x}$, as well as $(g,h) \in F'$ for which $h' \in U_{g,h,y}$, and since $x' \in V_{g,h,x}$ and $y' \in V_{g,h,y}$, it follows that $(g' \cdot x', h' \cdot y') \notin R$. ⊠

A *homomorphism* from $\rho \colon E \to G$ to $\Sigma \colon F \to \mathcal{P}(G)$ is a homomorphism from the function $P \colon E \to \mathcal{P}(G)$ given by $P(w,x) = \{\rho(w,x)\}$ to Σ. Given an equivalence relation E on X and a binary relation R on X, we say that a function $\phi \colon X_{\mathbf{g}} \to X$ is *doubly* $(R, (K_n)_{n \in \mathbb{N}})$*-expansive* with respect to a cocycle $P \colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ if it is a homomorphism from $\sim\mathbb{E}_{0,n}(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ to $\sim R^X_{K_n \mathrm{IP}(\mathbf{g}\upharpoonright n)} R R^X_{\mathrm{IP}(\mathbf{g}\upharpoonright n)^{-1} K_n}$ for all $n \in \mathbb{N}$. When R is equality on X, we say that ϕ is *doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive*. We say that $(\mathbf{g}, \mathbf{G})$ is *doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive* if $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1}\frown c}$ is $R^G_{(\mathrm{IP}(\mathbf{g}\upharpoonright n)^{-1}K_n \mathrm{IP}(\mathbf{g}\upharpoonright n))^2}$-discrete for all $n \in \mathbb{N}$ and $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$.

Proposition 1.2.5. *Suppose that* G *is a locally-compact separable group,* $(K_n)_{n\in\mathbb{N}}$ *is an exhaustive increasing sequence of compact subsets of* G*,* $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$*,* E *is an equivalence relation on a set* X*,* $P\colon E \to \mathcal{P}(G) \setminus \{\emptyset\}$ *is a cocycle, and* $\phi\colon X_{\mathbf{g}} \to X$ *is a doubly-*$(K_n)_{n\in\mathbb{N}}$*-expansive homomorphism from* $\wp_{\mathbf{g}}$ *to* P*. Then the function* $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ *given by* $\mathbf{G}_c = P(\phi(c), \phi(c))$ *is compatible with* $\mathbf{g}$*,* $(\mathbf{g}, \mathbf{G})$ *is doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive, and* ϕ *is a reduction of* $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ *to* P*.*

Proof. To see that $\mathbf{G}$ is compatible with $\mathbf{g}$, note that

$$\begin{aligned} \wp_{\mathbf{g}}(c,d)\mathbf{G}_d &= \wp_{\mathbf{g}}(c,d)P(\phi(d),\phi(d)) \\ &= P(\phi(c),\phi(d)) \\ &= P(\phi(c),\phi(c))\wp_{\mathbf{g}}(c,d) \\ &= \mathbf{G}_c\wp_{\mathbf{g}}(c,d). \end{aligned}$$

To see that $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, suppose that $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g})}$, $j, k \le |\mathbf{g}_n|$ are distinct, and $s, t \in X_{\mathbf{g}\upharpoonright n}$. The fact that $P(\phi(s \frown (j) \frown c), \phi(t \frown (k) \frown c))$ and $K_n \mathrm{IP}(\mathbf{g} \upharpoonright n)\mathrm{IP}(\mathbf{g} \upharpoonright n)^{-1}K_n$ are disjoint ensures that so too are $P(\phi(s \frown (j) \frown c), \phi((0)^{n+1} \frown c))$ and $K_n \mathrm{IP}(\mathbf{g} \upharpoonright n)\mathrm{IP}(\mathbf{g} \upharpoonright n)^{-1}K_n P(\phi(t \frown (k) \frown c), \phi((0)^{n+1} \frown c))$. As $\mathbf{g}^r(\overline{\mathbf{g}_n})_i\mathbf{G}_{(0)^{n+1}\frown c} = P(\phi(r \frown (i) \frown c), \phi((0)^{n+1} \frown c))$ for all $(r, i) \in \{(s, j), (t, k)\}$, it follows that $\overline{\mathbf{g}_n}\mathbf{G}_{(0)^{n+1}\frown c}$ is $R^G_{(\mathrm{IP}(\mathbf{g}\upharpoonright n)^{-1}K_n\mathrm{IP}(\mathbf{g}\upharpoonright n))^2}$-discrete.

To see that ϕ is a homomorphism from $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ to P, simply observe that if $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^n(\mathbf{g})}$, and $s, t \in X_{\mathbf{g}\upharpoonright n}$, then $\mathbb{P}_{\mathbf{g},\mathbf{G}}(s \frown c, t \frown c) = \wp_{\mathbf{g}}(s \frown c, t \frown c)P(\phi(t \frown c), \phi(t \frown c)) = P(\phi(s \frown c), \phi(t \frown c))$.

To see that ϕ is a homomorphism from $\sim\mathbb{E}_0(\mathbb{N}) \upharpoonright X_{\mathbf{g}}$ to $\sim E$, note that if $c, d \in X_{\mathbf{g}}$ are $\mathbb{E}_0(\mathbb{N})$-inequivalent but $\phi(c)\ E\ \phi(d)$, then $K_n \cap P(\phi(c), \phi(d)) = \emptyset$ for all $n \in \mathbb{N}$, a contradiction. ⊠

A *homomorphism parameter* for an action $G \curvearrowright X$ of a group by homeomorphisms of a Polish space is a sequence of the form $P = (d^P_X, (\epsilon^P_n)_{n\in\mathbb{N}}, \mathbf{g}^P, \mathcal{V}^P)$, where d^P_X is a compatible complete metric on X, $(\epsilon^P_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, $\mathbf{g}^P \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and $\mathcal{V}^P$ is a countable basis for X.

A *P-code* is a sequence $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, the following hold:

(1) $\forall k \le |\mathbf{g}^P_n|\ (\overline{\mathbf{g}^P_n})_k\overline{\mathbf{V}_{n+1}} \subseteq \mathbf{V}_n$.
(2) $\forall s \in X_{\mathbf{g}^P\upharpoonright\{0,\ldots,n\}}\ \mathrm{diam}_{d^P_X}((\mathbf{g}^P)^s\mathbf{V}_{n+1}) \le \epsilon^P_n$.

Condition (1) yields that if $c \in X_{\mathbf{g}^P}$ and $n \in \mathbb{N}$, then $(\mathbf{g}^P)^{c \restriction (n+1)} \overline{\mathbf{V}_{n+1}} = (\mathbf{g}^P)^{c \restriction n} (\overline{\mathbf{g}_n})_{c(n)} \overline{\mathbf{V}_{n+1}} \subseteq (\mathbf{g}^P)^{c \restriction n} \mathbf{V}_n$, so condition (2) implies that we obtain a continuous function $\phi^{P,\mathbf{V}} \colon X_{\mathbf{g}^P} \to X$ by letting $\phi^{P,\mathbf{V}}(c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} (\mathbf{g}^P)^{c \restriction n} \mathbf{V}_n$.

Proposition 1.2.6. *Suppose that $G \curvearrowright X$ is an action of a group by homeomorphisms of a Polish space, P is a homomorphism parameter, and $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ is a P-code. Then $\phi^{P,\mathbf{V}}$ is a homomorphism from $\rho_{\mathbf{g}^P}$ to P_G^X.*

Proof. Simply observe that

$$\begin{aligned}
\{\rho_{\mathbf{g}^P}&((0)^n \frown (k) \frown c, (0)^n \frown (0) \frown c) \cdot \phi^{P,\mathbf{V}}((0)^n \frown (0) \frown c)\} \\
&= \{(\overline{\mathbf{g}_n^P})_k \cdot \phi^{P,\mathbf{V}}((0)^n \frown (0) \frown c)\} \\
&= \textstyle\bigcap_{m \in \mathbb{N}} (\overline{\mathbf{g}_n^P})_k (\mathbf{g}^P)^{(0)^n \frown (0) \frown c \restriction m} \mathbf{V}_{n+1+m} \\
&= \textstyle\bigcap_{m \in \mathbb{N}} (\mathbf{g}^P)^{(0)^n \frown (k) \frown c \restriction m} \mathbf{V}_{n+1+m} \\
&= \{\phi^{P,\mathbf{V}}((0)^n \frown (k) \frown c)\}
\end{aligned}$$

for all $n \in \mathbb{N}$, $c \in X_{\mathfrak{s}^{n+1}(\mathbf{g}^P)}$, and $k \leq |\mathbf{g}_n^P|$. ⊠

An *embedding parameter* for an action $G \curvearrowright X$ of a σ-compact group by homeomorphisms of a Polish space is a sequence of the form $P = (d_X^P, (\epsilon_n^P)_{n \in \mathbb{N}}, \mathbf{g}^P, (K_n^P)_{n \in \mathbb{N}}, R^P, \mathcal{V}^P)$ with the property that the sequence $P' = (d_X^P, (\epsilon_n^P)_{n \in \mathbb{N}}, \mathbf{g}^P, \mathcal{V}^P)$ is a homomorphism parameter, $(K_n^P)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and R^P is a closed binary relation on X.

A *P-code* is a P'-code $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ such that $\overline{\mathbf{g}_n^P} \mathbf{V}_{n+1}$ is $R^X_{L_n^P} R^P R^X_{L_n^P}$-discrete, where $L_n^P = \mathrm{IP}(\mathbf{g}^P \restriction n)^{-1} K_n^P \mathrm{IP}(\mathbf{g}^P \restriction n)$, for all $n \in \mathbb{N}$.

Proposition 1.2.7. *Suppose that $G \curvearrowright X$ is an action of a σ-compact group by homeomorphisms of a Polish space, P is an embedding parameter, and $\mathbf{V} \in (\mathcal{V}^P)^{\mathbb{N}}$ is a P-code. Then $\phi^{P,\mathbf{V}}$ is doubly $(R^P, (K_n^P)_{n \in \mathbb{N}})$-expansive.*

Proof. If $n \in \mathbb{N}$, $m \geq n$, $c, d \in X_{\mathfrak{s}^{m+1}(\mathbf{g}^P)}$, $s, t \in X_{\mathbf{g}^P \restriction m}$, and $j, k \leq |\mathbf{g}_m^P|$ are distinct, then $\phi^{P,\mathbf{V}}(r \frown (i) \frown b) \in \mathrm{IP}(\mathbf{g}^P \restriction m) (\overline{\mathbf{g}_m^P})_i \mathbf{V}_{m+1}$ for all $i \in \{j, k\}$, $r \in \{s, t\}$, and $b \in \{c, d\}$, in which case $(\phi^{P,\mathbf{V}}(s \frown (j) \frown c), \phi^{P,\mathbf{V}}(t \frown (k) \frown d)) \notin R^X_{K_n^P \mathrm{IP}(\mathbf{g}^P \restriction n)} R^P R^X_{\mathrm{IP}(\mathbf{g}^P \restriction n)^{-1} K_n^P}$, as $\overline{\mathbf{g}_m^P} \mathbf{V}_{m+1}$ is $R^X_{L_m^P} R^P R^X_{L_m^P}$-discrete. ⊠

1.3. **Continuous disjoint unions.** We associate with each function $\mathbf{g} \colon I \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ the set $X_{\mathbf{g}} = \{(i, c) \in I \times \mathbb{N}^{\mathbb{N}} \mid c \in X_{\mathbf{g}(i)}\}$ and the cocycle $\rho_{\mathbf{g}} \colon (= \times \mathbb{E}_0(\mathbb{N})) \restriction X_{\mathbf{g}} \to G$ given by $\rho_{\mathbf{g}}((i, c), (i, d)) =$

$\wp_{\mathbf{g}(i)}(c,d)$. We say that a function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is *compatible* with $\mathbf{g}$ if $\mathbf{G}(i)\colon X_{\mathbf{g}(i)} \to \mathcal{S}(G)$ given by $\mathbf{G}(i)(c) = \mathbf{G}((i,c))$ is compatible with $\mathbf{g}(i)$ for all $i \in I$. In this case we define $\mathbb{P}_{\mathbf{g},\mathbf{G}}\colon ({=}\times\mathbb{E}_0(\mathbb{N})) \restriction X_{\mathbf{g}} \to \mathcal{S}(G)$ by $\mathbb{P}_{\mathbf{g},\mathbf{G}}((i,c),(i,d)) = \mathbb{P}_{\mathbf{g}(i),\mathbf{G}(i)}(c,d)$, and set $E_{\mathbf{g},\mathbf{G}} = E_{\overline{\mathbb{P}_{\mathbf{g},\mathbf{G}}}}$ and $\mathbb{X}_{\mathbf{g},\mathbf{G}} = (G \times X_{\mathbf{g}})/E_{\mathbf{g},\mathbf{G}}$. We say that $(\mathbf{g},\mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$*-expansive* if $(\mathbf{g}(i),\mathbf{G}(i))$ is $(K_n)_{n\in\mathbb{N}}$-expansive for all $i \in I$. We say that an action of a locally compact Polish group is a *continuous disjoint union* of actions obtained via expansive cutting and stacking if it is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where I is a Polish space, $\mathbf{g}\colon I \to (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is continuous, $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{G}$ and continuous, and $(\mathbf{g},\mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-expansive for some exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G.

Proposition 1.3.1. *Suppose that G is a locally-compact Polish group and $G \curvearrowright X$ is a continuous disjoint union of actions obtained via expansive cutting and stacking. Then X is Polish and $G \curvearrowright X$ is continuous.*

Proof. Fix an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G, a Polish space I, a continuous map $\mathbf{g}\colon I \to (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and a continuous function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ compatible with $\mathbf{g}$ for which $(\mathbf{g},\mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-expansive and $G \curvearrowright X$ is $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$. Note that $({=} \times \mathbb{E}_{0,n}(\mathbb{N})) \restriction X_{\mathbf{g}}$ is locally generated by continuous actions of compact groups, $(({=} \times \mathbb{E}_{0,n}(\mathbb{N})) \restriction X_{\mathbf{g}})$-saturations of open sets are open, and Proposition 1.2.1 ensures that $\mathbb{P}_{\mathbf{g},\mathbf{G}} \restriction (({=} \times \mathbb{E}_{0,n}(\mathbb{N})) \restriction X_{\mathbf{g}})$ is continuous for all $n \in \mathbb{N}$. Proposition 1.1.5 ensures that $E_{\mathbf{g},\mathbf{G}}$-saturations of open sets are open. As Proposition 1.2.2 implies that $\mathbb{P}_{\mathbf{g},\mathbf{G}}$ is $(({=}\times\mathbb{E}_{0,n}(\mathbb{N})) \restriction X_{\mathbf{g}}, K_n)_{n\in\mathbb{N}}$-expansive, Proposition 1.1.7 yields that $E_{\mathbf{g},\mathbf{G}}$ is closed, thus $\mathbb{X}_{\mathbf{g},\mathbf{G}}$ is Polish by Propositions 1.1.1 and 1.1.8. Proposition 1.1.3 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is continuous. ⊠

The *stabilizer function* associated with an action $G \curvearrowright X$ is given by $\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\}$ for all $x \in X$.

Proposition 1.3.2. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a Hausdorff space. Then the corresponding stabilizer function is upper semicontinuous.*

Proof. If $K \subseteq G$ is compact, then $\mathrm{Stab}^{-1}(\{F \subseteq G \mid F \cap K = \emptyset\}) = \{x \in X \mid \neg x \mathrel{R_K^X} x\}$, and the latter set is open by Proposition 1.2.4. ⊠

A function $\phi\colon X \to Y$ between topological spaces is *Baire class one* if the pre-image of every open subset of Y is F_σ. In the special case that Y is second countable, this is equivalent to the existence of a sequence

$(F_n)_{n\in\mathbb{N}}$ of closed subsets of X for which the pre-image of every open subset of Y is a union of sets along $(F_n)_{n\in\mathbb{N}}$.

Proposition 1.3.3. *Suppose that X is a topological space, Y is a locally-compact regular second-countable space, and $\phi\colon X \to \mathcal{F}(Y)$ is upper semicontinuous. Then ϕ is Baire class one.*

Proof. If $U \subseteq Y$ is open, then there are compact sets $K_n \subseteq Y$ with the property that $U = \bigcup_{n\in\mathbb{N}} K_n$, so $\phi^{-1}(\{F \subseteq Y \mid F \cap U \neq \emptyset\}) = \bigcup_{n\in\mathbb{N}} \phi^{-1}(\{F \subseteq Y \mid F \cap K_n \neq \emptyset\})$, and the latter set is F_σ. ⊠

Proposition 1.3.4. *Suppose that X is a locally-compact regular space. Then $\{(F_1, F_2) \in \mathcal{F}(X) \times \mathcal{F}(X) \mid F_1 \subseteq F_2\}$ is closed.*

Proof. If $F_1, F_2 \in \mathcal{F}(X)$ but there exists $x \in F_1 \setminus F_2$, then there is an open neighborhood $U \subseteq X$ of x whose closure is compact and disjoint from F_2, so $\{(F_1', F_2') \in \mathcal{F}(X) \times \mathcal{F}(X) \mid F_1' \cap U \neq \emptyset \text{ and } F_2' \cap \overline{U} = \emptyset\}$ is an open neighborhood of (F_1, F_2) disjoint from the set in question. ⊠

A *universal embedding parameter* for a Borel action $G \curvearrowright X$ of a locally-compact Polish group on a Polish space is a sequence of the form $P = (d_G^P, d_X^P, (\epsilon_n^P)_{n\in\mathbb{N}}, (F_n^P)_{n\in\mathbb{N}}, (K_n^P)_{n\in\mathbb{N}}, R^P, \mathcal{U}^P, \mathcal{V}^P, (W_n^P)_{n\in\mathbb{N}})$ for which there is a Polish topology τ on X such that X and (X, τ) have the same Borel sets, $G \curvearrowright (X, \tau)$ is continuous, d_G^P is a compatible complete metric on G, d_X^P is a compatible complete metric on (X, τ), $(\epsilon_n^P)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, $(F_n^P)_{n\in\mathbb{N}}$ is a sequence of closed subsets of (X, τ) such that the pre-image of every open subset of $\mathcal{F}(G)$ under the stabilizer function is a union of sets along $(F_n^P)_{n\in\mathbb{N}}$, $(K_n^P)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G containing 1_G, R^P is a closed binary relation on (X, τ), $\mathcal{U}^P$ is a countable basis for G, $\mathcal{V}^P$ is a countable basis for (X, τ), and $(W_n^P)_{n\in\mathbb{N}}$ is a sequence of dense open subsets of (X, τ) for which the topology $\bigcap_{n\in\mathbb{N}} W_n^P$ inherits from (X, τ) is finer than that it inherits from X.

For all $n \in \mathbb{N}$, $d \in (\mathbb{Z}^+)^n$, and $U \in \prod_{m<n} \mathcal{P}(G)^{\{0,\ldots,d_m\}}$, define $\mathrm{IP}(U) = \{U^s \mid s \in \prod_{m<n}\{0, \ldots, d_m\}\}$, where $U^s = \prod_{m<n}(U_m)_{s_m}$ for all $s \in \prod_{m<n}\{0, \ldots, d_m\}$.

A *P-code* is a pair $(\mathbf{U}, \mathbf{V}) \in (\prod_{n\in\mathbb{N}} \prod_{m<n} (\mathcal{U}^P)^{\{0,\ldots,d_m\}}) \times (\mathcal{V}^P)^{\mathbb{N}}$, where $d \in (\mathbb{Z}^+)^{\mathbb{N}}$, such that for all $n \in \mathbb{N}$, the following hold:

(1) $\forall m < n \forall k \leq d_m\ \overline{((\mathbf{U}_{n+1})_m)_k} \subseteq ((\mathbf{U}_n)_m)_k$.
(2) $\forall m \leq n \forall k \leq d_m\ \mathrm{diam}_{d_G^P}(((\mathbf{U}_{n+1})_m)_k) \leq \epsilon_n^P$.
(3) $\forall s \in \prod_{m\leq n}\{0, \ldots, d_m\}\ \overline{\mathbf{U}_{n+1}^s \mathbf{V}_{n+1}} \subseteq \mathbf{U}_n^{s\upharpoonright n} \mathbf{V}_n$.
(4) $\forall s \in \prod_{m\leq n}\{0, \ldots, d_m\}\ \mathrm{diam}_{d_X^P}(\mathbf{U}_{n+1}^s \mathbf{V}_{n+1}) \leq \epsilon_n^P$.

(5) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\}\ \mathbf{U}^s_{n+1}\mathbf{V}_{n+1} \subseteq W^P_n$.
(6) $\forall s \in \prod_{m \leq n}\{0, \ldots, d_m\} \exists F \in \{F^P_n, {\sim}F^P_n\}\ \mathbf{U}^s_{n+1}\mathbf{V}_{n+1} \subseteq F$.
(7) $(((\mathbf{U}_{n+1})_n)_k \mathbf{V}_{n+1})_{k \leq d_n}$ is $R^X_{L^{P,\mathbf{U}}_n} R^P R^X_{L^{P,\mathbf{U}}_n}$-discrete, where

$$L^{P,\mathbf{U}}_n = \mathrm{IP}(\mathbf{U}_{n+1} \restriction n)^{-1} K^P_n \mathrm{IP}(\mathbf{U}_{n+1} \restriction n).$$

(8) $\forall m \leq n\ 1_G \in ((\mathbf{U}_{n+1})_m)_0$.

Let I_P denote the set of all P-codes. Note that I_P is a closed subset of $(\prod_{n \in \mathbb{N}} \prod_{m<n} (\bigcup_{d \in \mathbb{Z}^+} (\mathcal{U}^P)^{\{0,\ldots,d\}})) \times (\mathcal{V}^P)^{\mathbb{N}}$. Conditions (1) and (2) ensure that we obtain a continuous function $\mathbf{g}^P \colon I_P \to (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ by letting $(\mathbf{g}^P_m(\mathbf{U}, \mathbf{V}))_k$ be the unique element of $\bigcap_{n>m}((\mathbf{U}_n)_m)_k$. Conditions (3) and (4) imply that we obtain a continuous function $\phi^P \colon X_{\mathbf{g}^P} \to (X, d^P_X)$ by letting $\phi^P((\mathbf{U}, \mathbf{V}), c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} \mathbf{U}^{c \restriction n}_n V_n$. Define $\mathbf{G}^P \colon X_{\mathbf{g}^P} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $\mathbf{G}^P = \mathrm{Stab} \circ \phi^P$.

Proposition 1.3.5. *Suppose that $G \curvearrowright X$ is a Borel action of a locally-compact Polish group on a Polish space and P is a universal embedding parameter. Then $\phi^P \colon X_{\mathbf{g}^P} \to X$ and $\mathbf{G}^P$ are continuous.*

Proof. As condition (5) ensures that $\phi^P(X_{\mathbf{g}^P}) \subseteq \bigcap_{n \in \mathbb{N}} W^P_n$, it follows that $\phi^P \colon X_{\mathbf{g}^P} \to X$ is continuous.

To see that $\mathbf{G}^P$ is continuous, note that if $(\mathbf{U}, \mathbf{V})$ is a P-code, $((\mathbf{U}, \mathbf{V}), c) \in X_{\mathbf{g}^P}$, and $U \subseteq \mathcal{F}(G)$ is an open neighborhood of the stabilizer of $\phi^P((\mathbf{U}, \mathbf{V}), c)$, then there exists $n \in \mathbb{N}$ with the property that $\phi^P((\mathbf{U}, \mathbf{V}), c) \in F^P_n$ and $\mathrm{Stab}(F^P_n) \subseteq U$, so condition (6) ensures that $\mathbf{U}^{c \restriction (n+1)}_{n+1} \mathbf{V}_{n+1} \subseteq F^P_n$, thus $\mathbf{G}^P(X_{\mathbf{g}^P} \cap ((\mathcal{N}_{\mathbf{U} \restriction (n+2)} \times \mathcal{N}_{\mathbf{V} \restriction (n+2)}) \times \mathcal{N}_{c \restriction (n+1)})) \subseteq U$. ⊠

2. Transience

2.1. **Basis theorems.** Given a sequence $d \in (\mathbb{Z}^+)^{\mathbb{N}}$, we say that a set $T \subseteq \bigcup_{n \in \mathbb{N}} \prod_{m<n}\{0, \ldots, d_m\}$ is *$\sqsubseteq$-dense* if for all $s \in \bigcup_{n \in \mathbb{N}} \prod_{m<n}[\{0, \ldots, d_m\}$, there exists $t \in T$ such that $s \sqsubseteq t$. For a set $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, we say that a sequence $\mathbf{g} \in \prod_{n \in \mathbb{N}} G^{\{1,\ldots,d_n\}}$ is *$\mathcal{S}$-dense* if for all $S \in \mathcal{S}$, there are densely-many $g \in G$ such that there are $\sqsubseteq$-densely-many $t \in T_{\mathbf{g}}$ for which $g\mathbf{g}^t \mathbf{g}_{|t|}(g\mathbf{g}^t)^{-1} \in S$.

Proposition 2.1.1. *Suppose that G is a topological group, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is $\mathcal{S}$-dense, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{S}(G)$ is compatible with $\mathbf{g}$, and $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive. Then $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is expansively $\mathcal{S}$-recurrent.*

Proof. Suppose that $K \subseteq G$ is compact, $S \in \mathcal{S}$, and $V \subseteq \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is a non-empty open set. Fix $s \in T_{\mathbf{g}}$ and a non-empty open set $U \subseteq G$ for which $U \times \mathcal{N}_s \subseteq \bigcup V$. Fix $n \in \mathbb{N}$ for which $g^{-1}Kg \subseteq K_n$. As $\mathbf{g}$ is $\mathcal{S}$-dense, there exists $g \in U$ and $t \in T_{\mathbf{g}}$, of length at least n, for which $s \sqsubseteq t$ and $g\mathbf{g}^t\mathbf{g}_{|t|}(g\mathbf{g}^t)^{-1} \in S$. Fix $c \in X_{\mathfrak{s}^{|t|+1}(\mathbf{g})}$. Then $\mathbb{P}_{\mathbf{g},\mathbf{G}}((g, t \frown (j) \frown c), (g, t \frown (k) \frown c)) = g\mathbf{g}^t(\overline{\mathbf{g}_{|t|}})_j\mathbf{G}_{(0)^{|t|+1}\frown c}(\overline{\mathbf{g}_{|t|}})_k^{-1}(g\mathbf{g}^t)^{-1}$ for all $j, k \leq |\mathbf{g}_{|t|}|$, by Proposition 1.2.1. But the latter set is disjoint from K whenever $j \neq k$, so the sequence $x \in V^{\{0,\ldots,|\mathbf{g}_{|t|}|\}}$ given by $x_k = [(g, t \frown (k) \frown c)]_{E_{\mathbf{g},\mathbf{G}}}$ is $R_K^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}$-discrete. It only remains to note that $x_k = g\mathbf{g}^t(\mathbf{g}_{|t|})_k(g\mathbf{g}^t)^{-1} \cdot x_0$ for all $1 \leq k \leq |\mathbf{g}_{|t|}|$, thus $\Delta_G^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}(\{y \in V^{\{0,\ldots,|\mathbf{g}_{|t|}|\}} \mid y \text{ is } R_K^{\mathbb{X}_{\mathbf{g},\mathbf{G}}}\text{-discrete}\}) \cap S \neq \emptyset$. ⊠

The following generalization of Pettis's Lemma easily implies that if a continuous action is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, then it is not expansively $\mathcal{S}$-recurrent:

Proposition 2.1.2. *Suppose that $G \curvearrowright X$ is an action of a group by homeomorphisms of a Baire space, $d \in \mathbb{Z}^+$, $R \subseteq X^{\{0,\ldots,d\}}$ is closed, $(V_k)_{k \leq d}$ is a sequence of non-empty open subsets of X, and $B_k \subseteq X$ is comeager in V_k for all $k \leq d$. Then $\Delta_G^X((\prod_{k \leq d} V_k) \setminus R) \subseteq \Delta_G^X((\prod_{k \leq d} B_k) \setminus R)$.*

Proof. If $g \in \Delta_G^X((\prod_{k \leq d} V_k) \setminus R)$, then there exists $x \in \bigcap_{k \leq d}(\overline{g}_k)^{-1}V_k$ such that $\overline{g} \cdot x \notin R$. Fix an open neighborhood $V \subseteq \bigcap_{k \leq d}(\overline{g}_k)^{-1}V_k$ of x with the property that $(\prod_{k \leq d} \overline{g}_k V) \cap R = \emptyset$. As $(\overline{g}_k)^{-1}B_k$ is comeager in $(\overline{g}_k)^{-1}V_k$ for all $k \leq d$, it follows that $\bigcap_{k \leq d}(\overline{g}_k)^{-1}B_k$ is comeager in $\bigcap_{k \leq d}(\overline{g}_k)^{-1}V_k$, and therefore intersects V, from which it follows that $g \in \Delta_G^X((\prod_{k \leq d} B_k) \setminus R)$. ⊠

If $\mathcal{S}$ is conjugation invariant and a continuous action is not σ-expansively $\mathcal{S}$-transient, then it is somewhere expansively $\mathcal{S}$-recurrent:

Proposition 2.1.3. *Suppose that $G \curvearrowright X$ is an action of a group by homeomorphisms of a second-countable topological space whose open subsets are F_σ, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, and X is not a union of countably-many expansively-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient closed sets. Then there is a G-invariant non-empty closed set $C \subseteq X$ such that $G \curvearrowright C$ is expansively $\mathcal{S}$-recurrent.*

Proof. As X is second countable, there is a maximal open set $V \subseteq X$ contained in a union of countably-many expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient closed sets. To see that the G-invariant non-empty closed set $C = {\sim}V$ is as desired, suppose that $W \subseteq C$ is an expansively

$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient open set, and fix an open set $W' \subseteq X$ such that $W = C \cap W'$, as well as closed sets $C_n \subseteq X$ for which $W' = \bigcup_{n \in \mathbb{N}} C_n$. As the sets $C \cap C_n$ are expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, the maximality of V ensures that it contains W', thus $W = \emptyset$. ⊠

Given a binary relation R on X, we say that a point $x \in X$ is *R-expansively $\mathcal{S}$-recurrent* if for all open neighborhoods $V \subseteq X$ of x, compact sets $K \subseteq G$, and $S \in \mathcal{S}$, there exists $g \in S$ such that $x \in \bigcap_{k \leq |g|} (\overline{g}_k)^{-1} V$ and $\overline{g} \cdot x$ is $R^X_{K^{-1}} R R^X_K$-discrete. In the special case that R is equality, we say that x is *expansively $\mathcal{S}$-recurrent*.

Proposition 2.1.4. *Suppose that G is a locally-compact separable group, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is countable, and $G \curvearrowright X$ is an expansively $\mathcal{S}$-recurrent continuous action on a second-countable topological space. Then there are comeagerly-many expansively $\mathcal{S}$-recurrent points.*

Proof. By Proposition 1.1.6, we need only show that if $V \subseteq X$ is a non-empty open set, $K \subseteq G$ is compact, and $S \in \mathcal{S}$, then there exist $g \in S$ and a non-empty open set $W \subseteq \bigcap_{k \leq |g|} (\overline{g}_k)^{-1} V$ for which $\overline{g} W$ is R^X_K-discrete. But this is a straightforward consequence of Proposition 1.2.4. ⊠

Let $\leq_{\text{lex}}$ denote the linear ordering of $\mathbb{N}^{<\mathbb{N}}$ given by $s <_{\text{lex}} t \iff (|s| < |t| \text{ or } (|s| = |t| \text{ and } s_{\delta(s,t)} < t_{\delta(s,t)}))$, where $\delta(s,t)$ is the least natural number for which $s_{\delta(s,t)} \neq t_{\delta(s,t)}$, and let $\langle \cdot \rangle \colon 2^{<\mathbb{N}} \to \mathbb{N}$ denote the isomorphism of $\leq_{\text{lex}} \upharpoonright 2^{<\mathbb{N}}$ with the usual ordering of $\mathbb{N}$. For all $d \in 2^{\mathbb{N}}$, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, and $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$, define both $\mathbf{g} * d \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ and $\mathbf{G} * d \colon X_{\mathbf{g} * d} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $(\mathbf{g} * d)_n = \mathbf{g}_{\langle d \upharpoonright n \rangle}$ and $(\mathbf{G} * d)_c = \mathbf{G}_{\phi_d(c)}$, where $\phi_d \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is given by $\phi_d(b) = \bigoplus_{n \in \mathbb{N}} b_n \frown (0)^{\langle d \upharpoonright (n+1) \rangle - \langle d \upharpoonright n \rangle - 1}$.

Proposition 2.1.5. *Suppose that $G \curvearrowright X$ is a Borel action of a locally-compact Polish group on a Polish space, P is a universal embedding parameter, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is countable, and $G \curvearrowright (X, d^P_X)$ has densely-many R^P-expansively $\mathcal{S}$-recurrent points. Then there is a P-code $(\mathbf{U}, \mathbf{V})$ such that $\mathbf{g}^P(\mathbf{U}, \mathbf{V}) * d$ is $\mathcal{S}$-dense for all $d \in 2^{\mathbb{N}}$.*

Proof. Fix a countable dense set $H \subseteq G$, and define $\mathcal{T} = \bigcup_{h \in H} h^{-1} \mathcal{S} h$.

Lemma 2.1.6. *There exists $\mathbf{T} \in \mathcal{T}^{\mathbb{N}}$ with the property that $\forall d \in 2^{\mathbb{N}} \forall T \in \mathcal{T} \exists^\infty n \in \mathbb{N}\ T = \mathbf{T}_{\langle d \upharpoonright n \rangle}$.*

Proof. Fix an enumeration $(T_n)_{n \in \mathbb{N}}$ of $\mathcal{T}$, as well as a sequence $(k_n)_{n \in \mathbb{N}}$ of natural numbers such that $\forall k \in \mathbb{N} \exists^\infty n \in \mathbb{N}\ k_n = k$, and define $\mathbf{T}_{\langle s \rangle} = T_{k_{|s|}}$ for all $s \in 2^{<\mathbb{N}}$. ⊠

Set $\mathbf{U}_0 = \emptyset$, $\mathbf{s}_0 = \emptyset$, and fix a non-empty set $\mathbf{V}_0 \in \mathcal{V}^P$. We will recursively find $\mathbf{c}_n \in \mathbb{Z}^+$, $\mathbf{s}_n \in \prod_{m<n}\{0,\ldots,\mathbf{c}_m\}$, $\mathbf{g}_n \in G^{\{1,\ldots,\mathbf{c}_n\}}$, sequences $(((\mathbf{U}_{n+1})_m)_k)_{k\leq\mathbf{c}_m, m\leq n}$ of non-empty sets in $\mathcal{U}^P$, and non-empty sets $\mathbf{V}_{n+1} \in \mathcal{V}^P$ such that:

1. $\forall m < n \forall k \leq \mathbf{c}_m$ $\overline{((\mathbf{U}_{n+1})_m)_k} \subseteq ((\mathbf{U}_n)_m)_k$.
2. $\forall m \leq n \forall k \leq \mathbf{c}_m$ $\operatorname{diam}_{d_G^P}(((\mathbf{U}_{n+1})_m)_k) \leq \epsilon_n^P$.
3. $\forall s \in X_{\mathbf{g}\restriction\{0,\ldots,n\}}$ $\overline{\mathbf{U}_{n+1}^s\mathbf{V}_{n+1}} \subseteq \mathbf{g}^{s\restriction n}\mathbf{V}_n$.
4. $\forall s \in X_{\mathbf{g}\restriction\{0,\ldots,n\}}$ $\operatorname{diam}_{d_X^P}(\mathbf{U}_{n+1}^s\mathbf{V}_{n+1}) \leq \epsilon_n^P$.
5. $\forall s \in X_{\mathbf{g}\restriction\{0,\ldots,n\}}$ $\mathbf{U}_{n+1}^s\mathbf{V}_{n+1} \subseteq W_n^P$.
6. $\forall s \in X_{\mathbf{g}\restriction\{0,\ldots,n\}} \exists F \in \{F_n^P, \sim F_n^P\}$ $\mathbf{U}_{n+1}^s\mathbf{V}_{n+1} \subseteq F$.
7. $(((\mathbf{U}_{n+1})_n)_k\mathbf{V}_{n+1})_{k\leq\mathbf{c}_n}$ is $R^X_{L_n^{P,\mathbf{U}}} R^P R^X_{L_n^{P,\mathbf{U}}}$-discrete.
8. $\forall m \leq n \forall k \leq \mathbf{c}_m$ $(\overline{\mathbf{g}_m})_k \in ((\mathbf{U}_{n+1})_m)_k$.
9. $\mathbf{s}_{n+1} = s \frown (0)^{\langle t\rangle - |s|}$, where $t \in 2^{<\mathbb{N}}$, $n+1 = \langle t\rangle$, and s is the $\leq_{\text{lex}}$-least element of $\bigcup_{n\leq|t|}\prod_{m<n}\{0,\ldots,\mathbf{c}_m\}$ such that $\operatorname{supp}(s) \subseteq \{\langle t \restriction \ell\rangle \mid \ell < |t|\}$ but there does not exist $\ell < |t|$ for which $s \sqsubseteq \mathbf{s}_{\langle t\restriction\ell\rangle}$ and $\mathbf{T}_{\langle t\restriction\ell\rangle} = \mathbf{T}_{\langle t\rangle}$.

Suppose that $n \in \mathbb{N}$ and we have already found $\mathbf{c} \restriction n$, $\mathbf{g} \restriction n$, $\mathbf{s} \restriction n+1$, $\mathbf{U}_n$, and $\mathbf{V}_n$. Fix an R^P-expansively $\mathcal{T}$-recurrent point $y_n \in \mathbf{g}^{\mathbf{s}_n}\mathbf{V}_n$, and define $L_n = \operatorname{IP}(\mathbf{g} \restriction n)^{-1}K_n^P\operatorname{IP}(\mathbf{g} \restriction n)$. Then there exists $g_n \in \mathbf{T}_n$ for which $y_n \in \bigcap_{k\leq|g_n|}(\overline{g_n})_k^{-1}\mathbf{g}^{\mathbf{s}_n}\mathbf{V}_n$ and $\overline{g_n} \cdot y_n$ is $R^X_{\mathbf{g}^{\mathbf{s}_n}L_n} R^P R^X_{L_n(\mathbf{g}^{\mathbf{s}_n})^{-1}}$-discrete. Set $\mathbf{c}_n = |g_n|$ and $\mathbf{g}_n = (\mathbf{g}^{\mathbf{s}_n})^{-1}g_n\mathbf{g}^{\mathbf{s}_n}$. Then the point $x_n = (\mathbf{g}^{\mathbf{s}_n})^{-1} \cdot y_n$ is in $\bigcap_{k\leq\mathbf{c}_n}(\overline{\mathbf{g}_n})_k^{-1}\mathbf{V}_n$ and $\overline{\mathbf{g}_n} \cdot x_n$ is $R^X_{L_n} R^P R^X_{L_n}$-discrete. For all $s \in \prod_{m\leq n}\{0,\ldots,\mathbf{c}_m\}$, the regularity of X and the fact that $\mathbf{g}^s \cdot x_n = \mathbf{g}^{s\restriction n}(\overline{\mathbf{g}_n})_{s(n)} \cdot x_n \in \mathbf{g}^{s\restriction n}\mathbf{V}_n$ yield an open neighborhood $W_s \subseteq X$ of $\mathbf{g}^s \cdot x_n$ whose closure is contained in $\mathbf{g}^{s\restriction n}\mathbf{V}_n$ and whose d_X^P-diameter is at most ϵ_n^P, and the continuity of $G \curvearrowright (X, d_X^P)$ yields open neighborhoods $U_{m,s} \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and an open neighborhood $V_s \subseteq X$ of x_n for which $(\prod_{m\leq n} U_{m,s})V_s \subseteq W_s$, in which case the intersections $((\mathbf{U}_{n+1})_m)_k$ of the sets $U_{m,s}$ where $s(m) = k$ and the intersection $\mathbf{V}_{n+1}$ of the sets V_s satisfy conditions (3) and (4). The regularity of G ensures that we can thin down the sets $((\mathbf{U}_{n+1})_m)_k$ to neighborhoods of $(\overline{\mathbf{g}_m})_k$ satisfying conditions (1) and (2). For all $s, t \in \prod_{m<n}\{0,\ldots,\mathbf{c}_m\}$ and $s', t' \in \prod_{m\leq n}\{0,\ldots,\mathbf{c}_m\}$ such that $s'(n) \neq t'(n)$, Proposition 1.2.4 yields open neighborhoods $(U_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and $(V_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{t(m)}$ for all $m < n$, $(U'_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{s'(m)}$ and $(V'_{s,s',t,t'})_m \subseteq G$ of $(\overline{\mathbf{g}_m})_{t'(m)}$ for all $m \leq n$, and $W_{s,s',t,t'} \subseteq X$ of x_n with the property that the product of $(\prod_{m<n}(U_{s,s',t,t'})_m)^{-1}(K_n^P)^{-1}(\prod_{m\leq n}(U'_{s,s',t,t'})_m)W_{s,s',t,t'}$ with $(\prod_{m<n}(V_{s,s',t,t'})_m)^{-1}K_n^P(\prod_{m\leq n}(V'_{s,s',t,t'})_m)W_{s,s',t,t'}$ is disjoint from R^P,

so we obtain sets satisfying condition (7) by replacing $((\mathbf{U}_{n+1})_m)_k$ with its intersection with the sets $(U_{s,s',t,t'})_m$ where $s(m) = k$, $(U'_{s,s',t,t'})_m$ where $s'(m) = k$, $(V_{s,s',t,t'})_m$ where $t(m) = k$, and $(V'_{s,s',t,t'})_m$ where $t'(m) = k$, and $\mathbf{V}_{n+1}$ with its intersection with the sets $W_{s,s',t,t'}$. As the intersection W_n of the sets $(\mathbf{g}^s)^{-1} W_n^P$ for $s \in \prod_{m \leq n} \{0, \ldots, \mathbf{c}_m\}$ is dense, there exists $x'_n \in \mathbf{V}_{n+1} \cap W_n$. For all $s \in \prod_{m \leq n} \{0, \ldots, \mathbf{c}_m\}$, the continuity of $G \curvearrowright X$ yields open neighborhoods $U'_{m,s} \subseteq G$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and $V'_s \subseteq X$ of x'_n for which $(\prod_{m \leq n} U'_{m,s}) V'_s \subseteq W_n^P$, in which case we obtain sets satisfying condition (5) by replacing each $((\mathbf{U}_{n+1})_m)_k$ with its intersection with the sets $U'_{m,s}$ where $s(m) = k$, and $\mathbf{V}_{n+1}$ with its intersection with the sets V'_s. Note that if $s \in X_{\mathbf{g} \restriction \{0,\ldots,n\}}$, then there is a non-empty open set $W'_s \subseteq \mathbf{g}^s \mathbf{V}_{n+1}$ contained in F_n^P or $\sim F_n^P$, and the continuity of $G \curvearrowright X$ yields neighborhoods $U''_{m,s} \subseteq ((\mathbf{U}_{n+1})_m)_{s(m)}$ of $(\overline{\mathbf{g}_m})_{s(m)}$ and a non-empty open set $V''_s \subseteq \mathbf{V}_{n+1}$ for which $(\prod_{m \leq n} U''_{m,s}) V''_s \subseteq W'_s$, so by replacing $((\mathbf{U}_{n+1})_m)_{s(m)}$ with $U''_{m,s}$ and $\mathbf{V}_{n+1}$ with V''_s, we obtain sets satisfying the instance of condition (6) at s. By recursively applying this observation to each $s \in \prod_{m \leq n} \{0, \ldots, \mathbf{c}_m\}$, we obtain sets satisfying condition (6). Replacing each of the sets $((\mathbf{U}_{n+1})_m)_k$ with a subneighborhood of $(\overline{\mathbf{g}_m})_k$ in $\mathcal{U}^P$, and $\mathbf{V}_{n+1}$ with a non-empty subset in $\mathcal{V}^P$ and choosing $\mathbf{s}_{n+1}$ according to condition (9) complete the construction.

Condition (9) implies that $\operatorname{supp}(\mathbf{s}_{\langle t \rangle}) \subseteq \{\langle t \restriction n \rangle \mid n < |t|\}$ for all $t \in 2^{<\mathbb{N}}$, and $\operatorname{supp}(s) \subseteq \{\langle d \restriction n \rangle \mid n \in \mathbb{N}\} \Longrightarrow \exists n \in \mathbb{N}\ (s \sqsubseteq \mathbf{s}_{\langle d \restriction n \rangle}$ and $T = \mathbf{T}_{\langle d \restriction n \rangle}$ for all $d \in 2^{\mathbb{N}}$, $s \in \bigcup_{n \in \mathbb{N}} \prod_{m < n} \{0, \ldots, \mathbf{c}_m\}$, and $T \in \mathcal{T}$.

In order to complete the proof, it only remains to note that $(\mathbf{U}, \mathbf{V})$ is a P-code, $\mathbf{g} = \mathbf{g}^P(\mathbf{U}, \mathbf{V})$, and $\mathbf{g}^P(\mathbf{U}, \mathbf{V}) * d$ is $\mathcal{S}$-dense for all $d \in 2^{\mathbb{N}}$. ⊠

We next characterize σ-expansive $(\bigcup_{g \in G} g \mathcal{S} g^{-1})$-transience:

Theorem 2.1.7. *Suppose that G is a locally-compact Polish group, I is a finite set, $(X_i)_{i \in I}$ is a sequence of Polish spaces, and $(G \curvearrowright X_i)_{i \in I}$ is a sequence of Borel actions such that* $\operatorname{Stab}(x_i) = \operatorname{Stab}(x_j)$ *for all distinct $i, j \in I$, $x_i \in X_i$, and $x_j \in X_j$, and $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is a countable non-empty set. Then the following are equivalent:*

1. *The action $G \curvearrowright \prod_{i \in I} X_i$ is not σ-expansively $(\bigcup_{g \in G} g \mathcal{S} g^{-1})$-transient.*
2. *There is an expansively $\mathcal{S}$-recurrent action, obtained via expansive cutting and stacking, that admits a Baire-measurable stabilizer-preserving homomorphism to each $G \curvearrowright X_i$.*
3. *There is an expansively $\mathcal{S}$-recurrent action, obtained via expansive cutting and stacking, that admits a continuous embedding into each $G \curvearrowright X_i$.*

Proof. Clearly (3) $\Longrightarrow$ (2).

To see $\neg$(1) $\Longrightarrow$ $\neg$(2), observe that if $G \curvearrowright X$ is a continuous action on a Polish space that admits a Baire-measurable stabilizer-preserving homomorphism to each $G \curvearrowright X_i$, then it admits a Baire-measurable stabilizer-preserving homomorphism to $G \curvearrowright \prod_{i \in I} X_i$, and since pullbacks of expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient sets through stabilizer-preserving homomorphisms are expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, it follows that if $G \curvearrowright \prod_{i \in I} X_i$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient, then $G \curvearrowright X$ admits an expansively $\mathcal{S}$-transient non-meager Baire-measurable set, in which case Proposition 2.1.2 ensures that $G \curvearrowright X$ is not expansively $\mathcal{S}$-recurrent.

To see (1) $\Longrightarrow$ (3), appeal to [BK96, Theorem 5.2.1] to obtain a Polish topology τ_i on each X_i for which X_i and (X_i, τ_i) have the same Borel sets and $G \curvearrowright (X_i, \tau_i)$ is continuous, and set $X = \prod_{i \in I} X_i$ and $\tau = \prod_{i \in I} \tau_i$. By Proposition 2.1.3, there is a G-invariant non-empty closed set $C \subseteq (X, \tau)$ such that $G \curvearrowright (C, \tau)$ is expansively $\mathcal{S}$-recurrent, so Proposition 2.1.4 ensures that $G \curvearrowright (C, \tau)$ has comeagerly-many expansively $\mathcal{S}$-recurrent points. Set $R = \bigcup_{i \in I} \{(x, y) \in X \times X \mid x_i = y_i\}$. As $\mathrm{Stab}(x_i) = \mathrm{Stab}(x_j)$ for all distinct $i, j \in I$, $x_i \in X_i$, and $x_j \in X_j$, it follows that every expansively $\mathcal{S}$-recurrent point is R-expansively $\mathcal{S}$-recurrent. As the "identity" function from (C, τ) to C is Borel, and therefore Baire measurable, there is a comeager subset of (C, τ) on which it is continuous, in which case the topology that the comeager subset inherits from τ is finer than that it inherits from X. In particular, it follows that there is a universal embedding parameter P for $G \curvearrowright C$ such that d_C^P is compatible with (C, τ) and $R^P = R$, in which case Proposition 2.1.5 yields a P-code $(\mathbf{U}, \mathbf{V})$ for which $\mathbf{g}^P(\mathbf{U}, \mathbf{V})$ is $\mathcal{S}$-dense. Proposition 1.3.5 ensures that $\mathbf{G}^P(\mathbf{U}, \mathbf{V})$ is continuous, Proposition 1.2.7 implies that each of the functions $\phi_i = \mathrm{proj}_{X_i} \circ \phi^P((\mathbf{U}, \mathbf{V}), \cdot)$ is a doubly-$(K_n^P)_{n \in \mathbb{N}}$-expansive homomorphism from $\mathbb{p}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V})}$ to $P_G^{X_i}$, and Proposition 1.2.5 yields that $\mathbf{G}^P(\mathbf{U}, \mathbf{V})$ is compatible with $\mathbf{g}^P(\mathbf{U}, \mathbf{V})$, $(\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V}))$ is $(K_n^P)_{n \in \mathbb{N}}$-expansive, and each ϕ_i is a reduction of $\mathbb{P}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ to $P_G^{X_i}$. Then $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is obtained via expansive cutting and stacking. Proposition 1.1.9 yields that each $(\phi_i)_G / E_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is an embedding of $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ into $G \curvearrowright X_i$, and Proposition 2.1.1 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U}, \mathbf{V}), \mathbf{G}^P(\mathbf{U}, \mathbf{V})}$ is expansively $\mathcal{S}$-recurrent. ⊠

The *σ-expansive-transience spectrum* of $G \curvearrowright X$ is the family of all countable non-empty sets $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1, \ldots, d\}})$ for which $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient.

Theorem 2.1.8. *Suppose that $G \curvearrowright X$ is a Borel (continuous) action of a locally-compact Polish group on a Polish space. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that has the same σ-expansive-transience spectrum as $G \curvearrowright X$ and admits a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By [BK96, Theorem 5.2.1], it is sufficient to establish the parenthetical (continuous) version of the theorem. Towards this end, fix a universal embedding parameter P for $G \curvearrowright X$ such that d_X^P is compatible with X and R^P is equality on X. Proposition 1.3.5 ensures that ϕ^P and $\mathbf{G}^P$ are continuous, Propositions 1.2.6 and 1.2.7 imply that $\phi^P((\mathbf{U},\mathbf{V}),\cdot)$ is a doubly-$(K_n^P)_{n\in\mathbb{N}}$-expansive homomorphism from $\wp_{\mathbf{g}^P(\mathbf{U},\mathbf{V})}$ to P_G^X for all P-codes $(\mathbf{U},\mathbf{V})$, and Proposition 1.2.5 yields that $\mathbf{G}^P$ is compatible with $\mathbf{g}^P$, $(\mathbf{g}^P,\mathbf{G}^P)$ is $(K_n^P)_{n\in\mathbb{N}}$-expansive, and $\phi^P((\mathbf{U},\mathbf{V}),\cdot)$ is a reduction of $\mathbb{P}_{\mathbf{g}^P(\mathbf{U},\mathbf{V}),\mathbf{G}^P(\mathbf{U},\mathbf{V})}$ to P_G^X for all P-codes $(\mathbf{U},\mathbf{V})$. It follows that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is a continuous disjoint union of actions obtained via expansive cutting and stacking, and Proposition 1.1.9 implies that $(\phi^P)_G/E_{\mathbf{g}^P,\mathbf{G}^P}$ is a stabilizer-preserving homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ to $G \curvearrowright X$. To see that the σ-expansive-transience spectrum of $G \curvearrowright X$ is contained in that of $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$, observe that if $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\dots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is σ-expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient, then the fact that pullbacks of expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient sets through stabilizer-preserving homomorphisms are themselves expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is σ-expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient. To see that the two spectra actually coincide, note that if $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\dots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is not σ-expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient, then Proposition 2.1.5 yields a P-code $(\mathbf{U},\mathbf{V})$ for which $\mathbf{g}^P(\mathbf{U},\mathbf{V})$ is $\mathcal{S}$-dense, in which case Propositions 2.1.1 and 2.1.2 ensure that $G \curvearrowright \mathbb{X}_{\mathbf{g}^P,\mathbf{G}^P}$ is not σ-expansively $(\bigcup_{g\in G} g\mathcal{S}g^{-1})$-transient. ⊠

2.2. **Anti-basis theorems.** When $\mathcal{T} \subseteq \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} X^{\{0,\dots,d\}}) \times \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\dots,d\}})$, we say that a set $Y \subseteq X$ is *$\mathcal{T}$-transient* if there exist $(R,S) \in \mathcal{T}$ for which $\Delta_G^X((\bigcup_{d\in\mathbb{Z}^+} Y^{\{0,\dots,d\}}) \setminus R) \cap S = \emptyset$. We say that a G-action by homeomorphisms of a topological space is *$\mathcal{T}$-recurrent* if no non-empty open set is $\mathcal{T}$-transient, and a Borel G-action on a standard Borel space X is *σ-$\mathcal{T}$-transient* if X is a union of countably-many $\mathcal{T}$-transient Borel sets. Let G act on $\mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} X^{\{0,\dots,d\}}) \times \bigcup_{d\in\mathbb{Z}^+} \mathcal{P}(G^{\{1,\dots,d\}})$ via $g \cdot (R,S) = (gR, gSg^{-1})$.

Associated with each action $G \curvearrowright X$ of a group by homeomorphisms of a topological space is the equivalence relation F_G^X on X given by $x \mathrel{F_G^X} y \iff \overline{Gx} = \overline{Gy}$. Recall that a subset of a topological space is G_δ if its complement is F_σ. A straightforward calculation reveals that if X is a Polish space, then F_G^X is G_δ, so each of its equivalence classes is G_δ, thus Polish (see, for example, [Kec95, Theorem 3.11]).

Proposition 2.2.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact separable group on a Polish space and $\mathcal{T} \subseteq (\bigcup_{d \in \mathbb{Z}^+} \mathcal{F}(X^{\{0,\ldots,d\}})) \times \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is a countable non-empty set. Then exactly one of the following holds:*

1. *The action $G \curvearrowright X$ is σ-$(G\mathcal{T})$-transient.*
2. *There exists $x \in X$ for which $G \curvearrowright [x]_{F_G^X}$ is $\mathcal{T}$-recurrent.*

Proof. Proposition 2.1.2 ensures that conditions (1) and (2) are mutually exclusive. To see $\neg(2) \implies (1)$, note that if $(R, S) \in \mathcal{T}$, $V \subseteq X$ is open, and $x \in X$, then the minimality of $G \curvearrowright [x]_{F_G^X}$ ensures that $V \cap [x]_{F_G^X} = \emptyset \iff V \cap Gx = \emptyset \iff x \notin GV$, so $V \cap [x]_{F_G^X}$ is $\{(R,S)\}$-transient $\iff \bigcup_{g \in S} \{y \in \bigcap_{k \leq |g|} (\overline{g}_k)^{-1} V \mid \overline{g} \cdot y \notin R\} \cap [x]_{F_G^X} = \emptyset \iff \bigcup_{g \in S} \{y \in \bigcap_{k \leq |g|} (\overline{g}_k)^{-1} V \mid \overline{g} \cdot y \notin R\} \cap Gx = \emptyset \iff x$ is not in the G-saturation of $\bigcup_{g \in S} \{y \in \bigcap_{k \leq |g|} (\overline{g}_k)^{-1} V \mid \overline{g} \cdot y \notin R\}$, thus the set of $x \in X$ for which $V \cap [x]_{F_G^X}$ is $\{(R,S)\}$-transient is closed. Fix enumerations $(R_m, S_m)_{m \in \mathbb{N}}$ of $\mathcal{T}$ and $(V_n)_{n \in \mathbb{N}}$ of a basis for X. For all $(m,n) \in \mathbb{N} \times \mathbb{N}$, let $V_{m,n}$ be the set of $x \in V_n$ for which $V_n \cap [x]_{F_G^X}$ is $\{(R_m, S_m)\}$-transient. Fix a countable dense set $H \subseteq G$, and observe that the sets of the form $hV_{m,n}$, where $h \in H$ and $m, n \in \mathbb{N}$, are $(G\mathcal{T})$-transient and cover X. ⊠

We use $\forall^* x \in X\ \phi(x)$ to indicate that $\{x \in X \mid \phi(x)\}$ is comeager, and $\exists^* x \in X\ \phi(x)$ to indicate that $\{x \in X \mid \phi(x)\}$ is not meager. We say that a function $\phi \colon X \to Y$ is *almost a homomorphism* from $G \curvearrowright X$ to $G \curvearrowright Y$ at a point $x \in X$ if $\forall^* g \in G\ \phi(g \cdot x) = g \cdot \phi(x)$.

Proposition 2.2.2. *Suppose that G is a non-empty Baire group, X and Y are sets, $G \curvearrowright X$ and $G \curvearrowright Y$ are actions, $\phi \colon X \to Y$ is a function, and $x \in X$ is a point at which ϕ is almost a homomorphism from $G \curvearrowright X$ to $G \curvearrowright Y$. Then* $\mathrm{Stab}(x) \subseteq \mathrm{Stab}(\phi(x))$.

Proof. Suppose that $h \in \mathrm{Stab}(x)$. As $\{g \in G \mid \phi(gh \cdot x) = gh \cdot \phi(x)\} = \{g \in G \mid \phi(g \cdot x) = g \cdot \phi(x)\}h^{-1}$, there are comeagerly-many $g \in G$ with the property that $\phi(g \cdot x) = g \cdot \phi(x)$ and $\phi(gh \cdot x) = gh \cdot \phi(x)$. As $\phi(g \cdot x) = \phi(gh \cdot x)$, it follows that $g \cdot \phi(x) = gh \cdot \phi(x)$, so $\phi(x) = h \cdot \phi(x)$, thus $h \in \mathrm{Stab}(\phi(x))$. ⊠

We say that ϕ is *almost a stabilizer-preserving homomorphism* from $G \curvearrowright X$ to $G \curvearrowright Y$ if there are comeagerly-many $x \in X$, at which ϕ is almost a homomorphism from $G \curvearrowright X$ to $G \curvearrowright Y$, such that $\mathrm{Stab}(\phi(x)) \subseteq \mathrm{Stab}(x)$.

Proposition 2.2.3. *Suppose that G is a Polish group, X is a Polish space, Y and Z are standard Borel spaces, $G \curvearrowright X$ is a continuous action, $G \curvearrowright Y$ is a Borel action, $\phi\colon Y \to Z$ is a G-invariant Borel function, and R is the set of $z \in Z$ for which there is a Borel function $\psi\colon X \to Y$ that is almost a stabilizer-preserving homomorphism from $G \curvearrowright X$ to $G \curvearrowright \phi^{-1}(\{z\})$. Then R is analytic.*

Proof. Fix a Polish topology on Y with the same Borel sets, as well as compatible metrics d_X and d_Y on X and Y. If $\psi\colon X \to Y$ is a Borel function, then there is a comeager set $C \subseteq X$ on which ψ is continuous (see, for example, [Kec95, Theorem 8.38]), in which case the separability of X yields an enumeration $(x_n)_{n\in\mathbb{N}}$ of a dense subset of C. Then there are trivially comeagerly-many $x \in X$ such that

$(1)_x$ $\forall \delta > 0 \exists n \in \mathbb{N}\ d_X(x, x_n) < \delta$.

Setting $y_n = \psi(x_n)$ for all $n \in \mathbb{N}$, the continuity of $\psi \restriction C$ yields comeagerly-many $x \in X$ for which there exists $y \in Y$ such that

$(2)_{x,y}$ $\forall \epsilon > 0 \exists \delta > 0 \forall n \in \mathbb{N}\ (d_X(x, x_n) < \delta \Longrightarrow d_Y(y, y_n) < \epsilon)$.

If $z \in Z$ and ψ is almost a stabilizer-preserving homomorphism from $G \curvearrowright X$ to $G \curvearrowright \phi^{-1}(\{z\})$, then there are comeagerly-many $x \in X$ for which $(1)_x$ holds and there exists $y \in Y$ such that $(2)_{x,y}$ holds, as do

(3) $\phi(y) = z$ and
(4) $\mathrm{Stab}(y) \subseteq \mathrm{Stab}(x)$.

As $\forall g \in G \forall^* x \in X\ g \cdot x \in C$, the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields that $\forall^* x \in X \forall^* g \in G\ g \cdot x \in C$, so there are comeagerly-many $x \in X$ for which there are comeagerly-many $g \in G$ such that $(1)_x$ and $(1)_{g\cdot x}$ hold and there exist $y, y_g \in Y$ such that $(2)_{x,y}$, $(2)_{g\cdot x, y_g}$, (3), and (4) hold, as does

(5) $g \cdot y = y_g$.

Conversely, suppose that $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$, let C be the set of $x \in X$ for which condition $(1)_x$ holds and there exists $y \in Y$ for which condition $(2)_{x,y}$ holds, and define $\psi\colon C \to Y$ by setting $\psi(x) = y$ if and only if condition $(2)_{x,y}$ holds. If $z \in Z$, there are comeagerly-many $x \in X$ for which there are comeagerly-many $g \in G$ such that $(1)_x$ and $(1)_{g\cdot x}$ hold and there exist $y, y_g \in Y$ such that $(2)_{x,y}$, $(2)_{g\cdot x, y_g}$, (3), (4), and (5) hold, and $B \subseteq C$ is a comeager Borel set consisting of such x, then any Borel extension of $\psi \restriction B$ to X is almost a stabilizer-preserving homomorphism from $G \curvearrowright X$ to $G \curvearrowright \phi^{-1}(\{z\})$.

As $\{(x, y) \in X \times Y \mid \mathrm{Stab}(y) \subseteq \mathrm{Stab}(x)\}$ is Borel (by Propositions 1.3.2–1.3.4), and a result of Novikov's ensures that the pointclass of analytic sets is closed under category quantification (see, for example, [Kec95, Theorem 29.22]), it follows that R is analytic. ⊠

The following observation yields our primary means of producing incompatible actions:

Proposition 2.2.4. *Suppose that G is a locally-compact Polish group, $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G with the property that $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n \in \mathbb{N}}$-expansive, and $d_0, d_1 \in 2^{\mathbb{N}}$ are distinct. Then no expansively-$\{G\}$-recurrent continuous action $G \curvearrowright X$ on a Polish space admits Borel functions $\phi_i \colon X \to \mathbb{X}_{\mathbf{g} * d_i, \mathbf{G} * d_i}$ that are almost stabilizer-preserving homomorphisms from $G \curvearrowright X$ to $G \curvearrowright \mathbb{X}_{\mathbf{g} * d_i, \mathbf{G} * d_i}$ for all $i < 2$.*

Proof. Suppose, towards a contradiction, that there are such functions. Then there is a compact set $L \subseteq G$ with the property that the set $B = \bigcap_{i<2} \phi_i^{-1}(L_i)$ is non-meager, where $L_i = (L \times X_{\mathbf{g} * d_i})/E_{\mathbf{g} * d_i, \mathbf{G} * d_i}$ for all $i < 2$. Fix $m \in \mathbb{N}$ sufficiently large that $L \cup L^{-1}L \subseteq K_m$ and $d_0 \restriction \{0, \ldots, m\} \neq d_1 \restriction \{0, \ldots, m\}$, set $K = K_m \mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, \langle (1)^m \rangle\})$, and fix a non-empty open set $V \subseteq X$ in which B is comeager. As $G \curvearrowright X$ is expansively $\{G\}$-recurrent, there exist $g \in G$ and $x \in V \cap g^{-1}V$ for which $\neg x \; R^X_{KK^{-1}} \; g \cdot x$. By Proposition 1.2.4, there are open neighborhoods $U \subseteq G$ of g and $W \subseteq V$ of x for which $UW \subseteq V$ and $R^X_{KK^{-1}} \cap (W \times UW) = \emptyset$. As $\forall h \in U \forall^* y \in W \; h \cdot y \in B$, the Kuratowski-Ulam theorem ensures that $\forall^* y \in W \forall^* h \in U \; h \cdot y \in B$, so Proposition 2.2.2 yields $h \in U$ and $y \in B \cap h^{-1}B$ with the property that $\neg y \; R^X_{KK^{-1}} \; h \cdot y$, $\mathrm{Stab}(\phi_i(y)) = \mathrm{Stab}(y)$ for all $i < 2$, and $\phi_i(h \cdot y) = h \cdot \phi_i(y)$ for all $i < 2$, thus $P_G^{\mathbb{X}_{\mathbf{g} * d_i, \mathbf{G} * d_i}}(\phi_i(h \cdot y), \phi_i(y)) = h\mathrm{Stab}(\phi_i(y)) = h\mathrm{Stab}(y) = P^X_G(h \cdot y, y)$ for all $i < 2$.

For all $i, j < 2$, fix $g_{i,j} \in L$ and $c_{i,j} \in X_{\mathbf{g} * d_i}$ such that $\phi_i(h^j \cdot y)$ is the $E_{\mathbf{g} * d_i, \mathbf{G} * d_i}$-class of $(g_{i,j}, c_{i,j})$. Note that for all $i < 2$, there exists $m_i \geq m$ for which $c_{i,0}(m_i) \neq c_{i,1}(m_i)$, since otherwise Proposition 1.2.1 ensures that $L\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, \langle (1)^m \rangle\})\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, \langle (1)^m \rangle\})^{-1} L^{-1} \cap P^X_G(y, h \cdot y) \neq \emptyset$, contradicting the fact that $\neg y \; R^X_{KK^{-1}} \; h \cdot y$.

For all $i < 2$, let m_i be the maximal natural number with the property that $c_{i,0}(m_i) \neq c_{i,1}(m_i)$, set $c_i = \mathbb{s}^{m_i+1}(c_{i,0}) = \mathbb{s}^{m_i+1}(c_{i,1})$ and $n_i = \langle d_i \restriction \{0, \ldots, m_i\} \rangle$, and fix $i < 2$ with the property that $n_i > n_{1-i}$. As $L\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, n_{1-i}\})\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, n_{1-i}\})^{-1} L^{-1} \cap P^X_G(y, h \cdot y) \neq \emptyset$

and $P_G^X(y, h \cdot y) \subseteq L\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, n_i - 1\})(\overline{\mathbf{g}_{n_i}})_{(c_{i,0})_{n_i}}(\mathbf{G} * d_i)_{(0)^{m_i+1} \frown c_i}$ $(\overline{\mathbf{g}_{n_i}})^{-1}_{(c_{i,1})_{n_i}}\mathrm{IP}(\mathbf{g} \restriction \{0, \ldots, n_i - 1\})^{-1}L^{-1}$ by Proposition 1.2.1, the fact that $n_i \geq \langle d \restriction \{0, \ldots, m\}\rangle \geq m$ contradicts the double $(K_n)_{n\in\mathbb{N}}$-expansivity of $(\mathbf{g}, \mathbf{G})$. ⊠

We now establish our primary anti-basis results:

Theorem 2.2.5. *Suppose that G is a locally-compact Polish group, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is a non-empty countable set, and $G \curvearrowright X$ is a non-σ-expansively-$\mathcal{S}$-transient Borel action on a standard Borel space. Then there is a family $\mathcal{B}$ of continuum-many G-invariant Borel subsets of X on which $G \curvearrowright X$ is not σ-expansively-$\mathcal{S}$-transient such that every non-σ-expansively-$\{G\}$-transient Borel action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action of the form $G \curvearrowright B$, where $B \in \mathcal{B}$.*

Proof. Fix an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G. By Proposition 2.1.5 and Theorem 2.1.7, we can assume that $G \curvearrowright X$ is of the form $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, where $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ and $\mathbf{g} * d$ is $\mathcal{S}$-dense for all $d \in 2^{\mathbb{N}}$, $\mathbf{G} \colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ is compatible with $\mathbf{g}$ and continuous, and $(\mathbf{g}, \mathbf{G})$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive. Proposition 2.2.4 then ensures that the family $\mathcal{B} = \{(G \times \phi_d(X_{\mathbf{g}*d}))/E_{\mathbf{g},\mathbf{G}} \mid d \in 2^{\mathbb{N}}\}$ is as desired. ⊠

Theorem 2.2.6. *Suppose that G is a locally-compact Polish group, $G \curvearrowright X$ is a Borel action on a standard Borel space, and $\mathcal{F}$ is a countable family of non-σ-expansively-$\{G\}$-transient Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and has the same σ-expansive-transience spectrum as $G \curvearrowright X$, but to which no action in $\mathcal{F}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Proposition 2.2.1, we can assume that each action in $\mathcal{F}$ is continuous and minimal. Fix a universal embedding parameter P, and let R be the set of pairs $((\mathbf{U}, \mathbf{V}), d) \in I_P \times 2^{\mathbb{N}}$ with the property that no action in $\mathcal{F}$ admits a Borel function that is almost a stabilizer-preserving homomorphism to $G \curvearrowright \mathbb{X}_{\mathbf{g}^P(\mathbf{U},\mathbf{V})*d, \mathbf{G}^P(\mathbf{U},\mathbf{V})*d}$.

Proposition 2.2.3 ensures that R is co-analytic, whereas Proposition 2.2.4 implies that every vertical section of R is co-countable. The usual uniformization results for co-analytic sets with large vertical sections (see, for example, [Kec95, Corollary 36.24]) therefore yield a Borel uniformization $\delta \colon I_P \to 2^{\mathbb{N}}$ of R. Define $\mathbf{g} \colon I_P \to (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$

and $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$ by $\mathbf{g}(\mathbf{U}, \mathbf{V}) = \mathbf{g}^P(\mathbf{U}, \mathbf{V}) * \delta(\mathbf{U}, \mathbf{V})$ and $\mathbf{G}(\mathbf{U}, \mathbf{V}) = \mathbf{G}^P(\mathbf{U}, \mathbf{V}) * \delta(\mathbf{U}, \mathbf{V})$.

The usual change-of-topology results (see, for example, [Kec95, §13]) and Proposition 1.3.1 ensure that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is a Borel action on a standard Borel space. Note that if $\phi\colon X_{\mathbf{g}*d} \to X$ is given by $\phi((\mathbf{U}, \mathbf{V}), c) = (\phi_P \circ \phi_{\delta(\mathbf{U},\mathbf{V})})(c)$, then $\overline{\phi}/E_{\mathbf{g},\mathbf{G}}$ is a stabilizer-preserving Borel homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ to $G \curvearrowright X$.

To see that the σ-expansive-transience spectrum of $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is contained in that of $G \curvearrowright X$, note that if $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$ is a countable non-empty set for which $G \curvearrowright X$ is not σ-expansively $\mathcal{S}$-transient, then Proposition 2.1.5 yields $(\mathbf{U}, \mathbf{V}) \in I_P$ for which $\mathbf{g}(\mathbf{U}, \mathbf{V})$ is $\mathcal{S}$-dense, so Proposition 2.1.1 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}(\mathbf{U},\mathbf{V}),\mathbf{G}(\mathbf{U},\mathbf{V})}$ is expansively $\mathcal{S}$-recurrent, thus Proposition 2.1.2 implies that $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ is not σ-expansively $\mathcal{S}$-transient.

To see that none of the actions $G \curvearrowright Y$ in $\mathcal{F}$ admit a Borel function ϕ that is almost a stabilizer-preserving homomorphism to $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, note that the minimality of $G \curvearrowright Y$ would otherwise yield $(\mathbf{U}, \mathbf{V}) \in I_P$ with the property that $\phi^{-1}(\mathbb{X}_{\mathbf{g}(\mathbf{U},\mathbf{V}),\mathbf{G}(\mathbf{U},\mathbf{V})})$ is comeager, contradicting the fact that $((\mathbf{U}, \mathbf{V}), \delta(\mathbf{U}, \mathbf{V})) \in R$. ⊠

3. Wandering

3.1. **Smoothness.** A *transversal* of an action $G \curvearrowright X$ is a set $Y \subseteq X$ containing exactly one point of every orbit. Burgess has shown that a Borel action of a Polish group on a standard Borel space is smooth if and only if it has a Borel transversal [Bur79].

Proposition 3.1.1. *A Borel action $G \curvearrowright X$ of a locally-compact Polish group on a standard Borel space is smooth if and only if it is σ-expansively $\{G\}$-transient.*

Proof. By [BK96, Theorem 5.2.1], we can assume that X is Polish and $G \curvearrowright X$ is continuous. Fix a compatible complete metric d on X.

To see ($\Longrightarrow$), fix a Borel transversal $B \subseteq X$ of $G \curvearrowright X$, and let s be the unique function from X to B whose graph is contained in E^X_G. As the graph of s is Borel, so too is s (see, for example, [Kec95, Theorem 14.12]). It follows that if $K \subseteq G$ is compact, then KB is Borel, for if H is a countable dense subset of K, then $x \in KB \iff x \in Ks(x) \iff \forall \epsilon > 0 \exists h \in H \; d(x, h \cdot s(x)) < \epsilon$ for all $x \in X$. But if $(K_n)_{n \in \mathbb{N}}$ is a sequence of compact subsets of G whose union is G, then $(K_n B)_{n \in \mathbb{N}}$ is a sequence of expansively $\{G\}$-transient Borel sets whose union is X.

To see ($\Longleftarrow$), suppose that $(B_n)_{n \in \mathbb{N}}$ is a sequence of expansively $\{G\}$-transient Borel sets whose union is X, and fix compact sets $K_n \subseteq G$

such that $E_G^X \upharpoonright B_n \subseteq R_{K_n}^X$ for all $n \in \mathbb{N}$. Then the uniformization theorem for Borel subsets of the plane with non-meager vertical sections (see, for example, [Kec95, Corollary 18.7]) ensures that the corresponding sets $C_n = \{x \in X \mid \exists^* g \in G \; g \cdot x \in B_n\}$ are Borel and there are Borel functions $\phi_n \colon C_n \to B_n$ whose graphs are contained in E_G^X. For all $n \in \mathbb{N}$, Proposition 1.2.4 ensures that $\overline{[x]_{E_G^X} \cap B_n} \subseteq [x]_{E_G^X}$ for all $x \in B_n$, which easily implies that $E_G^X \upharpoonright B_n$ is smooth (see, for example, the proof of [Kec95, Theorem 12.16]), thus so too is $G \curvearrowright C_n$. As the sets C_n are G-invariant and $X = \bigcup_{n \in \mathbb{N}} C_n$, it follows that $G \curvearrowright X$ is smooth. ⊠

We now establish our strengthening of the Glimm-Effros dichotomy for Borel actions of locally-compact Polish groups on Polish spaces:

Theorem 3.1.2. *Suppose that $G \curvearrowright X$ is a Borel action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) *The action $G \curvearrowright X$ is not smooth.*
(2) *There is a Baire-measurable stabilizer-preserving homomorphism from a G-action obtained via expansive cutting and stacking to $G \curvearrowright X$.*
(3) *There is a continuous embedding of a G-action obtained via expansive cutting and stacking into $G \curvearrowright X$.*

Proof. As the proof of Proposition 2.1.1 shows that every G-action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 2.1.7 and Proposition 3.1.1. ⊠

We now establish our anti-basis theorem for non-smooth Borel actions of locally-compact Polish groups on standard Borel spaces:

Theorem 3.1.3. *Suppose that $G \curvearrowright X$ a non-smooth Borel action of a locally-compact Polish group on a standard Borel space. Then there is a family $\mathcal{B}$ of continuum-many G-invariant Borel subsets of X on which $G \curvearrowright X$ is non-smooth such that every non-smooth Borel G-action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action of the form $G \curvearrowright B$, where $B \in \mathcal{B}$.*

Proof. Again appealing to the proof of Proposition 2.1.1 to see that every G-action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 2.2.5 and Proposition 3.1.1. ⊠

3.2. **Containing bases.** The following fact is a local refinement of our promised results on the robustness of the property of containing bases and its characterization via diagonal products:

Theorem 3.2.1. *Suppose that $G \curvearrowright X$ and $G \curvearrowright Y$ are Borel free actions of a locally-compact Polish group on Polish spaces. Then the following are equivalent:*

1. *The action $G \curvearrowright X \times Y$ is not smooth.*
2. *There is a Baire-measurable stabilizer-preserving homomorphism from a G-action obtained via expansive cutting and stacking to $G \curvearrowright X$ and $G \curvearrowright Y$.*
3. *There is a continuous embedding of a G-action obtained via expansive cutting and stacking into $G \curvearrowright X$ and $G \curvearrowright Y$.*

Proof. Once more appealing to the proof of Proposition 2.1.1 to see that every action obtained via expansive cutting and stacking is expansively $\{G\}$-recurrent, the desired result follows from Theorem 2.1.7 and Proposition 3.1.1. ⊠

When $\mathbf{g} \in G^{\mathbb{N}}$, we use $X_{\mathbf{g}}$, $E_{\mathbf{g}}$, and $\mathbb{X}_{\mathbf{g}}$ to denote $X_{\mathbf{h}}$, $E_{\mathbf{h}}$, and $\mathbb{X}_{\mathbf{h}}$, where $\mathbf{h} \in (G^{\{1\}})^{\mathbb{N}}$ is given by $(\mathbf{h}_n)_1 = \mathbf{g}_n$ for all $n \in \mathbb{N}$. In light of Theorems 3.1.2 and 3.2.1, the fact that every homomorphism between free actions is stabilizer preserving, and the fact that there is a continuous embedding of $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ whenever $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$, $(k_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, and $s_n \in T_{\mathbf{g}}$ is supported on $[k_n, k_{n+1})$ for all $n \in \mathbb{N}$, the following fact ensures that continuous free actions of locally-compact Polish groups on compact Polish spaces contain bases:

Proposition 3.2.2. *Suppose that G is a locally-compact Polish group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $G \curvearrowright X$ is a continuous action on a compact Polish space, and $\mathbf{g} \in (\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is $(K_n)_{n\in\mathbb{N}}$-expansive. Then there exist a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers, sequences $s_n \in T_{\mathbf{g}}$ with non-trivial support contained in $[k_n, k_{n+1})$ for all $n \in \mathbb{N}$, and a continuous homomorphism from $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n\in\mathbb{N}}}$ to $G \curvearrowright X$.*

Proof. The following fact will allow us to mimic the proof of the existence of G-invariant non-empty closed sets on which $G \curvearrowright X$ is minimal.

Lemma 3.2.3. *If $x \in X$ and $y \in \bigcap_{n\in\mathbb{N}} \overline{(IP(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x}$, then $\bigcap_{n\in\mathbb{N}} \overline{(IP(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot y} \subseteq \bigcap_{n\in\mathbb{N}} \overline{(IP(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x}$.*

Proof. It is sufficient to show that if $z \in \bigcap_{n\in\mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot y}$, $n \in \mathbb{N}$, and $W \subseteq X$ is an open neighborhood of z, then W intersects

$(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x$. Fix a sequence $s \in T_{\mathbb{s}^n(\mathbf{g})}$ with non-trivial support for which $\mathbb{s}^n(\mathbf{g})^s \cdot y \in W$. As $G \curvearrowright X$ is an action by homeomorphisms, there is an open neighborhood $V \subseteq X$ of y such that $\mathbb{s}^n(\mathbf{g})^s V \subseteq W$. Fix $t \in X_{\mathbb{s}^{n+|s|}(\mathbf{g})}$ with the property that $\mathbb{s}^{n+|s|}(\mathbf{g})^t \cdot x \in V$, and observe that $\mathbb{s}^n(\mathbf{g})^{s \frown t} \cdot x = \mathbb{s}^n(\mathbf{g})^s \mathbb{s}^{n+|s|}(\mathbf{g})^t \cdot x \in \mathbb{s}^n(\mathbf{g})^s V \subseteq W$ and the $(K_n)_{n \in \mathbb{N}}$-expansivity of $\mathbf{g}$ ensures that $\mathbb{s}^n(\mathbf{g})^{s \frown t} \neq 1_G$. ⊠

By Lemma 3.2.3, there is an ordinal λ for which there is a maximal sequence $(x_\alpha)_{\alpha<\lambda}$ such that $x_\alpha \in \bigcap_{\beta<\alpha} \bigcap_{n \in \mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\beta}$ but $\bigcap_{n \in \mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\alpha} \neq \bigcap_{\beta<\alpha} \bigcap_{n \in \mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\beta}$ for all $\alpha < \lambda$. Fix any point $x \in \bigcap_{\alpha<\lambda} \bigcap_{n \in \mathbb{N}} \overline{(\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\}) \cdot x_\alpha}$, and observe that x is $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g})) \setminus \{1_G\} \mid n \in \mathbb{N}\}$-recurrent.

Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, as well as a compatible complete metric on X, and set $k_0 = 0$ and $V_0 = X$. We will recursively construct k_{n+1}, s_n, and open neighborhoods V_{n+1} of x. Given $n \in \mathbb{N}$ for which we have already found k_n and V_n, fix a sequence $s_n \in T_{\mathbf{g}}$, whose support is non-empty and contained in $[k_n, \infty)$, for which $\mathbf{g}^{s_n} \cdot x \in V_n$, set $k_{n+1} = |s_n|$, and fix an open neighborhood $V_{n+1} \subseteq X$ of x such that $\overline{V_{n+1}} \subseteq V_n \cap (\mathbf{g}^{s_n})^{-1} V_n$ and $\mathrm{diam}(\mathbf{g}^s V_{n+1}) \leq \epsilon_n$ for all $s \in T_{\mathbf{g}}$ of length k_{n+1}.

Define a continuous function $\phi \colon X_{(\mathbf{g}^{s_n})_{n \in \mathbb{N}}} \to X$ by $\phi(c) =$ the unique element of $\bigcap_{n \in \mathbb{N}} (\mathbf{g}^{s_n})_{n \in \mathbb{N}}^{c \restriction n} V_n$. Then $\phi_G / E_{(\mathbf{g}^{s_n})_{n \in \mathbb{N}}}$ is a homomorphism from $G \curvearrowright \mathbb{X}_{(\mathbf{g}^{s_n})_{n \in \mathbb{N}}}$ to $G \curvearrowright X$ by the proof of Proposition 1.2.6. ⊠

In light of Theorem 3.2.1, the following fact ensures that Borel-probability-measure-preserving Borel free actions of locally-compact Polish groups on standard Borel spaces contain bases:

Proposition 3.2.4. *Suppose that G is a locally-compact Polish group, X and Y are standard Borel spaces, $G \curvearrowright X$ is a Borel action that is invariant with respect to a Borel probability measure μ on X, and $G \curvearrowright Y$ is a Borel action for which $G \curvearrowright X \times Y$ is free and smooth. Then $G \curvearrowright Y$ is smooth.*

Proof. Fix a Borel transversal $B \subseteq X \times Y$ of $G \curvearrowright X \times Y$, and define $\phi \colon X \times Y \to G$ by letting $\phi(x, y)$ be the unique $g \in G$ for which $g \cdot (x, y) \in B$. Let $P(G)$ denote the standard Borel space of Borel probability measures on G (see, for example, [Kec95, §17.E]), and define $\nu \colon Y \to P(G)$ by $\nu(y) = \phi(\cdot, y)_* \mu$. If $H \subseteq G$ is Borel and $y \in Y$, then

$$\begin{aligned} \nu(y)(H) &= \mu(\{x \in X \mid \exists h \in H \ h \cdot (x, y) \in B\}) \\ &= \mu(\{x \in X \mid (x, y) \in H^{-1} B\}), \end{aligned}$$

so the G-invariance of μ ensures that if $g \in G$, then

$$\begin{aligned}\nu(g \cdot y)(H) &= \mu(g^{-1} \cdot \{x \in X \mid (x, g \cdot y) \in H^{-1}B\}) \\ &= \mu(\{x \in X \mid (g \cdot x, g \cdot y) \in H^{-1}B\}) \\ &= \mu(\{x \in X \mid (x, y) \in (Hg)^{-1}B\}) \\ &= \nu(y)(Hg).\end{aligned}$$

But if $K \subseteq G$ is compact and $g \notin K^{-1}K$, then $K \cap Kg = \emptyset$, in which case $\{y \in Y \mid \nu(y)(K) > 1/2\}$ is σ-expansively $\{G\}$-transient, thus Theorem 3.1.1 ensures that $G \curvearrowright Y$ is smooth. ⊠

We next characterize expansive $\{G\}$-recurrence of products with free actions obtained via expansive cutting and stacking:

Proposition 3.2.5. *Suppose that G is a locally-compact Polish group, X is a Polish space, $G \curvearrowright X$ is a continuous free action, and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent $\iff$ $G \curvearrowright X$ is expansively $\{IP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrent.*

Proof. To see ($\Longrightarrow$), suppose that $K \subseteq G$ is compact, $n \in \mathbb{N}$, and $V \subseteq X$ is a non-empty open set, and fix an open neighborhood $U \subseteq G$ of 1_G with compact closure and a non-empty open set $V' \subseteq X$ for which $UV' \subseteq V$. As $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent, it follows that $\Delta_G^X(V' \times V') \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) \nsubseteq U^{-1}KU$. But $U\Delta_G^X(V' \times V')U^{-1} = \Delta_G^X(UV' \times UV')$ and Proposition 1.2.1 ensures that $\Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) = U^{-1}\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1}U$, so $\Delta_G^X(V \times V) \cap \mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \nsubseteq K$.

To see ($\Longleftarrow$), suppose that $K \subseteq G$ is compact, $s \in T_{\mathbf{g}}$, $U \subseteq G$ is a non-empty open set with compact closure, and $V \subseteq X$ is a non-empty open set. Then $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}V) \cap \mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1} \nsubseteq (U\mathbf{g}^s)^{-1}KU\mathbf{g}^s$ by expansive $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrence. But $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}V) = (U\mathbf{g}^s)^{-1}\Delta_G^X(V \times V)U\mathbf{g}^s$ and $U\mathbf{g}^s\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1}(U\mathbf{g}^s)^{-1} = \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}})$ by Proposition 1.2.1, so $\Delta_G^X(V \times V) \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}}) \nsubseteq K$. ⊠

We now establish a local version of the promised characterization of free actions containing bases in the abelian case:

Theorem 3.2.6. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $(K_m)_{m \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ is $(K_m)_{m \in \mathbb{N}}$-expansive.*

(1) *If* $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ *is smooth, then* $G \curvearrowright X$ *is* σ*-expansively* $(\bigcup_{g \in G} g\{IP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$*-transient.*
(2) *If* G *is abelian, then the converse holds.*

Proof. By [BK96, Theorem 5.2.1], we can assume that X is Polish and $G \curvearrowright X$ is continuous. Proposition 2.2.1 ensures that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient if and only if there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X}$ is expansively $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-recurrent, and Proposition 3.2.5 implies that the latter condition holds if and only if there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent. So it is enough to prove the analog of the theorem in which the σ-expansive $(\bigcup_{g \in G} g\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transience of $G \curvearrowright X$ is replaced with the condition that there is no $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent.

To see the analog of (1), appeal to Proposition 3.1.1 to see that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is σ-expansively $\{G\}$-transient, in which case Proposition 2.1.2 ensures that there does not exist $x \in X$ for which $G \curvearrowright [x]_{F_G^X} \times \mathbb{X}_{\mathbf{g}}$ is expansively $\{G\}$-recurrent.

To see the analog of (2), note that if $K \subseteq G$ is compact, $V \times W \subseteq X \times \mathbb{X}_{\mathbf{g}}$ is open, and $x \in X$, then the minimality of $G \curvearrowright [x]_{F_G^X}$ ensures that $V \cap [x]_{F_G^X} \neq \emptyset \iff V \cap Gx \neq \emptyset \iff x \in GV$, and the freeness of $G \curvearrowright X$ implies that $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \upharpoonright ((V \cap [x]_{F_G^X}) \times W) \subseteq R_K^{X \times \mathbb{X}_{\mathbf{g}}} \iff V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1}V \cap [x]_{F_G^X} = \emptyset \iff V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1}V \cap Gx = \emptyset \iff x \notin G(V \cap (\Delta_G^{\mathbb{X}_{\mathbf{g}}}(W \times W) \setminus K)^{-1}V)$, so the set of $x \in X$ for which $V \cap [x]_{F_G^X}$ is non-empty but $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \upharpoonright ((V \cap [x]_{F_G^X}) \times W) \subseteq R_K^{X \times \mathbb{X}_{\mathbf{g}}}$ is a difference of two G-invariant open sets. Fix an enumeration $(V_n \times W_n)_{n \in \mathbb{N}}$ of a basis for $X \times \mathbb{X}_{\mathbf{g}}$, and for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, let $U_{m,n}$ be the set of $x \in X$ for which $V_n \cap [x]_{F_G^X}$ is non-empty but $E_G^{X \times \mathbb{X}_{\mathbf{g}}} \upharpoonright ((V_n \cap [x]_{F_G^X}) \times W_n) \subseteq R_{K_m}^{X \times \mathbb{X}_{\mathbf{g}}}$. Fix a countable dense set $H \subseteq G$. Then the sets of the form $U_{m,n} \cap (gV_n \times hW_n)$, where $g, h \in H$ and $m, n \in \mathbb{N}$, cover $X \times \mathbb{X}_{\mathbf{g}}$, and the fact that G is abelian ensures that they are expansively $\{G\}$-transient, so Proposition 3.1.1 implies that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is smooth. ⊠

The promised basis theorem easily follows:

Theorem 3.2.7. *Suppose that* $G \curvearrowright X$ *is a Borel (continuous) free action of an abelian locally-compact Polish group on a Polish space that contains a basis. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that contains a basis* *and admits a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorems 2.1.8, 3.1.2, and 3.2.6. ⊠

We similarly obtain the promised anti-basis theorem:

Theorem 3.2.8. *Suppose that $G \curvearrowright X$ is a Borel free action of an abelian locally-compact Polish group on a standard Borel space containing a basis, and $\mathcal{F}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and contains a basis, but to which no action in $\mathcal{F}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorems 2.2.6, 3.1.2, and 3.2.6. ⊠

3.3. **Weak wandering.** The following straightforward observation ensures that the notions of completeness and σ-completeness with respect to continuous actions $G \curvearrowright X$ of separable groups on topological spaces are equivalent for sets of the form $\bigcup_{n \in \mathbb{N}} U_n \setminus B_n$, where each of the sets $B_n \subseteq X$ is G-invariant and each of the sets $U_n \subseteq X$ is open:

Proposition 3.3.1. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a topological space, $H \subseteq G$ is dense, and $U \subseteq X$ is open. Then $GU = HU$.*

Proof. Note that if $g \in G$ and $x \in U$, then $g^{-1} g \cdot x \in U$, so there exists $h \in H$ for which the point $y = h^{-1} g \cdot x$ is in U. But $g \cdot x = h \cdot y$. ⊠

When $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, we say that a set $Y \subseteq X$ is *$\mathcal{S}$-transient* if there exists $S \in \mathcal{S}$ with the property that $\Delta_G^X(\bigcup_{d \in \mathbb{Z}^+} Y^{\{0,\ldots,d\}}) \cap S = \emptyset$. Note that if $S \subseteq G$, then a set $Y \subseteq X$ is S-wandering if and only if it is $\{SS^{-1} \setminus \{1_G\}\}$-transient. We say that a G-action by homeomorphisms of a topological space is *$\mathcal{S}$-recurrent* if no non-empty open set is $\mathcal{S}$-transient, and a Borel G-action on a standard Borel space X is *σ-$\mathcal{S}$-transient* if X is a union of countably-many $\mathcal{S}$-transient Borel sets.

The following fact ensures that if $G \curvearrowright X$ is a minimal continuous action, then the existence of a weakly-wandering σ-complete Borel set is equivalent to the existence of a cover by countably-many weakly-wandering Borel sets:

Proposition 3.3.2. *Suppose that G is a separable group, $\mathcal{S} \subseteq \mathcal{P}(\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is a σ-$(\bigcup_{g \in G} g\mathcal{S}g^{-1})$-transient minimal continuous action on a Baire space. Then there exists $S \in \mathcal{S}$ for which there is an $\{S\}$-transient complete open set.*

Proof. Fix an $\mathcal{S}$-transient non-meager Borel set $B \subseteq X$, as well as $S \in \mathcal{S}$ for which B is $\{S\}$-transient, and a non-empty open set $V \subseteq X$ in which B is comeager. Then Proposition 2.1.2 ensures that V is $\{S\}$-transient, and the minimality of $G \curvearrowright X$ implies that it is complete. ⊠

We say that a sequence $\mathbf{g} \in G^{\mathbb{N}}$ is $\mathcal{S}$*-dense*, $(K_n)_{n\in\mathbb{N}}$*-expansive*, or *doubly* $(K_n)_{n\in\mathbb{N}}$*-expansive* if the sequence $\mathbf{h} \in (G^{\{1\}})^{\mathbb{N}}$, given by $(\mathbf{h}_n)_1 = \mathbf{g}_n$ for all $n \in \mathbb{N}$, has the corresponding property. The following fact ensures that the above assumption of minimality is necessary:

Proposition 3.3.3. *Suppose that G is a locally-compact Polish group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, and $\mathbf{g} \in G^{\mathbb{N}}$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive. Then there is a continuous disjoint union $G \curvearrowright X$ of free actions obtained via expansive cutting and stacking, a continuous surjective homomorphism $\phi\colon X \to 2^{\mathbb{N}}$ from E_G^X to equality, and a complete open set $V \subseteq X$ such that $V \cap \phi^{-1}(\{d\})$ is IP$(\mathbf{g} * d)$-wandering for all $d \in 2^{\mathbb{N}}$, but for all sets $S \subseteq G$ with non-compact closure, there is at most one $d \in 2^{\mathbb{N}}$ with the property that $G \curvearrowright \phi^{-1}(\{d\})$ is σ-expansively $\{S\}$-transient.*

Proof. We first note a pair of lemmas:

Lemma 3.3.4. *Suppose that $d, e \in 2^{\mathbb{N}}$ are distinct, $K, L \subseteq G$ are compact, and $S \subseteq K\mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1}K^{-1}$. Then the closure of $L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \cap S$ is compact.*

Proof. Let s be the maximal common initial segment of d and e. As $\mathbf{g}$ is doubly $(K_n)_{n\in\mathbb{N}}$-expansive, there is a natural number $n > \langle s\rangle$ such that $\mathbf{g}_m \notin (\mathrm{IP}(\mathbf{g} \restriction m)^{-1}(K^{-1}L)^{\pm 1}\mathrm{IP}(\mathbf{g} \restriction m))^2$ for all $m \geq n$, in which case a straightforward calculation reveals that

$$\begin{aligned} &L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \cap S \\ &\quad\subseteq K\mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1}K^{-1} \cap L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1} \\ &\quad\subseteq K\mathrm{IP}(\mathbf{g} \restriction n)\mathrm{IP}(\mathbf{g} \restriction n)^{-1}K^{-1} \cap L\mathrm{IP}(\mathbf{g} \restriction n)\mathrm{IP}(\mathbf{g} \restriction n)^{-1}L^{-1}, \end{aligned}$$

so it only remains to note that the latter set is compact. ⊠

Lemma 3.3.5. *Suppose that $K \subseteq G$ is compact, but the closure of $S \subseteq G$ is not compact. Then there exists $d \in 2^{\mathbb{N}}$ such that for all $e \neq d$, there is a $(K_n)_{n\in\mathbb{N}}$-expansive $\{S\}$-dense sequence $\mathbf{g}_e \in G^{\mathbb{N}}$ for which $K\mathrm{IP}(\mathbf{g}_e)\mathrm{IP}(\mathbf{g}_e)^{-1}K^{-1} \cap \mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} = \{1_G\}$.*

Proof. Fix a countable dense set $H \subseteq G$, as well as a sequence $\mathbf{h} \in H^{\mathbb{N}}$ such that $\forall h \in H \exists^\infty n \in \mathbb{N}\ h = \mathbf{h}_n$, and a sequence $\mathbf{s} \in \prod_{n\in\mathbb{N}} 2^n$ such that $\{\mathbf{s}_n \mid n \in \mathbb{N} \text{ and } h = \mathbf{h}_n\}$ is $\sqsubseteq$-dense for all $h \in H$. As the closure of S is not compact, Lemma 3.3.4 yields $d \in 2^{\mathbb{N}}$ such that

$S \not\subseteq L\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1}L^{-1}$ for all $e \neq d$ and compact sets $L \subseteq G$, in which case a simple recursive construction yields $\mathbf{g}_e \in G^{\mathbb{N}}$ such that:

(1) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \notin \mathrm{IP}(\mathbf{g}_e \upharpoonright n)^{-1}K_n^{\pm 1}\mathrm{IP}(\mathbf{g}_e \upharpoonright n)$.
(2) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \in (\mathbf{g}_e^{\mathbf{s}_n})^{-1}\mathbf{h}_n^{-1}S\mathbf{h}_n\mathbf{g}_e^{\mathbf{s}_n}$.
(3) $\forall n \in \mathbb{N}\ (\mathbf{g}_e)_n \notin \mathrm{IP}(\mathbf{g}_e \upharpoonright n)^{-1}K^{-1}\mathrm{IP}(\mathbf{g}*e)\mathrm{IP}(\mathbf{g}*e)^{-1}K\mathrm{IP}(\mathbf{g}_e \upharpoonright n)$.

The first condition ensures that $\mathbf{g}_e$ is $(K_n)_{n\in\mathbb{N}}$-expansive, the second condition implies that $\mathbf{g}_e$ is $\{S\}$-dense, and the third condition yields that $K\mathrm{IP}(\mathbf{g}_e)\mathrm{IP}(\mathbf{g}_e)^{-1}K^{-1} \cap \mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} = \{1_G\}$. ⊠

Fix an open neighborhood $U \subseteq G$ of 1_G with compact closure, let I be the set of $(d, \mathbf{g}_d) \in 2^{\mathbb{N}} \times G^{\mathbb{N}}$ for which $\mathbf{g}_d$ is $(K_n)_{n\in\mathbb{N}}$-expansive and $U\mathrm{IP}(\mathbf{g}_d)\mathrm{IP}(\mathbf{g}_d)^{-1}U^{-1} \cap \mathrm{IP}(\mathbf{g} * d)\mathrm{IP}(\mathbf{g} * d)^{-1} = \{1_G\}$, and define $X = \mathbb{X}_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}$. Then $G \curvearrowright X$ is a continuous disjoint union of actions obtained via expansive cutting and stacking, the function $\phi \colon X \to 2^{\mathbb{N}}$ given by $\phi([(g, ((d, \mathbf{g}_d), c))]_{E_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}}) = d$ is a homomorphism from E_G^X to equality, and $V = (U \times X_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I})/E_{\mathrm{proj}_{G^{\mathbb{N}}} \upharpoonright I}$ is a complete open set.

Proposition 1.2.1 ensures that $V \cap \phi^{-1}(\{d\})$ is $\mathrm{IP}(\mathbf{g}*d)$-wandering for all $d \in 2^{\mathbb{N}}$. If the closure of $S \subseteq G$ is not compact, then Lemma 3.3.5 yields $d \in 2^{\mathbb{N}}$ such that for all $e \neq d$, there is an $\{S\}$-dense sequence $\mathbf{g}_e \in I_e$, thus $G \curvearrowright \phi^{-1}(\{e\})$ is not σ-expansively $\{S\}$-transient by Propositions 2.1.1 and 2.1.2. If $d \in 2^{\mathbb{N}}$, $e \neq d$, and $\mathbf{g}_e \in I_e$, then $G \curvearrowright \mathbb{X}_{\mathbf{g}_e}$ is not $\{\mathrm{IP}(\mathbf{g} * e)\mathrm{IP}(\mathbf{g} * e)^{-1} \setminus \{1_G\}\}$-recurrent, and therefore not expansively $\{(\mathbf{g} * e)(\mathbb{N})\}$-recurrent, so Proposition 2.1.1 ensures that $\mathbf{g}_e$ is not $\{(\mathbf{g} * e)(\mathbb{N})\}$-dense, thus Lemma 3.3.5 implies that there is a $\{(\mathbf{g} * e)(\mathbb{N})\}$-dense sequence $\mathbf{g}_d \in I_d$, hence ϕ is surjective. ⊠

Even without minimality, a similar idea can be used to show that the existence of an S-wandering σ-complete Borel set is equivalent to σ-$(\bigcup_{g\in G} g\{SS^{-1} \setminus 1_G\}g^{-1})$-transience:

Proposition 3.3.6. *Suppose that G is a Polish group, $S \in \mathcal{P}(\bigcup_{d\in\mathbb{Z}^+} G^{\{1,\dots,d\}})$, and $G \curvearrowright X$ is a σ-$(\bigcup_{g\in G} g\{S\}g^{-1})$-transient continuous action on a Polish space. Then $G \curvearrowright X$ admits an $\{S\}$-transient σ-complete F_σ set.*

Proof. Proposition 2.1.2 ensures that there is no $x \in X$ for which $G \curvearrowright [x]_{F_G^X}$ is $\{S\}$-recurrent. Note that if $V \subseteq X$ is open and $x \in X$, then the minimality of $G \curvearrowright [x]_{F_G^X}$ ensures that $V \cap [x]_{F_G^X} \neq \emptyset \iff V \cap Gx \neq \emptyset \iff x \in GV$, and $V \cap [x]_{F_G^X}$ is $\{S\}$-transient $\iff \bigcup_{g\in S} \bigcap_{k\leq|g|} (\overline{g}_k)^{-1}V \cap [x]_{F_G^X} = \emptyset \iff \bigcup_{g\in S} \bigcap_{k\leq|g|} (\overline{g}_k)^{-1}V \cap Gx = \emptyset \iff x \notin G(\bigcup_{g\in S} \bigcap_{k\leq|g|} (\overline{g}_k)^{-1}V)$, so the set of $x \in X$ for which $V \cap [x]_{F_G^X}$ is non-empty but $\{S\}$-transient is a difference of two G-invariant open sets. Fix an enumeration $(V_n)_{n\in\mathbb{N}}$ of a basis for X, and for all $x \in X$, let $n(x)$ be the least natural number for which $V_{n(x)} \cap [x]_{F_G^X}$ is non-empty but $\{S\}$-transient. Then the set $B = \bigcup_{n\in\mathbb{N}}\{x \in V_n \mid n(x) = n\}$ is F_σ, $\{S\}$-transient, and σ-complete by Proposition 3.3.1. ⊠

We next note a restriction on the sets $S \subseteq G$ appearing in the definition of weak wandering in the topological setting:

Proposition 3.3.7. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, $S \subseteq G$, and there is an S-wandering non-empty open set $U \subseteq X$. Then S is closed and discrete.*

Proof. Otherwise, there is an injective sequence $(g_n)_{n\in\mathbb{N}}$ of elements of S that converges to some $g \in G$, so $g_n g^{-1} \to 1_G$. But if $x \in U$, then $g_n g^{-1} \cdot x \to x$, so there exists $n \in \mathbb{N}$ such that $g_m g^{-1} \cdot x \in U$ for all $m \geq n$, thus $g^{-1} \cdot x \in \bigcap_{m\geq n} g_m^{-1} U$, a contradiction. ⊠

Proposition 3.3.8. *Suppose that G is a locally-compact Polish group and the closure of $S \subseteq G$ is not compact. Then there is an infinite set $T \subseteq S$ for which TT^{-1} is closed and discrete.*

Proof. Fix an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of G with compact closures whose union is G, and recursively construct $g_n \in S\setminus(U_n^{\pm 1}\{g_i \mid i < n\})$ for all $n \in \mathbb{N}$. To see that the set $T = \{g_n \mid n \in \mathbb{N}\}$ is as desired, note that for all $g \in G$, there exists $n \in \mathbb{N}$ such that $g \in U_n$, but $TT^{-1} \cap U_n \subseteq \{g_i g_j^{-1} \mid i, j < n\}$. ⊠

Clearly $\{S\setminus\{1_G\}\}$-transience implies expansive $\{S\}$-transience. When S is closed and discrete, a natural weakening of the converse also holds:

Proposition 3.3.9. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $S \subseteq G$ is closed and discrete, and $B \subseteq X$ is an expansively $\{S\}$-transient Borel set. Then B is a union of finitely-many $\{S \setminus \{1_G\}\}$-transient Borel sets.*

Proof. Fix a compact set $K \subseteq G$ for which $R_S^B \subseteq R_K^B$. As $G \curvearrowright X$ is free, it follows that $R_S^B \subseteq R_{K\cap S}^B$. As S is closed and discrete, it follows that $K \cap S$ is finite. Set $F = (K \cap S)^{\pm 1} \setminus \{1_G\}$, and note that R_F^X is a Borel graph of vertex degree $|F|$, and therefore has a Borel $(|F| + 1)$-coloring (see [KST99, Proposition 4.6]), so B is the union of $(|F| + 1)$-many $\{S \setminus \{1_G\}\}$-transient Borel sets. ⊠

In light of Proposition 3.3.6, the following facts characterize both the existence of a weakly-wandering σ-complete Borel set and the existence of a cover by weakly-wandering Borel sets:

Proposition 3.3.10. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space and $\mathcal{S} \subseteq \mathcal{P}(G)$. Then the following are equivalent:*

1. *There are infinite sets $S_n \in \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ and S_n-wandering Borel sets $B_n \subseteq X$ for which $X = \bigcup_{n \in \mathbb{N}} B_n$.*
2. *There are infinite sets $T_n \in \bigcup_{S \in \mathcal{S}} \mathcal{P}(S)$ for which $T_n T_n^{-1}$ is closed and discrete with the property that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{T_n T_n^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient.*

Proof. To see (1) $\implies$ (2), note first that we can assume that X is Polish and $G \curvearrowright X$ is continuous by [BK96, Theorem 5.2.1]. Proposition 2.1.2 then ensures that for all $x \in X$, there exists $n \in \mathbb{N}$ for which $G \curvearrowright [x]_{F_G^X}$ is not $\{S_n S_n^{-1} \setminus \{1_G\}\}$-recurrent, in which case Proposition 3.3.7 implies that S_n is closed and discrete. Define $N = \{n \in \mathbb{N} \mid S_n \text{ is closed and discrete}\}$, and for all $n \in N$, appeal to Proposition 3.3.8 to obtain an infinite set $T_n \subseteq S_n$ for which $T_n T_n^{-1}$ is closed and discrete. Then Proposition 2.2.1 ensures that $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{T_n T_n^{-1} \setminus \{1_G\} \mid n \in N\}g^{-1})$-transient, and therefore σ-expansively $(\bigcup_{g \in G} g\{T_n T_n^{-1} \mid n \in N\}g^{-1})$-transient.

To see (2) $\implies$ (1), appeal to Proposition 3.3.9 to see that $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{T_n T_n^{-1} \setminus \{1_G\} \mid n \in \mathbb{N}\}g^{-1})$-transient. ⊠

The following fact easily follows from the proof of Proposition 3.3.10.

Proposition 3.3.11. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space. Then the following are equivalent:*

1. *The action $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{SS^{-1} \setminus \{1_G\}\}g^{-1})$-transient for some infinite set $S \in \mathcal{P}(G)$.*
2. *There is an infinite set T for which TT^{-1} is closed and discrete with the property that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{TT^{-1}\}g^{-1})$-transient.*

We next note that finite changes to S have little influence on the existence of large S-wandering Borel sets:

Proposition 3.3.12. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $g \in G$, $S \subseteq G$ is countable, and $B \subseteq X$ is an S-wandering Borel set. Then B is a union of countably-many $(\{g\} \cup S)$-wandering Borel sets.*

Proof. We can assume that $g \notin S$. Note that for all $x \in B$, there is at most one pair $(h, y) \in S \times B$ for which $g^{-1} \cdot x = h^{-1} \cdot y$. Let $\phi \colon B \rightharpoonup B$ be the partial function sending x to y. The freeness of

$G \curvearrowright X$ ensures that ϕ is fixed-point free, in which case $\mathrm{graph}(\phi)^{\pm 1}$ is a graph generated by a Borel function, and therefore has a Borel $\aleph_0$-coloring (see [KST99, Proposition 4.5]), thus B is a union of countably-many $(\{g\} \cup S)$-wandering Borel sets. ⊠

In light of Theorem 3.1.2, the following fact ensures that if a free Borel action does not contain a basis, then it admits a weakly-wandering σ-complete Borel set:

Proposition 3.3.13. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathbf{g} \in G^{\mathbb{N}}$ is $(K_n)_{n \in \mathbb{N}}$-expansive, and $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is smooth. Then $G \curvearrowright X$ admits a $\mathbf{g}(\mathbb{N})$-wandering σ-complete Borel set.*

Proof. Appeal first to Theorem 3.2.6 to see that $G \curvearrowright X$ is σ-expansively $(\bigcup_{g \in G} g\{\mathbf{g}(\mathbb{N} \setminus n)\mathbf{g}(\mathbb{N} \setminus n)^{-1} \mid n \in \mathbb{N}\}g^{-1})$-transient. The $(K_n)_{n \in \mathbb{N}}$- expansivity of $\mathbf{g}$ yields that $\mathbf{g}(\mathbb{N})\mathbf{g}(\mathbb{N})^{-1}$ is closed and discrete, so $G \curvearrowright X$ is σ-$(\bigcup_{g \in G} g\{\mathbf{g}(\mathbb{N} \setminus n)\mathbf{g}(\mathbb{N} \setminus n)^{-1} \setminus \{1_G\} \mid n \in \mathbb{N}\}g^{-1})$-transient by Proposition 3.3.9, thus σ-$(\bigcup_{g \in G} g\{\mathbf{g}(\mathbb{N})\mathbf{g}(\mathbb{N})^{-1} \setminus \{1_G\}\}g^{-1})$-transient by Proposition 3.3.12, in which case Proposition 3.3.6 yields a $\mathbf{g}(\mathbb{N})$-wandering σ-complete Borel set. ⊠

The following fact yields a sufficient condition for the existence of a non-smooth restriction with a suitably transient complete Borel set:

Proposition 3.3.14. *Suppose that G is a locally-compact Polish group, $(K_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathbf{g} \in G^{\mathbb{N}}$ is $(K_n)_{n \in \mathbb{N}}$-expansive, $S \subseteq G$ is disjoint from a neighborhood of 1_G, and there is no compact set $K \subseteq G$ with the property that $IP(\mathbf{g})IP(\mathbf{g})^{-1} \subseteq K^{-1}SK$. Then there is a G-action obtained via expansive cutting and stacking that admits a continuous embedding into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ and an $\{S\}$-transient non-empty open set.*

Proof. Note that for all compact sets $K \subseteq G$ and $n \in \mathbb{N}$, there exist $s_0, s_1 \in 2^{<\mathbb{N}}$ for which $\mathfrak{s}^n(\mathbf{g})^{s_1}(\mathfrak{s}^n(\mathbf{g})^{s_0})^{-1} \notin K \cup K^{-1}SK$. Fix an open neighborhood $U \subseteq G$ of 1_G with the property that $\overline{U}$ is compact and $S \cap UU^{-1} = \emptyset$, recursively find $\ell_n \in \mathbb{N}$ and $s_{0,n}, s_{1,n} \in 2^{\ell_n}$ such that $\mathbf{h}_n \notin \mathrm{IP}(\mathbf{h} \upharpoonright n)^{-1}(K_n^{\pm 1} \cup U^{-1}SU)\mathrm{IP}(\mathbf{h} \upharpoonright n)$ for all $n \in \mathbb{N}$, where $\mathbf{h}_n = \mathbf{g}^{\bigoplus_{m<n} s_{0,m}} \mathfrak{s}^{\sum_{m<n} \ell_m}(\mathbf{g})^{s_{1,n}} (\mathfrak{s}^{\sum_{m<n} \ell_m}(\mathbf{g})^{s_{0,n}})^{-1} (\mathbf{g}^{\bigoplus_{m<n} s_{0,m}})^{-1}$ for all $n \in \mathbb{N}$, and define $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\phi(c) = \bigoplus_{n \in \mathbb{N}} s_{c(n),n}$. Then $\mathbf{h}$ is $(K_n)_{n \in \mathbb{N}}$-expansive, so $G \curvearrowright \mathbb{X}_{\mathbf{h}}$ is obtained via expansive cutting and stacking, ϕ_G factors over $E_{\mathbf{h}}$ and $E_{\mathbf{g}}$ to a continuous embedding of $G \curvearrowright \mathbb{X}_{\mathbf{h}}$ into $G \curvearrowright \mathbb{X}_{\mathbf{g}}$, and Proposition 1.2.1 ensures that $(U \times 2^{\mathbb{N}})/E_{\mathbf{h}}$ is an $\{S\}$-transient non-empty open set. ⊠

For each set N, let $[N]^{\aleph_0}$ denote the family of countably-infinite subsets of N, and for each sequence of sets $(X_n)_{n\in N}$, define $\limsup_{n\in N} X_n = \{x \mid \exists^\infty n \in N\ x \in X_n\}$. We say that a sequence $\mathbf{h} \in G^{\mathbb{N}}$ is *sufficiently* $(K_n)_{n\in\mathbb{N}}$*-expansive* if the following hold, where $H_n = \{\mathbf{h}_m \mid m < n\}$:

(1) $\forall n \in \mathbb{N}\ \mathbf{h}_n \notin (K_n H_n H_n^{-1})^3 K_n H_n$.
(2) $\forall n \in \mathbb{N} \forall m > n$
$$\mathbf{h}_n \notin K_n \mathbf{h}_m H_n^{-1} K_n H_n \mathbf{h}_m^{-1} K_n H_n \cup K_n H_n \mathbf{h}_m^{-1} K_n \mathbf{h}_m H_n^{-1} K_n H_n \cup K_n^{-1} H_n \mathbf{h}_m^{-1} K_n^{-1} H_n H_n^{-1} K_n^{-1} \mathbf{h}_m \cup K_n H_n H_n^{-1} K_n H_n \mathbf{h}_m^{-1} K_n \mathbf{h}_m.$$
(3) $\forall K \subseteq G$ compact$\forall N \in [\mathbb{N} \times \mathbb{N}]^{\aleph_0} \exists M \in [N]^{\aleph_0}$ $\overline{\limsup_{(m,n)\in M} K \mathbf{h}_m \mathbf{h}_n^{-1} K \mathbf{h}_n \mathbf{h}_m^{-1} K}$ is compact.

Proposition 3.3.15. *Suppose that G is a non-compact locally-compact Polish group that admits a compatible two-sided-invariant metric, and $(K_n)_{n\in\mathbb{N}}$ is an increasing sequence of compact subsets of G. Then there is a sufficiently-$(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{h} \in G^{\mathbb{N}}$.*

Proof. The primary observation is as follows:

Lemma 3.3.16. *Suppose that $K \subseteq G$ is compact and $H \in [G]^{\aleph_0}$. Then there exists $H' \in [H]^{\aleph_0}$ such that $\overline{\limsup_{g\in H'} KgKg^{-1}K}$ is compact.*

Proof. By [Kle52, 1.5], there is a conjugation-invariant open neighborhood $U \subseteq G$ of 1_G with compact closure. Fix a finite set $F \subseteq G$ for which $K \subseteq FU$. By a straightforward induction, it is sufficient to show that for all $f \in F$ and $H \in [G]^{\aleph_0}$, there exists $H' \in [H]^{\aleph_0}$ for which $\overline{\limsup_{g\in H'} Kgfg^{-1}KU}$ is compact. Towards this end, we can assume that there is a set $H' \in [H]^{\aleph_0}$ for which $\bigcap_{g\in H'} Kgfg^{-1}KU \neq \emptyset$. Fix $h \in H'$, and note that $\forall g \in H'\ gfg^{-1} \in K^{-1}Khfh^{-1}KK^{-1}UU^{-1}$, so $\bigcup_{g\in H'} Kgfg^{-1}KU \subseteq KK^{-1}Khfh^{-1}KK^{-1}KUU^{-1}U$. As the latter set has compact closure, so too does $\limsup_{g\in H'} Kgfg^{-1}KU$. ⊠

As G is not compact, there is a discrete set $G_0 \in [G]^{\aleph_0}$. Given $n \in \mathbb{N}$, $G_n \in [G_0]^{\aleph_0}$, and $\mathbf{h} \upharpoonright n$, set $H_n = \{\mathbf{h}_m \mid m < n\}$ and define

$$L_{g,n} = K_n g H_n^{-1} K_n H_n g^{-1} K_n H_n \cup K_n H_n g^{-1} K_n g H_n^{-1} K_n H_n \cup K_n^{-1} H_n g^{-1} K_n^{-1} H_n H_n^{-1} K_n^{-1} g \cup K_n H_n H_n^{-1} K_n H_n g^{-1} K_n g$$

for all $g \in G_n$, and observe that four successive applications of Lemma 3.3.16 yield a set $G_n' \in [G_n]^{\aleph_0}$ with the property that the closure of $\limsup_{g\in G_n'} L_{g,n}$ is compact. As G_n is discrete and infinite, there exists $\mathbf{h}_n \in G_n \setminus ((K_n H_n H_n^{-1})^3 K_n H_n \cup \limsup_{g\in G_n'} L_{g,n})$, in which case the set $G_{n+1} = \{g \in G_n' \mid \mathbf{h}_n \notin L_{g,n}\}$ is infinite. Clearly $\mathbf{h}$ is as desired. ⊠

The following observation ensures that one can obtain a Borel free action $G \curvearrowright X$ that contains a basis and admits a weakly-wandering σ-complete Borel set by fixing an exhaustive increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G and a sufficiently-$(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{h} \in G^{\mathbb{N}}$, and taking a continuous disjoint union of the actions and weakly-wandering sets obtained by applying Proposition 3.3.14 to every $(K_n)_{n\in\mathbb{N}}$-expansive sequence $\mathbf{g} \in G^{\mathbb{N}}$ with $S = \mathbf{h}(\mathbb{N})\mathbf{h}(\mathbb{N})^{-1} \setminus \{1_G\}$:

Proposition 3.3.17. *Suppose that G is a locally-compact Polish group, $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G, $\mathbf{g} \in G^{\mathbb{N}}$ is $(K_n)_{n\in\mathbb{N}}$-expansive, and $\mathbf{h} \in G^{\mathbb{N}}$ is sufficiently $(K_n)_{n\in\mathbb{N}}$-expansive. Then there is no compact set $K \subseteq G$ with the property that $IP(\mathbf{g})IP(\mathbf{g})^{-1} \subseteq K^{-1}\mathbf{h}(\mathbb{N})\mathbf{h}(\mathbb{N})^{-1}K$.*

Proof. Suppose, towards a contradiction, that there is such a K, and set $H_n = \{\mathbf{h}_m \mid m < n\}$ for all $n \in \mathbb{N}$. The $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{g}$ ensures that $\mathbf{g}(\mathbb{N})$ is closed, discrete, and infinite, so by passing to a subsequence of $\mathbf{g}$, we can assume that there is a strictly increasing sequence $k \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbf{g}_n \in K^{-1}(\mathbf{h}_{k_n}H_{k_n}^{-1})^{\pm 1}K$ for all $n \in \mathbb{N}$. By passing to a terminal segment of $\mathbf{g}$, we can assume that $KK^{-1} \subseteq K_{k_0}$.

Lemma 3.3.18. *For all $n \in \mathbb{N}$, the set $IP(\mathbf{g} \upharpoonright n)\mathbf{g}_n(IP(\mathbf{g} \upharpoonright n))^{-1}$ is contained in $K^{-1}(\mathbf{h}_{k_n}H_{k_n}^{-1})^{\pm 1}K$.*

Proof. Granting that we have established the lemma below n, suppose that $s, t \in 2^n$, fix $k \in \mathbb{N}$ for which $\mathbf{g}^s\mathbf{g}_n(\mathbf{g}^t)^{-1} \in K^{-1}(\mathbf{h}_kH_k^{-1})^{\pm 1}K$, and note that $\mathbf{g}^s\mathbf{g}_n(\mathbf{g}^t)^{-1} \in K^{-1}H_{k_n}H_{k_n}^{-1}KK^{-1}(\mathbf{h}_{k_n}H_{k_n}^{-1})^{\pm 1}KK^{-1}H_{k_n}H_{k_n}^{-1}K$. A simple calculation then reveals that if $k \neq k_n$ and $\ell = \max(k, k_n)$, then $\mathbf{h}_\ell \in (K_\ell H_\ell H_\ell^{-1})^3K_\ell H_\ell$, contradicting the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

Lemma 3.3.19. *Suppose that $k, m \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ and $t \in 2^n$ such that $\forall s \in 2^m$ $\mathbf{g}^{s\frown t\frown(1)} \in K^{-1}(\mathbf{h}_{k_{m+n}}(H_{k_{m+n}} \setminus H_k)^{-1})^{\pm 1}K$.*

Proof. Suppose that the lemma fails, and fix $n \in \mathbb{N}$ for which $k_{m+n} \geq k$. Then there exist $i \in \{\pm 1\}$, $s_0, s_1 \in 2^m$, and distinct $t_0, t_1 \in 2^2$ such that $\forall j < 2$ $\mathbf{g}^{s_j\frown(0)^n\frown t_j\frown(1)} \in K^{-1}(\mathbf{h}_{k_{m+n+2}}H_k^{-1})^iK$, and $\ell \in \{m+n, m+n+1\}$ for which $\mathbf{g}^{s_0\frown(0)^n\frown t_0}(\mathbf{g}^{s_1\frown(0)^n\frown t_1})^{-1} \in K^{-1}(\mathbf{h}_{k_\ell}H_{k_\ell}^{-1})^{\pm 1}K$. A simple calculation then yields that $\mathbf{h}_{k_\ell} \in K_{k_\ell}\mathbf{h}_{k_{m+n+2}}H_{k_\ell}^{-1}K_{k_\ell}H_{k_\ell}\mathbf{h}_{k_{m+n+2}}^{-1}K_{k_\ell}H_{k_\ell} \cup K_{k_\ell}H_{k_\ell}\mathbf{h}_{k_{m+n+2}}^{-1}K_{k_\ell}\mathbf{h}_{k_{m+n+2}}H_{k_\ell}^{-1}K_{k_\ell}H_{k_\ell}$, which contradicts the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

In particular, there exist sequences $s_n \in 2^{<\mathbb{N}}$ such that $\mathbf{g}^{\phi(t\frown(1))} \in K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}}(H_{k_{n+\sum_{m\leq n}|s_m|}} \setminus H_{k_{n+\sum_{m<n}|s_m|}})^{-1})^{\pm 1}K$ for all $n \in \mathbb{N}$ and $t \in 2^n$, where $\phi\colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ is given by $\phi(t) = \bigoplus_{n<|t|} s_n \frown t_n$.

Lemma 3.3.20. *Suppose that* $i \in \{\pm 1\}$, $n \in \mathbb{N}$, $\ell_0, \ell_1 \in [k_{n+\sum_{m<n}|s_m|}, k_{n+\sum_{m\leq n}|s_m|})$, $t_0, t_1 \in 2^n$, *and* $\mathbf{g}^{\phi(t_j \frown (1))} \in K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_j}^{-1})^i K$ *for all* $j < 2$. *Then* $\ell_0 = \ell_1$.

Proof. Observe that if $\ell_0 \neq \ell_1$, $k = k_{n+\sum_{m\leq n}|s_m|}$, and $\ell = \max(\ell_0, \ell_1)$, then $K^{-1}H_\ell H_\ell^{-1}KK^{-1}(\mathbf{h}_k H_\ell^{-1})^i K \cap K^{-1}(\mathbf{h}_k \mathbf{h}_\ell^{-1})^i K \neq \emptyset$, so a straightforward calculation reveals that $\mathbf{h}_\ell \in K_\ell^{-1} H_\ell \mathbf{h}_k^{-1} K_\ell^{-1} H_\ell H_\ell^{-1} K_\ell^{-1} \mathbf{h}_k \cup K_\ell H_\ell H_\ell^{-1} K_\ell H_\ell \mathbf{h}_k^{-1} K_\ell \mathbf{h}_k$, contradicting the sufficient $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{h}$. ⊠

In particular, there are integers $\ell_{i,n} \in [k_{n+\sum_{m<n}|s_m|}, k_{n+\sum_{m\leq n}|s_m|})$ such that $\mathbf{g}^{\phi(t\frown(1))} \in \bigcup_{i\in\{\pm1\}} K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^i K$ for all $n \in \mathbb{N}$ and $t \in 2^n$. Fix $N \in [\mathbb{N}]^{\aleph_0}$ with the property that the closure L_i of $\limsup_{n\in N} K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^i K K^{-1}(\mathbf{h}_{k_{n+\sum_{m\leq n}|s_m|}} \mathbf{h}_{\ell_{i,n}}^{-1})^{-i} K$ is compact for all $i \in \{\pm 1\}$, as well as $n \in \mathbb{N}$ such that $L_{-1} \cup L_1 \subseteq K_n$, and $i \in \{\pm1\}$, $N' \in [N \setminus (n+2)]^{\aleph_0}$, and distinct $t_0, t_1 \in 2^2$ with the property that $\mathbf{g}^{\phi((0)^n \frown t_j \frown (0)^{n'-n-2} \frown (1))} \in K^{-1}(\mathbf{h}_{k_{n'+\sum_{m\leq n'}|s_m|}} \mathbf{h}_{\ell_{i,n'}}^{-1})^i K$ for all $j < 2$ and $n' \in N'$. Then $\mathbf{g}^{\phi((0)^n \frown t_0)}(\mathbf{g}^{\phi((0)^n \frown t_1)})^{-1} \in L_i$, contradicting the $(K_n)_{n\in\mathbb{N}}$-expansivity of $\mathbf{g}$. ⊠

We now establish our basis and anti-basis theorems for our two notions of admitting large weakly-wandering Borel sets:

Theorem 3.3.21. *Suppose that $G \curvearrowright X$ is a Borel (continuous) free action of a locally-compact Polish group on a Polish space that does not admit a weakly-wandering σ-complete Borel set. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that does not admit a weakly-wandering σ-complete Borel set but does admit a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorem 2.1.8, Proposition 3.3.6, and Proposition 3.3.11. ⊠

Theorem 3.3.22. *Suppose that $G \curvearrowright X$ is a Borel (continuous) free action of a locally-compact Polish group on a Polish space that does not admit a cover by countably-many weakly-wandering Borel set. Then there is a continuous disjoint union of actions obtained via expansive cutting and stacking that does not admit a cover by countably-many weakly-wandering Borel sets but does admit a Borel (continuous) stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. By Theorem 2.1.8 and Proposition 3.3.10. ⊠

Theorem 3.3.23. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space that does not admit a weakly-wandering σ-complete Borel set, and $\mathcal{F}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and does not admit a weakly-wandering σ-complete Borel set, but to which no action in $\mathcal{F}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorem 2.2.6 and Propositions 3.1.1, 3.3.6, and 3.3.11. ⊠

Theorem 3.3.24. *Suppose that $G \curvearrowright X$ is a Borel free action of a locally-compact Polish group on a standard Borel space that does not admit a cover by countably-many weakly-wandering Borel sets, and $\mathcal{F}$ is a countable family of non-smooth Borel actions on standard Borel spaces. Then there is a Borel G-action on a standard Borel space that admits a Borel stabilizer-preserving homomorphism to $G \curvearrowright X$ and does not admit a cover by countably-many weakly-wandering Borel sets, but to which no action in $\mathcal{F}$ admits a Borel almost stabilizer-preserving-homomorphism.*

Proof. By Theorem 2.2.6 and Propositions 3.1.1 and 3.3.10. ⊠

We say that a set $Y \subseteq X$ is *locally very-weakly-wandering* if for all $n \in \mathbb{N}$ and $x \in X$, there is a set $S \subseteq G$ of cardinality n such that $Gx \cap Y$ is S-wandering.

Proposition 3.3.25. *Suppose that $\mathbf{g}_n = 3^n$ for all $n \in \mathbb{N}$. Then there is neither a $\mathbb{Z}$-invariant Borel probability measure on $\mathbb{X}_\mathbf{g}$ nor a smooth Borel superequivalence relation F of $E_\mathbb{Z}^{\mathbb{X}_\mathbf{g}}$ such that $\mathbb{Z} \curvearrowright [x]_F$ admits a locally-very-weakly-wandering complete Borel set for all $x \in \mathbb{X}_\mathbf{g}$.*

Proof. It is easy to see that the sets $B_k^n = (\{k\} \times \mathcal{N}_{(0)^n})/E_\mathbf{g}$ for $k \in [0, 3^n)$ are pairwise disjoint, for every $n \in \mathbb{N}$. If μ is a $\mathbb{Z}$-invariant Borel measure on $\mathbb{X}_\mathbf{g}$, then a straightforward calculation shows that $\mu(\mathbb{X}_\mathbf{g}) \geq (3/2)^n \mu((\{0\} \times 2^\mathbb{N})/E_\mathbf{g})$ for all $n \in \mathbb{N}$. It follows that $\mu(\mathbb{X}_\mathbf{g}) \in \{0, \infty\}$.

Suppose that F is a smooth Borel superequivalence relation of $E_\mathbb{Z}^{\mathbb{X}_\mathbf{g}}$. As Proposition 1.2.3 ensures that $\mathbb{Z} \curvearrowright \mathbb{X}_\mathbf{g}$ is minimal, every F-invariant set with the Baire property is comeager or meager, so there exists $x \in \mathbb{X}_\mathbf{g}$ for which $[x]_F$ is comeager. Suppose that $B \subseteq [x]_F$ is a Borel set that is complete with respect to $G \curvearrowright [x]_F$. Then the countability of $\mathbb{Z}$ ensures that B is non-meager, so there exist $n \in \mathbb{Z}$ and $s \in 2^{<\mathbb{N}}$ for which B is comeager in the open set $U = (\{n\} \times \mathcal{N}_s)/E_\mathbf{g}$.

We will make use of the following well-known fact, which we provide a proof of for the reader's convenience.

Lemma 3.3.26. *For all $z \in \mathbb{Z}$ there exist finite disjoint sets P_z and N_z of $\mathbb{N}$ such that $z = \sum_{k \in P_z} 3^k - \sum_{k \in N_z} 3^k$.*

Proof. Fix $m \in \mathbb{N}$ such that $|z| \leq \sum_{k<m} 3^k$. Let $t \in 3^m$ be the base three representation of $z + \sum_{k<m} 3^k$ of length m, and define P_z, N_z by $k \in P_z \iff t_k = 2$ and $k \in N_z \iff t_k = 0$. Then $z = \sum_{k \in P_z} 3^k - \sum_{k \in N_z} 3^k$. ⊠

The lemma implies that $\Delta_{\mathbb{Z}}^{\mathbb{X}_{\mathbf{g}}}(U^{\{0,1\}}) = 3^{|s|}\mathbb{Z}$. Fix $y \in [x]_F \setminus \mathbb{Z}(U \setminus B)$ and note that $\Delta_{\mathbb{Z}}^{\mathbb{X}_{\mathbf{g}}}((U \cap \mathbb{Z}y)^{\{0,1\}}) = \Delta_{\mathbb{Z}}^{\mathbb{X}_{\mathbf{g}}}(U^{\{0,1\}}) = 3^{|s|}\mathbb{Z}$. Suppose that $S \subseteq \mathbb{Z}$ is a set of cardinality strictly greater than $3^{|s|}$. Then there exist distinct $j, k \in S$ such that $j - k \in 3^{|s|}\mathbb{Z}$, thus $\Delta_{\mathbb{Z}}^{\mathbb{X}_{\mathbf{g}}}((U \cap \mathbb{Z}y)^{\{0,1\}}) \cap ((S - S) \setminus \{0\}) \neq \emptyset$. Hence $\mathbb{Z} \curvearrowright [x]_F$ does not admit a locally-very-weakly-wandering complete Borel set. ⊠

Remark 3.3.27. The *odometer* on $3^{\mathbb{N}}$ is the isometry $\sigma \colon 3^{\mathbb{N}} \to 3^{\mathbb{N}}$ given by $\sigma((2)^n \frown (i) \frown c) = (0)^n \frown (i+1) \frown c$, where $c \in 3^{\mathbb{N}}$ and $i < 2$. It is easy to see that the above action $\mathbb{Z} \curvearrowright \mathbb{X}_{\mathbf{g}}$ is Borel isomorphic to that generated by the restriction of σ to the saturation of $2^{\mathbb{N}}$.

4. Mixing

4.1. **Weak mixing.** For our current purpose we restrict ourselves to families of the form $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$ and say that an action $G \curvearrowright X$ by homeomorphisms of a topological space is *$\mathcal{S}$-transitive* if $\Delta_G^X(\prod_{k \leq d} V_k) \cap S \neq \emptyset$ for all $d \in \mathbb{Z}^+$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and sequences $(V_k)_{k \leq d}$ of non-empty open subsets of X.

Proposition 4.1.1. *Suppose that G is a group, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is an $\mathcal{S}$-transitive action by homeomorphisms of a topological space. Then $G \curvearrowright X$ is $\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}}(\mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}}))G$-transitive.*

Proof. Note that if $d \in \mathbb{Z}^+$, $g \in G^{\{0,\ldots,d\}}$, $h \in G^{\{1,\ldots,d\}}$, and $(X_k)_{k \leq d}$ is a sequence of subsets of X, then

$$\begin{aligned} h \in \Delta_G^X(\textstyle\prod_{k \leq d} g_k X_k) &\iff g_0 X_0 \cap \textstyle\bigcap_{1 \leq k \leq d} h_k^{-1} g_k X_k \neq \emptyset \\ &\iff X_0 \cap \textstyle\bigcap_{1 \leq k \leq d} (g_k^{-1} h_k g_0)^{-1} X_k \neq \emptyset \\ &\iff (g_k^{-1} h_k g_0)_{1 \leq k \leq d} \in \Delta_G^X(\textstyle\prod_{k \leq d} X_k) \\ &\iff h \in (g_k)_{1 \leq k \leq d} \Delta_G^X(\textstyle\prod_{k \leq d} X_k) g_0^{-1}. \end{aligned}$$

It follows that if $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$ and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X, then the fact that $\Delta_G^X(\prod_{k \leq d} g_k U_k) \cap S \neq \emptyset$ ensures that $\Delta_G^X(\prod_{k \leq d} U_k) \cap (g_k^{-1})_{1 \leq k \leq d} S g_0 \neq \emptyset$. ⊠

Proposition 4.1.2. *Suppose that G is a topological group, X is a topological space, $H \subseteq G$ is dense, $\mathcal{S} \subseteq \bigcup_{d \in \mathbb{Z}^+} \mathcal{P}(G^{\{1,\ldots,d\}})$, and $G \curvearrowright X$ is continuous, $\bigcup_{d \in \mathbb{Z}^+} H^{\{1,\ldots,d\}}(\mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}}))$-recurrent, and topologically transitive. Then $G \curvearrowright X$ is $\mathcal{S}$-transitive.*

Proof. Suppose that $d \in \mathbb{Z}^+$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X. Set $V_0 = U_0$, and construct $h \in H^{\{1,\ldots,d\}}$ by recursively appealing to the topological transitivity of $G \curvearrowright X$ to obtain $h_{k+1} \in H$ such that the set $V_{k+1} = h_{k+1}U_{k+1} \cap V_k$ is non-empty for all $k < d$. As $\Delta_G^X(V_d^{\{0,\ldots,d\}}) \cap hS \neq \emptyset$, the same calculation as in the proof of Proposition 4.1.1 reveals that $\Delta_G^X((\overline{h})^{-1}V_d) = h^{-1}\Delta_G^X(V_d^{\{0,\ldots,d\}})$, so $\Delta_G^X((\overline{h})^{-1}V_d) \cap S \neq \emptyset$. As $(\overline{h}_k)^{-1}V_d \subseteq U_k$ for all $k \leq d$, it follows that $\Delta_G^X(\prod_{k \leq d} U_k) \cap S \neq \emptyset$. ⊠

Observe that if $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, $x \in X$, and Gx is non-meager, then there is a compact set $K \subseteq G$ for which Kx is non-meager, and therefore comeager in some non-empty open set $U \subseteq X$, in which case the fact that Kx is closed ensures that $U \subseteq Kx$, thus $Gx = GU$ is an σ-expansively-$\{G\}$-transient open orbit.

Proposition 4.1.3. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a Hausdorff space with no open orbits, $K \subseteq G$ is compact, $d \in \mathbb{Z}^+$, and $(U_k)_{k \leq d}$ is a sequence of non-empty open subsets of X. Then there are non-empty open sets $V_k \subseteq U_k$ for which $(V_k)_{k \leq d}$ is R_K^X-discrete.*

Proof. By the obvious induction, it is sufficient to show that for all distinct $j, k \leq d$, there are non-empty open sets $V_j \subseteq U_j$ and $V_k \subseteq U_k$ such that $V_j \cap KV_k = \emptyset$. Towards this end, fix $x_k \in U_k$, and note that $U_j \nsubseteq Gx_k$, since otherwise $GU_j = Gx_k$, contradicting the fact that Gx_k is not open. Fix $x_j \in U_j \setminus Kx_k$, and observe that Proposition 1.2.4 yields open neighborhoods $V_j \subseteq U_j$ of x_j and $V_k \subseteq U_k$ of x_k such that $V_j \cap KV_k = \emptyset$. ⊠

Along similar lines, we say that $G \curvearrowright X$ is *expansively $\mathcal{S}$-transitive* if $\Delta_G^X(\{y \in \prod_{k \leq d} V_k \mid y \text{ is } R_K^X\text{-discrete}\}) \cap \mathcal{S} \neq \emptyset$ for all $d \in \mathbb{Z}^+$, compact sets $K \subseteq G$, $S \in \mathcal{S} \cap \mathcal{P}(G^{\{1,\ldots,d\}})$, and sequences $(V_k)_{k \leq d}$ of non-empty open subsets of X.

Proposition 4.1.4. *Suppose that $d \in \mathbb{Z}^+$, $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space, and $H \subseteq G$ is dense. Then $G \curvearrowright X$ is topologically d-transitive and has no open orbits if and only if it is topologically transitive and expansively*

$(\bigcup_{g \in G} gH^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_{2i+1} = g_1 g_{2i}\}g^{-1})$*-recurrent.*

Proof. Clearly $G \curvearrowright X^d$ is topologically transitive if and only if $G \curvearrowright X$ is $\{\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}\}$-transitive. By Proposition 4.1.1, the latter condition holds if and only if $G \curvearrowright X$ is $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-transitive. By Proposition 4.1.3 and the comment immediately preceding it, the conjunction of this with the inexistence of open orbits is equivalent to the expansive $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-transitivity of $G \curvearrowright X$. And this holds if and only if $G \curvearrowright X$ is expansively $H^{\{1,\ldots,2d-1\}}\{g \in G^{\{1,\ldots,2d-1\}} \mid \forall 0 < i < d \; g_1 g_{2i} = g_{2i+1}\}$-recurrent and topologically transitive, by Proposition 4.1.2. ⊠

We now establish our basis theorem for weakly-mixing continuous actions of Polish groups:

Theorem 4.1.5. *Suppose that $G \curvearrowright X$ is a topologically-transitive continuous action of a locally-compact Polish group on a Polish space with no open orbits. Then the following are equivalent:*

1. *The action $G \curvearrowright X$ is weakly mixing.*
2. *There is a Baire-measurable stabilizer-preserving homomorphism from a weakly-mixing G-action obtained via expansive cutting and stacking to $G \curvearrowright X$.*
3. *There is a continuous embedding of a weakly-mixing G-action obtained via expansive cutting and stacking into $G \curvearrowright X$.*

Proof. By Theorem 2.1.7 and Proposition 4.1.4. ⊠

We now establish our anti-basis theorem for weakly-mixing continuous actions of Polish groups:

Theorem 4.1.6. *Suppose that $G \curvearrowright X$ is a weakly-mixing continuous action of a locally-compact Polish group on a Polish space. Then there is a family $\mathcal{A}$ of continuum-many weakly-mixing continuous G-actions on Polish spaces that admit continuous embeddings into $G \curvearrowright X$ such that every non-smooth Borel G-action on a standard Borel space admits a Borel stabilizer-preserving homomorphism to at most one action in $\mathcal{A}$.*

Proof. By Theorem 2.2.5 and Propositions 3.1.1 and 4.1.4. ⊠

We now establish the promised equivalence of the measure-theoretic and topological notions of weak mixing:

Theorem 4.1.7. *Suppose that $G \curvearrowright X$ is a continuous action of an abelian locally-compact Polish group on a Polish space. Then the following are equivalent:*

1. *There is a G-invariant σ-finite Borel measure μ on X with respect to which $G \curvearrowright X$ is weakly mixing.*
2. *There is a G-invariant closed set $C \subseteq X$ for which $G \curvearrowright C$ is weakly mixing.*

Proof. To see (1) $\Longrightarrow$ (2), let C be the complement of the union of all μ-null non-empty open sets $U \subseteq X$, and observe that if $U, U', V, V' \subseteq C$ are non-empty open sets, then the G-saturations of $U \times V$ and $U' \times V'$ are $(\mu \times \mu)$-conull, thus $\Delta_G^{C \times C}((U \times V) \times (U' \times V')) \neq \emptyset$.

To see (2) $\Longrightarrow$ (1), we first note the following:

Lemma 4.1.8. *Suppose that $x \in C$ and Gx is an open subset of C. Then x is the unique element of Gx, and therefore of C.*

Proof. Note that if $g \in G$ and $U \subseteq G$, then $R_{gU}^{Gx} = (g, 1_G) R_U^{Gx}$. It follows that if $H \subseteq G$ is a countable dense set, $H' = H \times \{1_G\}$, and $U \subseteq G$ is a non-empty open set, then $Gx \times Gx = \bigcup_{h \in H} R_{hU}^{Gx} = H' R_U^{Gx}$, so R_U^{Gx} is not meager.

Proposition 1.1.2 easily implies that $G/\mathrm{Stab}(x)$ is a Hausdorff space. It follows that if x is not the unique element of Gx, in which case $\mathrm{Stab}(x) \neq G$, then there are disjoint non-empty open sets $U, V \subseteq G/\mathrm{Stab}(x)$. As G is abelian, it follows that $R_{\bigcup U}^{Gx}$ and $R_{\bigcup V}^{Gx}$ are disjoint G-invariant non-meager sets with the Baire property, contradicting the fact that $G \curvearrowright C \times C$ is topologically transitive. ⊠

If C is a singleton, then any finite Borel measure concentrating on C is as desired. Otherwise, fix a countable dense subgroup H of G, as well as an exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G. By the proof of Theorem 4.1.5 and Lemma 4.1.8, we can assume that $G \curvearrowright X$ is of the form $G \curvearrowright \mathbb{X}_{\mathbf{h}}$, where $\mathbf{h} \in (H^{\{1,2,3\}})^{\mathbb{N}}$ is $(K_n)_{n \in \mathbb{N}}$-expansive and $\forall h \in H \exists^\infty n \in \mathbb{N}\ h(\mathbf{h}_n)_1 (\mathbf{h}_n)_2 = (\mathbf{h}_n)_3$.

For each $n \in \mathbb{N}$, let G_n denote the digraph on 2^n consisting of all pairs $(s, t) \in 2^n \times 2^n$ such that $\mathrm{supp}(s) \subseteq \mathrm{supp}(t)$ and $\mathrm{supp}(t) \setminus \mathrm{supp}(s)$ is a singleton.

Lemma 4.1.9. *Suppose that $n \in \mathbb{N}$. Then there there is a partial injection $\phi \colon 2^n \rightharpoonup 2^n$ whose graph is contained in G_n and whose domain has cardinality $2^n - \binom{n}{\lceil n/2 \rceil}$.*

Proof. Set $\phi_0 = \emptyset$ and recursively define

$$\phi_{n+1}(s \frown (i)) = \begin{cases} \phi_n(s) \frown (i) & \text{if } s \in \mathrm{dom}(\phi_n^{i+1}) \text{ and} \\ s \frown (1) & \text{if } i = 0 \text{ and } s \notin \mathrm{dom}(\phi_n). \end{cases}$$

To see that the injectivity of ϕ_n yields that of ϕ_{n+1}, suppose that $i \le j \le 1$, $s \frown (i)$ and $t \frown (j)$ are distinct elements of $\mathrm{dom}(\phi_{n+1})$, and $\phi_{n+1}(s \frown (i))(n) = \phi_{n+1}(t \frown (j))(n)$. Let k be the latter quantity. If $i = j = k = 0$ or $i = j = 1$, then $s \neq t$, so the injectivity of ϕ_n ensures that $\phi_{n+1}(s \frown (i)) = \phi_n(s) \frown (i) \neq \phi_n(t) \frown (j) = \phi_{n+1}(t \frown (j))$. Similarly, if $i = j = 0$ and $k = 1$, then $s \neq t$, and it immediately follows that $\phi_{n+1}(s \frown (i)) = s \frown (1) \neq t \frown (1) = \phi_{n+1}(t \frown (j))$. Finally, if $i < j$, then $s \notin \mathrm{dom}(\phi_n)$ and $\phi_n(t) \in \mathrm{dom}(\phi_n)$, so $s \neq \phi_n(t)$, thus $\phi_{n+1}(s \frown (i)) = s \frown (1) \neq \phi_n(t) \frown (1) = \phi_{n+1}(t \frown (j))$.

For all $n \in \mathbb{N}$ and $s \in 2^n$, define $m(s) = \max\{m \in \mathbb{N} \mid s \in \mathrm{dom}(\phi_n^m)\}$. As the definition of ϕ_{n+1} ensures that $m(s \frown (0)) = m(s) + 1$ and $m(s \frown (1)) \ge m(s) - 1$, a straightforward inductive argument reveals that $m(s) \ge n - 2|\mathrm{supp}(s)|$, so the set $T_n = \{t \in 2^n \mid |\mathrm{supp}(t)| = \lceil n/2 \rceil\}$ is a transversal of the orbit equivalence relation generated by ϕ_n. As the complement of $\mathrm{dom}(\phi_n)$ is also a transversal of this equivalence relation, it follows that $|T_n| = |{\sim}\mathrm{dom}(\phi_n)|$, in which case $|\mathrm{dom}(\phi_n)| = 2^n - |{\sim}\mathrm{dom}(\phi_n)| = 2^n - |T_n| = 2^n - \binom{n}{\lceil n/2 \rceil}$. ⊠

Let μ be the $\mathbb{N}$-fold power of the uniform probability measure on $\{0, 1, 2, 3\}$.

Lemma 4.1.10. *Suppose that $\epsilon > 0$, $n \in \mathbb{N}$, $h \in H$, and $s, t \in 4^n \times 4^n$. Then there exist a clopen set $C \subseteq \mathcal{N}_{s_0} \times \mathcal{N}_{s_1}$ and continuous functions $\phi_i \colon C \to \mathcal{N}_{t_i}$ with the property that $\phi_0 \times \phi_1$ is injective, $(\mu \times \mu)(C) \ge (1 - \epsilon)(\mu \times \mu)(\mathcal{N}_{s_0} \times \mathcal{N}_{s_1})$, and $\wp_{\mathbf{h}}(c_0, \phi_0(c_0, c_1))h = \wp_{\mathbf{h}}(c_1, \phi_1(c_0, c_1))$ for all $(c_0, c_1) \in C$.*

Proof. It is well known that $\binom{k}{\lceil k/2 \rceil}/2^k$ converges to zero, so there exists $k \in \mathbb{N}$ for which $\binom{k}{\lceil k/2 \rceil}/2^k < \epsilon$. For all $\ell \in \mathbb{N}$, appeal to Lemma 4.1.9 to obtain a partial injection $\phi_\ell \colon 2^\ell \rightharpoonup 2^\ell$ whose graph is contained in G_ℓ and whose domain has cardinality $2^\ell - \binom{\ell}{\lceil \ell/2 \rceil}$. For all $(u_0, u_1) \in 4^{\le \mathbb{N}} \times 4^{\le \mathbb{N}}$, let $K_{(u_0, u_1)}$ be the set of $k \in \bigcap_{i<2} \mathrm{dom}(u_i)$ with the property that $h^{-1}(\mathbf{h}^{s_0})^{-1}\mathbf{h}^{t_0}(\mathbf{h}^{t_1})^{-1}\mathbf{h}^{s_1}(\mathbf{h}_{k+n})_1(\mathbf{h}_{k+n})_2 = (\mathbf{h}_{k+n})_3$ and $((u_0)_k, (u_1)_k) \in \{(0, 2), (1, 3)\}$. As $K_{(c_0, c_1)}$ is infinite for $(\mu \times \mu)$-almost every $(c_0, c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}}$, there exists $m \in \mathbb{N}$ such that $(\mu \times \mu)(\{(c_0, c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}} \mid |K_{(c_0 \restriction m, c_1 \restriction m)}| < k\}) + \binom{k}{\lceil k/2 \rceil}/2^k < \epsilon$. For all $K \subseteq m$, let $(k_i^K)_{i<|K|}$ be the strictly increasing enumeration of K. For all $r_0, r_1 \in 4^{m \setminus K}$, set $U_{K,(r_0, r_1)} = \{(u_0, u_1) \in 4^m \times 4^m \mid K = K_{(u_0, u_1)} \text{ and } \forall i < 2\ r_i \sqsubseteq u_i\}$, and define $\psi_{K,(r_0,r_1)} \colon U_{K,(r_0,r_1)} \to 2^{|K|}$ by $\psi_{K,(r_0,r_1)}(u_0,u_1)_i = (u_0)_{k_i^K}$. Define $\pi \colon 4^m \times 4^m \rightharpoonup 4^m \times 4^m$ by $\pi(u_0,u_1) = (\psi^{-1}_{K,(r_0,r_1)} \circ \phi_{|K|} \circ \psi_{K,(r_0,r_1)})(u_0,u_1)$, where $K = K_{(u_0,u_1)}$ and $r_i = u_i \restriction (m \setminus K)$ for all $i < 2$, and observe that the partial function $(s_i \frown u_i \frown c_i)_{i<2} \mapsto (t_i \frown \pi_i(u_0,u_1) \frown c_i)_{i<2}$ is as desired, by Proposition 1.2.1. ⊠

Fix a Haar measure μ_G on G. Clearly $G \curvearrowright G \times 4^{\mathbb{N}}$ is invariant with respect to $\mu_G \times \mu$, and the latter is $E_{\mathbf{h}}$-invariant.

Lemma 4.1.11. *Suppose that $B \subseteq (G \times 4^{\mathbb{N}}) \times (G \times 4^{\mathbb{N}})$ is G-invariant and $(E_{\mathbf{h}} \times E_{\mathbf{h}})$-invariant. Then B or $\sim B$ is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-null.*

Proof. Suppose that B is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-positive. Then Fubini's theorem (see, for example, [Kec95, §17.A]) yields $g_0 \in G$ such that the set $B_{(g_0,g_1)} = \{(c_0,c_1) \in 4^{\mathbb{N}} \times 4^{\mathbb{N}} \mid ((g_0,c_0),(g_1,c_1)) \in B\}$ is $(\mu \times \mu)$-positive for a μ_G-positive set of $g_1 \in G$. Lemma 4.1.10 ensures that if $\epsilon > 0$, $g_1 \in G$, $h \in H$, $s \in \bigcup_{n \in \mathbb{N}} 4^n \times 4^n$, and $B_{(g_0,g_1)}$ has density strictly greater than $1 - \epsilon$ in $\mathcal{N}_{s_0} \times \mathcal{N}_{s_1}$, then $B_{(g_0,g_1 h)}$ has density strictly greater than $1 - \epsilon$ in $4^{\mathbb{N}} \times 4^{\mathbb{N}}$ for all $h \in H$. It follows that if $g_1 \in G$ and $(\mu \times \mu)(B_{(g_0,g_1)}) > 0$, then $(\mu \times \mu)(B_{(g_0,g_1 h)}) = 1$ for all $h \in H$, so $(\mu \times \mu)(B_{(g_0,g_1)}) = 1$ for μ_G-almost all $g_1 \in G$, since the uniqueness of Haar measure up to a scaling factor ensures that $H \curvearrowright G$ is ergodic with respect to μ_G. As B is G-invariant, it follows that B is $(\mu_G \times \mu) \times (\mu_G \times \mu)$-conull. ⊠

It follows that the restriction of $\mu_G \times \mu$ to any Borel transversal of $E_{\mathbf{h}}$ induces the desired measure on $\mathbb{X}_{\mathbf{h}}$. ⊠

Remark 4.1.12. While the above arguments work just as well for topological d-transitive when $d > 2$, this does not yield any greater generality, as these notions coincide with weak mixing for abelian groups.

We next turn our attention to anti-basis theorems for strengthenings of weak mixing. The primary observation we will use to obtain such results is the following:

Proposition 4.1.13. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric, $G \curvearrowright X$ is a continuous action on a non-empty Polish space, $G \curvearrowright Y$ is a continuous action on a Polish space with at least two elements, and $G \curvearrowright X \times Y$ is topologically transitive. Then there exist $x \in X$ and a G-invariant dense G_δ set $C \subseteq Y$ for which there is no continuous homomorphism $\phi \colon X \to Y$ from $G \curvearrowright X$ to $G \curvearrowright Y$ with the property that $\phi(x) \in C$.*

Proof. Fix a compatible complete metric on X, positive real numbers $\epsilon_n \to 0$, non-empty open sets $W_0, W_1 \subseteq Y$ with disjoint closures, and open neighborhoods $U \subseteq G$ of 1_G and non-empty open sets $W'_0, W'_1 \subseteq Y$ such that $UW'_i \subseteq W_i$ for all $i < 2$. By [Kle52, 1.5], we can assume that U is conjugation invariant. Fix natural numbers $i_n < 2$ and non-empty open sets $V_n \subseteq Y$ such that for all $i < 2$ and non-empty open sets $V \subseteq Y$, there are infinitely many $n \in \mathbb{N}$ for which $i_n = i$ and $V_n \subseteq V$.

Set $U_0 = X$. Given $n \in \mathbb{N}$ and a non-empty open set $U_n \subseteq X$, fix $g_n \in \Delta_G^{X \times Y}((U_n \times V_n) \times (U_n \times W'_{i_n}))$ and non-empty open sets $U_{n+1} \subseteq X$ and $V'_n \subseteq V_n$ such that $\operatorname{diam}(U_{n+1}) \leq \epsilon_n$, $\overline{U_{n+1}} \cup g_n U_{n+1} \subseteq U_n$, and $g_n V'_n \subseteq W'_{i_n}$.

Let x be the unique point of $\bigcap_{n \in \mathbb{N}} U_n$. Note that for all $i < 2$ and $n \in \mathbb{N}$, the open set $V_{i,n} = \bigcup_{i=i_m, m \geq n} V'_m$ is dense, thus so too is the G_δ set $D = \bigcap_{i<2, n \in \mathbb{N}} V_{i,n}$. Fix a countable dense set $H \subseteq G$, and observe that the G_δ set $D_H = \bigcap_{h \in H} h^{-1}D$ is also dense. Noting that $\forall g \in G \forall^* y \in Y \; g \cdot y \in D_H$, the Kuratowski-Ulam theorem ensures that the G-invariant set $C = \{y \in Y \mid \forall^* g \in G \; g \cdot y \in D_H\}$ is comeager. By [Vau75, Corollary 1.8], it is also G_δ.

Suppose now that $\phi \colon X \to Y$ is a continuous homomorphism from $G \curvearrowright X$ to $G \curvearrowright Y$. To see that $\phi(x) \notin C$, it is sufficient to show that if $y \in C$, then $g_n \cdot y \not\to y$, since $g_n \cdot x \to x$. Towards this end, fix $i < 2$ for which $y \notin \overline{W_i}$, as well as $g \in G$ for which $g \cdot y \in D_H$. As $G = UH$, there exists $h \in H$ for which $g^{-1} \in Uh$. As the set $N = \{n \in \mathbb{N} \mid hg \cdot y \in V'_n \text{ and } i = i_n\}$ is infinite, it only remains to note that if $n \in N$, then $g_n \cdot y \in g_n Uhg \cdot y = Ug_n hg \cdot y \subseteq W_i$. ⊠

In order to apply this result to obtain lower bounds on the cardinalities of bases consisting solely of weakly mixing actions, we will need the following straightforward observation:

Proposition 4.1.14. *Suppose that G is a group, $G \curvearrowright X$ is a weakly-mixing action by homeomorphisms of a topological space, $G \curvearrowright Y$ is a minimal action by homeomorphisms of a topological space, and there is a continuous homomorphism $\phi \colon X \to Y$ from $G \curvearrowright X$ to $G \curvearrowright Y$. Then $G \curvearrowright X \times Y$ is topologically transitive.*

Proof. Suppose that $U \times V, U' \times V' \subseteq X \times Y$ are non-empty open sets. As $G \curvearrowright Y$ is minimal, the sets $\phi^{-1}(V)$ and $\phi^{-1}(V')$ are non-empty. As $G \curvearrowright X$ is weakly mixing, the set $\Delta_G^{X \times X}((U \times \phi^{-1}(V)) \times (U' \times \phi^{-1}(V')))$ is non-empty. But the fact that ϕ is a homomorphism ensures that this set is contained in $\Delta_G^{X \times Y}((U \times V) \times (U' \times V'))$. ⊠

As a corollary, we obtain the following:

Theorem 4.1.15. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric and $\mathcal{A}$ is a non-empty class of* *minimal continuous G-actions on Polish spaces of cardinality at least two that is closed under restrictions to G-invariant dense G_δ sets. Then any basis $\mathcal{B}$ for $\mathcal{A}$ under continuous homomorphism consisting solely of weakly-mixing actions has cardinality at least the additivity of the meager ideal.*

Proof. Fix an action $G \curvearrowright X$ in $\mathcal{A}$, and suppose, towards a contradiction, that there is an enumeration $(G \curvearrowright X_\alpha)_{\alpha<\kappa}$ of $\mathcal{B}$ of length strictly less than the additivity of the meager ideal. For all $\alpha < \kappa$, Proposition 4.1.14 ensures that $G \curvearrowright X \times X_\alpha$ is topologically transitive, so Proposition 4.1.13 yields a G-invariant dense G_δ set $C_\alpha \subseteq X$ for which there is no continuous homomorphism from $G \curvearrowright X_\alpha$ to $G \curvearrowright C_\alpha$. Fix a dense G_δ set $C \subseteq \bigcap_{\alpha<\kappa} C_\alpha$. Then $\forall g \in G \forall^* x \in X \; g \cdot x \in C$, so the Kuratowski-Ulam theorem ensures that $\forall^* x \in X \forall^* g \in G \; g \cdot x \in C$, in which case $B = \{x \in X \mid \forall^* g \in G \; g \cdot x \in C\}$ is a G-invariant dense G_δ set for which no action in $\mathcal{B}$ admits a continuous homomorphism to $G \curvearrowright B$, the desired contradiction. ⊠

4.2. **Mild mixing.** We begin this section with an alternative characterization of mild mixing:

Proposition 4.2.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space with no open orbits and $(K_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of compact subsets of G. Then $G \curvearrowright X$ is mildly mixing if and only if $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive for all $(K_n)_{n\in\mathbb{N}}$-expansive sequences $\mathbf{g} \in G^{\mathbb{N}}$.*

Proof. By Proposition 1.2.3, it is sufficient to show ($\Longleftarrow$). Towards this end, suppose that $G \curvearrowright Y$ is a topologically-transitive continuous G-action with no open orbits, and fix $y \in Y$ for which $[y]_{F_G^Y}$ is comeager. The minimality of $G \curvearrowright [y]_{F_G^Y}$ ensures that it is topologically transitive. It also ensures that it has no open orbits, since otherwise $[y]_{F_G^Y}$ would itself be an orbit of $G \curvearrowright Y$, and since it is non-meager in Y, it would necessarily be open in Y.

Given $\mathbf{g} \in G^{\mathbb{N}}$ and $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$, we say that $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-*expansive* if $(\mathbf{h}, \mathbf{G})$ is $(K_n)_{n\in\mathbb{N}}$-expansive, where $\mathbf{h} \in (G^{\{1\}})^{\mathbb{N}}$ is given by $(\mathbf{h}_n)_1 = \mathbf{g}_n$ for all $n \in \mathbb{N}$.

Lemma 4.2.2. *There exist a $(K_n)_{n\in\mathbb{N}}$-exhaustive sequence $\mathbf{g} \in G^{\mathbb{N}}$ and a continuous homomorphism $\phi\colon \mathbb{X}_{\mathbf{g}} \to [y]_{F_G^Y}$ from $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ to $G \curvearrowright Y$.*

Proof. While it is easy enough to establish this directly, we will use the tools at hand: By Theorem 2.1.7 and Proposition 4.1.4, there exist a sequence $\mathbf{g} \in G^{\mathbb{N}}$, a continuous function $\mathbf{G}\colon X_{\mathbf{g}} \to \mathcal{F}(G) \cap \mathcal{S}(G)$

compatible with $\wp_{\mathbf{g}}$ for which $(\mathbf{g}, \mathbf{G})$ is $(K_n)_{n \in \mathbb{N}}$-expansive, and a continuous embedding $\psi \colon \mathbb{X}_{\mathbf{g},\mathbf{G}} \to [y]_{F_G^Y}$ from $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$ to $G \curvearrowright Y$. As the function $\pi \colon \mathbb{X}_{\mathbf{g}} \to \mathbb{X}_{\mathbf{g},\mathbf{G}}$ given by $\pi([(g,x)]_{E_{\mathbf{g}}}) = [(g,x)]_{E_{\mathbf{g},\mathbf{G}}}$ is a homomorphism from $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ to $G \curvearrowright \mathbb{X}_{\mathbf{g},\mathbf{G}}$, the function $\phi = \psi \circ \pi$ is as desired. ⊠

Suppose now that $U_0, U_1 \subseteq X$ and $V_0, V_1 \subseteq Y$ are non-empty open sets. The fact that $[y]_{F_G^Y}$ is comeager ensures that it intersects each V_i, so the fact that $G \curvearrowright [y]_{F_G^Y}$ is minimal implies that the pull-back of each V_i through ϕ is non-empty. The topological transitivity of $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ therefore implies that $\Delta_G^{X \times \mathbb{X}_{\mathbf{g}}}(\prod_{i<2} U_i \times \phi^{-1}(V_i))$ is non-empty, and since ϕ is a homomorphism, this set is contained in $\Delta_G^{X \times Y}(\prod_{i<2} U_i \times V_i)$, so the latter set is non-empty as well. ⊠

In light of Proposition 4.2.1, the following facts can be viewed as local refinements of further alternative characterizations of mild mixing:

Proposition 4.2.3. *Suppose that $G \curvearrowright X$ is a continuous action of a topological group on a topological space and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive if and only if $G \curvearrowright X$ is $\{IP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-transitive.*

Proof. Note that if $G \curvearrowright Y$ is topologically transitive, then $G \curvearrowright X \times Y$ is topologically transitive if and only if $\Delta_G^X(U \times V) \cap \Delta_G^Y(W \times W) \neq \emptyset$ for all non-empty open sets $U, V \subseteq X$ and $W \subseteq Y$, since $\Delta_G^X(U \times V) \cap \Delta_G^Y(W \times gW) = g(\Delta_G^X(U \times g^{-1}V) \cap \Delta_G^Y(W \times W))$ for all $g \in G$. In particular, this holds when $Y = \mathbb{X}_{\mathbf{g}}$, since Proposition 1.1.4 ensures that $G \curvearrowright \mathbb{X}_{\mathbf{g}}$ is minimal.

To see ($\Longrightarrow$), suppose that $n \in \mathbb{N}$ and $V, W \subseteq X$ are non-empty open sets, and fix an open neighborhood $U \subseteq G$ of 1_G and non-empty open sets $V', W' \subseteq X$ such that $UV' \subseteq V$ and $UW' \subseteq W$. Then $\Delta_G^X(V' \times W') \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) \neq \emptyset$. But $U\Delta_G^X(V' \times W')U^{-1} = \Delta_G^X(UV' \times UW')$, and it follows from Proposition 1.2.1 that $\Delta_G^{\mathbb{X}_{\mathbf{g}}}((U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}} \times (U^{-1} \times \mathcal{N}_{(0)^n})/E_{\mathbf{g}}) = U^{-1}\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1}U$, so $\Delta_G^X(V \times W) \cap \mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \neq \emptyset$.

To see ($\Longleftarrow$), suppose that $s \in T_{\mathbf{g}}$, and $U \subseteq G$ and $V, W \subseteq X$ are non-empty open sets, and observe that $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}W) \cap \mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))\mathrm{IP}(\mathbb{s}^{|s|}(\mathbf{g}))^{-1} \neq \emptyset$. Noting that $U\mathbf{g}^s\mathrm{IP}(\mathbb{s}^n\mathbf{g})\mathrm{IP}(\mathbb{s}^n\mathbf{g})^{-1}(U\mathbf{g}^s)^{-1} \subseteq \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}})$ by Proposition 1.2.1, the fact that $\Delta_G^X((U\mathbf{g}^s)^{-1}V \times (U\mathbf{g}^s)^{-1}W) = (U\mathbf{g}^s)^{-1}\Delta_G^X(V \times W)U\mathbf{g}^s$ ensures that $\Delta_G^X(V \times W) \cap \Delta_G^{\mathbb{X}_{\mathbf{g}}}((U \times \mathcal{N}_s)/E_{\mathbf{g}} \times (U \times \mathcal{N}_s)/E_{\mathbf{g}}) \neq \emptyset$. ⊠

Proposition 4.2.4. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space and $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$. Then the following are equivalent:*

1. *The action $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive and the action $G \curvearrowright X$ has no open orbits.*
2. *The action $G \curvearrowright X$ is topologically transitive and expansively $\{gIP(\mathbb{s}^n(\mathbf{g}))IP(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-recurrent.*

Proof. Note that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive if and only if $G \curvearrowright X$ is $\{\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid n \in \mathbb{N}\}$-transitive, by Proposition 4.2.3. The latter condition holds if and only if $G \curvearrowright X$ is $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-transitive, by Proposition 4.1.1. The conjunction of this with the inexistence of open orbits is equivalent to the expansive $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-transitivity of $G \curvearrowright X$, by Proposition 4.1.3 and the comment immediately preceding it. And the latter condition holds if and only if $G \curvearrowright X$ is expansively $\{g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1} \mid g \in G$ and $n \in \mathbb{N}\}$-recurrent and topologically transitive, by Proposition 4.1.2. ⊠

As a consequence, we obtain a necessary and sufficient condition for an intransitive minimal continuous action to be mildly mixing:

Theorem 4.2.5. *Suppose that $G \curvearrowright X$ is an intransitive minimal continuous action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

1. *The action $G \curvearrowright X$ is mildly mixing.*
2. *There is a continuous disjoint union of actions that is obtained via expansive cutting and stacking that is not σ-expansively $\{\bigcup_{g \in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{mm}\}$-transient but admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. Fix an exhaustive increasing sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G.

To see (1) $\Longrightarrow$ (2), note that if $S \in \bigcup \boldsymbol{\mathcal{S}}_{mm}$, then there exist $g \in G$ and a $(K_n)_{n \in \mathbb{N}}$-expansive sequence $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ for which $g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1} \subseteq S$. As the intransitivity and minimality of $G \curvearrowright X$ rule out the existence of open orbits, Proposition 4.2.4 ensures that $G \curvearrowright X$ is expansively $\{g\mathrm{IP}(\mathbf{g})\mathrm{IP}(\mathbf{g})^{-1}\}$-recurrent, so Proposition 2.1.2 implies that $G \curvearrowright X$ is not σ-expansively $(\bigcup_{g \in G} g\{S\}g^{-1})$-transient, thus Theorem 2.1.8 yields the desired disjoint union and embedding.

To see (2) $\Longrightarrow$ (1), given a sequence $\mathbf{g} \in (\bigcup_{d \in \mathbb{Z}^+} G^{\{1,\ldots,d\}})^{\mathbb{N}}$ that is $(K_n)_{n \in \mathbb{N}}$-expansive, observe that if $g \in G$, $n \in \mathbb{N}$, and $S = g\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))$ $\mathrm{IP}(\mathbb{s}^n(\mathbf{g}))^{-1}$, then $G \curvearrowright X$ is not σ-expansively $\bigcup_{g \in G} g\{S\}g^{-1}$-transient, so the minimality of $G \curvearrowright X$ ensures that it is expansively $\{S\}$-recurrent. As $G \curvearrowright X$ is topologically transitive, Proposition 4.2.4 implies that $G \curvearrowright X \times \mathbb{X}_{\mathbf{g}}$ is topologically transitive, so Proposition 4.2.1 yields that $G \curvearrowright X$ is mildly mixing. ⊠

We now establish the corresponding anti-basis theorem:

Theorem 4.2.6. *Suppose that G is a Polish group that admits a compatible two-sided-invariant metric and $\mathcal{A}$ is a non-empty class of mildly-mixing minimal continuous G-actions on Polish spaces of cardinality at least two that is closed under restrictions to G-invariant dense G_δ sets. Then any basis $\mathcal{B}$ for $\mathcal{A}$ under continuous homomorphism has cardinality at least the additivity of the meager ideal.*

Proof. Exactly as in the proof of Theorem 4.1.15, albeit without the need for Proposition 4.1.14. ⊠

4.3. **Strong mixing.** We begin this section with two local refinements of characterizations of strong mixing:

Proposition 4.3.1. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a topological space. Then $G \curvearrowright X$ is strongly mixing if and only if it is $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitive.*

Proof. This is a straightforward consequence of the fact that a closed subset of G is compact if and only if it does not contain a closed discrete infinite subset. ⊠

Proposition 4.3.2. *Suppose that $G \curvearrowright X$ is a continuous action of a locally-compact Polish group on a Polish space. Then $G \curvearrowright X$ is strongly mixing and has no open orbits if and only if it is topologically transitive and expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent.*

Proof. Note that $G \curvearrowright X$ is strongly mixing if and only if it is $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitive, by Proposition 4.3.1. The conjunction of the latter condition with the inexistence of open orbits is equivalent to the expansive $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-transitivity of $G \curvearrowright X$, by Proposition 4.1.3 and the comment immediately preceding it. And the latter condition holds if and only if $G \curvearrowright X$ is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent and topologically transitive, by Propositions 4.1.2 and 4.1.3. ⊠

As a consequence, we obtain a necessary and sufficient condition for an intransitive minimal continuous action to be strongly mixing:

Theorem 4.3.3. *Suppose that $G \curvearrowright X$ is an intransitive minimal continuous action of a locally-compact Polish group on a Polish space. Then the following are equivalent:*

(1) *The action $G \curvearrowright X$ is strongly mixing.*
(2) *There is a continuous disjoint union of actions obtained via expansive cutting and stacking that is not σ-expansively $\{\bigcup_{g\in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient but admits a continuous stabilizer-preserving homomorphism to $G \curvearrowright X$.*

Proof. To see (1) $\implies$ (2), note that the intransitivity and minimality of $G \curvearrowright X$ ensures that there are no open orbits, in which case Proposition 4.3.2 implies that $G \curvearrowright X$ is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent, so Proposition 2.1.2 implies that $G \curvearrowright X$ is not σ-expansively $\{\bigcup_{g\in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient, thus Theorem 2.1.8 yields the desired disjoint union and continuous stabilizer-preserving homomorphism.

To see (2) $\implies$ (1), observe that $G \curvearrowright X$ is not σ-expansively $\{\bigcup_{g\in G} g\mathcal{S}g^{-1} \mid \mathcal{S} \in \boldsymbol{\mathcal{S}}_{sm}\}$-transient, so the minimality of $G \curvearrowright X$ ensures that it is expansively $(\bigcup \boldsymbol{\mathcal{S}}_{sm})$-recurrent. As $G \curvearrowright X$ is topologically transitive, Proposition 4.3.2 implies that it is strongly mixing. ⊠

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