THE CLASSIFICATION OF FINITE BOREL EQUIVALENCE RELATIONS ON $2^{\mathbb{N}}/E_0$

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We study the Borel structure of quotient spaces of the form X/E, where X is a Polish space and E is a countable Borel equivalence relation on X. Our main result is the classification of finite Borel equivalence relations on the non-smooth hyperfinite quotient space $2^{\mathbb{N}}/E_0$. In particular, we see that for each natural number $n \in \mathbb{N}$, there are only finitely many Borel equivalence relations on $2^{\mathbb{N}}/E_0$ whose classes are all of cardinality n, up to Borel isomorphism. We achieve our main result by classifying Borel cocycles from hyperfinite equivalence relations into finite groups, up to Borel reducibility. This, in turn, depends on a parameterized family of embedding theorems in the style of Glimm-Effros and Dougherty-Jackson-Kechris.

1. Introduction

A topological space X is *Polish* if it is separable and completely metrizable. The *Borel subsets* of such a space are those which can be obtained from the open sets via countable unions and complements. A function $f : X \to Y$ between Polish spaces is said to be *Borel* if the preimages of open sets under f are Borel.

The study of Borel sets and functions on Polish spaces is a central focus of descriptive set theory. Part of the motivation for this study comes from the fact that numerous spaces appearing throughout mathematics are Polish. There are other objects, however, which are naturally realized not as Polish spaces, but as quotients of the form X/E, where X is a Polish space and E is a countable Borel equivalence relation on X.

The space X/E inherits a Borel structure from X. We say that a set $B \subseteq X/E$ is *Borel* if its lifting $\tilde{B} = \{x \in X : [x]_E \in B\}$ is Borel. A *transversal* of E is a set $B \subseteq X$ which intersects every E-class in exactly one point. We say that E is *smooth* if it admits a Borel transversal. If E is smooth, then the quotient Borel structure on X/E is *standard*, i.e., it is induced by a Polish topology on X/E. If E is non-smooth, then Theorem 1 of Harrington-Kechris-Louveau [2] can be used to show that the quotient Borel structure on X/E is not even countably generated.

Suppose now that E and F are countable Borel equivalence relations on Polish spaces X and Y. We say that a set $R \subseteq X/E \times Y/F$ is *Borel* if its lifting $\tilde{R} =$

$\{(x,y) \in X \times Y : ([x]_E, [y]_F) \in R\}$ is Borel. It is important here that we do not simply take the *product* of the σ-algebras on X/E and Y/F (i.e., the σ-algebra generated by the sets of the form $A \times B$, where $A \subseteq X/E$ and $B \subseteq Y/F$ are Borel), for if either E or F is non-smooth, then the product of the σ-algebras is far too small (e.g., if E is non-smooth, then it does not even contain the graph of a countable-to-one function).

We say that a function $f : X/E \to Y/F$ is *Borel* if its graph is Borel. This is equivalent to the existence of a Borel *lifting*, i.e., a Borel function $\tilde{f} : X \to Y$ such that $\forall x \in X\ (\tilde{f}(x) \in f([x]_E))$. One could also ask that the pre-image of Borel subsets of Y/F are necessarily Borel subsets of X/E. While the former requirement implies the latter, as soon as X is uncountable and F is non-smooth, it is consistent with ZFC that the converse is false.

We say that E is *hyperfinite* if it is of the form $\bigcup_{n\in\mathbb{N}} F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ are finite Borel equivalence relations on X. The typical example is the equivalence relation E_0 on $2^{\mathbb{N}}$, which is given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (\alpha(m) = \beta(m)).$$

Theorem 1 of Dougherty-Jackson-Kechris [1] implies that if E and F are non-smooth, hyperfinite equivalence relations, then X/E and Y/F are *Borel isomorphic*, i.e., there is a Borel bijection $\pi : X/E \to Y/F$. In particular, all such quotient spaces are Borel isomorphic to $2^{\mathbb{N}}/E_0$. We therefore refer to this as the (non-smooth) *hyperfinite quotient space*. Theorem 1 of Harrington-Kechris-Louveau [2] implies that $2^{\mathbb{N}}/E_0$ is the minimal quotient space whose Borel structure is not countably generated, in the sense that if E is non-smooth, then there is a Borel injection of $2^{\mathbb{N}}/E_0$ into X/E.

Suppose now that $E \subseteq F$ are countable Borel equivalence relations on X. We use F/E to denote the equivalence relation on X/E given by

$$[x]_E (F/E) [y]_E \Leftrightarrow xFy.$$

Given $E_i \subseteq F_i$ on X_i, we say that F_1/E_1 is *isomorphic* to F_2/E_2, or $F_1/E_1 \cong_B F_2/E_2$, if there is a Borel isomorphism $\pi : X_1/E_1 \to X_2/E_2$ such that

$$\forall x_1, y_1 \in X_1/E_1\ (x_1(F_1/E_1)y_1 \Leftrightarrow \pi(x_1)(F_2/E_2)\pi(y_1)).$$

The main goal of this paper is to give a complete classification of finite Borel equivalence relations on $2^{\mathbb{N}}/E_0$, up to Borel isomorphism.

Given a positive natural number n, we say that F is of *index* n over E, or $[F : E] = n$, if every equivalence class of F/E is of cardinality n. In this case, we say that F/E is *smooth* if X/E can be partitioned into n Borel transversals of F/E. It follows easily from Theorem 1 of Dougherty-Jackson-Kechris [1] that all smooth equivalence relations on the hyperfinite quotient space whose classes are of cardinality n are Borel isomorphic. As it turns out, one obtains a complete invariant for Borel equivalence relations on the hyperfinite quotient space whose classes are of cardinality n by measuring the fashion in which the equivalence relation in question fails to be smooth.

Suppose that $E \subseteq F$ are countable Borel equivalence relations on a Polish space X and $[F : E] = n$. We define $\mathrm{Enum}(E, F) \subseteq X^n$ by

$$\mathrm{Enum}(E, F) = \{(x_1, \ldots, x_n) \in X^n : [x_1]_F = [x_1]_E \cup \cdots \cup [x_n]_E\}.$$

We say that a set $S \subseteq S_n$ is an *essential value of* (E, F) if, for every cover $\mathcal{B}$ of $\mathrm{Enum}(E, F)$ by countably many Borel sets, there exists $B \in \mathcal{B}$ such that

$$\forall \sigma \in S\, \exists (x_1, \ldots, x_n) \neq (y_1, \ldots, y_n) \in B\, \forall 1 \leq i \leq n\ (x_i E y_{\sigma^{-1}(i)}).$$

We use $\mathrm{Ess}(E, F)$ to denote the family of all essential values of (E, F). Our main theorem is that this family is a complete invariant for Borel isomorphism:

Theorem A. *Suppose that X_1 and X_2 are Polish spaces, $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$ are non-smooth, hyperfinite equivalence relations on X_1 and X_2, and $[F_1 : E_1] = [F_2 : E_2] = n$. Then*

$$F_1/E_1 \cong_B F_2/E_2 \Leftrightarrow \mathrm{Ess}(E_1, F_1) = \mathrm{Ess}(E_2, F_2).$$

As there are only finitely many possibilities for $\mathrm{Ess}(F)$, we therefore obtain:

Theorem B. *There are only finitely many Borel equivalence relations on the non-smooth hyperfinite quotient space whose classes are of cardinality n.*

Our proof of Theorem A relies primarily upon an investigation of Borel cocycles. Suppose that G is a countable group. A function $\rho : E \to G$ is a *cocycle* if $\forall x E y E z\ (\rho(x, z) = \rho(x, y)\rho(y, z))$. For each $B \subseteq X$ and $x \in B$, let

$$\mathrm{Val}(\rho, B, x) = \{\rho(x, y) : x E y \text{ and } x \neq y \text{ and } y \in B\},$$

and set $\mathrm{Val}(\rho, B) = \bigcup_{x \in B} \mathrm{Val}(\rho, B, x)$. We say that a set $H \subseteq G$ is an *essential value* of a Borel cocycle $\rho : E \to G$ if, for every cover $\mathcal{B}$ of X by countably many Borel sets, there exists $B \in \mathcal{B}$ such that $H \subseteq \mathrm{Val}(\rho, B)$. We use $\mathrm{Ess}(\rho)$ to denote the family of all essential values of ρ.

We use E^* to denote the equivalence relation on $\mathrm{Enum}(E, F)$ which is given by $(x_1, \ldots, x_n) E^* (y_1, \ldots, y_n) \Leftrightarrow x_1 E y_1$, and we use F^* to denote the equivalence relation on $\mathrm{Enum}(E, F)$ which is given by $(x_1, \ldots, x_n) F^* (y_1, \ldots, y_n) \Leftrightarrow x_1 F y_1$. Then the projection $(x_1, \ldots, x_n) \mapsto x_1$ induces a Borel isomorphism of F^*/E^* with F/E. The advantage of considering F^*/E^* in place of F/E is that it comes equipped with a cocycle into S_n, which is defined by setting $\rho_{(E,F)}((x_1, \ldots, x_n), (y_1, \ldots, y_n))$ equal to the unique $\sigma \in S_n$ such that $\forall 1 \leq i \leq n\ (x_i E y_{\sigma^{-1}(i)})$. It is easy to see that the families of essential values of F/E and $\rho_{(E,F)}$ are one and the same.

Suppose that $\rho_1 : E_1 \to G_1$ and $\rho_2 : E_2 \to G_2$ are Borel cocycles. We say that ρ_1 is *Borel reducible* to ρ_2, or $\rho_1 \leq_B \rho_2$, if there is a Borel $\pi : X_1 \to X_2$ such that:

1. $\forall x, y \in X_1 \ (xE_1y \Leftrightarrow \pi(x)E_2\pi(y))$;
2. $\forall xE_1y \ (\rho_1(x, y) = \rho_2(\pi(x), \pi(y)))$.

We say that $\rho : E \to G$ has *everywhere full range* if $\mathrm{Val}(\rho, X, x) = G$, for all $x \in X$. We actually obtain Theorem A as a corollary of the following fact:

Theorem C. *Suppose that X_1 and X_2 are Polish spaces, E_1 and E_2 are non-smooth hyperfinite equivalence relations on X_1 and X_2, G is a finite group, and $\rho_1 : E_1 \to G$ and $\rho_2 : E_2 \to G$ are Borel cocycles with everywhere full ranges. Then*

$$\rho_1 \leq_B \rho_2 \Leftrightarrow \mathrm{Ess}(\rho_1) \subseteq \mathrm{Ess}(\rho_2).$$

We say that ρ_1 is *Borel embeddable* into ρ_2, or $\rho_1 \sqsubseteq_B \rho_2$, if there is an injective Borel reduction of ρ_1 into ρ_2. The proof of Theorem C consists essentially of two separate embedding theorems. The first is a Glimm-Effros style theorem which describes the circumstances under which a Borel cocycle $\rho : E \to G$ has a given set $H \subseteq G$ as an essential value, in terms of whether ρ contains a copy of a canonical Borel cocycle $\rho_H : E_0 \to H$, whose somewhat technical description we give in §2:

Theorem D. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a countable group, $\rho : E \to G$ is a Borel cocycle, and $H \leq G$. Then the following are equivalent:*

1. *H is an essential value of ρ;*
2. *$\rho_H \sqsubseteq_B \rho$.*

The second ingredient in the proof of Theorem C is the following Dougherty-Jackson-Kechris style embedding theorem:

Theorem E. *Suppose that X is a Polish space, E is a hyperfinite equivalence relation on X, G is a countable group, $H \leq G$, and $\rho : E \to H$ is a Borel cocycle. Then $\rho \sqsubseteq_B \rho_G$.*

We prove Theorem D in §2, Theorem E in §3, Theorem C in §4, and Theorems A and B in §5. In §6, we turn our attention to a measure-theoretic problem. The main result of Shelah-Weiss [5] is a characterization of the circumstances under there is an atomless, E-ergodic, E-quasi-invariant probability measure. For $2 \leq [F : E] < \aleph_0$, we characterize the circumstances under which there is an atomless, E-ergodic, F-quasi-invariant probability measure.

2. Glimm-Effros-style embeddings

In this section, we describe the circumstances under which a given set is among the essential values of a Borel cocycle. We begin by giving "combinatorially simple" examples of cocycles with a given essential value.

For each countable group H, fix a sequence $\langle g_0^H, g_1^H, \ldots\rangle$ of elements of H in which every element appears infinitely often. Let $g_H^{\emptyset} = 1_H$, and set

$$g_H^s = (g_0^H)^{s(0)}(g_1^H)^{s(1)} \cdots (g_n^H)^{s(n)},$$

for each $n \in \mathbb{N}$ and $s \in 2^{n+1}$. Define a Borel cocycle $\rho_H : E_0 \to H$ by setting

$$\rho_H(s\alpha, t\alpha) = g_H^s (g_H^t)^{-1},$$

where $s, t \in 2^{<\mathbb{N}}$ are of the same length and $\alpha \in 2^{\mathbb{N}}$.

Proposition 1. *Suppose that H is a countable group and $B \subseteq 2^{\mathbb{N}}$ is a non-meager Borel set. Then $H \subseteq \mathrm{Val}(\rho_H, B)$, thus H is an essential value of ρ_H.*

Proof. Suppose that $h \in H$, fix $s \in 2^{<\mathbb{N}}$ such that B is comeager in $\mathcal{N}_s$, and fix $k \in \mathbb{N}$ such that

$$g_{k+|s|}^H = (g_H^s)^{-1} h^{-1} g_H^s.$$

As B is comeager in $\mathcal{N}_s$, there exists $\alpha \in 2^{\mathbb{N}}$ such that $s0^k0\alpha, s0^k1\alpha \in B$. Then

$$\begin{aligned} \rho_H(s0^k0\alpha, s0^k1\alpha) &= g_H^{s0^k0}(g_H^{s0^k1})^{-1} \\ &= g_H^s (g_{k+|s|}^H)^{-1} (g_H^s)^{-1} \\ &= h, \end{aligned}$$

so $h \in \mathrm{Val}(\rho_H, B)$. As $h \in H$ was arbitrary, it follows that $H \subseteq \mathrm{Val}(\rho_H, B)$. $\square$

Remark 2. The above argument really shows that $H = \mathrm{Val}(\rho_H, B, x)$, for all but meagerly many $x \in B$. We will not need this stronger fact, however.

We will now show that ρ_H is the minimal Borel cocycle with essential value H:

Theorem 3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a countable group, $\rho : E \to G$ is a Borel cocycle, and $H \leq G$. Then the following are equivalent:*

1. *H is an essential value of ρ;*
2. *$\rho_H \sqsubseteq_B \rho$.*

Proof. To see (2) $\Rightarrow$ (1), suppose that $\pi : 2^{\mathbb{N}} \to X$ is a Borel embedding of ρ_H into ρ, and $\mathcal{B}$ is a cover of X by countable many Borel sets. Then $\pi^{-1}[\mathcal{B}]$ is a cover of $2^{\mathbb{N}}$ by countably many Borel sets, thus Proposition 1 ensures that there exists $B \in \mathcal{B}$ such that $H \subseteq \mathrm{Val}(\rho_H, \pi^{-1}[B]) = \mathrm{Val}(\rho, B)$, thus H is an essential value of ρ.

It remains to prove (1) $\Rightarrow$ (2). Let $\mathcal{I}_H$ denote the σ-ideal generated by Borel sets $B \subseteq X$ such that $H \not\subseteq \mathrm{Val}(\rho, B)$. Fix a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$, as well as an increasing sequence $\Delta_0 \subseteq \Delta_1 \subseteq \cdots$ of finite, symmetric neighborhoods of 1_Γ such that $\Gamma = \bigcup_{n \in \mathbb{N}} \Delta_n$. By standard change of topology results (see, for example, §13 of Kechris [3]), we can assume that X is a zero-dimensional Polish space, Γ acts by homeomorphisms, and for each $\gamma \in \Gamma$ and $h \in H$, the set $\{x \in X : \rho(x, \gamma \cdot x) = h\}$ is clopen. We will find clopen sets $A_n \subseteq X$ and $\gamma_n \in \Gamma$, from which we define $\delta_s : X \to X$ by $\delta_\emptyset = \mathrm{id}$ and

$$\delta_s = \gamma_0^{s(0)} \dots \gamma_n^{s(n)},$$

for $s \in 2^{n+1}$. We will ensure that, for all $n \in \mathbb{N}$, the following conditions hold:

(a) $A_n \notin \mathcal{I}_H$;

(b) $A_{n+1} \subseteq A_n \cap \gamma_n^{-1}[A_n]$;

(c) $\forall x \in A_{n+1}\ (\rho(\gamma_n \cdot x, x) = g_n^H)$;

(d) $\forall s, t \in 2^n\ \forall \delta \in \Delta_n\ (\delta\delta_s[A_{n+1}] \cap \delta_t \gamma_n [A_{n+1}] = \emptyset)$;

(e) $\forall s \in 2^{n+1}\ (\mathrm{diam}(\delta_s[A_{n+1}]) \leq 1/n)$.

We begin by setting $A_0 = X$. Suppose now that we have found $\langle A_i \rangle_{i \leq n}$ and $\langle \gamma_i \rangle_{i<n}$. For each $\gamma \in \Gamma$, let U_γ denote the set of $x \in A_n \cap \gamma^{-1}[A_n]$ such that

$$\rho(\gamma \cdot x, x) = g_n^H \text{ and } \forall s, t \in 2^n\ \forall \delta \in \Delta_n\ (\gamma \cdot x \neq \delta_t^{-1} \delta \delta_s \cdot x).$$

Our choice of topology ensures that each of these sets is open.

Lemma 4. *There exists $\gamma \in \Gamma$ such that $U_\gamma \notin \mathcal{I}_H$.*

Proof. It is enough to show that the set

$$B = A_n \setminus \bigcup_{\gamma \in \Gamma} U_\gamma$$

is in $\mathcal{I}_H$. Observe that if $x, y \in B$, xEy, and $\rho(y, x) = g_n^H$, then there exists $\gamma \in \Gamma$ such that $\gamma \cdot x = y$, and since $x \notin U_\gamma$, we can find $s, t \in 2^n$ and $\delta \in \Delta_n$ such that $y = \gamma \cdot x = \delta_t^{-1} \delta \delta_s \cdot x$. It follows that for each $x \in B$, there are at most $k = 2^{2n} |\Delta_n|$ points $y \in B \cap [x]_E$ such that $\rho(y, x) = g_n^H$. Define a directed graph $\mathcal{G}$ on B by setting

$$(x, y) \in \mathcal{G} \Leftrightarrow \rho(y, x) = g_n^H.$$

Then the vertex degree of $\mathcal{G}$ is at most k, thus Proposition 4.6 of Kechris-Solecki-Todorčević [4] ensures that there is a partition of B into Borel sets $B_0, B_1, \dots, B_k$ which are $\mathcal{G}$-discrete. This implies that

$$g_n^H \notin \mathrm{Val}(\rho, B_0) \cup \dots \cup \mathrm{Val}(\rho, B_k),$$

and it follows that $B \in \mathcal{I}_H$. □

By Lemma 4, there exists $\gamma \in \Gamma$ such that $U_\gamma \notin \mathcal{I}_H$. Set $\gamma_n = \gamma$. As Γ acts by homeomorphisms, we can write U_γ as the union of countably many clopen sets U which satisfy the following two conditions:

(d′) $\forall s,t \in 2^n \, \forall \delta \in \Delta_n \; (\delta\delta_s[U] \cap \delta_t\gamma_n[U] = \emptyset)$;

(e′) $\forall s \in 2^{n+1} \; (\mathrm{diam}(\delta_s[U]) \leq 1/n)$.

Fix such a U which is not in $\mathcal{I}_H$, and set $A_{n+1} = U$.

This completes the recursive construction. For each $s \in 2^n$, put $B_s = \delta_s[A_n]$. Conditions (b) and (e) ensure that, for each $\alpha \in 2^{\mathbb{N}}$, the sets $B_{\alpha|0}, B_{\alpha|1}, \ldots$ are decreasing and of vanishing diameter, and since they are clopen, they have singleton intersection. Define $\pi : 2^{\mathbb{N}} \to X$ by

$$\pi(\alpha) = \text{ the unique element of } \bigcap_{n\in\mathbb{N}} B_{\alpha|n}.$$

It follows from conditions (d) and (e) that π is a continuous injection.

Lemma 5. *Suppose that $n \in \mathbb{N}$, $s \in 2^n$, and $\alpha \in 2^{\mathbb{N}}$. Then $\pi(s\alpha) = \delta_s \cdot \pi(0^n\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}
\{\pi(s\alpha)\} &= \bigcap_{i\geq n} B_{(s\alpha)|i} \\
&= \bigcap_{i\geq 0} \delta_s\delta_{0^n(\alpha|i)}[A_{i+n}] \\
&= \delta_s \left[\bigcap_{i\geq 0} \delta_{0^n(\alpha|i)}[A_{i+n}] \right] \\
&= \delta_s \left[\bigcap_{i\geq n} B_{(0^n\alpha)|i} \right] \\
&= \{\delta_s \cdot \pi(0^n\alpha)\},
\end{aligned}$$

thus $\pi(s\alpha) = \delta_s \cdot \pi(0^n\alpha)$. □

To see that π is an embedding of ρ_H into ρ, we must check the following:

(i) $\forall \alpha,\beta \in 2^{\mathbb{N}} \; (\alpha E_0 \beta \Rightarrow \pi(\alpha) E \pi(\beta))$;

(ii) $\forall \alpha,\beta \in 2^{\mathbb{N}} \; (\pi(\alpha) E \pi(\beta) \Rightarrow \alpha E_0 \beta)$;

(iii) $\forall (\alpha,\beta) \in E_0 \; (\rho_H(\alpha,\beta) = \rho(\pi(\alpha),\pi(\beta)))$.

To see (i), suppose that $\alpha E_0 \beta$, and fix $n \in \mathbb{N}$ such that $\forall m \geq n \; (\alpha(m) = \beta(m))$. Lemma 5 then ensures that $\delta_{\alpha|n}^{-1} \cdot \pi(\alpha) = \delta_{\beta|n}^{-1} \cdot \pi(\beta)$, so $\pi(\alpha) E \pi(\beta)$.

To see (ii), it is enough to show that if $\alpha(n) \neq \beta(n)$, then there is no $\delta \in \Delta_n$ such that $\delta \cdot \pi(\alpha) = \pi(\beta)$. As Δ_n is symmetric, we can assume that $\alpha(n) = 0$ and $\beta(n) = 1$. Suppose, towards a contradiction, that there exists $\delta \in \Delta_n$ with $\delta \cdot \pi(\alpha) = \pi(\beta)$. Then $\pi(\alpha) \in \delta_{\alpha|n}[A_{n+1}]$ and $\pi(\beta) \in \delta_{\beta|n}\gamma_n[A_{n+1}]$, so $\pi(\beta) \in \delta\delta_{\alpha|n}[A_{n+1}] \cap \delta_{\beta|n}\gamma_n[A_{n+1}]$, which contradicts condition (d).

To see (iii), suppose that $\alpha E_0 \beta$, and fix $n \in \mathbb{Z}^+$ such that $\forall m \geq n\ (\alpha(m) = \beta(m))$. Lemma 5 then ensures that $\delta_{\alpha|n}^{-1} \cdot \pi(\alpha) = \delta_{\beta|n}^{-1} \cdot \pi(\beta)$, and condition (c) ensures that

$$\begin{aligned}
\rho(\pi(\alpha), \pi(\beta)) &= \rho(\pi(\alpha), \delta_{\alpha|n}^{-1} \cdot \pi(\alpha))\rho(\delta_{\beta|n}^{-1} \cdot \pi(\beta), \pi(\beta)) \\
&= \rho(\delta_{\alpha|0}^{-1} \cdot \pi(\alpha), \delta_{\alpha|1}^{-1} \cdot \pi(\alpha)) \cdots \rho(\delta_{\alpha|(n-1)}^{-1} \cdot \pi(\alpha), \delta_{\alpha|n}^{-1} \cdot \pi(\alpha)) \\
&\quad\ \rho(\delta_{\beta|n}^{-1} \cdot \pi(\beta), \delta_{\beta|(n-1)}^{-1} \cdot \pi(\beta)) \cdots \rho(\delta_{\beta|1}^{-1} \cdot \pi(\beta), \delta_{\beta|0}^{-1} \cdot \pi(\beta)) \\
&= (g_0^H)^{\alpha(0)} \cdots (g_{n-1}^H)^{\alpha(n-1)} (g_{n-1}^H)^{-\beta(n-1)} \cdots (g_0^H)^{\beta(0)} \\
&= g_H^{\alpha|n} (g_H^{\beta|n})^{-1} \\
&= \rho_H(\alpha, \beta),
\end{aligned}$$

which completes the proof of the theorem. □

3. Dougherty-Jackson-Kechris-style embeddings

In this section, we show that the cocycles ρ_G of §2 contain copies of all Borel cocycles from hyperfinite equivalence relations into G.

A *selector* for an equivalence relation E on X is a function $f : X \to X$ such that:

1. $\forall x, y \in X\ (xEy \Rightarrow f(x) = f(y))$;
2. $\forall x \in X\ (xEf(x))$.

A *coherent sequence of Borel selectors* for $F_0 \subseteq F_1 \subseteq \cdots$ is a sequence $\langle f_0, f_1, \ldots \rangle$, where f_i is a Borel selector for F_i, such that $f_0[X] \supseteq f_1[X] \supseteq \cdots$.

Proposition 6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, H is a countable group, and $\rho : E \to H$ is a Borel cocycle. Then there are finite Borel equivalence relations $\Delta(X) = F_0 \subseteq F_1 \subseteq \cdots$ and a coherent sequence of Borel selectors $\langle f_0, f_1, \ldots \rangle$ such that:*

1. $E = \bigcup_{n \in \mathbb{N}} F_n$;
2. $\forall n \in \mathbb{N}\, \forall x \in X\, \exists y \in X\ ([x]_{F_{n+1}} = [x]_{F_n} \cup [y]_{F_n})$;
3. $\forall n \in \mathbb{N}\, \forall x \in X\ (f_{n+1}(x) \neq f_n(x) \Rightarrow \rho(f_{n+1}(x), f_n(x)) = g_n^H)$.

Proof. Fix finite Borel equivalence relations $F_0' \subseteq F_1' \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n'$, as well as a Borel linear ordering $\leq$ of X. We define F_n and f_n recursively, beginning with $F_0 = \Delta(X)$ and $f_0 = \mathrm{id}$.

Suppose that we have already found $F_0, \ldots, F_n$ and $f_0, \ldots, f_n$. For each $x \in X$, let $k_n(x)$ be the least natural number k such that $[x]_{F_k'} \not\subseteq [x]_{F_n}$, and put

$$k_n([x]_{F_n}) = \min\{k_n(y) : y \in [x]_{F_n}\}.$$

Let $\varphi_n(x)$ be the $\leq$-least $y \in [x]_{F_n}$ such that $k_n(y) = k_n([x]_{F_n})$, and let $\psi_n(x)$ be the $\leq$-least element of the $F_{k_n([x]_{F_n})}'$-class of $\varphi_n(x)$ which is not in $[x]_{F_n}$. We say that $(x, y) \in X \times X$ is a *good pair* if the following conditions are satisfied:

1. $x = \varphi_n(x) = \psi_n(y)$;
2. $y = \varphi_n(y) = \psi_n(x)$;
3. $x < y$;
4. $\rho(x, y) = g_n^H$.

Define F_n'' on X by

$$xF_n''y \Leftrightarrow x = y \text{ or } ((x, y) \text{ is a good pair}),$$

let F_{n+1} be the equivalence relation generated by F_n and F_n'', set

$$f_{n+1}'(x) = \begin{cases} \psi_n(x) & \text{if } x \text{ is the } 2^{\text{nd}} \text{ coordinate of a good pair,} \\ x & \text{otherwise,} \end{cases}$$

and define $f_{n+1} = f_{n+1}' \circ f_n$.

It is clear that $F_0 \subseteq F_1 \subseteq \cdots$ are finite Borel equivalence relations and $\bigcup_{n\in\mathbb{N}} F_n \subseteq E$. Suppose, towards a contradiction, that $E \neq \bigcup_{n\in\mathbb{N}} F_n$, let k be the least natural number such that $F_k' \not\subseteq \bigcup_{n\in\mathbb{N}} F_n$, fix $x \in X$ such that $[x]_{F_k'} \not\subseteq \bigcup_{n\in\mathbb{N}}[x]_{F_n}$, and fix $\ell \geq k$ sufficiently large that $[y]_{F_\ell} = \bigcup_{n\in\mathbb{N}}[y]_{F_n}$, for all $y \in [x]_{F_k'}$. Note that the definition of F_ℓ ensures that each such $[y]_{F_\ell}$ is contained in $[x]_{F_k'}$. Let y be the $\leq$-least element of $[x]_{F_k'}$, let z be the $\leq$-least element of $[x]_{F_k'} \setminus [y]_{F_\ell}$, and fix $m \geq \ell$ with $\rho(y, z) = g_m^H$. Then $yF_m''z$, thus $yF_{m+1}z$, the desired contradiction. □

We are now ready for the main result of this section:

Theorem 7. *Suppose that X is a Polish space, E is a hyperfinite equivalence relation on X, $H \leq G$ are countable groups, and $\rho : E \to H$ is a Borel cocycle. Then $\rho \sqsubseteq_B \rho_G$.*

Proof. As in Proposition 6, fix finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ and a coherent sequence $\langle f_0, f_1, \ldots\rangle$ of Borel selectors such that $E = \bigcup_{n\in\mathbb{N}} F_n$. Fix a separating family $U_0, U_1, \ldots$ for X. For each $x \in X$, let $x|0 = \emptyset$ and $x|(n+1) = \chi_{U_0}(x) \ldots \chi_{U_n}(x)$. For each $n \in \mathbb{N}$ and $s \in 2^n$, define $\varphi_s : X \to 2^n$ by

$$\varphi_s(x) = \begin{cases} y|n & \text{if } xF_ny \text{ and } \forall i < n \ (f_{i+1}(y) = f_i(y) \Leftrightarrow s(i) = 0), \\ 0^n & \text{if no such } y \text{ exists.} \end{cases}$$

Our embedding $\pi : X \to 2^{\mathbb{N}}$ will be of the form $\pi(x) = \bigcup \pi_n(x)$, where

$$\pi_n(x) = \bigoplus_{i<n} u_i(x)v_i(x)w_i(x).$$

Here, the function $u_n : X \to (2^n)^{2^n}$ is given by

$$u_n(x) = \bigoplus_{s\in 2^n} \varphi_s(x)|n,$$

and the function $v_n : X \to 2$ is given by

$$v_n(x) = \begin{cases} 0 & \text{if } f_{n+1}(x) = f_n(x), \\ 1 & \text{otherwise.} \end{cases}$$

We define $w_n : X \to 2^{k_{n+1}-k'_n}$ by induction on n. We begin by setting $k_0 = 0$. Given k_n, set $k'_n = k_n + n \cdot 2^n + 1$, put $\pi'_n(x) = \pi_n(x)u_n(x)v_n(x)$, and define

$$\psi_n(x) = (g_G^{\pi'_n(f_n(x))})^{-1}(g_n^H)^{-1}g_G^{\pi'_n(f_{n+1}(x))}.$$

As there are only finitely many possibilities for $\psi_n(x)$, it follows that there exists $k \geq k'_n$ such that every $g \in \psi_n[X]$ is of the form g_i^G, for some $k'_n \leq i < k$. Let k_{n+1} be the least such k. If $f_{n+1}(x) = f_n(x)$, set $w_n(x) = 0^{k_{n+1}-k'_n}$. Otherwise, let $\ell \geq k'_n$ be least such that $\psi_n(x) = g_\ell^G$, and put $w_n(x) = 0^{\ell-k'_n}10^{k_{n+1}-\ell-1}$.

To see that π is injective, suppose that $\pi(x) = \pi(y)$, and observe that for all $n \in \mathbb{N}$, if $s = \bigoplus_{i<n} v_i(x) = \bigoplus_{i<n} v_i(y)$, then $x|n = \varphi_s(x) = \varphi_s(y) = y|n$, so $x = y$, thus π is injective.

To see that $xEy \Rightarrow \pi(x)E_0\pi(y)$, simply note that if xEy, then there exists $n \in \mathbb{N}$ such that xF_ny. It then follows that $u_m(x)v_m(x)w_m(x) = u_m(y)v_m(y)w_m(y)$, for all $m \geq n$, thus $\pi(x)E_0\pi(y)$.

To see that $\pi(x)E_0\pi(y) \Rightarrow xEy$, suppose that $\pi(x)E_0\pi(y)$, and fix $n \in \mathbb{N}$ such that $u_m(x)v_m(x)w_m(x) = u_m(y)v_m(y)w_m(y)$, for all $m \geq n$. For each $m \geq n$, set $s_m = \bigoplus_{i<m} v_i(f_n(x)) = \bigoplus_{i<m} v_i(f_n(y))$, and observe that $f_n(x)|m = \varphi_{s_m}(x) = \varphi_{s_m}(y) = f_n(y)|m$, so $f_n(x) = f_n(y)$, thus xEy.

To see that $xEy \Rightarrow \rho(x,y) = \rho_G(\pi(x),\pi(y))$, note that if $xF_{n+1}y$, then

$$\rho(x,y) = \rho(f_0(x), f_1(x)) \cdots \rho(f_n(x), f_{n+1}(x))\rho(f_{n+1}(y), f_n(y)) \cdots \rho(f_1(y), f_0(y)),$$

so it is enough to show that

$$\rho(f_{n+1}(x), f_n(x)) = \rho_G(\pi(f_{n+1}(x)), \pi(f_n(x))),$$

for all $x \in X$ and $n \in \mathbb{N}$. Towards this end, observe that if $f_{n+1}(x) \neq f_n(x)$, then

$$\begin{aligned}
\rho_G(\pi(f_{n+1}(x)), \pi(f_n(x))) &= g_G^{\pi_{n+1}(f_{n+1}(x))}(g_G^{\pi_{n+1}(f_n(x))})^{-1} \\
&= g_G^{\pi'_n(f_{n+1}(x))}(g_G^{0^{k'_n}w_n(f_n(x))})^{-1}(g_G^{\pi'_n(f_n(x))})^{-1} \\
&= g_G^{\pi'_n(f_{n+1}(x))}\psi_n(x)^{-1}(g_G^{\pi'_n(f_n(x))})^{-1} \\
&= g_G^{\pi'_n(f_{n+1}(x))}(g_G^{\pi'_n(f_{n+1}(x))})^{-1}g_n^H \\
&\quad\; g_G^{\pi'_n(f_n(x))}(g_G^{\pi'_n(f_n(x))})^{-1} \\
&= g_n^H \\
&= \rho(f_{n+1}(x), f_n(x)).
\end{aligned}$$

It now follows that π is the desired embedding of ρ into ρ_G. $\square$

As a corollary, we obtain the following:

Theorem 8. *Suppose that* X_1 *and* X_2 *are Polish spaces,* E_1 *and* E_2 *are countable Borel equivalence relations on* X_1 *and* X_2*,* G *is a countable group,* $\rho_1 : E_1 \to G_1$ *and* $\rho_2 : E_2 \to G_2$ *are Borel cocycles,* E_1 *is hyperfinite, and* G_1 *is an essential value of* ρ_2*. Then* $\rho_1 \sqsubseteq_B \rho_2$*.*

Proof. Simply note that Theorem 7 implies that $\rho_1 \sqsubseteq_B \rho_{G_1}$, and Theorem 3 implies that $\rho_{G_1} \sqsubseteq_B \rho_2$. □

4. Reducibility of Borel cocycles

In this section, we classify Borel cocycles from hyperfinite equivalence relations into finite groups up to Borel reducibility. We need first several preliminaries.

We begin by noting that if H is an essential value of a Borel cocycle $\rho : E \to G$, then so too is the group $\langle H \rangle$ generated by H. This is a consequence of:

Proposition 9. *Suppose that* X *is a Polish space,* E *is a countable Borel equivalence relation on* X*,* G *is a countable group,* $H \subseteq G$ *is non-empty, and* $\rho : E \to G$ *is a Borel cocycle. Then* $\mathcal{I}_H = \mathcal{I}_{\langle H \rangle}$*.*

Proof. It is enough to show that $\mathcal{I}_{\langle H \rangle} \subseteq \mathcal{I}_H$, or equivalently, that

$$\forall B \subseteq X \text{ Borel } (\langle H \rangle \nsubseteq \mathrm{Val}(\rho, B) \Rightarrow B \in \mathcal{I}_H).$$

We will show that if $B \subseteq X$ is a Borel set which is not in $\mathcal{I}_H$, then $\langle H \rangle \subseteq \mathrm{Val}(\rho, B)$. As $\mathrm{Val}(\rho, B)$ contains H and is therefore non-empty, it is enough to show that if $h_1, \ldots, h_n \in H$ and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$, then $g = h_1^{\epsilon_1} \cdots h_n^{\epsilon_n}$ is in $\mathrm{Val}(\rho, B)$. Towards this end, fix a countable group Γ of Borel automorphisms of X which generates E.

Lemma 10. *There are Borel sets* $B_0, B_1, \ldots, B_n \subseteq B$ *and* $\gamma_1, \ldots, \gamma_n \in \Gamma$ *such that:*

1. $\forall i \leq n \ (B_i \notin \mathcal{I}_H)$*;*
2. $\forall i < n \, \forall x \in B_{i+1} \ (\rho(\gamma_{i+1} \cdot x, x) = h_{i+1}^{\epsilon_{i+1}})$*;*
3. $\forall i < n \ (B_{i+1}, \gamma_{i+1}[B_{i+1}] \subseteq B_i)$*;*
4. $\forall i < n \ (B_{i+1} \cap \gamma_{i+1}[B_{i+1}] = \emptyset)$*.*

Proof. We begin by setting $B_0 = B$. Suppose now that $i < n$ and we have found $B_0, \ldots, B_i$ and $\gamma_1, \ldots, \gamma_i$. Define $A_{i+1} \subseteq B_i$ by

$$A_{i+1} = \{x \in B_i : h_{i+1}^{-\epsilon_{i+1}} \in \mathrm{Val}(\rho, B_i, x)\}.$$

As $h_{i+1} \notin \mathrm{Val}(\rho, B_i \setminus A_{i+1})$, it follows that $A_{i+1} \notin \mathcal{I}_H$. For each $\gamma \in \Gamma$, set

$$A_{i+1}^{\gamma} = \{x \in A_{i+1} : \gamma \cdot x \in B_i \text{ and } x \neq \gamma \cdot x \text{ and } \rho(\gamma \cdot x, x) = h_{i+1}^{\epsilon_{i+1}}\},$$

and note that $A_{i+1} = \bigcup_{\gamma \in \Gamma} A_{i+1}^{\gamma}$, so there exists $\gamma_{i+1} \in \Gamma$ such that $A_{i+1}^{\gamma_{i+1}} \notin \mathcal{I}_H$. As this latter set is the union of countably many Borel sets A with the property that $A \cap \gamma_{i+1}[A] = \emptyset$, it follows that there is a Borel set $B_{i+1} \subseteq A_{i+1}^{\gamma_{i+1}}$, outside of $\mathcal{I}_H$, such that $B_{i+1} \cap \gamma_{i+1}[B_{i+1}] = \emptyset$. □

Fix $B = B_0 \supseteq \cdots \supseteq B_n$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ as in Lemma 10. As $B_n \notin \mathcal{I}_H$, it is non-empty, so we can fix some $x_n \in B_n$. For each $i < n$, set $x_i = \gamma_{i+1} \cdots \gamma_n \cdot x_n$, so that $x_i \in B_i \setminus B_{i+1}$, thus $x_0, \ldots, x_n$ are pairwise distinct. Finally, observe that

$$\begin{aligned} \rho(x_0, x_n) &= \rho(x_0, x_1) \cdots \rho(x_{n-1}, x_n) \\ &= h_1^{\epsilon_1} \cdots h_n^{\epsilon_n} \\ &= g, \end{aligned}$$

thus $g \in \mathrm{Val}(\rho, B)$. □

We will use the following fact to organize the domains of our reductions:

Proposition 11. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a finite group, and $\rho : E \to G$ is a Borel cocycle. Then there is a countable family $\mathcal{B}$ of Borel subsets of X such that:*

1. *The sets $[B]_E$, for $B \in \mathcal{B}$, partition X;*
2. *For each $B \in \mathcal{B}$, the set $\mathrm{Val}(\rho, B)$ is an essential value of $\rho|(E|B)$.*

Proof. We begin by defining a family $\mathcal{B}_B$ corresponding to each Borel set $B \subseteq X$. If $\mathrm{Val}(\rho, B)$ is an essential value of $\rho|(E|B)$, then we set $\mathcal{B}_B = \{B\}$. Otherwise, there are Borel sets $B_0, B_1, \ldots$ such that $B = \bigcup_{n \in \mathbb{N}} B_n$ and $\mathrm{Val}(\rho, B_n) \subsetneq \mathrm{Val}(\rho, B)$, for all $n \in \mathbb{N}$. In this case, we set $\mathcal{B}_B = \{B_n \setminus \bigcup_{m<n} [B_m]_E : n \in \mathbb{N}\}$.

Now set $\mathcal{B}_0 = \{X\}$, and recursively define $\mathcal{B}_{i+1} = \bigcup_{B \in \mathcal{B}_i} \mathcal{B}_B$. Letting $n = |G|$, we claim that $\mathcal{B}_n$ is as desired. To see this suppose, towards a contradiction, that there exists $B \in \mathcal{B}_n$ such that $\mathrm{Val}(\rho, B)$ is not an essential value of $\rho|(E|B)$. Then there exist $B_i \in \mathcal{B}_i$ such that $B = B_n \subseteq \cdots \subseteq B_1 \subseteq B_0 = X$ and $\mathrm{Val}(\rho, B_n) \subsetneq \cdots \subsetneq \mathrm{Val}(\rho, B_1) \subsetneq \mathrm{Val}(\rho, B_0)$, thus $\mathrm{Val}(\rho, B_n) = \emptyset$, the desired contradiction. □

We will use the following fact to organize the ranges of our reductions:

Proposition 12. *Suppose that X is an uncountable Polish space, E is a countable Borel equivalence relation on X, G is a finite group, and $\rho : E \to G$ is a Borel cocycle. Then there are uncountable Borel sets $B_H \subseteq X$, for each $H \in \mathrm{Ess}(\rho)$, such that:*

1. *The sets of the form $[B_H]_E$, for $H \in \mathrm{Ess}(\rho)$, are pairwise disjoint;*
2. *For each $H \in \mathrm{Ess}(\rho)$, the set H is an essential value of $\rho|(E|B_H)$.*

Proof. By a straightforward induction, it is enough to show that if $H \in \mathrm{Ess}(\rho)$, then there are uncountable Borel sets $B_0, B_1 \subseteq X$, with $[B_0]_E \cap [B_1]_E = \emptyset$, such that H is an essential value of both $\rho|(E|B_0)$ and $\rho|(E|B_1)$. By Proposition 9, it is enough to show this in the special case that H is a subgroup of G. In light of Theorem 3, it is enough to show this for the cocycle ρ_H. Fix disjoint infinite sets $S_0, S_1 \subseteq \mathbb{N}$ such that $\mathbb{N} = S_0 \cup S_1$ and each $h \in H$ is of the form g_n^H, for infinitely many $n \in S_0$ and infinitely many $n \in S_1$. The *support* of $\alpha \in 2^{\mathbb{N}}$ is given by $\mathrm{supp}(\alpha) = \{n \in \mathbb{N} : \alpha(n) \neq 0\}$. For each $i \in \{0, 1\}$, define $B_i \subseteq 2^{\mathbb{N}}$ by

$$B_i = \{\alpha \in 2^{\mathbb{N}} : \mathrm{supp}(\alpha) \subseteq S_i\}.$$

Then $[B_0]_{E_0} \cap [B_1]_{E_0}$ is the E_0-class of the eventually constant sequence. As throwing out a countable set has no effect on essential values, it only remains to prove that H is an essential value of $\rho_H|(E_0|B_i)$, for each $i \in \{0, 1\}$. For this, it is enough to produce a continuous embedding of ρ_H into $\rho_H|(E_0|B_i)$. Towards this end, fix a strictly increasing sequence of natural numbers $k_0, k_1, \ldots \in S_i$ such that $g^H_n = g^H_{k_n}$, and observe that the map $\pi_i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, given by

$$\pi(\alpha) = 0^{k_0}\alpha(0)0^{k_1-k_0-1}\alpha(1)0^{k_2-k_1-1}\alpha(2)\ldots,$$

is as desired. □

The following fact will allow us to paste together reductions:

Proposition 13. *Suppose that X_1 and X_2 are Polish spaces, E_1 and E_2 are countable Borel equivalence relations on X_1 and X_2, G is a countable group, $\rho_1 : E_1 \to G$ and $\rho_2 : E_2 \to G$ are Borel cocycles with everywhere full ranges, and $B^i_1, B^i_2, \ldots \subseteq X_i$, for $i \in \{1, 2\}$, are Borel sets such that:*

1. *$\bigcup_{n\in\mathbb{Z}^+} B^1_n$ is an E_1-complete section;*
2. *$[B^2_1]_{E_2}, [B^2_2]_{E_2}, \ldots$ are pairwise disjoint;*
3. *$\rho_1|(E_1|B^1_n) \leq_B \rho_2|(E_2|B^2_n)$, for all $n \in \mathbb{N}$.*

Then $\rho_1 \leq_B \rho_2$.

Proof. Fix Borel reductions $\pi_n : B^1_n \to B^2_n$ of $\rho_1|(E_1|B^1_n)$ to $\rho_2|(E_2|B^2_n)$. Set $A^1_n = B^1_n \setminus \bigcup_{m<n}[B^1_m]_{E_1}$ and $A^2_n = \pi_n[A^1_n]$. For each $i \in \{1, 2\}$, put $A_i = \bigcup_{n\in\mathbb{N}} A^i_n$. For each $x \in X_1$, let $n(x)$ denote the unique $n \in \mathbb{N}$ such that $x \in [A^1_n]_{E_1}$. Fix a Borel function $\varphi : X_1 \to A_1$ such that $\mathrm{graph}(\varphi) \subseteq E$, and for each $g \in G$, fix a Borel function $\psi_g : A_2 \to X_2$ such that $\mathrm{graph}(\psi_g) \subseteq E_2$ and $\forall x \in A_2\, \forall g \in G\ (\rho_2(x, \psi_g(x)) = g)$. Define $\pi : X_1 \to X_2$ by

$$\pi(x) = \psi_{\rho_1(\varphi(x),x)} \circ \pi_{n(x)} \circ \varphi(x).$$

It is clear that π is a reduction of E_1 into E_2, and if xE_1y, then

$$\begin{aligned}
\rho_2(\pi(x), \pi(y)) &= \rho_2(\pi(x), \pi_{n(x)} \circ \varphi(x))\rho_2(\pi_{n(x)} \circ \varphi(x), \pi_{n(y)} \circ \varphi(y)) \\
&\quad \rho_2(\pi_{n(y)} \circ \varphi(y), \pi(y)) \\
&= \rho_1(\varphi(x), x)^{-1}\rho_1(\varphi(x), \varphi(y))\rho_1(\varphi(y), y) \\
&= \rho_1(x, y),
\end{aligned}$$

thus π is a Borel reduction of ρ_1 into ρ_2. □

We our now ready for the main result of this section:

Theorem 14. *Suppose that X_1 and X_2 are uncountable Polish spaces, E_1 and E_2 are hyperfinite equivalence relations on X_1 and X_2, G is a finite group, and $\rho_1 : E_1 \to G$ and $\rho_2 : E_2 \to G$ are Borel cocycles with everywhere full ranges. Then*

$$\rho_1 \leq_B \rho_2 \Leftrightarrow \mathrm{Ess}(\rho_1) \subseteq \mathrm{Ess}(\rho_2).$$

Proof. To see ($\Rightarrow$), suppose that $\pi : X_1 \to X_2$ is a Borel reduction of ρ_1 to ρ_2, fix $H \in \mathrm{Ess}(\rho_1)$, and fix a Borel set $Y \subseteq X_1$ such that H is an essential value of $\rho_1|(E_1|Y)$ and $\pi|Y$ is injective. If $\mathcal{B}$ is a countable family of Borel sets which cover X_2, then $\pi^{-1}[\mathcal{B}]$ is a countable family of Borel sets which cover Y. As $H \in \mathrm{Ess}(\rho_1, Y)$, it follows that there exists $B \in \mathcal{B}$ such that $H \subseteq \mathrm{Val}(\rho_1, \pi^{-1}[B] \cap Y)$, thus $H \subseteq \mathrm{Val}(\rho_2, B)$. As $H \in \mathrm{Ess}(\rho_1)$ was arbitrary, it follows that $\mathrm{Ess}(\rho_1) \subseteq \mathrm{Ess}(\rho_2)$.

To see ($\Leftarrow$), note first that Proposition 11 implies that there is a countable family $\mathcal{B}$ of Borel subsets of X_1 such that:

1. The sets $[B]_{E_1}$, for $B \in \mathcal{B}$, partition X_1;
2. For each $B \in \mathcal{B}$, the set $\mathrm{Val}(\rho_1, B)$ is an essential value of $\rho_1|(E_1|B)$.

For each $H \in \mathrm{Ess}(\rho_1)$, let $B_1^H = \bigcup\{B : B \in \mathcal{B} \text{ and } \mathrm{Val}(\rho_1, B) \subseteq H\}$. By Proposition 12, there are uncountable Borel sets $B_2^H \subseteq X_2$, for each $H \in \mathrm{Ess}(\rho_2)$, such that:

1. The sets of the form $[B_2^H]_{E_2}$, for $H \in \mathrm{Ess}(\rho_2)$, are pairwise disjoint;
2. For each $H \in \mathrm{Ess}(\rho_2)$, the group H is an essential value of $\rho|(E_2|B_2^H)$.

Theorem 8 implies that $\rho_1|(E_1|B_1^H) \sqsubseteq_B \rho_2|(E_2|B_2^H)$, for each $H \leq G$ in $\mathrm{Ess}(\rho_1)$ such that $B_1^H \neq \emptyset$, and Proposition 13 therefore implies that $\rho_1 \leq_B \rho_2$. □

We close this section by noting which subsets of a finite group G can occur as the set of essential values of a Borel cocycle from a non-smooth (hyperfinite) equivalence relation into G.

Proposition 15. *Suppose that G is a finite group and $\mathcal{F}$ is a family of subsets of G. Then the following are equivalent:*

1. *There is a Polish space X, a non-smooth hyperfinite equivalence relation E on X, and a Borel cocycle $\rho : E \to G$ such that $\mathcal{F} = \mathrm{Ess}(\rho)$;*
2. *$\mathcal{F}$ satisfies the following conditions:*
   (a) *The trivial group is in $\mathcal{F}$;*
   (b) *If $K \subseteq H$ and $H \in \mathcal{F}$, then $K \in \mathcal{F}$;*
   (c) *If $H \in \mathcal{F}$, then $\langle H \rangle \in \mathcal{F}$;*
   (d) *If $H \in \mathcal{F}$ and $g \in G$, then $gHg^{-1} \in \mathcal{F}$.*

Proof. To see (1) $\Rightarrow$ (2), note first that (a) follows from the non-smoothness of E, (b) holds trivially, and (c) follows from Proposition 9. To see (d), suppose that $H \in \mathrm{Ess}(\rho)$ and $g \in G$. For each set $B \subseteq X$, let

$$g[B] = \{x \in X : \exists y \in B \ (\rho(x, y) = g)\},$$

and observe that if $\mathcal{B}$ is a cover of X by countably many Borel sets, then the family $g^{-1}[\mathcal{B}]$ also covers X. Fix $B \in \mathcal{B}$ such that $H \subseteq \mathrm{Val}(\rho, g^{-1}[B])$, and observe that for each $h \in H$, there exist $x, y \in g^{-1}[B]$ such that $\rho(x, y) = h$. Fixing $x', y' \in B$ such that $\rho(x, x') = \rho(y, y') = g^{-1}$, it follows that

$$\rho(x', y') = \rho(x', x)\rho(x, y)\rho(y, y') = ghg^{-1},$$

so $gHg^{-1} \subseteq \mathrm{Val}(\rho, B)$, thus $gHg^{-1} \in \mathrm{Ess}(\rho)$.

To see (2) $\Rightarrow$ (1), let $\mathcal{F}' = \{H \in \mathcal{F} : H \leq G\}$, define

$$X = \{(H, g, \alpha) : H \in \mathcal{F} \text{ and } g \in G \text{ and } \alpha \in 2^{\mathbb{N}}\},$$

define an equivalence relation E on X by

$$(H_1, g_1, \alpha_1)E(H_2, g_2, \alpha_2) \Leftrightarrow H_1 = H_2 \text{ and } \alpha_1 E_0 \alpha_2,$$

and define a cocycle $\rho : E \to G$ by

$$\rho((H, g_1, \alpha_1), (H, g_2, \alpha_2)) = g_1 \rho_H(\alpha_1, \alpha_2) g_2^{-1}.$$

For each $H \in \mathcal{F}$, let $X_H = \{(H, g, \alpha) : g \in G \text{ and } \alpha \in 2^{\mathbb{N}}\}$ and $Y_H = \{(H, 1_G, \alpha) : \alpha \in 2^{\mathbb{N}}\}$. To see that $\mathcal{F} = \mathrm{Ess}(\rho)$, it is enough to show that $\mathrm{Ess}(\rho, X_H) = \{K \subseteq G : \exists g \in G \ (K \subseteq gHg^{-1})\}$. For this, it is enough to check that $\mathrm{Ess}(\rho, Y_H)$ is the powerset of H, and this follows from Proposition 1. $\square$

5. Finite Borel equivalence relations on $2^{\mathbb{N}}/E_0$

In this section, we classify finite Borel equivalence relations on the non-smooth hyperfinite quotient space up to Borel isomorphism.

We say that F_1/E_1 is *invariantly embeddable* into F_2/E_2, or $F_1/E_1 \sqsubseteq_B^i F_2/E_2$, if there is a Borel injection $\pi : X_1/E_1 \to X_2/E_2$, for which $\pi[X_1/E_1]$ is (F_2/E_2)-invariant, such that

$$\forall x_1, y_1 \in X_1/E_1 \ (x_1(F_1/E_1)y_1 \Leftrightarrow \pi(x_1)(F_2/E_2)\pi(y_1)).$$

Theorem 16. *Suppose that X_1 and X_2 are uncountable Polish spaces, $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$ are hyperfinite equivalence relations on X_1 and X_2, and $[F_1 : E_1] = [F_2 : E_2] = n$, for some $n \in \mathbb{N}$. Then the following are equivalent:*

1. $\mathrm{Ess}(E_1, F_1) \subseteq \mathrm{Ess}(E_2, F_2)$;
2. $F_1/E_1 \sqsubseteq_B^i F_2/E_2$.

Proof. In light of Theorem 14, it is enough to show that

$$\rho_{(E_1,F_1)} \leq_B \rho_{(E_2,F_2)} \Leftrightarrow F_1/E_1 \sqsubseteq_B^i F_2/E_2.$$

To see ($\Rightarrow$), simply note that if $\pi : \mathrm{Enum}(E_1, F_1) \to \mathrm{Enum}(E_2, F_2)$ is a Borel reduction of $\rho_{(E_1,F_1)}$ to $\rho_{(E_2,F_2)}$, then π is a reduction of E_1^* to E_2^* and of F_1^* to F_2^*, and therefore induces a Borel embedding of F_1^*/E_1^* into F_2^*/E_2^*. As $F_i/E_i \cong_B F_i^*/E_i^*$ and every equivalence class of these equivalence relations is of cardinality n, it follows that $F_1/E_1 \sqsubseteq_B^i F_2/E_2$.

To see ($\Leftarrow$), suppose that $\pi : X_1/E_1 \to X_2/E_2$ is a Borel embedding of F_1/E_1 into F_2/E_2, and fix a Borel lifting $\tilde{\pi} : X_1 \to X_2$ of π. Then $\tilde{\pi}$ is a reduction of E_1 to E_2 and of F_1 to F_2, and it follows that the map $(x_1, \ldots, x_n) \mapsto (\tilde{\pi}(x_1), \ldots, \tilde{\pi}(x_n))$ is a Borel reduction of $\rho_{(E_1,F_1)}$ to $\rho_{(E_2,F_2)}$. □

As a corollary, we obtain the main result of the paper:

Theorem 17. *Suppose that X_1 and X_2 are uncountable Polish spaces, $E_1 \subseteq F_1$ and $E_2 \subseteq F_2$ are hyperfinite equivalence relations on X_1 and X_2, and $[F_1 : E_1] = [F_2 : E_2] = n$, for some $n \in \mathbb{N}$. Then*

$$F_1/E_1 \cong_B F_2/E_2 \Leftrightarrow \mathrm{Ess}(E_1, F_1) = \mathrm{Ess}(E_2, F_2).$$

Proof. In light of Theorem 16, it is enough to show that if $F_1/E_1 \sqsubseteq_B^i F_2/E_2$ and $F_2/E_2 \sqsubseteq_B^i F_1/E_1$, then $F_1/E_1 \cong_B F_2/E_2$, and this follows from a straightforward Schröder-Bernstein argument. □

6. Measures

A (Borel) probability measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-null or μ-conull, and *E-quasi-invariant* if the family of μ-null sets is closed under E-saturation. The following well known consequence of the Glimm-Effros dichotomy characterizes the circumstances under which E admits such a measure:

Theorem 18 (Glimm, Effros, Shelah-Weiss). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*

1. *There is a Borel transversal of E;*
2. *There is an atomless, E-ergodic, E-quasi-invariant probability measure on X.*

Here we prove an analogous theorem for pairs of equivalence relations:

Theorem 19. *Suppose that X is a Polish space, $E \subseteq F$ are countable Borel equivalence relations on X, and $[F : E] = n$, for some natural number $n \geq 2$. Then the following are equivalent.*

1. *There is no E-invariant Borel set $B \subseteq X$ such that both B and $X \setminus B$ are F-complete sections;*

2. *There is a group $G \in \mathrm{Ess}(E, F)$ which acts transitively on $\{1, \ldots, n\}$;*

3. *There is an atomless, E-ergodic, F-quasi-invariant probability measure on X.*

Proof. To see (3) $\Rightarrow$ (1), fix an E-ergodic, F-quasi-invariant probability measure μ on X, and suppose that $B \subseteq X$ is an E-invariant Borel set. Then, by reversing the roles of B and $X \setminus B$ if necessary, we can assume that $\mu(B) = 0$, so $\mu([B]_F) = 0$, thus B is not an F-complete section.

To see $\neg(2) \Rightarrow \neg(1)$, appeal to Proposition 11 to find a countable family $\mathcal{B}$ of Borel subsets of $\mathrm{Enum}(E, F)$ such that:

1. The sets $[B]_{F^*}$, for $B \in \mathcal{B}$, partition $\mathrm{Enum}(E, F)$;

2. For each $B \in \mathcal{B}$, the set $\mathrm{Val}(\rho_{(E,F)}, B)$ is an essential value of $\rho_{(E,F)}|(F^*|B)$.

For each $B \in \mathcal{B}$, observe that because the group $\mathrm{Ess}(\rho_{(E,F)}, B)$ does not act transitively on $\{1, \ldots, n\}$, the sets $[B]_{E^*}$ and $[B]_{F^*} \setminus [B]_{E^*}$ are $(F^*|[B]_{F^*})$-complete sections. Letting $B_0, B_1, \ldots$ be an enumeration of the elements of $\mathcal{B}$, it follows that

$$B = \bigcup_{n \in \mathbb{N}} \left([B_n]_{E^*} \setminus \bigcup_{m < n} [B_m]_{F^*} \right)$$

is an E^*-invariant Borel set such that both B and $\mathrm{Enum}(E, F) \setminus B$ are F^*-complete sections, and since $F/E \cong_B F^*/E^*$, it follows that there is an E-invariant Borel set $B \subseteq X$ such that both B and $X \setminus B$ are F-complete sections.

To see (2) $\Rightarrow$ (3), suppose that $G \in \mathrm{Ess}(E, F)$ acts transitively on $\{1, \ldots, n\}$. The idea of the proof is to push the product measure on $2^{\mathbb{N}}$ through to X via the (continuous) embedding of ρ_G into $\rho_{(E,F)}$ given by Theorem 3. However, it will be a bit simpler to use a slightly different cocycle.

Let $E_0(G)$ denote the equivalence relation on $G^{\mathbb{N}}$ given by

$$\alpha E_0(G) \beta \Leftrightarrow \exists n \in \mathbb{N} \, \forall m \geq n \ (\alpha(m) = \beta(m)),$$

and define $\rho'_G : E_0(G) \to G$ by

$$\rho'_G(s\alpha, t\beta) = (s(0) \cdots s(n))(t(0) \cdots t(n))^{-1},$$

where $n \in \mathbb{N}$, $s, t \in G^{n+1}$, and $\alpha \in G^{\mathbb{N}}$. As $E_0(G)$ is obviously hyperfinite, Theorem 8 implies that there is a Borel embedding $\varphi : G^{\mathbb{N}} \to \mathrm{Enum}(E, F)$ of ρ'_G into $\rho_{(E,F)}$.

Define a subequivalence relation $E'_0(G)$ of $E_0(G)$ by

$$\alpha E'_0(G) \beta \Leftrightarrow \alpha E_0(G) \beta \text{ and } [\rho'_G(\alpha, \beta)](1) = 1,$$

and let μ denote the product measure on $G^{\mathbb{N}}$ obtained from the the uniform probability measure on G.

Lemma 20. *The measure μ is $E'_0(G)$-ergodic.*

Proof. We must show that if $B \subseteq X$ is a non-null, $E_0'(G)$-invariant Borel set and $\epsilon > 0$, then $\mu(B) \geq 1 - \epsilon$. By the ultrametric analog of the Lebesgue density theorem, there exists $s \in G^{<\mathbb{N}}$ such that $\mu(\mathcal{N}_s \setminus B)/\mu(\mathcal{N}_s) < \epsilon/|G|$. Let $A = \{sh\alpha \in B : \exists g \in G\ (sg\alpha \in B)\}$, and observe that

$$\begin{aligned} \mu(\mathcal{N}_s \setminus A) &\leq \sum_{g \in G} |G| \mu(\mathcal{N}_{sg} \setminus B) \\ &= |G| \mu(\mathcal{N}_s \setminus B) \\ &\leq \epsilon \mu(\mathcal{N}_s), \end{aligned}$$

thus $\mu(A)/\mu(\mathcal{N}_s) \geq 1 - \epsilon$. As B is $E_0'(G)$-invariant, it follows that $[A]_{E_0(G)} \subseteq B$, thus $\mu(B \cap \mathcal{N}_t)/\mu(\mathcal{N}_t) \geq 1 - \epsilon$, for all $t \in 2^{|s|}$, so $\mu(B) \geq 1 - \epsilon$. □

Note that φ is a reduction of $E_0'(G)$ into E^* and of $E_0(G)$ into F^*. By composing φ with the function $(x_1, \ldots, x_n) \mapsto x_1$, we therefore obtain a continuous function $\varphi : 2^{\mathbb{N}} \to X$ which reduces $E_0'(G)$ to E and $E_0(G)$ to F. Fix a group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms of X which generates E, and define ν on X by

$$\nu(B) = \sum_{n \in \mathbb{N}} \mu(\gamma_n^{-1}[B])/2^{n+1}.$$

It is easily verified that ν is the desired probability measure. □

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