FORCELESS, INEFFECTIVE, POWERLESS PROOFS OF DESCRIPTIVE DICHOTOMY THEOREMS

AN INTRODUCTION

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ABSTRACT. We discuss some of the motivation behind work described in lectures given at Université Paris 6 and 7 in July 2009.

Set theory was born in 1873 with Cantor's realization that there is no injection of the real numbers into the natural numbers [5]. He soon became convinced that no set has cardinality strictly between. This conjecture is now known as *Cantor's Continuum Hypothesis*, or CH, and the question of its truth appeared as Hilbert's first problem [14].

Whereas set theory strives to determine the nature of sets in general, work of Baire [2, 3], Borel [4], and Lebesgue [16, 17] at the turn of the century focused on properties of definable sets, an area which has since come to be known as *descriptive set theory*. The first theorem of the subject was actually established somewhat earlier when Cantor showed that closed subsets of Polish spaces are either countable or contain a non-empty perfect subset, and therefore satisfy CH [6].

Cantor's result was generalized by Alexandrov [1] and Hausdorff [13] to Borel subsets of Polish spaces, and then by Souslin to analytic Hausdorff spaces [22]. Since then, the search for dichotomy theorems has played a fundamental role in the development of the subject.

Over the next half century, the proofs of these theorems followed the same basic outline. Roughly speaking, they used a derivative to reduce the general theorem to a topologically simple special case which could be handled in a straightforward manner. Of course, the crux of these problems was to find the appropriate notion of derivative. Although this was not always an easy task, one was nevertheless led to the belief that the richness of the collection of derivatives in the mathematical universe is the force underlying the great variety of dichotomy theorems in descriptive set theory.

This point of view was dealt a serious blow in the early 1970s when Silver [21], answering a question of Martin, showed that every co-analytic equivalence relation on a Hausdorff space has either countably many equivalence classes or at least perfectly many. Unlike the proofs of earlier dichotomy theorems, Silver's argument was a technical *tour de force*, relying on a number of techniques from mathematical logic, as well as a much larger fragment of `ZF` than typical. In light of then-recent results of H. Friedman [9] and Martin [20] on Borel determinacy, it was natural to conjecture that this large fragment is necessary. Nevertheless, this conjecture was refuted a few years later with Harrington's proof [10] of Silver's theorem, which while comparably simple, still relied upon the recursion-theoretic refinement of descriptive set theory known as *effective descriptive set theory*, as well as the method of forcing, which was initially developed by Cohen [7, 8] in his proof of the independence of `CH`. Harrington-Shelah [12] later discovered another forcing proof which allowed one to generalize Silver's theorem to co-κ-Souslin equivalence relations satisfying a technical forcing hypothesis.

Over the next thirty years, the techniques of Harrington and Harrington-Shelah were applied in the discovery of an astonishing number of structural properties of definable sets. Although some of these were relatively straightforward generalizations of Silver's theorem, others used progressively more sophisticated and technically difficult refinements of the original arguments.

Despite the abundance of progress, a question remained lurking in the background. Is the use of effective descriptive set theory and forcing really necessary to derive these purely classical results? Could it be that the old intuition was correct, and it is really this rich collection of derivatives that underlies even the more recent dichotomy theorems? Even if derivatives alone are insufficient to explain the abundance of dichotomy theorems, is there another unifying explanation of these results? Of course, the hope is that positive answers to these questions might lead to simpler proofs and generalizations of known results, as well as to entirely new theorems.

Our goal here is to describe recent research leading towards a positive answer to these questions. This work is built on the backbone of a natural family of dichotomy theorems for definable graphs generalizing the Kechris-Solecki-Todorcevic characterization [15] of the family of analytic graphs with uncountable Borel chromatic numbers. The Borel instantiations of these theorems all have classical proofs using little more than the sorts of derivatives that appeared in Cantor's work, and they can be combined with classical Baire category arguments to obtain new proofs of many other descriptive set-theoretic dichotomy theorems. In particular, this has led to the first classical proofs of the theorems of Silver [21] and Harrington-Kechris-Louveau [11]. It has also led to classical proofs of analogs of these theorems for κ-Souslin structures which are ω-universally Baire.

As the work described here is quite new and much of it remains unfinished, one should take care to avoid making drastic conclusions. For instance, it is far too soon (as it will most likely always be) to assert that the new techniques make the old ones obsolete. In addition to the glaring fact that several dichotomy theorems continue to resist classical proofs, there are certain advantages to the old arguments. For instance, in applying the effective theory, one needs only a sufficient amount of faith that the effective arguments encompass all that they should, and one can proceed to go about the business of establishing new dichotomy theorems. In the new arguments, this faith is replaced with a new belief that these graph-theoretic dichotomy theorems encompass all they should about the descriptive set-theoretic objects in which we are interested. There is a new technical detail, however, in that one must determine the graph-theoretic dichotomy which underlies the question at hand. In some sense, this points to a shortcoming of the new techniques. On the other hand, the fact that one is forced to understand more precisely the graph-theoretic underpinnings of descriptive set-theoretic dichotomy theorems in order to prove them can be viewed as a boon.

The following four lectures follow the same general setup. They begin with a proof of a single graph-theoretic dichotomy theorem using nothing more than a derivative and the first separation theorem. While some of these graph-theoretic dichotomy theorems imply others, and it is indeed possible to isolate a single (albeit somewhat unnatural) theorem which implies them all, we have instead decided to isolate the precise graph-theoretic theorems that are necessary to establish the main result of each lecture, which is exactly what we do once our graph-theoretic results have been established. Although for the most part we confine our attention to analytic structures, we do spend some time discussing κ-Souslin structures, especially in the first lecture. We close each lecture with a number of exercises suggesting the great variety of results to which similar arguments apply.

Acknowledgements. I would like to offer my sincere thanks to the participants in the seminars at Paris 6 and 7 where I gave these lectures. Their interest and patience during these marathon sessions were very much appreciated, as were their many fine suggestions. I also owe a special debt to Dominique Lecomte, who both arranged my visit to Paris and provided many hours of fruitful discussion.

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