# FORCELESS, INEFFECTIVE, POWERLESS PROOFS OF DESCRIPTIVE DICHOTOMY THEOREMS

## LECTURE II: HJORTH'S THEOREM

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Abstract. We give a classical proof of a generalization of the Kechris-Solecki-Todorcevic dichotomy theorem [4] characterizing analytic graphs of uncountable Borel chromatic number. Using this, we give a classical proof of a generalization of Hjorth's theorem [2] characterizing smooth treeable equivalence relations.

In §1, we give two straightforward corollaries of the first separation theorem. In §2, we give the promised classical proof of a generalization of the Kechris-Solecki-Todorcevic [4] theorem. In §3, we use this to establish a dichotomy theorem for metrized equivalence relations, and derive from this Hjorth's theorem [2]. In §4, we give as exercises several results that can be obtained in a similar fashion.

### 1. Corollaries of separation

Suppose that $X_0$ and $X_1$ are sets and $(R_n)_{n\in\omega}$ is a sequence of subsets of $X_0 \times X_1$. A pair $(A_0, A_1)$ is *eventually* $(R_n)_{n\in\omega}$*-discrete* if $A_0 \subseteq X_0$, $A_1 \subseteq X_1$, and $(A_0 \times A_1) \cap R_n = \emptyset$ for all but finitely many $n \in \omega$.

**Proposition 1.** *Suppose that $X_0$ and $X_1$ are Hausdorff spaces, $(R_n)_{n\in\omega}$ is a sequence of analytic subsets of $X_0 \times X_1$, and $(A_0, A_1)$ is an eventually $(R_n)_{n\in\omega}$-discrete pair of analytic sets. Then there is an eventually $(R_n)_{n\in\omega}$-discrete pair $(B_0, B_1)$ of Borel sets with the property that $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$.*

*Proof.* Fix $m \in \omega$ such that $(A_0, A_1)$ is $R_n$-discrete for all $n \in \omega \setminus m$. For each such $n$, fix an $R_n$-discrete pair $(B_{0,n}, B_{1,n})$ of Borel sets such that $A_0 \subseteq B_{0,n}$ and $A_1 \subseteq B_{1,n}$. Clearly the sets $B_0 = \bigcap_{n\in\omega\setminus m} B_{0,n}$ and $B_1 = \bigcap_{n\in\omega\setminus m} B_{1,n}$ are as desired. ⊠

Suppose that $X$ is a set and $(\mathcal{G}_n)_{n\in\omega}$ is a sequence of graphs on $X$. A set $A \subseteq X$ is *eventually* $(\mathcal{G}_n)_{n\in\omega}$*-discrete* if it is $\mathcal{G}_n$-discrete for all but finitely many $n \in \omega$.

**Proposition 2.** *Suppose that $X$ is a Hausdorff space, $(\mathcal{G}_n)_{n\in\omega}$ is a sequence of analytic graphs on $X$, and $A \subseteq X$ is an eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete analytic set. Then there is an eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete Borel set $B \subseteq X$ such that $A \subseteq B$.*

*Proof.* By Proposition 1, there is an eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete pair $(B_0, B_1)$ of Borel subsets of $X$ such that $A \subseteq B_0$ and $A \subseteq B_1$. It is easily verified that the set $B = B_0 \cap B_1$ is as desired. ⊠

## 2. A generalization of the Kechris-Solecki-Todorcevic theorem

For each set $I \subseteq {}^{<\omega}2$, let $\mathcal{G}_{I,n}$ denote the graph on ${}^{\omega}2$ consisting of all pairs $(s^\frown i^\frown x, s^\frown \bar{i}^\frown x)$, where $i \in 2$, $s \in I \cap {}^{n}2$, and $x \in {}^{\omega}2$.

**Proposition 3.** *Suppose that $I \subseteq {}^{<\omega}2$ is dense and $A \subseteq {}^{\omega}2$ is nonmeager and has the Baire property. Then the set $A$ is not eventually $(\mathcal{G}_{I,n})_{n\in\omega}$-discrete.*

*Proof.* Fix $s \in {}^{<\omega}2$ such that $A$ is comeager in $\mathcal{N}_s$. Given any $m \in \omega$, there exists $n \in \omega \setminus m$ and $t \in I \cap {}^{n}2$ such that $s \sqsubseteq t$. Then there exists $x \in {}^{\omega}2$ such that $t^\frown 0^\frown x, t^\frown 1^\frown x \in A$. As $(t^\frown 0^\frown x, t^\frown 1^\frown x) \in \mathcal{G}_{I,n}$, it follows that $A$ is not eventually $(\mathcal{G}_{I,n})_{n\in\omega}$-discrete. ⊠

Fix sequences $s_n \in {}^{n}2$ such that the set $I = \{s_n \mid n \in \omega\}$ is dense. Define $\mathcal{G}_{0,n} = \mathcal{G}_{I,n}$. A ($\kappa$-)*coloring* of $(\mathcal{G}_n)_{n\in\omega}$ is a function $c\colon X \to \kappa$ such that $c^{-1}(\{\alpha\})$ is eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete for all $\alpha \in \kappa$.

Suppose that $\zeta \in {}^{\omega}\omega$ and $\eta \in {}^{\leq\omega}\omega$. We say that $\eta$ is $\zeta$-*fast* if $\eta(n) > \max_{m\in n} \zeta \circ \eta(m)$ for all $n \in \omega$. For $\eta \in {}^{\omega}\omega$, an $\eta$-*homomorphism* from $(\mathcal{G}_n)_{n\in\omega}$ to $(\mathcal{H}_n)_{n\in\omega}$ is a homomorphism from $(\mathcal{G}_n)_{n\in\omega}$ to $(\mathcal{H}_{\eta(n)})_{n\in\omega}$. A $\zeta$-*fast homomorphism* from $(\mathcal{G}_n)_{n\in\omega}$ to $(\mathcal{H}_n)_{n\in\omega}$ is an $\eta$-homomorphism from $(\mathcal{G}_n)_{n\in\omega}$ to $(\mathcal{H}_n)_{n\in\omega}$, where $\eta$ is $\zeta$-fast.

**Theorem 4.** *Suppose that $\zeta \in {}^{\omega}\omega$, $X$ is a Hausdorff space and $(\mathcal{G}_n)_{n\in\omega}$ is a sequence of analytic graphs on $X$. Then exactly one of the following holds:*

1. *There is a Borel $\omega$-coloring of $(\mathcal{G}_n)_{n\in\omega}$.*
2. *There is a continuous $\zeta$-fast homomorphism from the sequence $(\mathcal{G}_{0,n})_{n\in\omega}$ to the sequence $(\mathcal{G}_n)_{n\in\omega}$.*

*Proof.* To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $c\colon X \to \omega$ is an $\omega$-universally Baire coloring of $(\mathcal{G}_n)_{n\in\omega}$, $\eta \in {}^{\omega}\omega$, and $\pi\colon {}^{\omega}2 \to X$ is a Baire measurable $\eta$-homomorphism from $(\mathcal{G}_{0,n})_{n\in\omega}$ to $(\mathcal{G}_n)_{n\in\omega}$. Then $c \circ \pi$ is a Baire measurable coloring of $(\mathcal{G}_{0,n})_{n\in\omega}$, so there exists $k \in \omega$ such that the set

$(c \circ \pi)^{-1}(\{k\})$ is non-meager and eventually $(\mathcal{G}_{0,n})_{n \in \omega}$-discrete, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that each $\mathcal{G}_n$ is non-empty. Fix continuous functions $\varphi_{\mathcal{G}_n} \colon {}^\omega\omega \to X \times X$ such that $\mathcal{G}_n = \varphi_{\mathcal{G}_n}({}^\omega\omega)$ for all $n \in \omega$, as well as a continuous function $\varphi_X \colon {}^\omega\omega \to X$ such that $\bigcup_{n \in \omega} \mathrm{dom}(\mathcal{G}_n) \subseteq \varphi_X({}^\omega\omega)$.

A *global* $(n\text{-})$*approximation* is a triple of the form $p = (e^p, u^p, v^p)$, where $e^p \in {}^n\omega$ is $\zeta$-fast, $u^p \colon {}^n 2 \to {}^n\omega$, and $v^p \colon {}^{<n}2 \to {}^n\omega$. Fix an enumeration $(p_n)_{n \in \omega}$ of the set of all global approximations.

An *extension* of a global $m$-approximation $p$ is a global $n$-approximation $q$ with the property that $e^p \sqsubseteq e^q$, $s_p \sqsubseteq s_q \Longrightarrow u^p(s_p) \sqsubseteq u^q(s_q)$, and $t_p \sqsubseteq t_q \Longrightarrow v^p(t_p) \sqsubseteq v^q(t_q)$ for all $s_p \in {}^m 2$, $s_q \in {}^n 2$, $t_p \in {}^{<m}2$, and $t_q \in {}^{<n}2$ with $n - m = |t_q| - |t_p|$. When $n = m + 1$, we say that $q$ is a *one-step extension* of $p$.

A *local* $(n\text{-})$*approximation* is a triple of the form $l = (e^l, f^l, g^l)$, where $e^l \in {}^n\omega$ is $\zeta$-fast, $f^l \colon {}^n 2 \to {}^\omega\omega$, and $g^l \colon {}^{<n}2 \to {}^\omega\omega$, such that

$$\varphi_{\mathcal{G}_{e^l(k)}} \circ g^l(t) = (\varphi_X \circ f^l(s_k \frown 0 \frown t), \varphi_X \circ f^l(s_k \frown 1 \frown t))$$

for all $k \in n$ and $t \in {}^{n-(k+1)}2$. We say that $l$ is *compatible* with a global $n$-approximation $p$ if $e^p = e^l$, $u^p(s) \sqsubseteq f^l(s)$, and $v^p(t) \sqsubseteq g^l(t)$ for all $s \in {}^n 2$ and $t \in {}^{<n}2$. We say that $l$ is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l({}^n 2) \subseteq Y$.

Suppose now that $\alpha$ is a countable ordinal, $Y \subseteq X$ is a Borel set, and $c \colon Y^c \to \omega \cdot \alpha$ is a Borel coloring of $(\mathcal{G}_n)_{n \in \omega} \upharpoonright Y^c$. Associated with each global $n$-approximation $p$ is the set $L_n(p, Y)$ of local $n$-approximations which are compatible with both $p$ and $Y$.

A global $n$-approximation $p$ is $Y$*-terminal* if $L_{n+1}(q, Y) = \emptyset$ for all one-step extensions $q$ of $p$. Let $T(Y)$ denote the set of such approximations. Set $A(p, Y) = \bigcup_{n \in \omega} \{\varphi_X \circ f^l(s_n) \mid l \in L_n(p, Y)\}$.

**Lemma 5.** *Suppose that $p$ is a global approximation and $A(p, Y)$ is not eventually $(\mathcal{G}_n)_{n \in \omega}$-discrete. Then $p \notin T(Y)$.*

*Proof of lemma.* Fix $n \in \omega$ such that $p$ is a global $n$-approximation, as well as $e > \max_{m \in n} \zeta \circ e^p(m)$ and local $n$-approximations $l_0, l_1 \in L_n(p, Y)$ with $(\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n)) \in \mathcal{G}_e$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_{\mathcal{G}_e}(x) = (\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n))$. Let $l$ denote the local $(n+1)$-approximation given by $e^l \upharpoonright n = e^p$, $e^l(n) = e$, $f^l(s \frown i) = f^{l_i}(s)$, $g^l(\emptyset) = x$, and $g^l(t \frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n 2$, and $t \in {}^{<n}2$. Then $l$ is compatible with a one-step extension of $p$, thus $p$ is not $Y$-terminal. ⊠

Proposition 2 and Lemma 5 ensure that for each $p \in T(Y)$, there is an eventually $(\mathcal{G}_n)_{n \in \omega}$-discrete Borel set $B(p, Y) \subseteq X$ with $A(p, Y) \subseteq$

$B(p, Y)$. Set $Y' = Y \setminus \bigcup\{B(p, Y) \mid p \in T(Y)\}$. For each $y \in Y \setminus Y'$, put $n(y) = \min\{n \in \omega \mid p_n \in T(Y) \text{ and } y \in B(p_n, Y)\}$. Define $c' \colon (Y')^c \to \omega \cdot (\alpha + 1)$ by

$$c'(y) = \begin{cases} c(y) & \text{if } y \in Y^c \text{ and} \\ \omega \cdot \alpha + n(y) & \text{otherwise.} \end{cases}$$

**Lemma 6.** *The function $c'$ is a coloring of $(\mathcal{G}_n)_{n\in\omega} \restriction (Y')^c$.*

*Proof of lemma.* Note that if $\beta \in \omega \cdot \alpha$ then $(c')^{-1}(\{\beta\}) = c^{-1}(\{\beta\})$, and if $\beta \in \omega \cdot (\alpha+1) \setminus \omega \cdot \alpha$ then there exists $n \in \omega$ with $\beta = \omega \cdot \alpha + n$, so $p_n \in T(Y)$ and $(c')^{-1}(\{\beta\}) \subseteq B(p_n, Y)$. Then $(c')^{-1}(\{\beta\})$ is eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete for all $\beta \in \omega \cdot (\alpha + 1)$, thus $c'$ is a coloring of the sequence $(\mathcal{G}_n)_{n\in\omega} \restriction (Y')^c$. ⊠

**Lemma 7.** *Suppose that $p$ is a global approximation whose one-step extensions are all $Y$-terminal. Then $p$ is $Y'$-terminal.*

*Proof of lemma.* Fix $n \in \omega$ such that $p$ is a global $n$-approximation. Suppose, towards a contradiction, that there is a one-step extension $q$ of $p$ for which there exists $l \in L_{n+1}(q, Y')$. Then $\varphi_X \circ f^l(s_{n+1}) \in B(q, Y)$ and $B(q, Y) \cap Y' = \emptyset$, thus $\varphi_X \circ f^l(s_{n+1}) \notin Y'$, a contradiction. ⊠

Recursively define Borel sets $Y_\alpha \subseteq X$ and Borel colorings $c_\alpha \colon Y_\alpha^c \to \omega \cdot \alpha$ of $(\mathcal{G}_n)_{n\in\omega} \restriction Y_\alpha^c$ by

$$(Y_\alpha, c_\alpha) = \begin{cases} (X, \emptyset) & \text{if } \alpha = 0, \\ (Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta\in\alpha} Y_\beta, \lim_{\beta\to\alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many approximations, there exists $\alpha \in \omega_1$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

Let $p^0$ denote the unique global 0-approximation. As $\mathrm{dom}(\mathcal{G}) \cap Y_\alpha \subseteq A(p^0, Y_\alpha)$, it follows that if $p^0$ is $Y_\alpha$-terminal, then $c_\alpha$ extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of $(\mathcal{G}_n)_{n\in\omega}$, thus there is a Borel $\omega$-coloring of $(\mathcal{G}_n)_{n\in\omega}$.

Otherwise, by repeatedly applying Lemma 7 we obtain global $n$-approximations $p^n = (e^n, u^n, v^n)$ with the property that $p^{n+1}$ is a one-step extension of $p^n$ for all $n \in \omega$. Note that the sequence $\eta = \lim_{n\to\infty} e^n$ is $\zeta$-fast, and define continuous functions $\pi \colon {}^\omega 2 \to {}^\omega\omega$ and $\pi_k \colon {}^\omega 2 \to {}^\omega\omega$ for $k \in \omega$ by

$$\pi(x) = \lim_{n\to\omega} u^n(x \restriction n) \text{ and } \pi_k(x) = \lim_{n\to\omega} v^{k+n+1}(x \restriction n).$$

To see that $\varphi_X \circ \pi$ is an $\eta$-homomorphism from $(\mathcal{G}_{0,n})_{n\in\omega}$ to $(\mathcal{G}_n)_{n\in\omega}$, it is enough to show that

$$\varphi_{\mathcal{G}_{\eta(k)}} \circ \pi_k(x) = (\varphi_X \circ \pi(s_k{}^\frown 0^\frown x), \varphi_X \circ \pi(s_k{}^\frown 1^\frown x))$$

for all $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k{}^\frown 0^\frown x), \pi(s_k{}^\frown 1^\frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_{\mathcal{G}_{\eta(k)}}(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \upharpoonright n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k{}^\frown 0^\frown (x \upharpoonright n))} \times \mathcal{N}_{u^{k+n+1}(s_k{}^\frown 1^\frown (x \upharpoonright n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, Y_\alpha)$, and observe that $z = g^l(x \upharpoonright n)$, $z_0 = f^l(s_k{}^\frown 0^\frown (x \upharpoonright n))$, and $z_1 = f^l(s_k{}^\frown 1^\frown (x \upharpoonright n))$ are as desired. ⊠

## 3. Hjorth's theorem

Suppose that $X$ is a set and $E$ is an equivalence relation on $X$. An *$E$-quasi-metric* is a function $d \colon E \to [0, \infty)$ such that:

1. $\forall x \in X \ (d(x, x) = 0)$.
2. $\forall x, y \in X \ (d(x, y) = d(y, x))$.
3. $\forall x, y, z \in X \ (xEyEz \Longrightarrow d(x, z) \leq d(x, y) + d(y, z))$.

We say that $d$ is an *$E$-metric* if $\forall x, y \in X \ (x = y \Longleftrightarrow d(x, y) = 0)$.

We say that a set $A \subseteq X$ is *$d$-bounded* if $d(E \upharpoonright A) \subseteq n$ for some $n \in \omega$. Let $d_0$ denote the $E_0$-metric given by

$$d_0(x, y) = \min\{n \in \omega \mid \forall m \in \omega \setminus n \ (x(m) = y(m))\}.$$

**Proposition 8.** *Suppose that $A \subseteq {}^\omega 2$ is $d_0$-bounded and has the Baire property. Then $A$ is meager.*

*Proof.* Fix $n \in \omega$ such that $d_0(E \upharpoonright A) \subseteq n$, and note that for each $s \in {}^n 2$, the set $A_s = A \cap \mathcal{N}_s$ is a partial transversal of $E_0$. As any such set is meager, so too is $A$. ⊠

We say that a homomorphism $\pi \colon {}^\omega 2 \to X$ from $E_0$ to $E$ is *$d$-expansive* if $d_0(x, y) \leq d(\pi(x), \pi(y))$ for all $(x, y) \in E_0$.

**Theorem 9.** *Suppose that $X$ is a Hausdorff space, $E$ is an analytic equivalence relation on $X$, and $d$ is an $E$-quasi-metric such that $d^{-1}(n, \infty)$ is analytic for all $n \in \omega$. Then exactly one of the following holds:*

1. *There is a cover of $X$ by countably many $d$-bounded Borel sets.*
2. *There is a continuous $d$-expansive embedding of $E_0$ into $E$.*

*Proof.* Proposition 8 easily implies that (1) and (2) are mutually exclusive. For each $n \in \omega$, set $\mathcal{G}_n = \{(x, y) \in X \times X \mid x \neq y \text{ and } d(x, y) = n\}$. As every eventually $(\mathcal{G}_n)_{n\in\omega}$-discrete set is $d$-bounded, it follows that if there is a Borel $\omega$-coloring of $\mathcal{G}$, then $X$ is the union of countably many $d$-bounded Borel sets.

Define $\zeta \in {}^\omega\omega$ by $\zeta(n) = 8n$. By Theorem 4, we can assume that there is a $\zeta$-fast sequence $\eta \in {}^\omega\omega$ for which there exists a continuous $\eta$-homomorphism $\varphi$ from $(\mathcal{G}_{0,n})_{n\in\omega}$ to $(\mathcal{G}_n)_{n\in\omega}$.

**Lemma 10.** *Suppose that* $(x, y) \in E_0 \setminus \Delta(X)$. *Then*

$$d(\varphi(x), \varphi(y)) \leq 2\eta(d_0(x, y) - 1).$$

*Proof of lemma.* By induction on $n = d_0(x, y)$. To handle the case $n = 1$, observe that if $d_0(x, y) = 1$, then $(x, y) \in \mathcal{G}_{0,0}$, so it follows that $d(\varphi(x), \varphi(y)) = \eta(0)$. Suppose now that $n \in \omega \setminus 1$ and we have established the lemma for $d_0(x, y) \leq n$. Given $u, v \in {}^n2$ and $z \in {}^\omega2$, set $x = u^\frown 0^\frown z$ and $y = v^\frown 1^\frown z$. The triangle inequality and two applications of the induction hypothesis ensure that

$$\begin{aligned} d(\varphi(x), \varphi(y)) \leq\ & d(\varphi(u^\frown 0^\frown z), \varphi(s_n{}^\frown 0^\frown z)) + \\ & d(\varphi(s_n{}^\frown 0^\frown z), \varphi(s_n{}^\frown 1^\frown z)) + \\ & d(\varphi(s_n{}^\frown 1^\frown z), \varphi(v^\frown 1^\frown z)) \\ \leq\ & 2\eta(n-1) + \eta(n) + 2\eta(n-1) \\ \leq\ & 2\eta(n), \end{aligned}$$

which completes the proof. ⊠

It is clear that $\varphi$ is a homomorphism from $E_0$ to $E$.

**Lemma 11.** *The homomorphism* $\varphi$ *is* $d$*-expansive.*

*Proof of lemma.* Suppose that $(x, y) \in E_0 \setminus \Delta({}^\omega2)$ and set $n = d_0(x, y)$. Clearly we can assume that $n \geq 2$. After reversing the roles of $x$ and $y$ if necessary, we can assume that there exist $u, v \in {}^n2$ and $z \in {}^\omega2$ with $x = u^\frown 0^\frown z$ and $y = v^\frown 1^\frown z$. The triangle inequality and two applications of Lemma 10 ensure that

$$\begin{aligned} \eta(n) &= d(\varphi(s_n{}^\frown 0^\frown z), \varphi(s_n{}^\frown 1^\frown z)) \\ &\leq d(\varphi(s_n{}^\frown 0^\frown z), \varphi(u^\frown 0^\frown z)) + \\ &\quad\ d(\varphi(u^\frown 0^\frown z), \varphi(v^\frown 1^\frown z)) + \\ &\quad\ d(\varphi(v^\frown 1^\frown z), \varphi(s_n{}^\frown 1^\frown z)) \\ &\leq 2\eta(n-1) + d(\varphi(x), \varphi(y)) + 2\eta(n-1), \end{aligned}$$

so $d(\varphi(x), \varphi(y)) \geq \eta(n)/2$, thus $\varphi$ is $d$-expansive. ⊠

Set $F = (\varphi \times \varphi)^{-1}(E)$ and $e(x, y) = d(\varphi(x), \varphi(y))$.

**Lemma 12.** *The equivalence relation $F$ is meager.*

*Proof of lemma.* By the Kuratowski-Ulam theorem, it suffices to show that every $F$-class is meager. Suppose, towards a contradiction, that there exists $x \in {}^{\omega}2$ such that $[x]_F$ is non-meager. Then there exists $n \in \omega$ such that the set

$$A = \{y \in [x]_F \mid e(x, y) = n\}$$

is non-meage, so there exists $(y, z) \in \mathcal{G}_{0,m} \restriction A$ for some $m \in \omega \setminus (2n+1)$. Then $e(y, z) > 2n$, so the triangle inequality implies that $e(x, y) > n$ or $e(x, z) > n$, the desired contradiction. ⊠

Lemma 12 easily implies that there is a continuous $d_0$-expansive homomorphism $\psi$ from $(E_0, E_0^c)$ into $(E_0, F^c)$, and it follows that $\varphi \circ \psi$ is a continuous $d$-expansive embedding of $E_0$ into $E$. ⊠

Suppose that $\mathcal{G}$ is a graph on $X$ and $n \in \omega$. A *$\mathcal{G}$-path* of length $n$ is a sequence $(x_i)_{i \in n+1} \in {}^{n+1}X$ such that $(x_i, x_{i+1}) \in \mathcal{G}$ for all $i \in n$. We say that $\mathcal{G}$ is *acyclic* if there is no $\mathcal{G}$-path of length at least three whose initial and terminal points are the same. We use $E_{\mathcal{G}}$ to denote the equivalence relation consisting of those pairs which are the initial and terminal points of a $\mathcal{G}$-path.

We say that $E$ is *analytic treeable* if there is an acyclic analytic graph $\mathcal{T}$ such that $E = E_{\mathcal{T}}$. A *transversal* of $E$ is a set which intersects every $E$-class in exactly one point.

**Theorem 13.** *Suppose that $X$ is a Hausdorff space and $E$ is an analytic treeable analytic equivalence relation on $X$. Then at least one of the following holds:*

1. *There is a co-analytic transversal of $E$.*
2. *There is a continuous embedding of $E_0$ into $E$.*

*Proof.* Fix an analytic treeing $\mathcal{T}$ of $E$, and let $d$ denote the $E$-metric obtained by putting $d(x, y) = n$ whenever there is an injective $\mathcal{T}$-path from $x$ to $y$ of length $n$.

**Lemma 14.** *Suppose that $B \subseteq X$ is a $d$-bounded Borel set. Then there is an $E$-invariant analytic set $A \subseteq X$ such that $B \subseteq A$ and $E \restriction A$ has a co-analytic transversal.*

*Proof of lemma.* We say that a set $A \subseteq X$ is *$(d, n)$-bounded* if $d(E \restriction A) \subseteq n$. Fix $n \in \omega$ such that $B$ is $(d, 2n)$-bounded.

Set $B_0 = B$. Given a $2(n-i)$-bounded Borel set $B_i \subseteq X$, let $A_{i+1}$ denote the set of points which lie strictly $\mathcal{T}$-between two points of $B_i$, and observe that the set

$$A'_{i+1} = \{x \in A_{i+1} \mid |A_{i+1} \cap \mathcal{T}_x| \geq 2\}$$

is $2(n-(i+1))$-bounded. Fix a $2(n-(i+1))$-bounded Borel set $B_{i+1} \subseteq X$ such that $A'_{i+1} \subseteq B_{i+1}$.

Fix a Borel linear ordering $\leq$ of $X$, and for $i \in n$ define $C_i \subseteq B_i$ by

$$C_i = \{x \in B_i \setminus \bigcup_{j \in \omega \setminus (i+1)} [B_j]_E \mid \forall y \in B_i \ (xEy \implies x \leq y)\}$$

It is clear that the set $C = \bigcup_{i \in n} C_i$ is the desired transversal. ⊠

By Theorem 9, it is enough to show that if $X$ is the union of countably many $d$-bounded Borel sets, then there is a co-analytic partial transversal of $E$, which follows from Lemma 14. ⊠

We say that $E$ is *Borel treeable* if there is an acyclic Borel graph $\mathcal{T}$ such that $E = E_{\mathcal{T}}$.

**Theorem 15** (Hjorth)**.** *Suppose that $X$ is a Polish space and $E$ is a Borel-treeable equivalence relation on $X$. Then exactly one of the following holds:*

(1) *There is a Borel transversal of $E$.*
(2) *There is a continuous embedding of $E_0$ into $E$.*

*Proof.* It is clear that (1) and (2) are mutually exclusive. Fix an acyclic Borel graph $\mathcal{T}$ such that $E = E_{\mathcal{T}}$, and let $d$ denote the $E$-metric obtained by putting $d(x,y) = n$ whenever there is an injective $\mathcal{T}$-path from $x$ to $y$. By Theorem 9, it is enough to show that if $B$ is a bounded Borel set, then there is a Borel transversal of $E \restriction [B]_E$.

Towards this end, suppose that $B \subseteq X$ is a Borel set which is $2n$-*bounded*, i.e., the distance between any two $E$-related elements of $B$ is strictly less than $2n$. Set $B_0 = B$. Given a $2(n-i)$-bounded Borel set $B_i \subseteq X$, let $A_{i+1}$ denote the set of points which lie $\mathcal{T}$-between two points of $B_i$, and observe that the set $A'_{i+1} = \{x \in A_{i+1} \mid |A_{i+1} \cap \mathcal{T}_x| \geq 2\}$ is $2(n-(i+1))$-bounded. By the first reflection theorem, there is a $2(n-(i+1))$-bounded Borel set $B_{i+1} \supseteq A'_{i+1}$.

Set $C_i = B_i \setminus \bigcup_{j \in (n+1) \setminus (i+1)} [C_j]_E$. It is clear that $C_i$ intersects every $E$-class in at most two points, thus the Borel treeability of $E$ ensures that $C_i$ is Borel. It follows that $C = \bigcup_{i \in n+1} C_i$ is a Borel set which intersects every $E$-class in at most two points. As $B \subseteq [C]_E$, it follows that there is a Borel partial transversal of $E \restriction [B]_E$. ⊠

## 4. Exercises

**Exercise 16.** Show that if $X$ and $Y$ are Hausdorff spaces and $(\mathcal{G}_n)_{n \in \omega}$ is a sequence of analytic subsets of $X \times (Y \times Y)$ whose vertical sections are graphs, then exactly one of the following holds:

(1) There is a Borel function $c \colon X \times Y \to \omega$ such that for all $x \in X$, the map $c_x(y) = c(x, y)$ is a coloring of $((\mathcal{G}_n)_x)_{n \in \omega}$.
(2) For some $x \in X$, there is a continuous homomorphism from $(\mathcal{G}_{0,n})_{n \in \omega}$ to $((\mathcal{G}_n)_x)_{n \in \omega}$.

We say that $E$ is *smooth* if there is a Borel reduction of $E$ to $\Delta({}^\omega 2)$. We say that $E$ is *hypersmooth* if it is of the form $\bigcup_{n \in \omega} F_n$, where $(F_n)_{n \in \omega}$ is an increasing sequence of smooth equivalence relations.

**Exercise 17** (a special case of Harrington-Kechris-Louveau [1])**.** Show that if $X$ is a Hausdorff space and $E$ is a hypersmooth analytic equivalence relation on $X$, then exactly one of the following hold:

(1) The equivalence relation $E$ is smooth.
(2) There is a continuous embedding of $E_0$ into $E$.

**Exercise 18.** State and prove generalizations of all of the results mentioned thus far to $\kappa$-Souslin $\omega$-universally Baire structures.

*Hint*: The proofs are virtually identical!

## References

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