FORCELESS, INEFFECTIVE, POWERLESS PROOFS OF DESCRIPTIVE DICHOTOMY THEOREMS

LECTURE III: THE HARRINGTON-KECHRIS-LOUVEAU THEOREM

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Abstract. We give a classical proof of a generalization of the Kechris-Solecki-Todorcevic dichotomy theorem [4] characterizing analytic graphs of uncountable Borel chromatic number. Using this, we give a classical proof of the Harrington-Kechris-Louveau theorem [2] characterizing non-smooth Borel equivalence relations.

In §1, we give two straightforward corollaries of the first separation theorem. In §2, we establish a local version of the Kechris-Solecki-Todorcevic theorem [4]. In §3, we use this to give a classical proof of the Harrington-Kechris-Louveau theorem [2]. In §4, we give as exercises several results that can be obtained in a similar fashion.

1. Corollaries of separation

Suppose that X is a set, $A \subseteq X$, and E is an equivalence relation on X. The *E-saturation* of A is given by $[A]_E = \{x \in X \mid \exists y \in A \ (xEy)\}$. The set A is *E-invariant* if $A = [A]_E$.

Proposition 1. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, and (A_0, A_1) is an E-discrete pair of analytic subsets of X. Then there is an E-discrete pair (B_0, B_1) of E-invariant Borel subsets of X such that $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$.*

Proof. Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$. Suppose now that we have an E-discrete pair $(A_{0,n}, A_{1,n})$ of analytic subsets of X. Then there is an E-discrete pair $(B_{0,n}, B_{1,n})$ of Borel subsets of X such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$. Set $A_{0,n+1} = [B_{0,n}]_E$ and $A_{1,n+1} = [B_{1,n}]_E$. Clearly the sets $B_0 = \bigcup_{n\in\omega} B_{0,n}$ and $B_1 = \bigcup_{n\in\omega} B_{1,n}$ are as desired. ⊠

Proposition 2. *Suppose that X is a Hausdorff space, E is a bi-analytic equivalence relation on X, F is an analytic equivalence relation on X, $E \subseteq F$, and $A \subseteq X$ is an $(F \setminus E)$-discrete analytic set. Then there is an E-invariant $(F \setminus E)$-discrete Borel set $B \subseteq X$ such that $A \subseteq B$.*

Proof. Set $A_0 = A$. Suppose now that we have an $(F \setminus E)$-discrete analytic set $A_n \subseteq X$. Then there is an $(F \setminus E)$-discrete Borel set $B_n \subseteq X$ such that $A_n \subseteq B_n$. Set $A_{n+1} = [B_n]_E$. Clearly the set $B = \bigcup_{n \in \omega} B_n$ is as desired. ⊠

2. A local generalization of the Kechris-Solecki-Todorcevic theorem

For each set $J \subseteq \bigcup_{n \in \omega} {}^n 2 \times {}^n 2$, let $\mathcal{H}_J$ denote the graph on ${}^\omega 2$ consisting of all pairs of the form $(s(i) ^\frown i ^\frown x, s(\bar{\imath}) ^\frown \bar{\imath} ^\frown x)$, where $i \in 2$, $s \in J$, and $x \in {}^\omega 2$. We use E_J to denote the equivalence relation whose classes are the connected components of $\mathcal{H}_J$. We say that J is *dense* if $\forall s \in {}^{<\omega} 2 \times {}^{<\omega} 2 \exists t \in J \forall i \in 2 \ (s(i) \sqsubseteq t(i))$.

Proposition 3. *Suppose that $J \subseteq \bigcup_{n \in \omega} {}^n 2 \times {}^n 2$ is dense and $\varphi \colon {}^\omega 2 \to {}^\omega 2$ is a Baire measurable homomorphism from E_J to $\Delta({}^\omega 2)$. Then there exists $x \in {}^\omega 2$ such that $\varphi^{-1}(\{x\})$ is comeager.*

Proof. An equivalence relation is *generically ergodic* if every invariant set with the Baire property is meager or comeager.

Lemma 4. *The equivalence relation E_J is generically ergodic.*

Proof of lemma. Suppose, towards a contradiction, that $B \subseteq {}^\omega 2$ is an E_J-invariant set with the Baire property which is neither meager nor comeager. Fix $s \in {}^{<\omega} 2 \times {}^{<\omega} 2$ such that B is comeager in $\mathcal{N}_{s(0)}$ and meager in $\mathcal{N}_{s(1)}$. Then there exists $t \in J$ with $s(0) \sqsubseteq t(0)$ and $s(1) \sqsubseteq t(1)$, so the fact that B is comeager in $\mathcal{N}_{t(0) ^\frown 0}$ ensures that it is also comeager in $\mathcal{N}_{t(1) ^\frown 1}$, thus non-meager in $\mathcal{N}_{s(1)}$, a contradiction. ⊠

Lemma 4 ensures that for each $n \in \omega$, there is a unique sequence $s_n \in {}^n 2$ such that $\varphi^{-1}(\mathcal{N}_{s_n})$ is comeager. Setting $x = \lim_{n \to \omega} s_n$, it follows that $\varphi^{-1}(\{x\})$ is comeager. ⊠

Fix sequences $s_{2n} \in {}^{2n} 2$ and pairs $s_{2n+1} \in {}^{2n+1} 2 \times {}^{2n+1} 2$ for $n \in \omega$ such that the sets $I = \{s_{2n} \mid n \in \omega\}$ and $J = \{s_{2n+1} \mid n \in \omega\}$ are dense. Define $\mathcal{G}_0(\text{even}) = \mathcal{G}_I$, $\mathcal{H}_0(\text{odd}) = \mathcal{H}_J$, and $E_0(\text{odd}) = E_J$.

We say that E is *smooth* if there is a Borel reduction of E to $\Delta({}^\omega 2)$.

Theorem 5. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, and $\mathcal{G}$ is an analytic graph on X. Then exactly one of the following holds:*

1. *There is a Borel ω-coloring of $F \cap \mathcal{G}$, for some smooth equivalence relation $F \supseteq E$.*
2. *There is a continuous homomorphism $\pi \colon {}^\omega 2 \to X$ from the pair $(\mathcal{G}_0(\text{even}), E_0(\text{odd}))$ to the pair $(\mathcal{G}, E)$.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $F \supseteq E$ is an equivalence relation, $\varphi \colon X \to {}^{\omega}2$ is an ω-universally Baire reduction of F to $\Delta({}^{\omega}2)$, $c \colon X \to \omega$ is an ω-universally Baire measurable ω-coloring of $F \cap \mathcal{G}$, and $\pi \colon {}^{\omega}2 \to X$ is a Baire measurable homomorphism from $(\mathcal{G}_0(\text{even}), E_0(\text{odd}))$ to $(\mathcal{G}, E)$. Then $\varphi \circ \pi$ is a Baire measurable homomorphism from $E_0(\text{odd})$ to $\Delta({}^{\omega}2)$, so Proposition 3 ensures the existence of $x \in {}^{\omega}2$ such that the set $C = (\varphi \circ \pi)^{-1}(\{x\})$ is comeager. Note that $\pi(C)$ is a single F-class, so $c \restriction \pi(C)$ is a coloring of $\mathcal{G} \restriction \pi(C)$, thus $(c \circ \pi) \restriction C$ is a coloring of $\mathcal{G}_0(\text{even})$, the desired contradiction.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that $\mathcal{G}$ is non-empty, in which case there are continuous functions $\varphi_{\mathcal{G}}, \varphi_E \colon {}^{\omega}\omega \to X \times X$ such that $\mathcal{G} = \varphi_{\mathcal{G}}({}^{\omega}\omega)$ and $E = \varphi_E({}^{\omega}\omega)$. Fix a continuous function $\varphi_X \colon {}^{\omega}\omega \to X$ such that $\text{dom}(\mathcal{G}) \subseteq \varphi_X({}^{\omega}\omega)$.

A *global* (n-)*approximation* is a pair of the form $p = (u^p, v^p)$, where $u^p \colon {}^{n}2 \to {}^{n}\omega$ and $v^p \colon {}^{<n}2 \to {}^{n}\omega$. Fix an enumeration $(p_n)_{n \in \omega}$ of the set of all global approximations.

An *extension* of a global m-approximation p is a global n-approximation q with the property that $s_p \sqsubseteq s_q \implies u^p(s_p) \sqsubseteq u^q(s_q)$ and $t_p \sqsubseteq t_q \implies v^p(t_p) \sqsubseteq v^q(t_q)$ for all $s_p \in {}^{m}2$, $s_q \in {}^{n}2$, $t_p \in {}^{<m}2$, and $t_q \in {}^{<n}2$ with $n - m = |t_q| - |t_p|$. When $n = m + 1$, we say that q is a *one-step extension* of p.

A *local* (n-)*approximation* is a pair of the form $l = (f^l, g^l)$, where $f^l \colon {}^{n}2 \to {}^{\omega}\omega$ and $g^l \colon {}^{<n}2 \to {}^{\omega}\omega$, such that

$$\varphi_{\mathcal{G}} \circ g^l(t) = (\varphi_X \circ f^l(s_k {}^\frown 0 {}^\frown t), \varphi_X \circ f^l(s_k {}^\frown 1 {}^\frown t))$$

for all even $k \in n$ and $t \in {}^{n-k-1}2$, and

$$\varphi_E \circ g^l(t) = (\varphi_X \circ f^l(s_k(0) {}^\frown 0 {}^\frown t), \varphi_X \circ f^l(s_k(1) {}^\frown 1 {}^\frown t))$$

for all odd $k \in n$ and $t \in {}^{n-k-1}2$. We say that l is *compatible* with a global n-approximation p if $u^p(s) \sqsubseteq f^l(s)$ and $v^p(t) \sqsubseteq g^l(t)$ for all $s \in {}^{n}2$ and $t \in {}^{<n}2$. We say that l is *compatible* with an equivalence relation F on X if $\varphi_X \circ f^l({}^{n}2)$ is contained in a single F-class. We say that l is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l({}^{n}2) \subseteq Y$.

Suppose now that α is a countable ordinal, $F \supseteq E$ is a smooth Borel equivalence relation, $Y \subseteq X$ is a Borel set, and $c \colon Y^c \to \omega \cdot \alpha$ is a Borel coloring of $(F \cap \mathcal{G}) \restriction Y^c$. Associated with each global n-approximation p is the set $L_n(p, F, Y)$ of local n-approximations which are compatible with p, F, and Y.

A global n-approximation p is (F, Y)-*terminal* if $L_{n+1}(q, F, Y) = \emptyset$ for all one-step extensions q of p. Let $T_n(F, Y)$ denote the set of all such global n-approximations, and set $T_{\text{even}}(F,Y) = \bigcup_{n \in \omega} T_{2n}(F,Y)$, $T_{\text{odd}}(F,Y) = \bigcup_{n \in \omega} T_{2n+1}(F,Y)$, and $T(F,Y) = \bigcup_{n \in \omega} T_n(F,Y)$.

When n is even, we use $A(p,F,Y)$ to denote the set of points of the form $\varphi_X \circ f^l(s_n)$, where $l \in L_n(p,F,Y)$.

Lemma 6. *Suppose that $n \in \omega$ is even, p is a global n-approximation, and the set $A(p,F,Y)$ is not $(F \cap \mathcal{G})$-discrete. Then $p \notin T_n(F,Y)$.*

Proof of lemma. Fix local n-approximations $l_0, l_1 \in L_n(p,F,Y)$ with $(\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n)) \in F \cap \mathcal{G}$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_{\mathcal{G}}(x) = (\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n))$. Let l denote the local $(n+1)$-approximation given by $f^l(s \frown i) = f^{l_i}(s)$, $g^l(\emptyset) = x$, and $g^l(t \frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n 2$, and $t \in {}^{<n}2$. Then l is compatible with a one-step extension of p, thus p is not (F,Y)-terminal. ⊠

Lemma 6 ensures that for each $p \in T_{\text{even}}(F,Y)$, there is an $(F \cap \mathcal{G})$-discrete Borel set $B(p,F,Y) \subseteq X$ with $A(p,F,Y) \subseteq B(p,F,Y)$. Set $Y' = Y \setminus \bigcup \{B(p,F,Y) \mid p \in T_{\text{even}}(F,Y)\}$. For each $y \in Y \setminus Y'$, put $n(y) = \min\{n \in \omega \mid p_n \in T_{\text{even}}(F,Y) \text{ and } y \in B(p_n,F,Y)\}$. Define $c' \colon (Y')^c \to \omega \cdot (\alpha + 1)$ by

$$c'(y) = \begin{cases} c(y) & \text{if } y \in Y^c \text{ and} \\ \omega \cdot \alpha + n(y) & \text{otherwise.} \end{cases}$$

Lemma 7. *The function c' is a coloring of $(F \cap \mathcal{G}) \restriction (Y')^c$.*

Proof of lemma. Note that if $\beta \in \omega \cdot \alpha$ then $(c')^{-1}(\{\beta\}) = c^{-1}(\{\beta\})$, and if $\beta \in \omega \cdot (\alpha + 1) \setminus \omega \cdot \alpha$ then there exists $n \in \omega$ with $\beta = \omega \cdot \alpha + n$, so $p_n \in T_{\text{even}}(F,Y)$ and $(c')^{-1}(\{\beta\}) \subseteq B(p_n,F,Y)$. Then $(c')^{-1}(\{\beta\})$ is $(F \cap \mathcal{G})$-discrete for all $\beta \in \omega \cdot (\alpha+1)$, thus c' is a coloring of $(F \cap \mathcal{G}) \restriction (Y')^c$. ⊠

When $i \in 2$ and n is odd, we use $A_i(p,F,Y)$ to denote the set of points of the form $\varphi_X \circ f^l \circ s_n(i)$, where $l \in L_n(p,F,Y)$.

Lemma 8. *Suppose that $n \in \omega$ is odd, p is a global n-approximation, and $(A_0(p,F,Y), A_1(p,F,Y))$ is not E-discrete. Then $p \notin T_n(F,Y)$.*

Proof of lemma. Fix local n-approximations $l_0, l_1 \in L(p,F,Y)$ with $\varphi_X \circ f^{l_0} \circ s_n(0) E \varphi_X \circ f^{l_1} \circ s_n(1)$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_E(x) = (\varphi_X \circ f^{l_0} \circ s_n(0), \varphi_X \circ f^{l_1} \circ s_n(1))$. Let l denote the local $(n+1)$-approximation given by $f(s \frown i) = f^{l_i}(s)$, $g(\emptyset) = x$, and $g(t \frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n 2$, and $t \in {}^{<n}2$. Then l is compatible with a one-step extension of p, and it follows that $p \notin T_n(F,Y)$. ⊠

Proposition 1 and Lemma 8 ensure that for each $p \in T_{\mathrm{odd}}(F, Y)$, there is an E-discrete pair $(B_0(p, F, Y), B_1(p, F, Y))$ of E-invariant Borel sets with $A_0(p, F, Y) \subseteq B_0(p, F, Y)$ and $A_1(p, F, Y) \subseteq B_1(p, F, Y)$. Define $\psi \colon X \to {}^{\omega}2$ by

$$\psi(x)(n) = \begin{cases} \chi_{B_0(p_n, S, Y)}(x) & \text{if } p_n \in T_n(S, Y) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let F' denote the smooth equivalence relation given by

$$xF'y \iff xFy \text{ and } \psi(x) = \psi(y).$$

Lemma 9. *The equivalence relation F' contains E.*

Proof of lemma. This follows from the E-invariance of $B_i(p, F, Y)$ for $i \in 2$ and $p \in T_{\mathrm{odd}}(F, Y)$, as well as the fact that $E \subseteq F$. ⊠

Lemma 10. *Suppose that p is a global approximation whose one-step extensions are all (F, Y)-terminal. Then $p \in T(F', Y')$.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation. Suppose, towards a contradiction, that there is a one-step extension q of p for which there exists $l \in L_{n+1}(q, F', Y')$.

If n is odd, then $\varphi_X \circ f^l(s_{n+1}) \in B(q, F, Y)$ and $B(q, F, Y) \cap Y' = \emptyset$, so $\varphi_X \circ f^l(s_{n+1}) \notin Y'$, a contradiction.

If n is even, then $\varphi_X \circ f^l \circ s_{n+1}(0) \in B_0(p, F, Y)$ and $\varphi_X \circ f^l \circ s_{n+1}(1) \in B_1(p, F, Y)$. As $(B_0(p, F, Y), B_1(p, F, Y))$ is E-discrete, it follows that $(\varphi_X \circ f^l \circ s_{n+1}(0), \varphi_X \circ f^l \circ s_{n+1}(1)) \notin E$, a contradiction. ⊠

Recursively define smooth Borel equivalence relations F_α, Borel sets $Y_\alpha \subseteq X$, and Borel colorings $c_\alpha \colon Y_\alpha^c \to \omega \cdot \alpha$ of $(F_\alpha \cap \mathcal{G}) \restriction Y_\alpha^c$ by

$$(F_\alpha, Y_\alpha, c_\alpha) = \begin{cases} (X \times X, X, \emptyset) & \text{if } \alpha = 0, \\ (F'_\beta, Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta \in \alpha} F_\beta, \bigcap_{\beta \in \alpha} Y_\beta, \lim_{\beta \to \alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many approximations, there exists $\alpha \in \omega_1$ such that $T(F_\alpha, Y_\alpha) = T(F_{\alpha+1}, Y_{\alpha+1})$.

Let p^0 denote the unique global 0-approximation. As $\mathrm{dom}(\mathcal{G}) \cap Y_\alpha \subseteq A(p^0, F_\alpha, Y_\alpha)$, it follows that if p^0 is (F_α, Y_α)-terminal, then c_α extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of $F_\alpha \cap \mathcal{G}$, thus there is a Borel ω-coloring of $F_\alpha \cap \mathcal{G}$.

Otherwise, by repeatedly applying Lemma 10 we obtain global n-approximations $p^n = (u^n, v^n)$ with the property that p^{n+1} is a one-step extension of p^n for all $n \in \omega$. Define continuous functions $\pi \colon {}^{\omega}2 \to {}^{\omega}\omega$

and $\pi_k \colon {}^\omega 2 \to {}^\omega\omega$ for $k \in \omega$ by

$$\pi(x) = \lim_{n\to\omega} u^n(x \upharpoonright n) \text{ and } \pi_k(x) = \lim_{n\to\omega} v^{k+n+1}(x \upharpoonright n).$$

To see that $\varphi_X \circ \pi$ is a homomorphism from $\mathcal{G}_0(\text{even})$ to $\mathcal{G}$, it is enough to show that $\varphi_\mathcal{G} \circ \pi_k(x) = (\varphi_X \circ \pi(s_k \frown 0 \frown x), \varphi_X \circ \pi(s_k \frown 1 \frown x))$ for all even $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k \frown 0 \frown x), \pi(s_k \frown 1 \frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_\mathcal{G}(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \upharpoonright n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k \frown 0 \frown (x \upharpoonright n))} \times \mathcal{N}_{u^{k+n+1}(s_k \frown 1 \frown (x \upharpoonright n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, F_\alpha, Y_\alpha)$, and observe that $z = g^l(x \upharpoonright n)$, $z_0 = f^l(s_k \frown 0 \frown (x \upharpoonright n))$, and $z_1 = f^l(s_k \frown 1 \frown (x \upharpoonright n))$ are as desired.

To see that $\varphi_X \circ \pi$ is a homomorphism from $E_0(\text{odd})$ to E, it is enough to show that $\varphi_E \circ \pi_k(x) = (\varphi_X \circ \pi(s_k(0) \frown 0 \frown x), \varphi_X \circ \pi(s_k(1) \frown 1 \frown x))$ for all odd $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k(0) \frown 0 \frown x), \pi(s_k(1) \frown 1 \frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_E(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \upharpoonright n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k(0) \frown 0 \frown (x \upharpoonright n))} \times \mathcal{N}_{u^{k+n+1}(s_k(1) \frown 1 \frown (x \upharpoonright n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, F_\alpha, Y_\alpha)$, and observe that $z = g^l(x \upharpoonright n)$, $z_0 = f^l(s_k(0) \frown 0 \frown (x \upharpoonright n))$, and $z_1 = f^l(s_k(1) \frown 1 \frown (x \upharpoonright n))$ are as desired. ⊠

3. The Harrington-Kechris-Louveau theorem

We begin this section with a Mycielski-style theorem.

Proposition 11. *Suppose that $J \subseteq \bigcup_{n\in\omega} {}^n 2 \times {}^n 2$ is dense, E is a meager equivalence relation on ${}^\omega 2$ which contains $\mathcal{H}_J$, and $C \subseteq E$ is closed. Then there is a continuous homomorphism $\pi \colon {}^\omega 2 \to {}^\omega 2$ from $(\Delta({}^\omega 2)^c, E_0^c, E_0)$ to (C^c, E^c, E).*

Proof. Fix a decreasing sequence $(U_n)_{n\in\omega}$ of dense open subsets of C^c such that $E \cap \bigcap_{n\in\omega} U_n = \emptyset$. An *$n$-approximation* is a pair (k, u), where $k \colon n+1 \to \omega$ and $u \colon {}^n 2 \to {}^{k(n)} 2$, such that

$$s \upharpoonright [m, n) = t \upharpoonright [m, n) \Longrightarrow u(s) \upharpoonright [k(m), k(n)) = u(t) \upharpoonright [k(m), k(n))$$

for all $m \in n$ and $s, t \in {}^n 2$. A *refinement* of (k, u) is an approximation (k', u') such that $k \upharpoonright n = k' \upharpoonright n$ and $u(s) \sqsubseteq u'(s)$ for all $s \in {}^n 2$.

Lemma 12. *Suppose that $n \in \omega$, (k, u) is an $(n+1)$-approximation, and $s \in {}^n 2 \times {}^n 2$. Then there is a refinement (k', u') of (k, u) such that $\mathcal{N}_{u'(s(0) \frown 0)} \times \mathcal{N}_{u'(s(1) \frown 1)} \subseteq U_{n+1}$.*

Proof of lemma. Fix $l \in \omega \setminus k(n+1)$ and $t \in {}^l2 \times {}^l2$ with $u \circ s(0) \sqsubseteq t(0)$, $u \circ s(1) \sqsubseteq t(1)$, and $\mathcal{N}_{t(0)} \times \mathcal{N}_{t(1)} \subseteq U_{n+1}$. Then the refinement of (k, u) given by $k'(n+1) = l$, $u'(s(0)^\frown 0) = t(0)$, and $u'(s(1)^\frown 1) = t(1)$ is clearly as desired. ⊠

Let (k_0, u_0) denote the 0-approximation given by $k_0(0) = 0$ and $u_0 = \emptyset$. Given an n-approximation (k_n, u_n), let (k, u) denote the $(n+1)$-approximation given by $k \restriction (n+1) = k_n$, $k(n+1) = k_n(n)$, and $u(s^\frown i) = u_n(s)$ for $i \in 2$ and $s \in {}^n2$. By applying Lemma 12 finitely many times, we obtain a refinement (k', u') such that $\mathcal{N}_{u'(s(0)^\frown 0)} \times \mathcal{N}_{u'(s(1)^\frown 1)} \subseteq U_{n+1}$ for all $s \in {}^n2 \times {}^n2$. Fix $s \in J$ such that $u'(1^{n\frown}0) \sqsubseteq s(0)$ and $u'(0^{n\frown}1) \sqsubseteq s(1)$, and let (k_{n+1}, u_{n+1}) denote the refinement given by $k_{n+1}(n+1) = |s(0)| + 1 = |s(1)| + 1$, $u_{n+1}(1^{n\frown}0) = s(0)^\frown 0$, and $u_{n+1}(0^{n\frown}1) = s(1)^\frown 1$.

Define $\pi \colon {}^\omega2 \to {}^\omega2$ by $\pi(x) = \lim_{n \to \omega} u_n(x \restriction n)$. Clearly π is continuous. Note now that if $n \in \omega$, $x, y \in {}^\omega2$, and $x(n) \neq y(n)$, then $(\pi(x), \pi(y)) \in \mathcal{N}_{u_{n+1}(x \restriction (n+1))} \times \mathcal{N}_{u_{n+1}(y \restriction (n+1))} \subseteq U_{n+1}$. In particular, it follows that π is a homomorphism from $(\Delta({}^\omega2)^c, E_0^c)$ to (C^c, E^c).

Finally, observe that if $n \in \omega$ and $x \in {}^\omega2$, then there exist $s \in J$ and $y \in {}^\omega2$ with $(\pi(1^{n\frown}0^\frown x), \pi(0^{n\frown}1^\frown x)) = (s(0)^\frown 0^\frown y, s(1)^\frown 1^\frown y) \in \mathcal{H}_J \subseteq E$. As E_0 is the smallest equivalence relation containing all pairs of the form $(1^{n\frown}0^\frown x, 0^{n\frown}1^\frown x)$ for $n \in \omega$ and $x \in {}^\omega2$, it follows that π is a homomorphism from E_0 to E. ⊠

We are now ready for our main result.

Theorem 13 (Harrington-Kechris-Louveau)**.** *Suppose that X is a Hausdorff space and E is a bi-analytic equivalence relation on X. Then exactly one of the following holds:*

1. *The equivalence relation E is smooth.*
2. *There is a continuous embedding $\pi \colon {}^\omega2 \to X$ of E_0 into E.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $\varphi \colon X \to {}^\omega2$ is an ω-universally Baire measurable reduction of E to $\Delta({}^\omega2)$ and $\pi \colon {}^\omega2 \to X$ is a Baire measurable embedding of E_0 into E. Then $\varphi \circ \pi$ is a Baire measurable reduction of E_0 to $\Delta({}^\omega2)$, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. Towards this end, set $\mathcal{G} = E^c$ and suppose that there is a Borel ω-coloring $c \colon X \to \omega$ of $F \cap \mathcal{G}$, for some smooth equivalence relation $F \supseteq E$. Proposition 2 ensures that for each $n \in \omega$, there is an E-invariant $(F \setminus E)$-discrete Borel set $B_n \subseteq X$ such that $c^{-1}(\{n\}) \subseteq B_n$. Define $f \colon X \to {}^\omega2$ by $f(x)(n) = \chi_{B_n}(x)$, and observe that

$$xEy \iff xFy \text{ and } f(x) = f(y),$$

thus E is smooth.

By Theorem 5, we can assume that there is a continuous homomorphism $\varphi\colon {}^\omega 2 \to X$ from $(\mathcal{G}_0(\text{even}), E_0(\text{odd}))$ to $(\mathcal{G}, E)$. Set $C = (\varphi \times \varphi)^{-1}(\Delta(X))$ and $F = (\varphi \times \varphi)^{-1}(E)$. Then C is closed and F is meager, so Proposition 11 ensures that there is a continuous homomorphism $\psi\colon {}^\omega 2 \to {}^\omega 2$ from $(\Delta({}^\omega 2)^c, E_0^c, E_0)$ to (C^c, F^c, F). Then the map $\pi = \varphi \circ \psi$ is the desired continuous embedding of E_0 into E. ⊠

4. Exercises

Exercise 14. Show that if X and Y are Hausdorff spaces, $E \subseteq X \times (Y \times Y)$ is an analytic set whose vertical sections are equivalence relations, and $\mathcal{G} \subseteq X \times (Y \times Y)$ is an analytic set whose vertical sections are graphs, then exactly one of the following holds:

(1) There is a set $F \supseteq E$, a Borel function $\varphi\colon X \times Y \to {}^\omega 2$, and a Borel function $c\colon X \times Y \to \omega$ such that for all $x \in X$, the map $\varphi_x(y) = \varphi(x, y)$ is a reduction of F_x to $\Delta({}^\omega 2)$ and the map $c_x(y) = c(x, y)$ is a coloring of $F_x \cap \mathcal{G}_x$.

(2) For some $x \in X$, there is a continuous homomorphism from $(\mathcal{G}_0(\text{even}), E_0(\text{odd}))$ to $(\mathcal{G}_x, E_x)$.

Exercise 15. Show that if X is a Hausdorff space, E is a co-analytic equivalence relation on X, and F is an analytic subequivalence relation of E, then exactly one of the following holds:

(1) There is a smooth equivalence relation between F and E.

(2) There is a continuous embedding $\pi\colon {}^\omega 2 \to X$ of the pair (E_0, E_0) into the pair (E, F).

Exercise 16. State and prove a version of Exercise 15 for κ-Souslin ω-universally Baire structures.

Hint: To give a classical proof of a weak generalization, first establish a weak κ-Souslin analog of Theorem 5 by removing all uses of separation from the argument given in §2. Note that the resulting theorem is a true dichotomy in $\mathtt{ZF} + \mathtt{BP}$, and implies analogous results of Ditzen [1] and Foreman-Magidor (unpublished) under $\mathtt{AD}$.

Hint: To give a strong generalization, adapt the techniques of Kanovei [3] to first establish a strong κ-Souslin analog of Theorem 5. Although the resulting proof is not classical, the resulting theorem has the advantage that it is a true generalization of Exercise 15.

References

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