FORCELESS, INEFFECTIVE, POWERLESS PROOFS OF DESCRIPTIVE DICHOTOMY THEOREMS

LECTURE I: SILVER'S THEOREM

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ABSTRACT. We give a classical proof of the Kechris-Solecki-Todorcevic dichotomy theorem [9] characterizing analytic graphs of uncountable Borel chromatic number. Using this, we give a classical proof of a generalization of Silver's theorem [16] characterizing co-analytic equivalence relations which admit perfect sets of inequivalent elements.

In §1, we briefly review the definitions of several pointclasses that will be used throughout, and we also mention without proof some of their basic properties. In §2, we give two straightforward corollaries of the first separation theorem. In §3, we give the promised classical proof of the Kechris-Solecki-Todorcevic [9] theorem. In §4, we sketch a simplification of our argument that yields a weak form of Kanovei's generalization [6] of the Kechris-Solecki-Todorcevic theorem [9] to κ-Souslin graphs. In §5, we use this weak generalization to give a classical proof of a generalization of the theorems of Burgess [1] and Silver [16] to co-κ-Souslin, ω-universally Baire equivalence relations. In §6, we give as exercises several results that can be obtained in a similar fashion.

1. Preliminaries

Suppose that X is a Hausdorff space. A set $A \subseteq X$ is *κ-Souslin* if it is the continuous image of a closed subset of ${}^{\omega}\kappa$. It is easy to see that non-empty κ-Souslin sets are in fact continuous images of ${}^{\omega}\kappa$ itself. A set is *analytic* if it is ω-Souslin. A set $B \subseteq X$ is *κ^+-Borel* if it is in the closure of the topology of X under complements and intersections of length κ. A set is *Borel* if it is ω_1-Borel.

We will use two basic facts concerning the connection between these two types of sets, both due to Souslin [17]. One is that every Borel subset of a κ-Souslin Hausdorff space is κ-Souslin. The other, typically referred to as the *first separation theorem*, is that any two disjoint κ-Souslin subsets A_0, A_1 of a Hausdorff space X can be *separated* by a κ^+-Borel set $B \subseteq X$, meaning that $A_0 \subseteq B$ and $A_1 \cap B = \emptyset$. While the reader might have encountered these facts in a somewhat different context (most likely in Polish spaces as in [7] or [18]), he is assured that the proofs he knows and loves easily go through in this generality.

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\varphi \colon {}^{\omega}\omega \to X$, the set $\varphi^{-1}(B)$ has the Baire property. For our arguments, we actually need only the weaker assertion that for every continuous function $\varphi \colon {}^{\omega}2 \to X$, the set $\varphi^{-1}(B)$ has the Baire property, but the former notion is somewhat more pleasant to work with, as the composition of an ω-universally Baire measurable function with a Baire measurable function on a Polish space is always Baire measurable.

Of course, the sets in the σ-algebra generated by the analytic subsets of a Hausdorff space are ω-universally Baire, as are the C-measurable sets. While strong hypotheses such as $\mathtt{AD}$ ensure that every set is ω-universally Baire, one should recall Shelah's result [14] that the consistency of the latter statement follows from that of $\mathtt{ZF}$. When dealing with ω-universally Baire sets, we will invoke the basic results on Baire category such as the Kuratowski-Ulam theorem and Mycielski's theorem as needed (see [7] or [18]).

2. Corollaries of separation

Suppose that X_0 and X_1 are sets and $R \subseteq X_0 \times X_1$. A pair (A_0, A_1) is *R-discrete* if $A_0 \subseteq X_0$, $A_1 \subseteq X_1$, and $(A_0 \times A_1) \cap R = \emptyset$.

Proposition 1. *Suppose that X_0 and X_1 are Hausdorff spaces, $R \subseteq X_0 \times X_1$ is analytic, and (A_0, A_1) is an R-discrete pair of analytic sets. Then there is an R-discrete pair (B_0, B_1) of Borel sets with the property that $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$.*

Proof. Set $A_0' = \{x_0 \in X_0 \mid \exists x_1 \in A_1 \ ((x_0, x_1) \in R)\}$. Then A_0' is analytic and $A_0 \cap A_0' = \emptyset$, so the first separation theorem ensures that there is a Borel set $B_0 \subseteq X_0$ which separates A_0 from A_0'.

Now set $A_1' = \{x_1 \in X_1 \mid \exists x_0 \in B_0 \ ((x_0, x_1) \in R)\}$. One must take slight care here in declaring that A_1' is analytic, since the existential quantifier runs over elements of B_0, and Borel subsets of arbitrary Hausdorff spaces need not be analytic. Fortunately, it follows that $A_1' = \{x_1 \in X_1 \mid \exists x_0 \in B_0 \cap \mathrm{proj}_{X_0}(R) \ ((x_0, x_1) \in R)\}$, and since Borel subsets of analytic Hausdorff spaces are analytic, it follows that A_1' is analytic. As $A_1 \cap A_1' = \emptyset$, another application of the first separation theorem yields a Borel set $B_1 \subseteq X_1$ which separates A_1 from A_1'. It is easily verified that the pair (B_0, B_1) is as desired. ⊠

A *graph* on X is an irreflexive symmetric set $\mathcal{G} \subseteq X \times X$. The *restriction* of $\mathcal{G}$ to a set $A \subseteq X$ is the graph on A given by $\mathcal{G} \restriction A = \mathcal{G} \cap (A \times A)$. A set $A \subseteq X$ is *$\mathcal{G}$-discrete* if $\mathcal{G} \restriction A = \emptyset$.

Proposition 2. *Suppose that X is a Hausdorff space, $\mathcal{G}$ is an analytic graph on X, and $A \subseteq X$ is a $\mathcal{G}$-discrete analytic set. Then there is a $\mathcal{G}$-discrete Borel set $B \subseteq X$ such that $A \subseteq B$.*

Proof. By Proposition 1, there is a $\mathcal{G}$-discrete pair (B_0, B_1) of Borel subsets of X such that $A \subseteq B_0$ and $A \subseteq B_1$. It is easily verified that the set $B = B_0 \cap B_1$ is as desired. ⊠

3. The Kechris-Solecki-Todorcevic theorem

Define $\bar{\imath} = 1 - i$. For each set $I \subseteq {}^{<\omega}2$, let $\mathcal{G}_I$ denote the graph on ${}^{\omega}2$ consisting of all pairs of the form $(s \frown i \frown x, s \frown \bar{\imath} \frown x)$, where $i \in 2$, $s \in I$, and $x \in {}^{\omega}2$. We say that I is *dense* if $\forall s \in {}^{<\omega}2 \exists t \in I\ (s \sqsubseteq t)$.

Proposition 3. *Suppose that $I \subseteq {}^{<\omega}2$ is dense and $A \subseteq {}^{\omega}2$ is non-meager and has the Baire property. Then A is not $\mathcal{G}_I$-discrete.*

Proof. Fix $s \in {}^{<\omega}2$ such that A is comeager in $\mathcal{N}_s$. Fix $t \in I$ such that $s \sqsubseteq t$. Then there exists $x \in {}^{\omega}2$ such that $t \frown 0 \frown x, t \frown 1 \frown x \in A$. As $(t \frown 0 \frown x, t \frown 1 \frown x) \in \mathcal{G}_I$, it follows that A is not $\mathcal{G}_I$-discrete. ⊠

Fix sequences $s_n \in {}^{n}2$ such that the set $I = \{s_n \mid n \in \omega\}$ is dense. Note that I contains exactly one sequence of each finite length. Define $\mathcal{G}_0 = \mathcal{G}_I$. A ($\kappa$-)*coloring* of $\mathcal{G}$ is a function $c \colon X \to \kappa$ with the property that $c^{-1}(\{\alpha\})$ is $\mathcal{G}$-discrete for all $\alpha \in \kappa$.

Theorem 4 (Kechris-Solecki-Todorcevic)**.** *Suppose that X is a Hausdorff space and $\mathcal{G}$ is an analytic graph on X. Then exactly one of the following holds:*

1. *There is a Borel ω-coloring of $\mathcal{G}$.*
2. *There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $c \colon X \to \omega$ is an ω-universally Baire measurable coloring of $\mathcal{G}$ and $\pi \colon {}^{\omega}2 \to X$ is a Baire measurable homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. Then $c \circ \pi$ is a Baire measurable coloring of $\mathcal{G}_0$, so there exists $k \in \omega$ such that the set $(c \circ \pi)^{-1}(\{k\})$ is non-meager and $\mathcal{G}_0$-discrete, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that $\mathcal{G}$ is non-empty. The *domain* of $\mathcal{G}$, or $\mathrm{dom}(\mathcal{G})$, is the union of the two projections of $\mathcal{G}$. Fix a continuous function

$\varphi_{\mathcal{G}} \colon {}^\omega\omega \to X \times X$ such that $\mathcal{G} = \varphi_{\mathcal{G}}({}^\omega\omega)$, as well as a continuous function $\varphi_X \colon {}^\omega\omega \to X$ such that $\mathrm{dom}(\mathcal{G}) \subseteq \varphi_X({}^\omega\omega)$.

We will next describe a pair of ways of approximating the desired homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. These two different types of approximations will work hand-in-hand throughout the proof. One will work *globally*, in the sense that it deals with large analytic subsets of the space, while the other will work *locally*, in the sense that it deals with individual points of the space. The reader is cautioned that our terminology does not indicate any form of homogeneity or lack thereof.

A *global* (n-)*approximation* is a pair of the form $p = (u^p, v^p)$, where $u^p \colon {}^n2 \to {}^n\omega$ and $v^p \colon {}^{<n}2 \to {}^n\omega$. Fix an enumeration $(p_n)_{n \in \omega}$ of the set of all global approximations.

An *extension* of a global m-approximation p is a global n-approximation q with the property that $s_p \sqsubseteq s_q \implies u^p(s_p) \sqsubseteq u^q(s_q)$ and $t_p \sqsubseteq t_q \implies v^p(t_p) \sqsubseteq v^q(t_q)$ for all $s_p \in {}^m2$, $s_q \in {}^n2$, $t_p \in {}^{<m}2$, and $t_q \in {}^{<n}2$ with $n - m = |t_q| - |t_p|$. When $n = m + 1$, we say that q is a *one-step extension* of p.

A *local* (n-)*approximation* is a pair of the form $l = (f^l, g^l)$, where $f^l \colon {}^n2 \to {}^\omega\omega$ and $g^l \colon {}^{<n}2 \to {}^\omega\omega$, with the property that $\varphi_{\mathcal{G}} \circ g^l(t) = (\varphi_X \circ f^l(s_k \frown 0 \frown t), \varphi_X \circ f^l(s_k \frown 1 \frown t))$ for all $k \in n$ and $t \in {}^{n-(k+1)}2$. We say that l is *compatible* with a global n-approximation p if $u^p(s) \sqsubseteq f^l(s)$ and $v^p(t) \sqsubseteq g^l(t)$ for all $s \in {}^n2$ and $t \in {}^{<n}2$. We say that l is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l({}^n2) \subseteq Y$.

Suppose now that α is a countable ordinal, $Y \subseteq X$ is a Borel set, and $c \colon Y^c \to \omega \cdot \alpha$ is a Borel coloring of $\mathcal{G} \upharpoonright Y^c$. Associated with each global n-approximation p is the set $L_n(p, Y)$ of local n-approximations which are compatible with both p and Y.

A global n-approximation p is Y*-terminal* if $L_{n+1}(q, Y) = \emptyset$ for all one-step extensions q of p. Let $T(Y)$ denote the set of such approximations. Set $A(p, Y) = \bigcup_{n \in \omega} \{\varphi_X \circ f^l(s_n) \mid l \in L_n(p, Y)\}$.

Lemma 5. *Suppose that p is a global approximation and $A(p, Y)$ is not $\mathcal{G}$-discrete. Then $p \notin T(Y)$.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation, as well as local n-approximations $l_0, l_1 \in L_n(p, Y)$ with $(\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n)) \in \mathcal{G}$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_{\mathcal{G}}(x) = (\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n))$. Let l denote the local $(n+1)$-approximation given by $f^l(s \frown i) = f^{l_i}(s)$, $g^l(\emptyset) = x$, and $g^l(t \frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n2$, and $t \in {}^{<n}2$. Then l is compatible with a one-step extension of p, thus p is not Y-terminal. ⊠

Proposition 2 and Lemma 5 ensure that for each $p \in T(Y)$, there is a $\mathcal{G}$-discrete Borel set $B(p, Y) \subseteq X$ with $A(p, Y) \subseteq B(p, Y)$. Set $Y' = Y \setminus \bigcup\{B(p, Y) \mid p \in T(Y)\}$. For each $y \in Y \setminus Y'$, put $n(y) = \min\{n \in \omega \mid p_n \in T(Y) \text{ and } y \in B(p_n, Y)\}$. Define $c' \colon (Y')^c \to \omega \cdot (\alpha + 1)$ by

$$c'(y) = \begin{cases} c(y) & \text{if } y \in Y^c \text{ and} \\ \omega \cdot \alpha + n(y) & \text{otherwise.} \end{cases}$$

Lemma 6. *The function c' is a coloring of $\mathcal{G} \restriction (Y')^c$.*

Proof of lemma. Note that if $\beta \in \omega \cdot \alpha$ then $(c')^{-1}(\{\beta\}) = c^{-1}(\{\beta\})$, and if $\beta \in \omega \cdot (\alpha+1) \setminus \omega \cdot \alpha$ then there exists $n \in \omega$ with $\beta = \omega \cdot \alpha + n$, so $p_n \in T(Y)$ and $(c')^{-1}(\{\beta\}) \subseteq B(p_n, Y)$. Then $(c')^{-1}(\{\beta\})$ is $\mathcal{G}$-discrete for all $\beta \in \omega \cdot (\alpha + 1)$, thus c' is a coloring of $\mathcal{G} \restriction (Y')^c$. ⊠

Lemma 7. *Suppose that p is a global approximation whose one-step extensions are all Y-terminal. Then p is Y'-terminal.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation. Suppose, towards a contradiction, that there is a one-step extension q of p for which there exists $l \in L_{n+1}(q, Y')$. Then $\varphi_X \circ f^l(s_{n+1}) \in B(q, Y)$ and $B(q, Y) \cap Y' = \emptyset$, thus $\varphi_X \circ f^l(s_{n+1}) \notin Y'$, a contradiction. ⊠

Recursively define Borel sets $Y_\alpha \subseteq X$ and Borel colorings $c_\alpha \colon Y_\alpha^c \to \omega \cdot \alpha$ of $\mathcal{G} \restriction Y_\alpha^c$ by

$$(Y_\alpha, c_\alpha) = \begin{cases} (X, \emptyset) & \text{if } \alpha = 0, \\ (Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta \in \alpha} Y_\beta, \lim_{\beta \to \alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many approximations, there exists $\alpha \in \omega_1$ such that $T(Y_\alpha) = T(Y_{\alpha+1})$.

Let p^0 denote the unique global 0-approximation. As $\mathrm{dom}(\mathcal{G}) \cap Y_\alpha \subseteq A(p^0, Y_\alpha)$, it follows that if p^0 is Y_α-terminal, then c_α extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of $\mathcal{G}$, thus there is a Borel ω-coloring of $\mathcal{G}$.

Otherwise, by repeatedly applying Lemma 7 we obtain global n-approximations $p^n = (u^n, v^n)$ with the property that p^{n+1} is a one-step extension of p^n for all $n \in \omega$. Define continuous functions $\pi \colon {}^\omega 2 \to {}^\omega \omega$ and $\pi_k \colon {}^\omega 2 \to {}^\omega \omega$ for $k \in \omega$ by

$$\pi(x) = \lim_{n \to \omega} u^n(x \restriction n) \text{ and } \pi_k(x) = \lim_{n \to \omega} v^{k+n+1}(x \restriction n).$$

To see that $\varphi_X \circ \pi$ is a homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$, it is enough to show that $\varphi_{\mathcal{G}} \circ \pi_k(x) = (\varphi_X \circ \pi(s_k \frown 0 \frown x), \varphi_X \circ \pi(s_k \frown 1 \frown x))$ for all $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k \frown 0 \frown x), \pi(s_k \frown 1 \frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_{\mathcal{G}}(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \restriction n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k \frown 0 \frown (x \restriction n))} \times \mathcal{N}_{u^{k+n+1}(s_k \frown 1 \frown (x \restriction n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, Y_\alpha)$, and observe that $z = g^l(x \restriction n)$, $z_0 = f^l(s_k \frown 0 \frown (x \restriction n))$, and $z_1 = f^l(s_k \frown 1 \frown (x \restriction n))$ are as desired. ⊠

4. Kanovei's generalization

Kanovei [6] has established the natural generalization of the Kechris-Solecki-Todorcevic theorem [9] to κ-Souslin graphs:

Theorem 8 (Kanovei)**.** *Suppose that κ is an infinite aleph, X is a Hausdorff space, and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:*

1. *There is a κ^+-Borel κ-coloring of $\mathcal{G}$.*
2. *There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

We say that a set $A \subseteq X$ is *weakly κ^+-Souslin* if it is the continuous image of a κ^+-Borel subset of ${}^\omega\kappa$. Of course, it is tempting to simply plug in κ in place of ω in our previous argument to give a proof of the generalization as well. Unfortunately, this approach does not seem to work, as one is quickly faced with the need to apply the first separation theorem to weakly κ^+-Souslin sets.

Question 9. Can two disjoint weakly κ^+-Souslin subsets of a Hausdorff space always be separated by a κ^+-Borel set?

Of course, it just so happens that when $\kappa = \omega$ the classes of κ-Souslin sets and weakly κ-Souslin sets coincide. Under `AD`, this holds more generally at odd projective(-like) ordinals. In such cases, the classical proof does indeed go through. Unfortunately, the question of whether there is a classical proof of Kanovei's full theorem remains open.

On the bright side, for many applications one needs only the following weak form of Kanovei's result:

Theorem 10. *Suppose that κ is an infinite aleph, X is a Hausdorff space, and $\mathcal{G}$ is a κ-Souslin graph on X. Then at least one of the following holds:*

1. *There is a κ-coloring of $\mathcal{G}$.*
2. *There is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$.*

Proof. Proceed exactly as in the proof of Theorem 4, but replace every occurrence of ω with κ, and avoid the separation results altogether by replacing $B(p, Y)$ with $A(p, Y)$ in the definition of Y'. ⊠

5. Silver's theorem

Suppose that E is an equivalence relation on X and F is an equivalence relation on Y. A *reduction* of E to F is a function $\pi \colon X \to Y$ such that $x_0 E x_1 \iff \pi(x_0) F \pi(x_1)$ for all $(x_0, x_1) \in X \times X$. An *embedding* is an injective reduction.

Theorem 11. *Suppose that X is a Hausdorff space and E is a co-κ-Souslin, ω-universally Baire equivalence relation on X. Then at least one of the following holds:*

1. *The equivalence relation E has at most κ-many classes.*
2. *There is a perfect set of pairwise E-inequivalent points.*

Proof. It is clear that (1) and (2) are mutually exclusive. Set $\mathcal{G} = E^c$, and observe that if there is a κ-coloring of $\mathcal{G}$, then the pre-image of each singleton is contained in a single E-class, so E has at most κ-many equivalence classes.

By Theorem 10 it is enough to show that if there is a continuous homomorphism $\varphi \colon {}^{\omega}2 \to X$ from $\mathcal{G}_0$ to $\mathcal{G}$, then (2) holds. Note that the equivalence relation $F = (\varphi \times \varphi)^{-1}(E)$ has the Baire property.

Lemma 12. *The equivalence relation F is meager.*

Proof of lemma. By the Kuratowski-Ulam Theorem, it suffices to show that every equivalence class of F with the Baire property is meager. Suppose, towards a contradiction, that there exists $x \in {}^{\omega}2$ such that $[x]_F$ has the Baire property and is non-meager. Proposition 3 then ensures that there exists $(y, z) \in \mathcal{G}_0 \upharpoonright [x]_F$, in which case $(\varphi(y), \varphi(z)) \in \mathcal{G} \cap E$, the desired contradiction. ⊠

Lemma 12 and Mycielski's Theorem imply that there is a continuous embedding $\psi \colon {}^{\omega}2 \to {}^{\omega}2$ of the diagonal $\Delta({}^{\omega}2)$ into F, and it follows that the perfect set $\varphi \circ \psi({}^{\omega}2)$ is as desired. ⊠

Remark 13. By using Theorem 8 in place of Theorem 10, we can obtain the strengthening of Theorem 11 in which condition (1) is strengthened to the existence of a κ^+-Borel reduction of E to $\Delta(\kappa)$. Of course, the downside is that the proof is no longer classical.

Theorem 11 implies the theorems of Burgess [1] and Silver [16], as well as the generalization of the former to ω-universally Baire $\mathbf{\Pi}^1_2$ equivalence relations. This latter family includes all equivalence relations which are $\sigma(\mathbf{\Sigma}^1_1)$, C-measurable, or even absolutely $\mathbf{\Delta}^1_2$. Under $\mathtt{ZF}+\mathtt{BP}$, it includes every equivalence relation on an analytic Hausdorff space. In particular, it follows that the consistency of the full generalization of Silver's theorem to co-κ-Souslin equivalence relations follows from that of $\mathtt{ZF}$. However, the following remains open:

Question 14. What is the relationship between our assumption that E is ω-universally Baire and the Harrington-Shelah [5] assumption that E remains an equivalence relation after Cohen forcing?

6. Exercises

Exercise 15. Show that at the expensive of invoking $\mathtt{AC}_\omega$, one can eliminate the derivative in the proof of Theorem 4.

Exercise 16 (Souslin)**.** Show that if X is an analytic Hausdorff space, then exactly one of the following holds:

(1) The set X is countable.
(2) There is a continuous injection of ${}^\omega 2$ into X.

Hint: Do not give the usual proof! Use Theorem 4.

For each $n \in \omega$, we use ${}^n[X]$ to denote the family of n-element subsets of X equipped with the topology it inherits from ${}^n X$.

Exercise 17 (Feng [3])**.** Show that if X is an analytic Hausdorff space, $c \colon {}^2[X] \to 2$, and $c^{-1}(\{1\})$ is open, then exactly one of the following holds:

(1) The set X is the union of countably many 0-homogeneous Borel sets.
(2) There is a continuous injection of ${}^\omega 2$ into a 1-homogenous set.

Exercise 18. Show that if X and Y are Hausdorff spaces and $\mathcal{G} \subseteq X \times (Y \times Y)$ is an analytic set whose vertical sections are graphs, then exactly one of the following holds:

(1) There is a Borel function $c \colon X \times Y \to \omega$ such that for all $x \in X$, the map $c_x(y) = c(x, y)$ is a coloring of $\mathcal{G}_x$.
(2) For some $x \in X$, there is a continuous homomorphism from $\mathcal{G}_0$ to $\mathcal{G}_x$.

Hint: Do not given a parametrized version of the proof of the Kechris-Solecki-Todorcevic [9] theorem. This is a straightforward corollary.

A *partial uniformization* of a set $R \subseteq X \times Y$ is a subset of R all of whose vertical sections have at most one element.

Exercise 19 (Lusin-Novikov [12])**.** Show that if X and Y are Hausdorff spaces, $R \subseteq X \times Y$ is analytic, and no vertical section of R has a perfect subset, then R is the union of countably many relatively Borel partial uniformizations.

Hint: This is really a parametrized version of Exercise 16. Whenever Theorem 4 can be used to obtain dichotomy theorems, Exercise 18 can be used to obtain its parametrized counterpart.

A *transversal* of an equivalence relation E is a set which intersects every E-class in exactly one point. Let E_0 denote the equivalence relation on ${}^{\omega}2$ given by $xE_0y \iff \exists n \in \omega\ (x \upharpoonright [n, \omega) = y \upharpoonright [n, \omega))$.

Exercise 20 (Dougherty-Jackson-Kechris [2])**.** Show that if X is a Hausdorff space, E is an analytic equivalence relation on X, and no equivalence class of E has a perfect subset, then exactly one of the following holds:

(1) There is a Borel transversal of E.
(2) There is a continuous embedding of E_0 into E.

Let F_0 denote the equivalence relation on ${}^{\omega}2$ given by $xF_0y \iff \exists n \in \omega\ (x \upharpoonright [n, \omega) = y \upharpoonright [n, \omega)$ and $\sum_{i \in n} x(i) \equiv \sum_{i \in n} y(i) \pmod 2)$. A set $B \subseteq X/E$ is *Borel* if $\{x \in X \mid [x]_E \in B\}$ is Borel.

Exercise 21 (Louveau)**.** Show that if X is a Hausdorff space, E is an analytic equivalence relation on X, F is a relatively co-analytic subequivalence relation of E, and every E-class is the the disjoint union of two F-classes, then exactly one of the following holds:

(1) There is a Borel transversal of E/F.
(2) There is a continuous embedding of (E_0, F_0) into (E, F).

A *quasi-metric* is a function which satisfies the requirements for being a metric except that distinct points can be of distance zero apart.

Exercise 22 (Friedman-Harrington-Kechris [4, 8])**.** Show that if X is a Hausdorff space, d is a quasi-metric on X, and for all $\epsilon > 0$ the set $d^{-1}([0, \epsilon))$ is co-analytic, then exactly one of the following holds:

(1) There is a countable dense set.
(2) There is a continuous embedding of ${}^{\omega}2$ into an ϵ-discrete subspace of (X, d), for some $\epsilon > 0$.

Hint: First establish a similar dichotomy theorem in `ZF`, and then check that the two theorems are equivalent under $\mathtt{AC}_{\omega}$.

Exercise 23 (Louveau [10])**.** State and prove generalizations of Theorem 4 to digraphs and finite-dimensional hypergraphs.

A *quasi-order* on X is a reflexive transitive set $R \subseteq X \times X$. The *equivalence relation* associated with such a quasi-order is given by

$$x \equiv_R y \iff (x, y) \in R \text{ and } (y, x) \in R.$$

An *antichain* is a set in which no two points are R-related. The *lexicographic ordering* of ${}^{\omega}2$ is given by

$$(x, y) \in R_{\text{lex}} \iff x = y \text{ or } \exists n \in \omega \ (x \upharpoonright n = y \upharpoonright n \text{ and } x(n) < y(n)).$$

Exercise 24 (Louveau [11])**.** Show that if X is a Hausdorff space and R is a co-analytic quasi-order on X, then exactly one of the following holds:

(1) The equivalence relation $\equiv_R$ has at most countably many classes.
(2) At least one of the following holds:
  (a) There is a continuous injection of ${}^{\omega}2$ into an antichain.
  (b) There is a continuous embedding of R_{lex} into R.

Exercise 25. Show that if X is a Hausdorff space and R is a co-analytic quasi-order on X, then the following are equivalent:

(1) The set X is the union of countably many Borel chains.
(2) The set X is the union of countably many ω-universally Baire measurable chains.

A quasi-order is *linear* if any two points are comparable.

Exercise 26 (Friedman-Shelah [15])**.** Show that if X is a Hausdorff space and R is a co-analytic linear quasi-order on X, then exactly one of the following holds:

(1) There is a countable dense set.
(2) There is a continuous injection of ${}^{\omega}2$ into a pairwise disjoint family of non-empty open intervals.

Hint: First establish a similar dichotomy theorem in $\mathtt{ZF}$, and then check that the two theorems are equivalent under $\mathtt{DC}_{\omega}$. Under no circumstances should you use $\mathtt{DC}_{\omega_1}$.

Suppose that $D \colon \mathcal{P}(X) \to \omega \cup \{\infty\}$. The *span* of a set $A \subseteq X$ is given by $\operatorname{span}_D A = \{x \in X \mid D(A) = D(A \cup \{x\})\}$. We say that D is a *notion of dimension* if it satisfies the following conditions:

(1) $\forall x \in X \ (D(\{x\}) \leq 1)$.
(2) $\forall A \subseteq B \subseteq X \ (D(A) \leq D(B))$.
(3) $\forall A \subseteq X \ (D(A) = D(\operatorname{span}_D A))$.

We refer to $D(A)$ as the *dimension* of A.

Exercise 27 (van Engelen-Kunen-Miller [19] in several special cases)**.** Show that if $d \in \omega \setminus 2$, X is a Hausdorff space, and D is a notion of dimension on X with the property that $D^{-1}(k) \cap {}^{k}[X]$ is ω-universally Baire for all $k \in d$ and co-analytic for $k = d$, then exactly one of the following holds:

(1) The set X is the union of countably many Borel sets of dimension strictly less than d.
(2) There is a continuous injection of ${}^{\omega}2$ into a subset of X whose d-element subsets are all of dimension at least d.

Exercise 28. Show that if X is a Hausdorff space and D is a notion of dimension on X with the property that $D^{-1}(k) \cap {}^{k}[X]$ is co-analytic for all $k \in \omega$, then exactly one of the following holds:

(1) The set X is the union of countably many finite-dimensional Borel sets.
(2) There is a continuous injection of ${}^{\omega}2$ into a subset of X whose d-element subsets are all of dimension at least d for all $d \in \omega$.

What does this tell us about suitably definable vector spaces?

Exercise 29. State and prove generalizations of all of the results mentioned thus far to κ-Souslin ω-universally Baire structures.

Hint: The proofs are virtually identical!

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