FORCELESS, INEFFECTIVE, POWERLESS PROOFS OF DESCRIPTIVE DICHOTOMY THEOREMS

LECTURE IV: THE KANOVEI-LOUVEAU THEOREM

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Abstract. We give a classical proof of a generalization of the Kechris-Solecki-Todorcevic dichotomy theorem [5] characterizing analytic graphs of uncountable Borel chromatic number. Using this, we give a classical proof of a result of Kanovei-Louveau [4] which simultaneously generalizes results of Harrington-Kechris-Louveau [1] and Harrington-Marker-Shelah [2].

In §1, we give two straightforward corollaries of the first separation theorem. In §2, we establish a directed local version of the Kechris-Solecki-Todorcevic theorem [5]. In §3, we use this to give a classical proof of the Kanovei-Louveau characterization [4] of linearizable Borel quasi-orders which simultaneously generalizes the Harrington-Kechris-Louveau characterization [1] of smooth Borel equivalence relations and the Harrington-Marker-Shelah characterization [2] of linear Borel quasi-orders. In §4, we give as exercises several results that can be obtained in a similar fashion.

1. Corollaries of separation

Suppose that X is a set. A *quasi-order* on X is a reflexive transitive set $R \subseteq X \times X$. The *equivalence relation* associated with R is given by $x \equiv_R y \iff (xRy \text{ and } yRx)$. The *strict quasi-order* associated with R is given by $x <_R y \iff (xRy \text{ and } x \not\equiv_R y)$.

Suppose that $A \subseteq X$. The *upward R-saturation* of A is given by $[A]^R = \{x \in X \mid \exists y \in A \ ((y, x) \in R)\}$. The set A is *upward R-invariant* if $A = [A]^R$. The *downward R-saturation* of A is given by $[A]_R = \{x \in X \mid \exists y \in A \ ((x, y) \in R)\}$. The set A is *downward R-invariant* if $A = [A]_R$.

Proposition 1. *Suppose that X is a Hausdorff space, R is an analytic quasi-order on X, and (A_0, A_1) is an R-discrete pair of analytic subsets of X. Then there is an R-discrete pair (B_0, B_1) of Borel subsets of X*

such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is upward R-invariant, and B_1 is downward R-invariant.

Proof. Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$. Suppose now that we have an R-discrete pair $(A_{0,n}, A_{1,n})$ of analytic subsets of X. Then there is an R-discrete pair $(B_{0,n}, B_{1,n})$ of Borel subsets of X such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$. Set $A_{0,n+1} = [B_{0,n}]^R$ and $A_{1,n+1} = [B_{1,n}]_R$. The sets $B_0 = \bigcup_{n\in\omega} B_{0,n}$ and $B_1 = \bigcup_{n\in\omega} B_{1,n}$ are as desired. $\boxtimes$

Proposition 2. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, R is a bi-analytic quasi-order on X, and (A_0, A_1) is an $(E \setminus R)$-discrete pair of analytic sets. Then there is an $(E\setminus R)$-discrete pair (B_0, B_1) of Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is downward $(E\cap R)$-invariant, and B_1 is upward $(E\cap R)$-invariant.*

Proof. Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$. Suppose now that we have an $(E\setminus R)$-discrete pair $(A_{0,n}, A_{1,n})$ of analytic subsets of X. Then there is an $(E\setminus R)$-discrete pair $(B_{0,n}, B_{1,n})$ of Borel subsets of X such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$. Set $A_{0,n+1} = [B_{0,n}]_{E\cap R}$ and $A_{1,n+1} = [B_{1,n}]^{E\cap R}$. The sets $B_0 = \bigcup_{n\in\omega} B_{0,n}$ and $B_1 = \bigcup_{n\in\omega} B_{1,n}$ are as desired. $\boxtimes$

2. A directed local generalization of the Kechris-Solecki-Todorcevic theorem

For each set $I \subseteq {}^{<\omega}2$, let $\mathcal{G}_I$ denote the digraph on ${}^{\omega}2$ consisting of all pairs of the form $(s^\frown 0^\frown x, s^\frown 1^\frown x)$, where $s \in I$ and $x \in {}^{\omega}2$. We say that I is *dense* if $\forall s \in {}^{<\omega}2 \exists t \in I\ (s \sqsubseteq t)$.

Proposition 3. *Suppose that $I \subseteq {}^{<\omega}2$ is dense and $A \subseteq {}^{\omega}2$ is nonmeager and has the Baire property. Then A is not $\mathcal{G}_I$-discrete.*

Proof. Fix $s \in {}^{<\omega}2$ such that A is comeager in $\mathcal{N}_s$. Fix $t \in I$ such that $s \sqsubseteq t$. Then there exists $x \in {}^{\omega}2$ such that $t^\frown 0^\frown x, t^\frown 1^\frown x \in A$. As $(t^\frown 0^\frown x, t^\frown 1^\frown x) \in \mathcal{G}_I$, it follows that A is not $\mathcal{G}_I$-discrete. $\boxtimes$

For each set $J \subseteq \bigcup_{n\in\omega} {}^{n}2 \times {}^{n}2$, let $\mathcal{H}_J$ denote the digraph on ${}^{\omega}2$ consisting of all pairs of the form $(s(0)^\frown 0^\frown x, s(1)^\frown 1^\frown x)$, where $s \in J$ and $x \in {}^{\omega}2$. Let R_J denote the smallest quasi-order containing $\mathcal{H}_J$. We say that J is *dense* if $\forall s \in {}^{<\omega}2 \times {}^{<\omega}2 \exists t \in J \forall i \in 2\ (s(i) \sqsubseteq t(i))$.

Proposition 4. *Suppose that $J \subseteq \bigcup_{n\in\omega} {}^{n}2 \times {}^{n}2$ is dense and $R \subseteq {}^{\omega}2 \times {}^{\omega}2$ is a transitive set with the Baire property which contains $\mathcal{H}_J$. Then R is meager or comeager.*

Proof. Suppose, towards a contradiction, that there exist $u, v \in {}^{<\omega}2 \times {}^{<\omega}2$ with R comeager in $\mathcal{N}_{u(1)} \times \mathcal{N}_{v(0)}$ and meager in $\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)}$. Fix

$s, t \in J$ such that $u(i) \sqsubseteq s(i)$ and $v(i) \sqsubseteq t(i)$ for all $i \in 2$. Then

$$\forall^* x, y \in {}^\omega 2\ (s(0)^\frown 0^\frown x R s(1)^\frown 1^\frown x R t(0)^\frown 0^\frown y R t(1)^\frown 1^\frown y).$$

As $u(0) \sqsubseteq s(0)$ and $v(1) \sqsubseteq t(1)$, this contradicts our assumption that R is meager in $\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)}$. ⊠

Proposition 5. *Suppose that $J \subseteq \bigcup_{n\in\omega} {}^n 2 \times {}^n 2$ is dense, X is a Hausdorff space, R is an ω-universally Baire linear quasi-order on X, and $\varphi\colon {}^\omega 2 \to X$ is a Baire measurable homomorphism from R_J to R. Then there exists $x \in X$ such that $\varphi^{-1}([x]_{\equiv_R})$ is comeager.*

Proof. Set $S = (\varphi \times \varphi)^{-1}(R)$. As S is linear, it is necessarily non-meager, so Proposition 4 ensures that it is comeager. Then $\equiv_S$ is comeager and therefore has a comeager equivalence class. ⊠

Fix sequences $s_{2n} \in {}^{2n}2$ and pairs $s_{2n+1} \in {}^{2n+1}2 \times {}^{2n+1}2$ for $n \in \omega$ such that the sets $I = \{s_{2n} \mid n \in \omega\}$ and $J = \{s_{2n+1} \mid n \in \omega\}$ are dense. Define $\mathcal{G}_0(\text{even}) = \mathcal{G}_I$, $\mathcal{H}_0(\text{odd}) = \mathcal{H}_J$, and $R_0(\text{odd}) = R_J$.

For each ordinal α, the *lexicographic ordering* of ${}^\alpha 2$ is given by

$$x <_{R_{\text{lex}}(\alpha)} y \iff \exists \beta \in \alpha\ (x \upharpoonright (0,\beta) = y \upharpoonright (0,\beta) \text{ and } x(\beta) < y(\beta)).$$

We say that a quasi-order R is *lexicographically reducible* if it is Borel reducible to $R_{\text{lex}}(\alpha)$ for some countable ordinal α.

Theorem 6. *Suppose that X is a Hausdorff space, $\mathcal{G}$ is an analytic digraph on X, and R is an analytic quasi-order on X. Then exactly one of the following holds:*

1. *There is a Borel ω-coloring of $\equiv_S \cap \mathcal{G}$, for some lexicographically reducible quasi-order $S \supseteq R$.*
2. *There is a continuous homomorphism $\pi\colon {}^\omega 2 \to X$ from the pair $(\mathcal{G}_0(\text{even}), R_0(\text{odd}))$ to the pair $(\mathcal{G}, R)$.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that α is a countable ordinal, $S \supseteq R$ is a quasi-order, $\varphi\colon X \to {}^\alpha 2$ is an ω-universally Baire measurable reduction of S to $R_{\text{lex}}(\alpha)$, $c\colon X \to \omega$ is an ω-universally Baire measurable ω-coloring of $\equiv_S \cap \mathcal{G}$, and $\pi\colon {}^\omega 2 \to X$ is a Baire measurable homomorphism from $(\mathcal{G}_0(\text{even}), R_0(\text{odd}))$ to $(\mathcal{G}, R)$. Then $\varphi \circ \pi$ is a Baire measurable homomorphism from $R_0(\text{odd})$ to $R_{\text{lex}}(\alpha)$, so Proposition 5 ensures the existence of $x \in {}^\omega 2$ such that the set $C = (\varphi \circ \pi)^{-1}(\{x\})$ is comeager. Note that $\pi(C)$ is a single $\equiv_S$-class, so $c \upharpoonright \pi(C)$ is a coloring of $\mathcal{G} \upharpoonright \pi(C)$, thus $(c \circ \pi) \upharpoonright C$ is a coloring of $\mathcal{G}_0(\text{even})$. Then there exists $n \in \omega$ such that $c^{-1}(\{n\})$ is non-meager, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that $\mathcal{G}$ is non-empty, in which case there are continuous functions $\varphi_{\mathcal{G}}, \varphi_R \colon {}^\omega\omega \to X \times X$ such that $\mathcal{G} = \varphi_{\mathcal{G}}({}^\omega\omega)$ and $R = \varphi_R({}^\omega\omega)$. Fix a continuous function $\varphi_X \colon {}^\omega\omega \to X$ such that $\mathrm{dom}(\mathcal{G}) \subseteq \varphi_X({}^\omega\omega)$.

A *global* (n-)*approximation* is a pair of the form $p = (u^p, v^p)$, where $u^p \colon {}^n2 \to {}^n\omega$ and $v^p \colon {}^{<n}2 \to {}^n\omega$. Fix an enumeration $(p_n)_{n\in\omega}$ of the set of all global approximations.

An *extension* of a global m-approximation p is a global n-approximation q with the property that $s_p \sqsubseteq s_q \implies u^p(s_p) \sqsubseteq u^q(s_q)$ and $t_p \sqsubseteq t_q \implies v^p(t_p) \sqsubseteq v^q(t_q)$ for all $s_p \in {}^m2$, $s_q \in {}^n2$, $t_p \in {}^{<m}2$, and $t_q \in {}^{<n}2$ with $n - m = |t_q| - |t_p|$. When $n = m + 1$, we say that q is a *one-step extension* of p.

A *local* (n-)*approximation* is a pair of the form $l = (f^l, g^l)$, where $f^l \colon {}^n2 \to {}^\omega\omega$ and $g^l \colon {}^{<n}2 \to {}^\omega\omega$, such that

$$\varphi_{\mathcal{G}} \circ g^l(t) = (\varphi_X \circ f^l(s_k{}^\frown 0^\frown t), \varphi_X \circ f^l(s_k{}^\frown 1^\frown t))$$

for all even $k \in n$ and $t \in {}^{n-k-1}2$, and

$$\varphi_R \circ g^l(t) = (\varphi_X \circ f^l(s_k(0)^\frown 0^\frown t), \varphi_X \circ f^l(s_k(1)^\frown 1^\frown t))$$

for all odd $k \in n$ and $t \in {}^{n-k-1}2$. We say that l is *compatible* with a global n-approximation p if $u^p(s) \sqsubseteq f^l(s)$ and $v^p(t) \sqsubseteq g^l(t)$ for all $s \in {}^n2$ and $t \in {}^{<n}2$. We say that l is *compatible* with a quasi-order S on X if $\varphi_X \circ f^l({}^n2)$ is contained in a single $\equiv_S$-class. We say that l is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l({}^n2) \subseteq Y$.

Suppose now that α is a countable ordinal, $S \supseteq R$ is a lexicographically reducible quasi-order, $Y \subseteq X$ is a Borel set, and $c \colon Y^c \to \omega \cdot \alpha$ is a Borel coloring of $(\equiv_S \cap \mathcal{G}) \restriction Y^c$. Associated with each global n-approximation p is the set $L_n(p, S, Y)$ of local n-approximations which are compatible with p, S, and Y.

A global n-approximation p is (S,Y)-*terminal* if $L_{n+1}(q, S, Y) = \emptyset$ for all one-step extensions q of p. Let $T_n(S, Y)$ denote the set of all such global n-approximations, and set $T_{\mathrm{even}}(S, Y) = \bigcup_{n\in\omega} T_{2n}(S, Y)$, $T_{\mathrm{odd}}(S, Y) = \bigcup_{n\in\omega} T_{2n+1}(S, Y)$, and $T(S, Y) = \bigcup_{n\in\omega} T_n(S, Y)$.

When n is even, we use $A(p, S, Y)$ to denote the set of points of the form $\varphi_X \circ f^l(s_n)$, where $l \in L_n(p, S, Y)$.

Lemma 7. *Suppose that $n \in \omega$ is even, p is a global n-approximation, and the set $A(p, S, Y)$ is not $(\equiv_S \cap \mathcal{G})$-discrete. Then $p \notin T_n(S, Y)$.*

Proof of lemma. Fix local n-approximations $l_0, l_1 \in L_n(p, S, Y)$ with $(\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n)) \in {\equiv_S} \cap \mathcal{G}$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_{\mathcal{G}}(x) = (\varphi_X \circ f^{l_0}(s_n), \varphi_X \circ f^{l_1}(s_n))$. Let l denote the local $(n+1)$-approximation given by $f^l(s^\frown i) = f^{l_i}(s)$, $g^l(\emptyset) = x$, and $g^l(t^\frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n2$, and $t \in {}^{<n}2$. Then l is compatible with a one-step extension of p, thus p is not (S, Y)-terminal. ⊠

Lemma 7 ensures that for each $p \in T_{\text{even}}(S, Y)$, there is an $(\equiv_S \cap \mathcal{G})$-discrete Borel set $B(p, S, Y) \subseteq X$ with $A(p, S, Y) \subseteq B(p, S, Y)$. Set $Y' = Y \setminus \bigcup \{B(p, S, Y) \mid p \in T_{\text{even}}(S, Y)\}$. For each $y \in Y \setminus Y'$, put $n(y) = \min\{n \in \omega \mid p_n \in T_{\text{even}}(S, Y) \text{ and } y \in B(p_n, S, Y)\}$. Define $c' \colon (Y')^c \to \omega \cdot (\alpha + 1)$ by

$$c'(y) = \begin{cases} c(y) & \text{if } y \in Y^c \text{ and} \\ \omega \cdot \alpha + n(y) & \text{otherwise.} \end{cases}$$

Lemma 8. *The function c' is a coloring of $(\equiv_S \cap \mathcal{G}) \restriction (Y')^c$.*

Proof of lemma. Note that if $\beta \in \omega \cdot \alpha$ then $(c')^{-1}(\{\beta\}) = c^{-1}(\{\beta\})$, and if $\beta \in \omega \cdot (\alpha + 1) \setminus \omega \cdot \alpha$ then there exists $n \in \omega$ with $\beta = \omega \cdot \alpha + n$, so $p_n \in T_{\text{even}}(S, Y)$ and $(c')^{-1}(\{\beta\}) \subseteq B(p_n, S, Y)$. Then $(c')^{-1}(\{\beta\})$ is $(\equiv_S \cap \mathcal{G})$-discrete for all $\beta \in \omega \cdot (\alpha + 1)$, thus c' is a coloring of $(\equiv_S \cap \mathcal{G}) \restriction (Y')^c$. ⊠

When $i \in 2$ and n is odd, we use $A_i(p, S, Y)$ to denote the set of points of the form $\varphi_X \circ f^l \circ s_n(i)$, where $l \in L_n(p, S, Y)$.

Lemma 9. *Suppose that $n \in \omega$ is odd, p is a global n-approximation, and $(A_0(p, S, Y), A_1(p, S, Y))$ is not $(\equiv_S \cap R)$-discrete. Then $p \notin T_n(S, Y)$.*

Proof of lemma. Fix local n-approximations $l_0, l_1 \in L(p, S, Y)$ with $(\varphi_X \circ f^{l_0} \circ s_n(0), \varphi_X \circ f^{l_1} \circ s_n(1)) \in {\equiv_S} \cap R$. Then there exists $x \in {}^\omega\omega$ such that $\varphi_R(x) = (\varphi_X \circ f^{l_0} \circ s_n(0), \varphi_X \circ f^{l_1} \circ s_n(1))$. Let l denote the local $(n+1)$-approximation given by $f(s \frown i) = f^{l_i}(s)$, $g(\emptyset) = x$, and $g(t \frown i) = g^{l_i}(t)$ for $i \in 2$, $s \in {}^n2$, and $t \in {}^{<n}2$. Then l is compatible with a one-step extension of p, and it follows that $p \notin T_n(S, Y)$. ⊠

Proposition 1 and Lemma 9 ensure that for each $p \in T_{\text{odd}}(S, Y)$, there is an $(\equiv_S \cap R)$-discrete pair $(B_0(p, S, Y), B_1(p, S, Y))$ of Borel sets such that $A_0(p, S, Y) \subseteq B_0(p, S, Y)$, $A_1(p, S, Y) \subseteq B_1(p, S, Y)$, $B_0(p, S, Y)$ is upward $(\equiv_S \cap R)$-invariant, and $B_1(p, S, Y)$ is downward $(\equiv_S \cap R)$-invariant. Define $\psi \colon X \to {}^\omega 2$ by

$$\psi(x)(n) = \begin{cases} \chi_{B_0(p_n, S, Y)}(x) & \text{if } p_n \in T_{\text{odd}}(S, Y) \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Let S' denote the lexicographically reducible quasi-order given by

$$x S' y \iff x <_S y \text{ or } (x \equiv_S y \text{ and } \psi(x) R_{\text{lex}} \psi(y)).$$

Lemma 10. *The quasi-order S' contains R.*

Proof of lemma. This follows from the upward $(\equiv_S \cap R)$-invariance of the sets of the form $B_0(p, S, Y)$ and the fact that $R \subseteq S$. ⊠

Lemma 11. *Suppose that p is a global approximation whose one-step extensions are all (S, Y)-terminal. Then $p \in T(S', Y')$.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation. Suppose, towards a contradiction, that there is a one-step extension q of p for which there exists $l \in L_{n+1}(q, S', Y')$.

If n is odd, then $\varphi_X \circ f^l(s_{n+1}) \in A(q, S, Y)$ and $A(q, S, Y) \cap Y' = \emptyset$, so $\varphi_X \circ f^l(s_{n+1}) \notin Y'$, a contradiction.

If n is even, then $\varphi_X \circ f^l \circ s_{n+1}(0) \in A_0(p, S, Y)$ and $\varphi_X \circ f^l \circ s_{n+1}(1) \in A_1(p, S, Y)$. As $(A_0(p, S, Y), A_1(p, S, Y))$ is $(\equiv_S \cap R)$-discrete, it follows that $(\varphi_X \circ f^l \circ s_{n+1}(0), \varphi_X \circ f^l \circ s_{n+1}(1)) \notin \equiv_S \cap R$, a contradiction. ⊠

Recursively define lexicographically reducible quasi-orders S_α, Borel sets Y_α, and Borel colorings $c_\alpha \colon Y_\alpha^c \to \omega \cdot \alpha$ of $(\equiv_{S_\alpha} \cap \mathcal{G}) \restriction Y_\alpha^c$ by

$$(S_\alpha, Y_\alpha, c_\alpha) = \begin{cases} (X \times X, X, \emptyset) & \text{if } \alpha = 0, \\ (S'_\beta, Y'_\beta, c'_\beta) & \text{if } \alpha = \beta + 1, \text{ and} \\ (\bigcap_{\beta \in \alpha} S_\beta, \bigcap_{\beta \in \alpha} Y_\beta, \lim_{\beta \to \alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many approximations, there exists $\alpha \in \omega_1$ such that $T(S_\alpha, Y_\alpha) = T(S_{\alpha+1}, Y_{\alpha+1})$.

Let p^0 denote the unique global 0-approximation. As $\mathrm{dom}(\mathcal{G}) \cap Y_\alpha \subseteq A(p^0, S_\alpha, Y_\alpha)$, it follows that if p^0 is (S_α, Y_α)-terminal, then c_α extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of $\equiv_{S_\alpha} \cap \mathcal{G}$, thus there is a Borel ω-coloring of $\equiv_{S_\alpha} \cap \mathcal{G}$.

Otherwise, by repeatedly applying Lemma 11 we obtain global n-approximations $p^n = (u^n, v^n)$ with the property that p^{n+1} is a one-step extension of p^n for all $n \in \omega$. Define continuous functions $\pi \colon {}^\omega 2 \to {}^\omega \omega$ and $\pi_k \colon {}^\omega 2 \to {}^\omega \omega$ for $k \in \omega$ by

$$\pi(x) = \lim_{n \to \omega} u^n(x \restriction n) \text{ and } \pi_k(x) = \lim_{n \to \omega} v^{k+n+1}(x \restriction n).$$

To see that $\varphi_X \circ \pi$ is a homomorphism from $\mathcal{G}_0(\text{even})$ to $\mathcal{G}$, it is enough to show that $\varphi_\mathcal{G} \circ \pi_k(x) = (\varphi_X \circ \pi(s_k \frown 0 \frown x), \varphi_X \circ \pi(s_k \frown 1 \frown x))$ for all even $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k \frown 0 \frown x), \pi(s_k \frown 1 \frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_\mathcal{G}(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \restriction n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k \frown 0 \frown (x \restriction n))} \times \mathcal{N}_{u^{k+n+1}(s_k \frown 1 \frown (x \restriction n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, S_\alpha, Y_\alpha)$, and observe that $z = g^l(x \restriction n)$, $z_0 = f^l(s_k \frown 0 \frown (x \restriction n))$, and $z_1 = f^l(s_k \frown 1 \frown (x \restriction n))$ are as desired.

To see that $\varphi_X \circ \pi$ is a homomorphism from $R_0(\text{odd})$ to R, it is enough to show that $\varphi_R \circ \pi_k(x) = (\varphi_X \circ \pi(s_k(0) \frown 0 \frown x), \varphi_X \circ \pi(s_k(1) \frown 1 \frown x))$ for all odd $k \in \omega$ and $x \in {}^\omega 2$. By continuity, it is enough to show that every open neighborhood $U \times V$ of $(\pi_k(x), (\pi(s_k(0)^\frown 0^\frown x), \pi(s_k(1)^\frown 1^\frown x)))$ contains a point $(z, (z_0, z_1))$ such that $\varphi_R(z) = (\varphi_X(z_0), \varphi_X(z_1))$. Towards this end, fix $n \in \omega$ sufficiently large that $\mathcal{N}_{v^{k+n+1}(x \upharpoonright n)} \subseteq U$ and

$$\mathcal{N}_{u^{k+n+1}(s_k(0)^\frown 0^\frown (x\upharpoonright n))} \times \mathcal{N}_{u^{k+n+1}(s_k(1)^\frown 1^\frown (x\upharpoonright n))} \subseteq V.$$

Fix $l \in L_{k+n+1}(p^{k+n+1}, S_\alpha, Y_\alpha)$, and observe that $z = g^l(x \upharpoonright n)$, $z_0 = f^l(s_k(0)^\frown 0^\frown (x \upharpoonright n))$, and $z_1 = f^l(s_k(1)^\frown 1^\frown (x \upharpoonright n))$ are as desired. ⊠

3. The Kanovei-Louveau theorem

Let R_0 denote the partial order on ${}^\omega 2$ given by

$$x <_{R_0} y \iff \exists n \in \omega\ (x(n) < y(n) \text{ and } x \upharpoonright (n, \omega) = y \upharpoonright (n, \omega)).$$

A straightforward induction shows that the E_0-class of every non-eventually constant sequence is $\mathbb{Z}$-ordered by R_0.

Proposition 12. *Suppose that X is a Hausdorff space, R is an ω-universally Baire linear quasi-order on X, and $\varphi \colon {}^\omega 2 \to X$ is a Baire measurable homomorphism from R_0 to R. Then there exists $x \in X$ such that $\varphi^{-1}([x]_{\equiv_R})$ is comeager.*

Proof. Set $S = (\varphi \times \varphi)^{-1}(R)$. Fix $s \in {}^{<\omega}2$ such that the set $\{x \in {}^\omega 2 \mid \forall^* y \in \mathcal{N}_s\ (xSy)\}$ is non-meager. Then the set $\{x \in {}^\omega 2 \mid \forall y \in [x]_{E_0} \forall^* z \in \mathcal{N}_s\ (ySz)\}$ is also non-meager, so comeager, thus $\equiv_S$ has an equivalence class which is comeager in $\mathcal{N}_s$, and therefore comeager. ⊠

Proposition 13. *Suppose that $J \subseteq \bigcup_{n\in\omega} {}^n 2 \times {}^n 2$ is dense, $R \supseteq R_J$ is a meager quasi-order, and $C \subseteq R$ is closed. Then there is a continuous homomorphism $\pi \colon {}^\omega 2 \to {}^\omega 2$ from $(\Delta({}^\omega 2)^c, E_0^c, R_0)$ to (C^c, R^c, R).*

Proof. Fix a decreasing sequence $(U_n)_{n\in\omega}$ of dense open subsets of C^c such that $R \cap \bigcap_{n\in\omega} U_n = \emptyset$. An *$n$-approximation* is a pair (k, u), where $k \colon n+1 \to \omega$ and $u \colon {}^n 2 \to {}^{k(n)} 2$, such that

$$s \upharpoonright [m, n) = t \upharpoonright [m, n) \Longrightarrow u(s) \upharpoonright [k(m), k(n)) = u(t) \upharpoonright [k(m), k(n))$$

for all $m \in n$ and $s, t \in {}^n 2$. A *refinement* of (k, u) is an approximation (k', u') such that $k \upharpoonright n = k' \upharpoonright n$ and $u(s) \sqsubseteq u'(s)$ for all $s \in {}^n 2$.

Lemma 14. *Suppose that $n \in \omega$, (k, u) is an $(n+1)$-approximation, and $s \in {}^n 2 \times {}^n 2$. Then there is a refinement (k', u') of (k, u) such that $\mathcal{N}_{u'(s(0)^\frown 0)} \times \mathcal{N}_{u'(s(1)^\frown 1)} \subseteq U_{n+1}$.*

Proof of lemma. Fix $l \in \omega \setminus k(n+1)$ and $t \in {}^l 2 \times {}^l 2$ with $u \circ s(0) \sqsubseteq t(0)$, $u \circ s(1) \sqsubseteq t(1)$, and $\mathcal{N}_{t(0)} \times \mathcal{N}_{t(1)} \subseteq U_{n+1}$. Then the refinement of (k, u) given by $k'(n+1) = l$, $u'(s(0)^\frown 0) = t(0)$, and $u'(s(1)^\frown 1) = t(1)$ is clearly as desired. ⊠

Let (k_0, u_0) denote the 0-approximation given by $k_0(0) = 0$ and $u_0 = \emptyset$. Given an n-approximation (k_n, u_n), let (k, u) denote the $(n+1)$-approximation given by $k \restriction (n+1) = k_n$, $k(n+1) = k_n(n)$, and $u(s^\frown i) = u_n(s)$ for $i \in 2$ and $s \in {}^n2$. By applying Lemma 14 finitely many times, we obtain a refinement (k', u') such that $\mathcal{N}_{u'(s(0)^\frown 0)} \times \mathcal{N}_{u'(s(1)^\frown 1)} \subseteq U_{n+1}$ for all $s \in {}^n2 \times {}^n2$. Fix $s \in J$ such that $u'(1^{n\frown}0) \sqsubseteq s(0)$ and $u'(0^{n\frown}1) \sqsubseteq s(1)$, and let (k_{n+1}, u_{n+1}) denote the refinement given by $k_{n+1}(n+1) = |s(0)| + 1 = |s(1)| + 1$, $u_{n+1}(1^{n\frown}0) = s(0)^\frown 0$, and $u_{n+1}(0^{n\frown}1) = s(1)^\frown 1$.

Define $\pi \colon {}^\omega 2 \to {}^\omega 2$ by $\pi(x) = \lim_{n \to \omega} u_n(x \restriction n)$. Clearly π is continuous. Note now that if $n \in \omega$, $x, y \in {}^\omega 2$, and $x(n) \neq y(n)$, then $(\pi(x), \pi(y)) \in \mathcal{N}_{u_{n+1}(x \restriction (n+1))} \times \mathcal{N}_{u_{n+1}(y \restriction (n+1))} \subseteq U_{n+1}$. In particular, it follows that π is a homomorphism from $(\Delta({}^\omega 2)^c, E_0^c)$ to (C^c, R^c).

Finally, observe that if $n \in \omega$ and $x \in {}^\omega 2$, then there exist $s \in J$ and $y \in {}^\omega 2$ with $(\pi(1^{n\frown}0^\frown x), \pi(0^{n\frown}1^\frown x)) = (s(0)^\frown 0^\frown y, s(1)^\frown 1^\frown y) \in \mathcal{H}_J \subseteq R$. As R_0 is the smallest quasi-order containing all pairs of the form $(1^{n\frown}0^\frown x, 0^{n\frown}1^\frown x)$ for $n \in \omega$ and $x \in {}^\omega 2$, it follows that π is a homomorphism from R_0 to R. ⊠

Proposition 15. *Suppose that $C \subseteq {}^\omega 2$ is a non-meager G_δ set. Then there is a continuous embedding of R_0 into $R_0 \restriction C$.*

Proof. Fix $s_0 \in {}^{<\omega}2$ such that C is comeager in $\mathcal{N}_{s_0}$, as well as a decreasing sequence of dense open sets $U_n \subseteq \mathcal{N}_{s_0}$ such that $\bigcap_{n \in \omega} U_n \subseteq C$. An *$n$-approximation* is a pair (k, u), where $k \colon n + 1 \to \omega$ and $u \colon {}^n2 \to \{s \in {}^{k(n)}2 \mid s_0 \sqsubseteq s\}$, such that $s \restriction [m, n) = t \restriction [m, n) \implies u(s) \restriction [k(m), k(n)) = u(t) \restriction [k(m), k(n))$ for all $m \in n$ and $s, t \in {}^n2$. A *refinement* of (k, u) is an approximation (k', u') such that $k \restriction n = k' \restriction n$ and $u(s) \sqsubseteq u'(s)$ for all $s \in {}^n2$.

Lemma 16. *Suppose that $n \in \omega$, (k, u) is an $(n+1)$-approximation, and $s \in {}^{n+1}2$. Then there is a refinement (k', u') of (k, u) such that $\mathcal{N}_{u'(s)} \subseteq U_{n+1}$.*

Proof of lemma. As U_{n+1} is dense and open, there exist $l \in \omega \setminus k(n+1)$ and an extension $t \in {}^l2$ of $u(s)$ with $\mathcal{N}_t \subseteq U_{n+1}$. Then any refinement of (k, u) for which $k'(n+1) = l$ and $u'(s) = t$ is as desired. ⊠

Let (k_0, u_0) denote the 0-approximation given by $k_0(0) = |s_0|$ and $u_0(\emptyset) = s_0$. Given an n-approximation (k_n, u_n), let (k, u) denote the $(n+1)$-approximation given by $k \restriction (n+1) = k_n$, $k(n+1) = k_n(n) + 1$, and $u(s^\frown i) = u_n(s)^\frown i$ for $i \in 2$ and $s \in {}^n2$. By applying Lemma 16 finitely many times, we obtain a refinement (k_{n+1}, u_{n+1}) with the property that $\mathcal{N}_{u_{n+1}(s)} \subseteq U_{n+1}$ for all $s \in {}^{n+1}2$.

Define $\pi \colon {}^\omega 2 \to {}^\omega 2$ by $\pi(x) = \lim_{n \to \infty} u_n(x \upharpoonright n)$. Clearly π is continuous. Moreover, if $x \in {}^\omega 2$, then $\pi(x) \in \bigcap_{n \in \omega} \mathcal{N}_{u_n(x \upharpoonright n)} \subseteq \bigcap_{n \in \omega} U_n \subseteq C$, thus $\pi({}^\omega 2) \subseteq C$.

To see that π is an injective homomorphism from E_0^c to E_0^c, simply observe that if $x, y \in {}^\omega 2$ and $x(n) < y(n)$, then $\pi(x)(k_n(n)) < \pi(y)(k_n(n))$. Note also that if $x \upharpoonright (n, \omega) = y \upharpoonright (n, \omega)$, then $\pi(x) \upharpoonright (k_n(n), \omega) = \pi(y) \upharpoonright (k_n(n), \omega)$, thus π is a homomorphism from $(R_0, E_0 \setminus R_0)$ to $(R_0, E_0 \setminus R_0)$, and therefore an embedding of R_0 into $R_0 \upharpoonright C$. ⊠

Proposition 17. *Suppose that $J \subseteq \bigcup_{n \in \omega} {}^n 2 \times {}^n 2$ is dense, $R \supseteq R_J$ is a meager quasi-order, and $C \subseteq R$ is closed. Then there is a continuous function $\pi \colon {}^\omega 2 \to {}^\omega 2$ which is a homomorphism from $(\Delta({}^\omega 2)^c, E_0^c, E_0)$ or $(\Delta({}^\omega 2)^c, R_0^c, R_0)$ to (C^c, R^c, R).*

Proof. By Proposition 13, there is a continuous homomorphism $\varphi \colon {}^\omega 2 \to {}^\omega 2$ from $(\Delta({}^\omega 2)^c, E_0^c, R_0)$ to (C^c, R^c, R). Set $S = (\varphi \times \varphi)^{-1}(R)$, noting that $R_0 \subseteq S \subseteq E_0$.

For each $x \in {}^\omega 2 \setminus \{1^\omega\}$, let $\sigma(x)$ denote the immediate successor of x under R_0. Define $B = \{x \in {}^\omega 2 \setminus \{1^\omega\} \mid x <_S \sigma(x)\}$, noting that $S \upharpoonright B = R_0 \upharpoonright B$ and $S \upharpoonright [B]_{E_0}^c = E_0 \upharpoonright [B]_{E_0}^c$.

If B is meager, then there is a dense G_δ set $D \subseteq [B]_{E_0}^c$. Otherwise, there is a non-meager G_δ set $D \subseteq B$. By Proposition 15, there is a continuous embedding $\psi \colon {}^\omega 2 \to D$ from R_0 to $R_0 \upharpoonright D$. Set $\pi = \varphi \circ \psi$. If $D \subseteq [B]_{E_0}^c$, then π is a continuous embedding of E_0 into R. If $D \subseteq B$, then π is a continuous embedding of R_0 into R. ⊠

We are now ready for our main results.

Theorem 18 (Kanovei-Louveau)**.** *Suppose that X is a Hausdorff space and R is a bi-analytic quasi-order on X. Then exactly one of the following holds:*

1. *There is a lexicographically reducible quasi-order $S \supseteq R$ with the property that $\equiv_R = \equiv_S$.*
2. *There is a continuous embedding $\pi \colon {}^\omega 2 \to X$ of either E_0 or R_0 into R.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that α is a countable ordinal, $S \supseteq R$ is a quasi-order with $\equiv_R = \equiv_S$, $\varphi \colon X \to {}^\alpha 2$ is an ω-universally Baire reduction of S to $R_{\mathrm{lex}}(\alpha)$, and $\psi \colon {}^\omega 2 \to X$ is a Baire measurable reduction of E_0 or R_0 to R. In particular, it follows that ψ is a homomorphism from R_0 to R, so $\varphi \circ \psi$ is a Baire measurable homomorphism from R_0 to $R_{\mathrm{lex}}(\alpha)$, thus Proposition 12 ensures the existence of $x \in {}^\alpha 2$ such that the set $C = (\varphi \circ \psi)^{-1}(\{x\})$ is comeager. As $\pi(C)$ is a single $\equiv_S$-class, it is also a single $\equiv_R$-class, thus ψ sends comeagerly many E_0-classes to a single $\equiv_R$-class, the desired contradiction.

It remains to show that at least one of (1) and (2) holds. Towards this end, set $\mathcal{G} = R^c$ and suppose that there is a Borel ω-coloring $c \colon X \to \omega$ of $\equiv_S \cap \mathcal{G}$, for some lexicographically reducible quasi-order $S \supseteq R$. Proposition 2 ensures that for each $n \in \omega$, there is an $(\equiv_S \setminus R)$-discrete pair $(B_{n,0}, B_{n,1})$ of Borel sets such that $c^{-1}(\{n\}) \subseteq B_{n,0} \cap B_{n,1}$, $B_{n,0}$ is downward $(\equiv_S \cap R)$-invariant, and $B_{n,1}$ is upward $(\equiv_S \cap R)$-invariant. Define $\psi \colon X \to {}^\omega 2$ by $\psi(x)(n) = \chi_{B_{n,1}}(x)$, let T denote the lexicographically reducible quasi-order on X given by

$$xTy \iff x <_S y \text{ or } (x \equiv_S y \text{ and } \psi(x) R_{\mathrm{lex}} \psi(y)),$$

and observe that $R \subseteq T$ and $\equiv_R = \equiv_T$.

By Theorem 6, we can assume that there is a continuous homomorphism $\varphi \colon {}^\omega 2 \to X$ from $(\mathcal{G}_0(\text{even}), R_0(\text{odd}))$ to $(\mathcal{G}, R)$. Set $C = (\varphi \times \varphi)^{-1}(\Delta(X))$ and $S = (\varphi \times \varphi)^{-1}(R)$. If S is comeager, then so too is $\equiv_S$, which contradicts the fact that $\mathcal{G}_0(\text{even}) \cap S = \emptyset$. Proposition 4 therefore implies that S is meager. Proposition 17 now ensures that there is a continuous function $\psi \colon {}^\omega 2 \to {}^\omega 2$ which is a homomorphism of either $(\Delta({}^\omega 2)^c, E_0^c, E_0)$ or $(\Delta({}^\omega 2)^c, R_0^c, R_0)$ to (C^c, S^c, S), so the map $\pi = \varphi \circ \psi$ is a continuous embedding of E_0 or R_0 into R. ⊠

Theorem 19 (Harrington-Kechris-Louveau)**.** *Suppose that X is a Hausdorff space and E is a bi-analytic equivalence relation on X. Then exactly one of the following holds:*

1. *The equivalence relation E is smooth.*
2. *There is a continuous embedding $\pi \colon {}^\omega 2 \to X$ of E_0 into E.*

Proof. Note first that if S is a quasi-order and $\varphi \colon X \to Y$ is a reduction of S to a partial order on Y, then φ is also a Borel reduction of $\equiv_S$ to $\Delta(Y)$. Note also that no non-trivial partial order can be embedded into an equivalence relation. It follows that (1) of Theorem 18 is equivalent to our (1), and (2) of Theorem 18 is equivalent to our (2), thus the desired result follows from Theorem 18. ⊠

Theorem 20 (Harrington-Marker-Shelah)**.** *Every bi-analytic linear quasi-order on a Hausdorff space is lexicographically reducible.*

Proof. Suppose that X is a Hausdorff space and R is a bi-analytic linear quasi-order on X. By Theorem 18, we can assume that there is a continuous embedding $\varphi \colon {}^\omega 2 \to X$ from E_0 or R_0 to R. In particular, it follows that φ is a homomorphism from R_0 to R, so Proposition 12 ensures the existence of $x \in X$ such that $\varphi^{-1}([x]_{\equiv_R})$ is comeager.

It follows that φ sends comeagerly many E_0-classes to a single point, which contradicts the fact that φ is an embedding. $\boxtimes$

4. Exercises

Exercise 21. Show that if X and Y are Hausdorff spaces, $R \subseteq X \times (Y \times Y)$ is an analytic set whose vertical sections are quasi-orders, and $\mathcal{G} \subseteq X \times (Y \times Y)$ is an analytic set whose vertical sections are digraphs, then exactly one of the following holds:

1. There is a countable ordinal α, a set $S \supseteq R$, a Borel function $\varphi \colon X \times Y \to {}^{\alpha}2$, and a Borel function $c \colon X \times Y \to \omega$ such that for all $x \in X$, the map $\varphi_x(y) = \varphi(x, y)$ is a reduction of S_x to $R_{\mathrm{lex}}(\alpha)$ and the map $c_x(y) = c(x, y)$ is a coloring of $\equiv_{S_x} \cap \, \mathcal{G}_x$.
2. For some $x \in X$, there is a continuous homomorphism from $(\mathcal{G}_0(\mathrm{even}), R_0(\mathrm{odd}))$ to $(\mathcal{G}_x, R_x)$.

Exercise 22. Show that if X is a Hausdorff space, R is an analytic quasi-order on X, and $T \supseteq R$ is a co-analytic quasi-order on X, then exactly one of the following holds:

1. There is a lexicographically reducible quasi-order $T \supseteq R$ such that $\equiv_R \subseteq \equiv_S \subseteq \equiv_T$.
2. There is a continuous embedding $\pi \colon {}^{\omega}2 \to X$ of either (E_0, E_0), (R_0, E_0), or (R_0, R_0) into (R, T).

Exercise 23. Show that if X is a Hausdorff space, $\mathcal{G}$ is an analytic graph on X, and R is a bi-analytic quasi-order on X, then exactly one of the following holds:

1. There is a Borel ω-coloring of $\equiv_S \cap \mathcal{G}$, for some lexicographically reducible quasi-order S on X with $<_S \subseteq <_R$.
2. There is a continuous homomorphism $\pi \colon {}^{\omega}2 \to X$ from the pair $(\mathcal{G}_0(\mathrm{even}), \mathcal{H}_0(\mathrm{odd}))$ to the pair $(\mathcal{G}, <_R^c)$.

Exercise 24 (Harrington-Marker-Shelah)**.** Show that if X is a Hausdorff space and R is a bi-analytic quasi-order on X, then exactly one of the following holds:

1. The set X is the union of countably many Borel chains.
2. There is a perfect antichain.

Hint: First apply Exercise 23 with $\mathcal{G} = R^c$. In the case that one obtains the continuous homomorphism π, show that $\perp_R$ is non-meager in every non-empty basic open square (this takes some effort!), and use this to build the perfect antichain.

Exercise 25 (Harrington-Marker-Shelah)**.** Show that if X is a Hausdorff space and R is a bi-analytic linear quasi-order on X, then there exists $\alpha \in \omega_1$ such that R is Borel reducible to the lexicographic ordering on ${}^{\alpha}2$, and as a result R does not have a chain of length ω_1.

Exercise 26. State and prove versions of the above exercises for κ-Souslin ω-universally Baire structures.

Hint: To give a classical proof of a weak generalization, first establish a weak κ-Souslin analog of Theorem 6 by removing all uses of separation from the argument given in §2. Note that the resulting theorem is a true dichotomy in $\mathtt{ZF} + \mathtt{BP}$.

Hint: To give a strong generalization, adapt the techniques of Kanovei [3] to first establish a strong κ-Souslin analog of Theorem 6. Although the resulting proof is not classical, the resulting theorem is a true generalization of the Borel version.

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