A GENERALIZED MARKER LEMMA

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Abstract. We generalize the marker lemma from aperiodic countable Borel equivalence relations to transitive Borel binary relations with countably infinite vertical sections.

We say that a set $B \subseteq X$ is a *complete section* for a binary relation R on X if it intersects every vertical section of R, i.e., if $\forall x \in X \ (B \cap R_x \neq \emptyset)$. This clearly agrees with the usual notion of complete section for equivalence relations. Beyond this case, the notion is useful when considering the relation of lying in the forward orbit of a point under an action of a countable semigroup of Borel functions. Here we establish the generalization of the standard marker lemma to such relations:

Theorem 1. *Suppose that X is a Polish space and R is a transitive Borel binary relation on X whose vertical sections are all countably infinite. Then there are Borel R-complete sections $A_0 \supseteq A_1 \supseteq \cdots$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.*

Proof. Fix an enumeration $B_0, B_1, \ldots$ of a countable family of Borel subsets of X which separates points, and for each $s \in 2^{<\mathbb{N}}$, define $B_s \subseteq X$ by

$$B_s = \left(\bigcap_{s(i)=0} X \setminus B_i \right) \cap \left(\bigcap_{s(i)=1} B_i \right).$$

For each $n \in \mathbb{N}$, define $S_n : X \to \mathcal{P}(2^n)$ by

$$S_n(x) = \{ s \in 2^n : \forall y \in R_x \ (|B_s \cap R_y| = \aleph_0) \}.$$

Lemma 2. *Suppose that $x, y \in X$, $n \in \mathbb{N}$, $s \in 2^n$, and $i \in \{0, 1\}$. Then:*

1. *$(x, y) \in R \Rightarrow S_n(x) \subseteq S_n(y)$;*
2. *$si \in S_{n+1}(x) \Rightarrow s \in S_n(x)$.*

Proof. The first claim is a consequence of the transitivity of R, and the second is a trivial consequence of the definition of S_n. □

For each $s \in 2^n$, define $C_s \subseteq X$ by

$$C_s = \{ x \in X : \forall y \in R_x \ (s = \min_{\text{lex}} S_n(y)) \},$$

and for each $n \in \mathbb{N}$, define $D_n \subseteq X$ by

$$D_n = \bigcup_{s \in 2^n} B_s \cap C_s.$$

We will show that the sets $D_0, D_1, \ldots$ are nearly as desired.

Lemma 3. $\forall n \in \mathbb{N}\ (D_{n+1} \subseteq D_n)$.

Proof. Fix $n \in \mathbb{N}$ and suppose that $x \in D_{n+1}$. Then there exists $s \in 2^n$ and $i \in \{0,1\}$ such that $x \in B_{si} \cap C_{si}$. In particular, it follows that $x \in B_s$, so to see that $x \in D_n$, it is enough to show that $x \in C_s$. Suppose, towards a contradiction, that there exists $y \in R_x$ such that $s \neq t$, where $t = \min_{\text{lex}} S_n(y)$. As $x \in C_{si}$, it follows that $si \in S_{n+1}(x)$. As (1) ensures that $S_n(x) \subseteq S_n(y)$ and (2) ensures that $s \in S_n(x)$, it follows that $s \in S_n(y)$, thus $t <_{\text{lex}} s$. As $t0 <_{\text{lex}} si$ and $si = \min_{\text{lex}} S_{n+1}(y)$, it follows that $t0 \notin S_{n+1}(y)$, so there exists $z \in R_y$ such that $|B_{t0} \cap R_z| < \aleph_0$. Similarly, since $t1 <_{\text{lex}} si$ and $si = \min_{\text{lex}} S_{n+1}(z)$, it follows that $t1 \notin S_{n+1}(z)$, so there exists $w \in R_z$ such that $|B_{t1} \cap R_w| < \aleph_0$. As the transitivity of R ensures that $R_w \subseteq R_z$, this implies that $|B_t \cap R_w| < \aleph_0$. As the transitivity of R implies also that $(y,w) \in R$, this contradicts our assumption that $t \in S_n(y)$. □

While each D_n is an R-complete section, we will show something stronger:

Lemma 4. $\forall x \in X\ \forall n \in \mathbb{N}\ (|D_n \cap R_x| = \aleph_0)$.

Proof. Fix an enumeration $\langle s_i \rangle_{i<2^n}$ of $\{0,1\}^n$. For each $x \in X$, set $x_0 = x$, and given x_i, let x_{i+1} be any element of R_{x_i} such that $\min S_n(x_{i+1}) \neq \min S_n(x_i)$, if such an element exists. Otherwise, set $x_{i+1} = x_i$. Let $y = x_{2^n}$ and $s = \min S_n(y)$, and observe that $\forall z \in R_y\ (s = \min S_n(z))$, thus $y \in C_s$. As $s \in S_n(y)$, it follows that $|B_s \cap R_y| = \aleph_0$, and since $y \in C_s$, it follows that $B_s \cap R_y = B_s \cap C_s \cap R_y$, thus $|B_s \cap C_s \cap R_y| = \aleph_0$. As $B_s \cap C_s \subseteq D_n$ and the transitivity of R ensures that $R_y \subseteq R_x$, it follows that $|D_n \cap R_x| = \aleph_0$. □

Unfortunately, it need not be the case that the set $D = \bigcap_{n\in\mathbb{N}} D_n$ is empty. However, this is not so far from the truth:

Lemma 5. $\forall x, y \in D\ (x \neq y \Rightarrow (x,y) \notin R)$.

Proof. Suppose, towards a contradiction, that there are distinct points $x, y \in D$ such that $(x,y) \in R$. Fix $n \in \mathbb{N}$ and $s \in 2^n$ such that $x \in B_s$ and $y \notin B_s$. As $x \in D_n$, it follows that $S_n(x) = S_n(y) = s$, so $y \notin D_n$, thus $y \notin D$, the desired contradiction. □

Now define $A_n = D_n \setminus D$. Lemma 2 implies that these sets are decreasing, and they clearly have empty intersection, so it only remains to check that each A_n is an R-complete section. Towards this end, fix $x \in X$, and observe that two applications of Lemma 4 ensure that there are distinct points $y \in D_n \cap R_x$ and $z \in D_n \cap R_y$. Lemma 5 then ensures that $y \notin A_n \Rightarrow y \in D \Rightarrow z \notin D \Rightarrow z \in A_n$, and the transitivity of R then implies that $A_n \cap R_x \neq \emptyset$. □

Remark 6. It is not difficult to produce examples which show that the assumption of transitivity in Theorem 1 is necessary.