# Borel equivalence relations and everywhere faithful actions of free products

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**Abstract**

We study the circumstances under which an aperiodic countable Borel equivalence relation is generated by a Borel action of a free product of countable groups which is faithful on every equivalence class.

An action of a group $G$ on a set $X$ is *faithful* if $\forall g \in G \, \exists x \in X \ (g \cdot x \neq x)$. The *orbits* of a $G$-action are the sets of the form $[x]_G = \{g \cdot x : g \in G\}$. We say that an action is *everywhere faithful* if its restriction to each orbit is faithful. The *orbit equivalence relation* associated with a $G$-action is given by

$$x E_G^X y \Leftrightarrow \exists g \in G \ (g \cdot x = y).$$

We say that an equivalence relation $E$ on $X$ is *faithfully generated* by a $G$-action if $E = E_G^X$ and the $G$-action is everywhere faithful.

A *Polish space* is a separable, completely metrizable topological space. An equivalence relation on such a space is *countable* if each of its equivalence classes is countable, and *aperiodic* if each of its equivalence classes is infinite. Our main goal here is to provide some insight into the circumstances under which a given countable Borel equivalence relation on a Polish space is faithfully generated by a Borel action of a given non-trivial free product of groups.

In §2, we consider compressible equivalence relations. A Borel set $B \subseteq X$ is an *$E$-complete section* if it intersects every $E$-class, and $E$ is *compressible* if there is a Borel injection $f : X \to X$ such that $\mathrm{graph}(f) \subseteq E$ and $X \setminus f(X)$ is an $E$-complete section. The *full group* of $E$ is the group $[E]$ of Borel automorphisms $f : X \to X$ such that $\mathrm{graph}(f) \subseteq E$. A measure $\mu$ on $X$ is *$E$-invariant* if every element of $[E]$ is $\mu$-preserving. By a remarkable theorem of Nadkarni [10], a countable Borel equivalence relation is compressible if and only if it does not admit an invariant probability measure. In the absence of such measures, we can essentially always find the sorts of actions we desire:

**Theorem.** *Suppose that $G$ and $H$ are non-trivial countable groups such that $G * H \not\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then every compressible Borel equivalence relation is faithfully generated by a Borel action of $G * H$.*

An equivalence relation $E$ is *finite* if all of its equivalence classes are finite, and *hyperfinite* if there are finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n\in\mathbb{N}} F_n$. Our assumption above that $G * H \ncong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ is necessary, as an equivalence relation is faithfully generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ if and only if it is aperiodic and hyperfinite.

In §3, we prove a selection theorem which will allow us to perform certain constructions off of a set on which the equivalence relation in question is compressible. Although this fact has essentially appeared elsewhere (see Miller [8] and Miller [9]), we provide the proof here for the sake of completeness.

In §4, we turn our attention to incompressible hyperfinite equivalence relations. Let $E_0$ denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$xE_0y \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

The usual product measure $\mu_0$ on $2^{\mathbb{N}}$ is $E_0$-invariant, thus $E_0$ is incompressible. The *measure-theoretic full group* of $(E, \mu)$ is the group $[E]_\mu$ obtained from $[E]$ by identifying automorphisms which agree $\mu$-almost everywhere. It is clear that if there is a Borel action of $G * H$ that faithfully generates $E_0$, then both $G$ and $H$ embed into $[E_0]_{\mu_0}$. The converse also holds:

**Theorem.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic incompressible hyperfinite equivalence relation on $X$, and $G$ and $H$ are non-trivial countable groups. Then the following are equivalent:*

1. *$E$ is faithfully generated by a Borel action of $G * H$;*
2. *$G$ and $H$ embed into the measure-theoretic full group of $(E_0, \mu_0)$.*

A well known theorem of Ornstein-Weiss [11] implies that every countable amenable group can be embedded into $[E_0]_{\mu_0}$. As every countable group residually contained in $[E_0]_{\mu_0}$ can be embedded into $[E_0]_{\mu_0}$, it follows that every aperiodic hyperfinite equivalence relation is faithfully generated by a Borel action of every non-trivial free product of residually amenable groups.

In §5, we show that if an aperiodic countable Borel equivalence relation is generated by equivalence relations $E_n$ which are themselves faithfully generated by Borel actions of $G_n$, then $E$ is faithfully generated by a Borel action of $*_{n\in\mathbb{N}}\, G_n$. As a corollary, we obtain the following:

**Theorem.** *Suppose that $G_0, G_1, \ldots$ are non-trivial countable groups. Then the following are equivalent:*

1. *Every aperiodic countable Borel equivalence relation is faithfully generated by a Borel action of $*_{n\in\mathbb{N}}\, G_n$;*
2. *Each $G_n$ embeds into the measure-theoretic full group of $(E_0, \mu_0)$.*

*In particular, condition (1) holds if each $G_n$ is residually amenable.*

# 1 Compressible equivalence relations

In this section, we determine completely the circumstances under which a given compressible equivalence relation is faithfully generated by a Borel action of a given non-trivial free product.

We need first some notation. Let $I(X) = X \times X$ denote the maximal equivalence relation on $X$. The *product* of equivalence relations $E$ and $F$ on $X$ and $Y$ is the equivalence relation $E \times F$ on $X \times Y$ given by

$$(x_1, y_1) E \times F (x_2, y_2) \Leftrightarrow x_1 E x_2 \text{ and } y_1 F y_2.$$

The *join* of equivalence relations $E$ and $F$ on the same space is the smallest equivalence relation $E \vee F$ which contains both $E$ and $F$.

Before getting to the main results of this section, we consider first the only amenable non-trivial free product:

**Proposition 1.** *Suppose that $X$ is a Polish space and $E$ is an aperiodic countable Borel equivalence relation on $X$. Then the following are equivalent:*

1. *$E$ is freely generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.*
2. *$E$ is faithfully generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.*
3. *$E$ is generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.*
4. *$E$ is hyperfinite.*

*Proof.* It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To see (3) $\Rightarrow$ (4), let $i$ and $j$ be the generators of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, and observe that $\mathcal{L} = \text{graph}(i) \cup \text{graph}(j)$ is as in Remark 6.8 of Kechris-Miller [7], thus $E$ is hyperfinite. To see (4) $\Rightarrow$ (1), appeal to Proposition 7.4 of Kechris-Miller [7] to find a Borel equivalence relation $F \subseteq E$ whose classes are all of cardinality 2. Let $i : X \to X$ be the involution which sends $x$ to the other element of its $F$-class, fix a Borel linear ordering $\leq$ of $X$, and set $B = \{x \in X : x < i(x)\}$. By Theorem 6.6 of Kechris-Miller [7] (which is due to Slaman-Steel [12] and Weiss [13]), there is a Borel automorphism $f : B \to B$ generating $E|B$. Define $j : X \to X$ by

$$j(x) = \begin{cases} i \circ f^{-1}(x) & \text{if } x \in B, \\ f \circ i(x) & \text{otherwise.} \end{cases}$$

This clearly induces the desired action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. □

As $E$ is hyperfinite if and only if $E \times I(\mathbb{N})$ is hyperfinite, it follows that if $E$ is not hyperfinite, then $E \times I(\mathbb{N})$ is not generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. In contrast, we have the following:

**Proposition 2.** *Suppose that $X$ is a Polish space, $E$ is a countable Borel equivalence relation on $X$, and $G$ and $H$ are non-trivial countable groups such that $G * H \not\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then $E \times I(\mathbb{N})$ is faithfully generated by a Borel action of $G * H$.*

*Proof.* By reversing the roles of $G$ and $H$ if necessary, we can assume that $|H| \geq 3$. We say that equivalence relations $F_1$ and $F_2$ on $X$ are *independent* if for all $x_0, x_1, \ldots, x_{2n} \in X$ such that $x_0 F_1 x_1 F_2 \ldots F_2 x_{2n} = x_0$, there exists $i < 2n$ such that $x_i = x_{i+1}$.

**Lemma 3.** *There are independent equivalence relations $F_G$ and $F_H$ on $\mathbb{N} \times 3$ which satisfy the following conditions:*

1. *$I(\mathbb{N} \times 3) = F_G \vee F_H$;*
2. *Every $F_G$-class is of cardinality $|G|$;*
3. *The sets $\mathbb{N} \times \{0\}$, $\mathbb{N} \times \{1\}$, and $\mathbb{N} \times \{2\}$ are $F_H$-invariant;*
4. *Every equivalence class of $F_H|(\mathbb{N} \times \{0\})$ has cardinality $1$;*
5. *Every equivalence class of $F_H|(\mathbb{N} \times \{1\})$ has cardinality $|H| - 1$;*
6. *Every equivalence class of $F_H|(\mathbb{N} \times \{2\})$ has cardinality $|H|$;*
7. *For every $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that the $n$-fold iterated saturation $[[\ldots[[(k, 2)]_{F_G}]_{F_H} \ldots]_{F_G}]_{F_H}$ lies entirely within $\mathbb{N} \times \{2\}$;*
8. *$\mathbb{N} \times \{1\}$ contains infinitely many $F_H$-classes.*

*Proof.* This follows from a straightforward inductive construction. □

Fix $F_G$ and $F_H$ as in Lemma 3. Condition (8) ensures that we can recursively define $k_n \in \mathbb{N}$ by setting $k_0 = 0$ and

$$k_{n+1} = \min\{k \in \mathbb{N} : (k, 1) \notin \textstyle\bigcup_{i \leq k_n} [(i, 1)]_{F_H}\}.$$

By the proof of Theorem 1 of Feldman-Moore [3], there are Borel involutions $i_n : X \to X$ such that $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(i_n)$. Define $E_G = \Delta(X) \times F_G$, and let $E_H$ be the equivalence relation generated by $\Delta(X) \times F_H$ and the function $\varphi : X \times (\mathbb{N} \times \{0\}) \to X \times (\mathbb{N} \times \{1\})$ given by $\varphi(x, (n, 0)) = (i_n(x), (k_n, 1))$. Condition (1) ensures that $E \times I(\mathbb{N} \times 3) = E_G \vee E_H$. Condition (2) ensures that $E_G$ is freely generated by a Borel action of $G$, and conditions (3) — (6) ensure that $E_H$ is freely generated by a Borel action of $H$. Condition (7) and the independence of $F_G$ and $F_H$ then ensure that the corresponding action of $G * H$ on $X \times (\mathbb{N} \times 3)$ is everywhere faithful, and since $E \times I(\mathbb{N}) \cong_B E \times I(\mathbb{N} \times 3)$, the proposition follows. □

We say that $E$ is *(Borel) reducible* to $F$, or $E \leq_B F$, if there is a Borel function $\pi : X \to Y$ such that $\forall x_1, x_2 \in X$ $(x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2))$. We say that $E$ and $F$ are *(Borel) bi-reducible*, or $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$.

**Proposition 4.** *Suppose that $G$ and $H$ are non-trivial countable groups such that $G * H \not\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then every countable Borel equivalence relation is bi-reducible with one which is faithfully generated by a Borel action of $G * H$.*

*Proof.* As $E \sim_B E \times I(\mathbb{N})$, this follows from Proposition 2. $\square$

We are now ready for the main result of this section:

**Theorem 5.** *Suppose that $G$ and $H$ are non-trivial countable groups such that $G * H \not\cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Then every compressible equivalence relation is faithfully generated by a Borel action of $G * H$.*

*Proof.* We say that countable Borel equivalence relations $E$ and $F$ are *(Borel) isomorphic*, or $E \cong_B F$, if there is a Borel bijection $\pi : X \to Y$ such that $\forall x_1, x_2 \in X\ (x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2))$. By the proof of Lemma 4.4.1 of Becker-Kechris [1], a countable Borel equivalence relation $E$ is compressible if and only if $E \cong_B E \times I(\mathbb{N})$, so the theorem follows from Proposition 2. $\square$

We close by noting a much stronger fact in the hyperfinite case:

**Theorem 6.** *Suppose that $X$ is a Polish space and $E$ is countable Borel equivalence relation on $X$. Then the following are equivalent:*

1. *$E$ is freely generated by a Borel action of every countably infinite group.*
2. *$E$ is faithfully generated by a Borel action of every countably infinite group.*
3. *$E$ is compressible and hyperfinite.*

*Proof.* It is clear that (1) $\Rightarrow$ (2). To see (2) $\Rightarrow$ (3), note that $E$ must be aperiodic, since infinite groups cannot act faithfully on finite sets. Proposition 1 then implies that $E$ is hyperfinite. The proof of Proposition 4.14 of Kechris [6] implies that no aperiodic hyperfinite equivalence relation which carries an invariant probability measure is generated by a Borel action of every countable group. It follows that $E$ does not admit an invariant probability measure, thus the theorem of Nadkarni [10] implies that $E$ is compressible.

To see (3) $\Rightarrow$ (1), suppose that $E$ is a compressible and hyperfinite, and fix a countably infinite group $G$. We say that $E$ is *smooth* if it admits a Borel *transversal*, i.e., a set which intersects every $E$-class in exactly one point. As the case that $E$ is smooth is a straightforward consequence of the Lusin-Novikov uniformization theorem (see, for example, §18 of Kechris [5]), we can assume that $E$ is non-smooth. Let $X$ denote the free part of the action of $G$ on $2^G$. As $E_G^X$ is generically non-smooth, it follows from Theorem 12.1 (which is due to Hjorth-Kechris [4]) and Corollary 13.3 of Kechris-Miller [7], as well as the Dougherty-Jackson-Kechris [2] classification of hyperfinite equivalence relations, that there is a comeager, $E_G^X$-invariant Borel set $C \subseteq X$ such that $E \cong_B E_G^X | C$, and we obtain the desired action by pulling back through this isomorphism. $\square$

# 2 Selection

Let $[E]^{<\infty}$ denote the standard Borel space of all finite sets $S \subseteq X$ with the property that $\forall x_1, x_2 \in S\ (x_1 E x_2)$. We say that $\mathcal{B} \subseteq [E]^{<\infty}$ is *pairwise disjoint* if $\forall S, T \in \mathcal{B}\ (S \neq T \Rightarrow S \cap T = \emptyset)$. While the axiom of choice ensures the existence of maximal pairwise disjoint subsets of any given subset of $[E]^{<\infty}$, the following useful fact is perhaps a bit surprising:

**Proposition 7.** *Suppose that $X$ is a Polish space and $E$ is a countable Borel equivalence relation on $X$. Then every Borel subset of $[E]^{<\infty}$ has a maximal pairwise disjoint Borel subset.*

*Proof.* This is a rephrasing of Proposition 7.3 of Kechris-Miller [7]. $\square$

The *restriction* of $\mathcal{B} \subseteq [E]^{<\infty}$ to $B \subseteq X$ is given by $\mathcal{B}|B = \mathcal{B} \cap [E|B]^{<\infty}$. Although the following fact is essentially a rephrasing of Proposition 4.7 of Miller [9], it is sufficiently different that we include a proof here:

**Proposition 8.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic countable Borel equivalence relation on $X$, and $\mathcal{B}_0, \mathcal{B}_1, \ldots \subseteq [E]^{<\infty}$ are Borel. Then there is an $E$-invariant Borel set $B \subseteq X$ and pairwise disjoint Borel sets $B_0, B_1, \ldots \subseteq X$ such that:*

1. *$E|(X \setminus B)$ is compressible.*
2. *$\forall n \in \mathbb{N}\, \forall x \in B\ (\mathcal{B}_n|[x]_E \neq \emptyset \Rightarrow \mathcal{B}_n|(B_n \cap [x]_E) \neq \emptyset)$.*

*Proof.* Let $P(X)$ denote the standard Borel space of Borel probability measures on $X$. We say that such a measure $\mu$ is *$E$-invariant* if every element of $[E]$ is $\mu$-measure preserving, and we say that $\mu$ is *$E$-ergodic* if every $E$-invariant Borel set is $\mu$-null or $\mu$-conull. We use $\mathcal{I}_E$ to denote the set of all $E$-invariant probability measures, and we use $\mathcal{EI}_E$ to denote the set of such measures which are also $E$-ergodic. As we can clearly assume that $E$ is incompressible, it follows from Nadkarni [10] that $\mathcal{I}_E \neq \emptyset$. Fix a Farrell-Varadarajan-style ergodic decomposition $\pi : X \to \mathcal{EI}_E$ (see, for example, §3 of Kechris-Miller [7]).

By Proposition 7, we can assume that each of the sets $\mathcal{B}_n$ is pairwise disjoint. Define equivalence relations $E_n$ on $\mathcal{B}_n$ by setting

$$S E_n T \Leftrightarrow \exists x \in X\ (S \cup T \subseteq [x]_E).$$

Note that if $\mathcal{B} \subseteq \mathcal{B}_n$ is Borel and $E_n|\mathcal{B}$ is smooth, then $E|\bigcup \mathcal{B}$ is also smooth, thus $E|[\bigcup \mathcal{B}]_E$ is compressible. It follows that, after throwing out an $E$-invariant Borel set on which $E$ is compressible, we can assume that each of the equivalence relations $E_n$ is aperiodic. By Lemma 6.7 of Kechris-Miller [7], there are Borel $E_n$-complete sections $\mathcal{B}^n_0 \supseteq \mathcal{B}^n_1 \supseteq \cdots$ such that $\bigcap_{i \in \mathbb{N}} \mathcal{B}^n_i = \emptyset$.

For each $\mu \in \mathcal{EI}_E$, let $\mu_n$ be the (possibly trivial) measure on $\mathcal{B}_n$ given by

$$\mu_n(\mathcal{B}) = \mu(\bigcup \mathcal{B}).$$

While these measures need not be $E_n$-invariant, they are certainly *$E_n$-quasi-invariant*, i.e., the $E_n$-saturations of $\mu_n$-null sets are $\mu_n$-null. In particular, it follows that if $\mu_n(\mathcal{B}_n) > 0$, then $\mu_n(\mathcal{B}^n_i) > 0$, for all $i \in \mathbb{N}$.

Recursively define functions $k_n : \mathcal{EI}_E \to \mathbb{N}$ by letting $k_n(\mu)$ be the least natural number such that, for all natural numbers $i < n$,

$$\mu_i(\{S \in \mathcal{B}^i_{k_i(\mu)} : \exists T \in \mathcal{B}^n_{k_n(\mu)}\ (S \cap T \neq \emptyset)\}) < \mu_i(\mathcal{B}^i_{k_i(\mu)})/2.$$

Extend $\pi$ to $[E]^{<\infty}$ by setting $\pi(S) = \pi(x)$, for some (equivalently, all) $x \in S$, and for each $n \in \mathbb{N}$, define $\mathcal{A}_n \subseteq \mathcal{B}_n$ by

$$\mathcal{A}_n = \{S \in [E]^{<\infty} : S \in \mathcal{B}^n_{k_n(\pi(S))} \text{ and } \forall m > n\, \forall T \in \mathcal{B}^m_{k_m(\pi(S))}\ (S \cap T = \emptyset)\}.$$

Observe now that if $\mu \in \mathcal{EI}_E$ and $\mu(\mathcal{B}_n) > 0$, then $\mu(\mathcal{A}_n) > 0$ as well, so the set $\mathcal{C}_n = \mathcal{B}_n \setminus [\mathcal{A}_n]_{E_n}$ is $\mu_n$-null, thus the restriction of $E$ to the set $B = [\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{C}_n]_E$ admits no invariant, ergodic probability measure. The theorem of Nadkarni [10] then implies that $E|B$ is compressible, and it follows that the sets $B_n = \bigcup \mathcal{A}_n$ are as desired. □

# 3 Hyperfinite equivalence relations

In this section, we study the circumstances under which an aperiodic incompressible hyperfinite equivalence relation is faithfully generated by a Borel action of a free product of a given pair of countable groups. We begin by studying a weaker notion. We say that a $G$-action is *$E$-faithful* if $E^X_G \subseteq E$ and its restriction to each equivalence class of $E$ is faithful.

**Proposition 9.** *For every countable group $G$, the following are equivalent:*

1. *$G$ can be embedded into the measure-theoretic full group of $(E_0, \mu_0)$.*
2. *There is an $E_0$-faithful Borel action of $G$.*

*Proof.* To see (2) ⇒ (1), simply note that if $G$ acts $E_0$-faithfully on $X$, then the map which associates with each $g \in G$ the equivalence class of the function $x \mapsto g \cdot x$ is the desired embedding.

To see (1) ⇒ (2), suppose that $\pi : G \to [E_0]_{\mu_0}$ is an embedding, and for each $g \in G$, let $\varphi(g)$ be a Borel automorphism in the equivalence class of $\pi(g)$. Then the set

$$A = \{x \in X : \forall g, h \in G\ ([\varphi(gh)](x) = [\varphi(g)] \circ [\varphi(h)](x))\}$$

is of full measure. Let $G$ act on $A$ via $g \cdot x = [\varphi(g)](x)$, and observe that the set

$$B = \{x \in A : \forall g \in G \setminus \{1_G\}\, \exists y \in [x]_{E_0}\ (g \cdot y \neq y)\}$$

is also of full measure. As the action of $G$ on $B$ is $(E_0|B)$-faithful, it is enough to build an $E_0|(X \setminus B)$-faithful action of $G$. As $\mu_0$ is the unique $E_0$-invariant, $E_0$-ergodic probability measure, it follows that $E_0|(X \setminus B)$ does not admit an invariant probability measure. The theorem of Nadkarni [10] then implies that $E_0|(X \setminus B)$ is compressible. If $G$ is infinite, then Theorem 6 implies that

$E_0|(X \setminus B)$ is freely generated by a Borel action of $G$. If $G$ is finite, then Proposition 7.4 of Kechris-Miller [7] ensures that there is a Borel equivalence relation $F \subseteq E_0|(X \setminus B)$ whose classes are all of cardinality $|G|$. The Lusin-Novikov uniformization theorem implies that $F$ is freely generated by a Borel action of $G$, and any such action is necessarily $E_0|(X \setminus B)$-faithful. $\square$

We see next that the existence of $E$-faithful Borel actions is a notion that behaves nicely with respect to free products:

**Proposition 10.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic countable Borel equivalence relation on $X$, and $G$ and $H$ are countable groups. Then the following are equivalent:*

1. *There are $E$-faithful Borel actions of $G$ and $H$;*
2. *There is an $E$-faithful Borel action of $G * H$ with the property that, for every reduced $(G * H)$-word $w = g_k h_k \ldots g_1 h_1$ and every $x \in X$, there exists $y \in [x]_E$ such that the points $y, h_1 \cdot y, g_1 h_1 \cdot y, \ldots, g_k h_k \ldots g_1 h_1 \cdot y$ are pairwise distinct.*

*Proof.* It is enough to show (1) $\Rightarrow$ (2). By the proof of Theorem 5, it is enough to show that (2) holds off of an $E$-invariant Borel set on which $E$ is compressible.

For each $g \in G$, define $X_g \subseteq X$ by

$$X_g = \{x \in X : g \cdot x \neq x\},$$

and define $A_g \subseteq X$ by

$$A_g = \{x \in X : |X_g \cap [x]_E| < \aleph_0\}.$$

As the action of $G$ is $E$-faithful, it follows that $E|A_g$ is smooth. As $E$ is aperiodic, it follows that $E|[A_g]_E$ is compressible. By throwing out each of the sets $[A_g]_E$, we can therefore assume that for every $g \in G$ other than $1_G$, the set $X_g$ intersects each equivalence class of $E$ in an infinite set.

Similarly, we can assume that for every $h \in H$ other than $1_H$, the set

$$Y_h = \{y \in X : h \cdot y \neq y\}$$

intersects each equivalence class of $E$ in an infinite set.

We will assume also that both $G$ and $H$ are non-trivial, since otherwise the proposition trivializes.

Given a partial injection $\pi$ on $X$, $k \in G \cup H$, and $x \in X$, set

$$k^\pi \cdot x = \pi k \pi^{-1} \cdot x.$$

More generally, let $k_n^{\pi_n} k_{n-1}^{\pi_{n-1}} \cdots k_1^{\pi_1} \cdot x = k_n^{\pi_n} \cdot (k_{n-1}^{\pi_{n-1}} \cdot (\cdots (k_1^{\pi_1} \cdot x) \cdots))$.

Suppose that $w = g_k h_k \cdots g_1 h_1$ is a non-trivial reduced $(G * H)$-word. We say that a tuple $(S, x, \varphi, \psi)$ is a *$w$-witness* if it satisfies the following conditions:

1. $S \in [E]^{<\infty}$;

2. $\varphi$ and $\psi$ are permutations of $S$;

3. $x, h_1^\psi \cdot x, g_1^\varphi h_1^\psi \cdot x, \ldots, g_k^\varphi h_k^\psi \cdots g_1^\varphi h_1^\psi \cdot x$ are pairwise distinct elements of $S$.

Let $\mathcal{B}_w$ denote the Borel set of $S \in [E]^{<\infty}$ for which there exist $x \in S$ and permutations $\varphi$ and $\psi$ of $S$ such that $(S, x, \varphi, \psi)$ is a $w$-witness.

**Lemma 11.** *The set $\mathcal{B}_w$ covers $X$.*

*Proof.* Fix $x \in X$. We will recursively define pairwise distinct $y_0, x_1, y_1, \ldots, y_k \in [x]_E$, as well as finite partial injections $\varphi_0, \ldots, \varphi_k$ and $\psi_0, \ldots, \psi_k$, such that:

1. $\forall i \leq k$ $(y_i \notin \mathrm{range}(\psi_i))$.

2. $\forall i < k$ $(x_{i+1} = h_{i+1}^{\psi_{i+1}} \cdot y_i$ and $y_{i+1} = g_{i+1}^{\varphi_{i+1}} \cdot x_{i+1})$.

We begin by setting $y_0 = x$ and $\varphi_0 = \psi_0 = \emptyset$.

Suppose now that we have $y_0, x_1, y_1, \ldots, y_i$, as well as $\varphi_i$ and $\psi_i$, for some $i < k$. Since $[y_i]_E \cap Y_{h_{i+1}}$ is infinite, there exists

$$y_i' \in ([y_i]_E \cap Y_{h_{i+1}}) \setminus (\mathrm{dom}(\psi_i) \cup h_{i+1}^{-1}(\mathrm{dom}(\psi_i))),$$

and since $[y_i]_E$ is infinite, there exists

$$x_{i+1} \in [y_i]_E \setminus (\mathrm{range}(\psi_i) \cup \mathrm{range}(\varphi_i) \cup \{y_0, x_1, \ldots, y_i\}).$$

As $y_i', h_{i+1} \cdot y_i'$ are distinct points outside of $\mathrm{dom}(\psi_i)$, and $x_{i+1}, y_i$ are distinct points outside of $\mathrm{range}(\psi_i)$, we obtain a partial injection by setting

$$\psi_{i+1}(y) = \begin{cases} \psi_i(y) & \text{if } y \in \mathrm{dom}(\psi_i), \\ y_i & \text{if } y = y_i', \\ x_{i+1} & \text{if } y = h_{i+1} \cdot y_i'. \end{cases}$$

Similarly, since $[x_{i+1}]_E \cap X_{g_{i+1}}$ is infinite, there exists

$$x_{i+1}' \in ([x_{i+1}]_E \cap X_{g_{i+1}}) \setminus (\mathrm{dom}(\varphi_i) \cup g_{i+1}^{-1}(\mathrm{dom}(\varphi_i))),$$

and since $[x_{i+1}]_E$ is infinite, there exists

$$y_{i+1} \in [x_{i+1}]_E \setminus (\mathrm{range}(\varphi_i) \cup \mathrm{range}(\psi_{i+1}) \cup \{y_0, x_1, \ldots, y_i, x_{i+1}\}).$$

As $x_{i+1}', g_{i+1} \cdot x_{i+1}'$ are distinct points outside of $\mathrm{dom}(\varphi_i)$, and $x_{i+1}, y_{i+1}$ are distinct points outside of $\mathrm{range}(\varphi_i)$, we obtain a partial injection by setting

$$\varphi_{i+1}(x) = \begin{cases} \varphi_i(x) & \text{if } x \in \mathrm{dom}(\varphi_i), \\ x_{i+1} & \text{if } x = x_{i+1}', \\ y_{i+1} & \text{if } x = g_{i+1} \cdot x_{i+1}'. \end{cases}$$

This completes the recursive construction. Note that $y_{i+1} \notin \mathrm{range}(\psi_{i+1})$,

$$h_{i+1}^{\psi_{i+1}} \cdot y_i = \psi_{i+1} h_{i+1} \psi_{i+1}^{-1} \cdot y_i = x_{i+1},$$

and

$$g_{i+1}^{\varphi_{i+1}} \cdot x_{i+1} = \varphi_{i+1} g_{i+1} \varphi_{i+1}^{-1} \cdot x_{i+1} = y_{i+1}.$$

Let $S = \{y_0, x_1, y_1, \ldots, y_k\}$, fix extensions $\varphi$ and $\psi$ of $\varphi_k$ and $\psi_k$ to permutations of $S$, and observe that $(S, x, \varphi, \psi)$ is a $w$-witness. $\square$

Proposition 8 ensures that, after throwing away an $E$-invariant Borel set on which $E$ is compressible, there are pairwise disjoint Borel sets $B_w$, such that each $\mathcal{B}_w | B_w$ contains a subset of every equivalence class of $E$. By the Lusin-Novikov uniformization theorem, there is a Borel map $S \mapsto (x_S, \varphi_S, \psi_S)$ such that, for each $S \in \bigcup_w \mathcal{B}_w | B_w$, the tuple $(S, x_S, \varphi_S, \psi_S)$ is a $w$-witness. Fix $\varphi \in [E]$ and $\psi \in [E]$ which simultaneously extend each of the permutations $\varphi_S$ and $\psi_S$, respectively. Then the conjugates of the actions of $G$ and $H$ by $\varphi$ and $\psi$ yield the desired action of $G * H$. $\square$

We are now ready to connect the existence of $E$-faithful Borel actions with the existence of everywhere faithful Borel actions:

**Proposition 12.** *For non-trivial countable groups $G$ and $H$, the following are equivalent:*

1. *$G$ and $H$ can be embedded into the measure-theoretic full group of $(E_0, \mu_0)$.*
2. *$E_0$ is faithfully generated by a Borel action of $G * H$.*

*Proof.* Proposition 9 implies (2) $\Rightarrow$ (1), so it is enough to show (1) $\Rightarrow$ (2). By Proposition 9, there are $E_0$-faithful Borel actions of $G$ and $H$. By Proposition 10, we can fix an $E_0$-faithful Borel action of $G * H$ such that, for every reduced $(G * H)$-word $w = g_k h_k \ldots g_1 h_1$ and every $x \in X$, there exists $y \in [x]_E$ such that the points $y, h_1 \cdot y, g_1 h_1 \cdot y, \ldots, g_k h_k \ldots g_1 h_1 \cdot y$ are pairwise distinct.

For each reduced $(G*H)$-word $w = g_k h_k \ldots g_1 h_1$, let $\mathcal{B}_w$ denote the collection of sets $S \in [E]^{<\infty}$ which are made up of pairwise distinct points $x, h_1 \cdot x, g_1 h_1 \cdot x, \ldots, g_k h_k \ldots g_1 h_1 \cdot x, y_1, y_2, z_1, z_2$, where $y_1 E_G^X y_2$ and $z_1 E_H^X z_2$. By Proposition 8, after throwing out an $E$-invariant Borel set on which $E$ is compressible (which we are free to do by Theorem 5), there are pairwise disjoint Borel sets $B_w \subseteq X$ such that each $\mathcal{B}_w | B_w$ contains a subset of every $E$-class. Set $\mathcal{B} = \bigcup_w \mathcal{B}_w | B_w$, and let $\mathcal{E}$ denote the equivalence relation on $\mathcal{B}$ given by

$$S \mathcal{E} T \Leftrightarrow \exists x \in X \ (S \cup T \subseteq [x]_E).$$

Then $\mathcal{E} \sim_B E_0$, thus $\mathcal{E}$ is hyperfinite. As $\mathcal{E}$ is clearly aperiodic, it follows from Proposition 1 that $\mathcal{E}$ is freely generated by a Borel action of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. Let $a$ and $b$ denote the generators of $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

By the Lusin-Novikov uniformization theorem, there is a Borel map $S \mapsto (x^S, y_1^S, y_2^S, z_1^S, z_2^S)$ such that every $S \in \mathcal{B}$ is made up of the pairwise distinct points $x^S, h_1 \cdot x^S, g_1 h_1 \cdot x^S, \ldots, g_k h_k \ldots g_1 h_1 \cdot x^S, y_1^S, y_2^S, z_1^S, z_2^S$. Fix a Borel linear ordering $\leq$ of $\mathcal{B}$, define $\varphi \in [E]$ by

$$\varphi(x) = \begin{cases} y_1^S & \text{if } \exists S \in \mathcal{B} \ (x = x^S \text{ and } S < a \cdot S), \\ y_2^{a \cdot S} & \text{if } \exists S \in \mathcal{B} \ (x = x^S \text{ and } a \cdot S < S), \\ x & \text{otherwise,} \end{cases}$$

and similarly, define $\psi \in [E]$ by

$$\psi(x) = \begin{cases} z_1^S & \text{if } \exists w \, \exists S \in \mathcal{B}_w \ (x = w \cdot x^S \text{ and } S < b \cdot S), \\ z_2^{b \cdot S} & \text{if } \exists w \, \exists S \in \mathcal{B}_w \ (x = w \cdot x^S \text{ and } b \cdot S < S), \\ x & \text{otherwise.} \end{cases}$$

Now consider the conjugates of the actions of $G$ and $H$ by $\varphi^{-1}$ and $\psi^{-1}$, respectively. Let $B$ denote the set of $x \in X$ such that this new action of $G * H$ on $[x]_{G*H}$ is faithful. Then the set $B$ intersects every equivalence class of $E_0$, and as a consequence, the equivalence relation $F$ on $B$ generated by the new action is hyperfinite, incompressible, and faithfully generated by a Borel action of $G * H$. It follows from the Dougherty-Jackson-Kechris [2] classification of hyperfinite equivalence relations that $F$ is of the form $E_0 \times \Delta(Y)$, for some Polish space $Y$, and this implies that $E_0$ is faithfully generated by a Borel action of $G * H$. □

As a corollary, we obtain the main result of this section:

**Theorem 13.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic incompressible hyperfinite equivalence relation on $X$, and $G$ and $H$ are non-trivial countable groups. Then the following are equivalent:*

1. *$G$ and $H$ can be embedded into the measure-theoretic full group of $(E_0, \mu_0)$.*
2. *$E$ is faithfully generated by a Borel action of $G * H$.*

*Proof.* In light of Proposition 12, it is enough to check that $E$ is faithfully generated by a Borel action of $G * H$ if and only if $E_0$ is faithfully generated by a Borel action of $G * H$, and this follows from the Dougherty-Jackson-Kechris [2] classification of aperiodic hyperfinite equivalence relations. □

Of course, this theorem will become useful only when we have specified a reasonable collection of countable groups which can be embedded into the measure-theoretic full group of $(E_0, \mu_0)$.

**Proposition 14.** *Every amenable group can be embedded into the measure-theoretic full group of $(E_0, \mu_0)$.*

*Proof.* Suppose that $G$ is an amenable group. If $G$ is finite, then Proposition 7.4 of Kechris-Miller [7] ensures that there is a Borel equivalence relation $F \subseteq E_0$ whose classes are all of cardinality $|G|$, thus $F$ is freely generated by a Borel action of $G$. As any such action is necessarily $E_0$-faithful, it follows that $G$ can be embedded into $[E_0]_{\mu_0}$.

If $G$ is infinite, then let $G$ act on $X = 2^G$ via the shift, and let $\mu$ denote the usual product measure on $2^G$. The theorem of Ornstein-Weiss [11] ensures that there is an $E^X_G$-invariant Borel set $B \subseteq X$ of full measure such that $E^X_G | B$ is hyperfinite. The Dougherty-Jackson-Kechris [2] classification of aperiodic hyperfinite equivalence relations then implies that $E_0$ is freely generated by a Borel action of $G$, and the desired result follows from Proposition 9. □

Recall that if $P$ is a property of groups, then a group $G$ is said to be *residually* $P$ if, for every $g \neq 1_G$ in $G$, there is a group $H$ with property $P$ and an epimorphism $\varphi : G \to H$ such that $\varphi(g) \neq 1_H$. We prove next a descriptive analog of Proposition 4.13 of Kechris [6]:

**Proposition 15.** *Suppose that $G$ is a countable group which is residually contained in the measure-theoretic full group of $(E_0, \mu_0)$. Then $G$ embeds into the measure-theoretic full group of $(E_0, \mu_0)$.*

*Proof.* Fix an enumeration $g_0, g_1, \ldots$ of $G$, and for each $n \in \mathbb{N}$, fix a homomorphism $\varphi_n : G \to [E_0]_{\mu_0}$ such that $\varphi_n(g_n) \neq \mathrm{id}$. Set $H_n = \varphi_n(G)$ and $X_n = \mathcal{N}_{0^n 1}$. By Proposition 9, there are $(E_0 | X_n)$-faithful Borel actions of $H_n$. By pulling back the action of $H_n$ on $X_n$ through $\varphi_n$ and insisting that $G$ acts trivially on $0^\infty$, we obtain an $E_0$-faithful Borel action of $G$, and it follows from Proposition 9 that $G$ embeds into the measure-theoretic full group of $(E_0, \mu_0)$. □

As a corollary, we obtain the following:

**Theorem 16.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic hyperfinite equivalence relation on $X$, and $G$ and $H$ are non-trivial residually amenable groups. Then $E$ is faithfully generated by a Borel action of $G * H$.*

*Proof.* This follows from Theorem 13 and Propositions 14 and 15. □

# 4 The general case

In this section, we show that every aperiodic countable Borel equivalence relation is faithfully generated by a Borel action of every free product of infinitely many non-trivial countable groups. We note first the following fact:

**Proposition 17.** *Suppose that $X$ is a Polish space, $E$ is an aperiodic countable Borel equivalence relation on $X$, $G$ and $H$ are countable groups equipped with everywhere faithful Borel actions on $X$, and $E^X_G \vee E^X_H = E$. Then there is an $E$-invariant Borel set $B \subseteq X$ and conjugates of the actions of $G$ and $H$ by elements of the full group of $E$ such that:*

1. *$E | (X \setminus B)$ is compressible.*
2. *The corresponding action of $G * H$ on $B$ faithfully generates $E | B$.*

*Proof.* For each $g \in G$, define $X_g \subseteq X$ by

$$X_g = \{x \in X : g \cdot x \neq x\},$$

and define $A_g \subseteq X$ by

$$A_g = \{x \in X : |X_g \cap [x]_G| < \aleph_0\}.$$

As the action of $G$ is everywhere faithful, it follows that $E^X_G | A_g$ is smooth.

If $G$ is infinite, then the fact that the action of $G$ is everywhere faithful also ensures that $E^X_G$ is aperiodic. This easily implies that $E^X_G | A_g$ is compressible, thus $E|[A_g]_E$ is compressible. By throwing out each of the sets $[A_g]_E$, we can therefore assume that if $G$ is infinite, then for every $g \in G$ other than $1_G$, the set $X_g$ intersects each $G$-orbit in an infinite set.

Similarly, we can assume that if $H$ is infinite, then for every $h \in H$ other than $1_H$, the set

$$Y_h = \{y \in X : h \cdot y \neq y\}$$

intersects each $H$-orbit in an infinite set.

We will assume also that both $G$ and $H$ are non-trivial, since otherwise the proposition trivializes.

Suppose that $w = g_k h_k \cdots g_1 h_1$ is a non-trivial reduced $(G * H)$-word. We say that a tuple $(S, x, \varphi, \psi)$ is a *$w$-witness* if it satisfies the following conditions:

1. $S \in [E]^{<\infty}$;
2. $\varphi$ and $\psi$ are permutations of $S$;
3. $\mathrm{graph}(\varphi) \subseteq E^X_G$ and $\mathrm{graph}(\psi) \subseteq E^X_H$;
4. $x, h_1^\psi \cdot x, g_1^\varphi h_1^\psi \cdot x, \ldots, g_k^\varphi h_k^\psi \cdots g_1^\varphi h_1^\psi \cdot x \in S$;
5. $x \neq g_k^\varphi h_k^\psi \cdots g_1^\varphi h_1^\psi \cdot x$.

Let $\mathcal{B}_w$ denote the Borel set of $S \in [E]^{<\infty}$ for which there exist $x \in S$ and permutations $\varphi$ and $\psi$ of $S$ such that $(S, x, \varphi, \psi)$ is a $w$-witness.

**Lemma 18.** *The set $\mathcal{B}_w$ covers $X$.*

*Proof.* To see that a point $x \in X$ is contained in some element of $\mathcal{B}_w$, it is enough to find $y_0 \in [x]_E$ and finite partial injections $\varphi$ and $\psi$ of $[x]_E$, whose graphs are contained in $E^X_G$ and $E^X_H$, respectively, such that $g_k^\varphi h_k^\psi \cdots g_1^\varphi h_1^\psi \cdot y_0$ is defined and distinct from $y_0$. The exact manner in which we accomplish this depends upon whether $G$ and $H$ are infinite.

We handle first the case that both $G$ and $H$ are infinite. We recursively define $y_0, y_1, \ldots, y_k \in [x]_E$, as well as finite partial injections $\varphi_0, \ldots, \varphi_k$ and $\psi_0, \ldots, \psi_k$, such that:

1. $\forall i \leq k \ (y_i \notin \mathrm{range}(\psi_i))$.

2. $\forall i < k \ (g_{i+1}^{\varphi_{i+1}} h_{i+1}^{\psi_{i+1}} \cdot y_i = y_{i+1})$.

We begin by setting $y_0 = x$ and $\varphi_0 = \psi_0 = \emptyset$.

Suppose now that we have $y_0, y_1, \ldots, y_i$, as well as $\varphi_i$ and $\psi_i$, for some $i < k$. Since $[y_i]_H \cap Y_{h_{i+1}}$ is infinite, there exists

$$y_i' \in ([y_i]_H \cap Y_{h_{i+1}}) \setminus (\mathrm{dom}(\psi_i) \cup h_{i+1}^{-1}(\mathrm{dom}(\psi_i))),$$

and since $[y_i]_H$ is infinite, there exists

$$x_{i+1} \in [y_i]_H \setminus (\mathrm{range}(\psi_i) \cup \mathrm{range}(\varphi_i) \cup \{y_i\}).$$

As $y_i', h_{i+1} \cdot y_i'$ are distinct points outside of $\mathrm{dom}(\psi_i)$, and $x_{i+1}, y_i$ are distinct points outside of $\mathrm{range}(\psi_i)$, we obtain a partial injection by setting

$$\psi_{i+1}(y) = \begin{cases} \psi_i(y) & \text{if } y \in \mathrm{dom}(\psi_i), \\ y_i & \text{if } y = y_i', \\ x_{i+1} & \text{if } y = h_{i+1} \cdot y_i'. \end{cases}$$

Similarly, since $[x_{i+1}]_G \cap X_{g_{i+1}}$ is infinite, there exists

$$x_{i+1}' \in ([x_{i+1}]_G \cap X_{g_{i+1}}) \setminus (\mathrm{dom}(\varphi_i) \cup g_{i+1}^{-1}(\mathrm{dom}(\varphi_i))),$$

and since $[x_{i+1}]_G$ is infinite, there exists

$$y_{i+1} \in [x_{i+1}]_G \setminus (\mathrm{range}(\varphi_i) \cup \mathrm{range}(\psi_{i+1}) \cup \{y_0, x_{i+1}\}).$$

As $x_{i+1}', g_{i+1} \cdot x_{i+1}'$ are distinct points outside of $\mathrm{dom}(\varphi_i)$, and $x_{i+1}, y_{i+1}$ are distinct points outside of $\mathrm{range}(\varphi_i)$, we obtain a partial injection by setting

$$\varphi_{i+1}(x) = \begin{cases} \varphi_i(x) & \text{if } x \in \mathrm{dom}(\varphi_i), \\ x_{i+1} & \text{if } x = x_{i+1}', \\ y_{i+1} & \text{if } x = g_{i+1} \cdot x_{i+1}'. \end{cases}$$

This completes the construction. Note that $y_{i+1} \notin \mathrm{range}(\psi_{i+1})$ and

$$g_{i+1}^{\varphi_{i+1}} h_{i+1}^{\psi_{i+1}} \cdot y_i = \varphi_{i+1} g_{i+1} \varphi_{i+1}^{-1} \psi_{i+1} h_{i+1} \psi_{i+1}^{-1} \cdot y_i = y_{i+1}.$$

Set $\varphi = \varphi_k$ and $\psi = \psi_k$, and observe that $y_0 \neq y_k = g_k^{\varphi} h_k^{\psi} \cdots g_1^{\varphi} h_1^{\psi} \cdot y_0$.

We handle next the case that exactly one of $G$ and $H$ are infinite. By reversing the roles of $G$ and $H$ if necessary, we can assume that $G$ is finite and $H$ is infinite. We recursively define $y_0, y_1, \ldots, y_k \in [x]_E$, as well as finite partial injections $\varphi_0, \ldots, \varphi_k$ and $\psi_0, \ldots, \psi_k$, such that:

1. $\forall i \leq k \ (y_i \notin \mathrm{range}(\psi_i))$.
2. $\forall i < k \ (g_{i+1}^{\varphi_{i+1}} h_{i+1}^{\psi_{i+1}} \cdot y_i = y_{i+1})$.

We begin by setting $y_0 = x$ and $\varphi_0 = \psi_0 = \emptyset$.

Suppose now that we have $y_0, y_1, \ldots, y_i$, as well as $\varphi_i$ and $\psi_i$, for some $i < k$. Since $[y_i]_H \cap Y_{h_{i+1}}$ is infinite, there exists

$$y_i' \in ([y_i]_H \cap Y_{h_{i+1}}) \setminus (\mathrm{dom}(\psi_i) \cup h_{i+1}^{-1}(\mathrm{dom}(\psi_i))),$$

and since $[y_i]_H$ is infinite and $G$ is finite, there exists

$$x_{i+1} \in [y_i]_H \setminus [\mathrm{dom}(\varphi_i) \cup \mathrm{range}(\varphi_i) \cup \mathrm{range}(\psi_i) \cup \{y_0, y_i\}]_G.$$

As $y_i', h_{i+1} \cdot y_i'$ are distinct points outside of $\mathrm{dom}(\psi_i)$, and $x_{i+1}, y_i$ are distinct points outside of $\mathrm{range}(\psi_i)$, we obtain a partial injection by setting

$$\psi_{i+1}(y) = \begin{cases} \psi_i(y) & \text{if } y \in \mathrm{dom}(\psi_i), \\ y_i & \text{if } y = y_i', \\ x_{i+1} & \text{if } y = h_{i+1} \cdot y_i'. \end{cases}$$

Fix $x_{i+1}' \in [x_{i+1}]_G \cap X_{g_{i+1}}$ and $y_{i+1} \in [x_{i+1}]_G \setminus \{x_{i+1}\}$. As $x_{i+1}', g_{i+1} \cdot x_{i+1}'$ are distinct points outside of $\mathrm{dom}(\varphi_i)$, and $x_{i+1}, y_{i+1}$ are distinct points outside of $\mathrm{range}(\varphi_i)$, we obtain a partial injection by setting

$$\varphi_{i+1}(x) = \begin{cases} \varphi_i(x) & \text{if } x \in \mathrm{dom}(\varphi_i), \\ x_{i+1} & \text{if } x = x_{i+1}', \\ y_{i+1} & \text{if } x = g_{i+1} \cdot x_{i+1}'. \end{cases}$$

This completes the recursive construction. Note that $y_{i+1} \notin \mathrm{range}(\psi_{i+1})$ and

$$g_{i+1}^{\varphi_{i+1}} h_{i+1}^{\psi_{i+1}} \cdot y_i = \varphi_{i+1} g_{i+1} \varphi_{i+1}^{-1} \psi_{i+1} h_{i+1} \psi_{i+1}^{-1} \cdot y_i = y_{i+1}.$$

Set $\varphi = \varphi_k$ and $\psi = \psi_k$, and observe that $y_0 \neq y_k = g_k^\varphi h_k^\psi \cdots g_1^\varphi h_1^\psi \cdot y_0$.

It remains to handle the case that both $G$ and $H$ are finite. We say that there is a *$(G * H)$-path from $x$ to $y$ that avoids $S$* if there exist $g_1', \ldots, g_n' \in G$ and $h_1', \ldots, h_n' \in H$ such that $g_n' h_n' \cdots g_1' h_1' \cdot x = y$, and none of the points $h_1' \cdot x, g_1' h_1' \cdot x, \ldots, g_n' h_n' \cdots g_1' h_1' \cdot x$ are in $S$. Recursively define $y_0, x_1, \ldots, y_k \in [x]_E$ such that:

1. For all $i \leq k$, there are $(G * H)$-paths from $y_i$ to infinitely many points of $[x]_E$ which avoid $\bigcup_{1 \leq i \leq k} [x_i]_H \cup \bigcup_{i \leq k} [y_i]_G$;
2. For all $i \leq k$, there are $(G * H)$-paths from $x_i$ to infinitely many points of $[x]_E$ which avoid $\bigcup_{1 \leq i \leq k} [x_i]_H \cup \bigcup_{i < k} [y_i]_G$.

For each $i < k$, fix $y_i' \in [y_i]_H \cap Y_{h_{i+1}}$ and $x_i' \in [x_i]_G \cap X_{g_{i+1}}$. Set

$$\varphi(x) = \begin{cases} x_i & \text{if } x = x_i', \\ y_i & \text{if } x = g_{i+1} \cdot x_i', \end{cases}$$

and

$$\psi(y) = \begin{cases} y_i & \text{if } y = y_i', \\ x_{i+1} & \text{if } y = h_{i+1} \cdot y_i'. \end{cases}$$

It is clear that $\varphi$ and $\psi$ are as desired. $\square$

Proposition 8 ensures that, after throwing away an $E$-invariant Borel set on which $E$ is compressible, there are pairwise disjoint Borel sets $B_w$, such that each $\mathcal{B}_w|B_w$ contains a subset of every equivalence class of $E$. By the Lusin-Novikov uniformization theorem, there is a Borel map $S \mapsto (x_S, \varphi_S, \psi_S)$ which assigns to each $S$ in some $\mathcal{B}_w|B_w$ a triple $(x_S, \varphi_S, \psi_S)$ such that $(S, x_S, \varphi_S, \psi_S)$ is a $w$-witness. Fix $\varphi \in [E_G^X]$ and $\psi \in [E_H^X]$ which simultaneously extend all of these permutations. Then the conjugates of the actions of $G$ and $H$ on $X$ by $\varphi$ and $\psi$ still generate the same equivalence relations, and the corresponding action of $G * H$ on $X$ is everywhere faithful. $\square$

**Remark 19.** *Proposition 17 implies its strengthening in which we only conjugate the action of $H$ by an element of the full group of $E$. For if $\varphi$ and $\psi$ witness Proposition 17, then so too do* id *and* $\varphi^{-1}\psi$.

We are now ready for the main theorem of this section:

**Theorem 20.** *Suppose that $G_0, G_1, \ldots$ are non-trivial countable groups. Then the following are equivalent:*

1. *Every aperiodic countable Borel equivalence relation is faithfully generated by a Borel action of $*_{n\in\mathbb{N}} G_n$;*

2. *Each $G_n$ embeds into the measure-theoretic full group of $(E_0, \mu_0)$.*

*Proof.* It is enough to show (2) $\Rightarrow$ (1). Rewrite the groups as $G_0, H_0, G_1, H_1, \ldots$, and fix aperiodic hyperfinite equivalence relations $F_0, F_1, \ldots \subseteq E$ such that $E = \bigcup_{n\in\mathbb{N}} F_n$. Theorem 13 implies that $F_n$ is faithfully generated by a Borel action of $G_n * H_n$. By repeated application of Proposition 17 (and Remark 19), we can find $\mathrm{id} = \pi_0, \pi_1, \ldots \in [E]$ such that, for each $n \in \mathbb{N}$, the action of $(G_0 * H_0) * \cdots * (G_n * H_n)$ obtained by conjugating the action of $G_i * H_i$ by $\pi_i$, faithfully generates $F_0 \vee \cdots \vee F_n$. It follows that the action of $*_{n\in\mathbb{N}} G_n * H_n$, obtained by conjugating the action of $G_i * H_i$ by $\pi_i$, faithfully generates $E$. $\square$

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