The axiom of dependent choice and regular open algebras of Baire spaces

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Abstract

We show that the axiom of dependent choice holds if and only if every Boolean algebra admits a complete embedding into the regular open algebra of a Baire space.

The Stone representation theorem implies that every Boolean algebra is isomorphic to the clopen algebra of a compact, Hausdorff, zero-dimensional topological space. While this result is equivalent to the prime ideal theorem, no choice is required to show the weaker fact that every Boolean algebra admits a complete embedding into the regular open algebra of a topological space, as can be seen by considering the usual completion of the Boolean algebra in question. Here we describe the fragment of choice necessary to obtain a useful intermediate fact:

Theorem. *The following are equivalent:*

1. *For every Boolean algebra $\mathfrak{A}$, there is a Baire space (X, τ) and a complete embedding of $\mathfrak{A}$ into the regular open algebra of (X, τ).*
2. *The axiom of dependent choice.*
3. *For every topological space (X, τ), there is a Baire space $(\hat{X}, \hat{\tau})$ such that the regular open algebras of (X, τ) and $(\hat{X}, \hat{\tau})$ are isomorphic.*

Proof. To see (3) $\Rightarrow$ (1), suppose that $\mathfrak{A}$ is a Boolean algebra, fix a topological space (X, τ) for which there is a complete embedding of $\mathfrak{A}$ into the regular open algebra of (X, τ) (see, for example, the proof of Lemma 17.2 of [1]), and observe that (3) yields a Baire space $(\hat{X}, \hat{\tau})$ for which there is a complete embedding of $\mathfrak{A}$ into the regular open algebra of $(\hat{X}, \hat{\tau})$.

To see (1) $\Rightarrow$ (2), suppose that R is a binary relation on Y whose vertical sections are non-empty. Set $P_n = \{\langle y_i \rangle \in Y^n : \forall i < n - 1 \ (y_i R y_{i+1})\}$, and let $\leq$ denote extension on $P = \bigcup_{n \in \mathbb{N}} P_n$. Fix a Boolean algebra $\mathfrak{A}$ for which there is a complete embedding $\phi : P \to \mathfrak{A}$ (see, for example, the proof of Lemma 17.2 of [1]). By (1), there is a Baire space (X, τ) and a complete embedding ψ of $\mathfrak{A}$ into the regular open algebra of (X, τ). Set $\pi = \psi \circ \phi$, and observe that for each

$n \in \mathbb{N}$, the open set $U_n = \pi[P_n]$ is dense in $\pi(\emptyset)$, thus there exists $x \in \bigcap_{n\in\mathbb{N}} U_n$. For each $n \in \mathbb{N}$, let p_n denote the unique element of P_n such that $x \in \pi(p_n)$, set $y = \lim_{n\to\infty} p_n$, and observe that $y_n R y_{n+1}$, for all $n \in \mathbb{N}$.

To see (2) $\Rightarrow$ (3), suppose that (X, τ) is a topological space, let $\hat{\tau}$ denote the topology on $X^{\mathbb{N}}$ which is generated by the sets of the form

$$\hat{U} = \{\langle x_n\rangle \in X^{\mathbb{N}} : \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in U)\},$$

for $U \in \tau$, and define $\pi : \tau \to \hat{\tau}$ by $\pi(U) = \hat{U}$.

Lemma 1. *Suppose that $U, V \in \tau$ and $\langle U_i\rangle \in \tau^I$.*

1. $\pi(U \cap V) = \pi(U) \cap \pi(V)$.
2. $\pi(\mathrm{int}_\tau(X \setminus U)) = \mathrm{int}_{\hat{\tau}}(\hat{X} \setminus \pi(U))$.
3. *U is τ-dense in $V \Leftrightarrow \pi(U)$ is $\hat{\tau}$-dense in $\pi(V)$.*
4. $\pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\pi(U)))$.
5. *$\bigcup_{i\in I} \pi(U_i)$ is $\hat{\tau}$-dense in $\pi(\bigcup_{i\in I} U_i)$.*
6. $\pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(\bigcup_{i\in I} U_i))) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\bigcup_{i\in I} \pi(U_i)))$.

Proof. To see (1), note that

$$\begin{aligned}\pi(U \cap V) = &\ \{\langle x_n\rangle \in \hat{X} : \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in U \cap V)\}\\ = &\ \{\langle x_n\rangle \in \hat{X} : \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in U)\} \cap\\ &\ \{\langle x_n\rangle \in \hat{X} : \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in V)\}\\ = &\ \pi(U) \cap \pi(V).\end{aligned}$$

To see (2), observe that (1) implies that

$$\begin{aligned}\pi(\mathrm{int}_\tau(X \setminus U)) = &\ \{\langle x_n\rangle \in \hat{X} : \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in \mathrm{int}_\tau(X \setminus U))\}\\ = &\ \{\langle x_n\rangle \in \hat{X} : \exists V \in \tau\ (\pi(U \cap V) = \emptyset \text{ and}\\ &\qquad \exists n \in \mathbb{N} \forall m \geq n\ (x_m \in V))\}\\ = &\ \{\langle x_n\rangle \in \hat{X} : \exists V \in \tau\ (\pi(U) \cap \pi(V) = \emptyset \text{ and}\\ &\qquad \langle x_n\rangle \in \pi(V))\}\\ = &\ \mathrm{int}_{\hat{\tau}}(\hat{X} \setminus \pi(U)).\end{aligned}$$

To see (3), observe that (1) implies that

$$\begin{aligned}U \text{ is } \tau\text{-dense in } V &\Leftrightarrow \forall W \in \tau\ (V \cap W \neq \emptyset \Rightarrow U \cap W \neq \emptyset)\\ &\Leftrightarrow \forall W \in \tau\ (\pi(V \cap W) \neq \emptyset \Rightarrow \pi(U \cap W) \neq \emptyset)\\ &\Leftrightarrow \forall W \in \tau\ (\pi(V) \cap \pi(W) \neq \emptyset \Rightarrow \pi(U) \cap \pi(W) \neq \emptyset)\\ &\Leftrightarrow \pi(U) \text{ is } \hat{\tau}\text{-dense in } \pi(V).\end{aligned}$$

To see (4), observe that (3) implies that

$$\begin{aligned}\pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))) &= \{\langle x_n\rangle \in \hat{X} : \exists n \in \mathbb{N}\forall m \geq n\ (x_m \in \mathrm{int}_\tau(\mathrm{cl}_\tau(U)))\}\\ &= \{\langle x_n\rangle \in \hat{X} : \exists V \in \tau\ (U \text{ is } \tau\text{-dense in } V \text{ and}\\ &\qquad\qquad \exists n \in \mathbb{N}\forall m \geq n\ (x_m \in V))\}\\ &= \{\langle x_n\rangle \in \hat{X} : \exists V \in \tau\ (\pi(U) \text{ is } \hat{\tau}\text{-dense in } \pi(V) \text{ and}\\ &\qquad\qquad \langle x_n\rangle \in \pi(V))\}\\ &= \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\pi(U))).\end{aligned}$$

To see (5), we must show that if $\pi(U) \cap \pi(\bigcup_{i\in I} U_i)$ is non-empty, then so too is $\pi(U) \cap \bigcup_{i\in I} \pi(U_i)$. Towards this end, note first that if $\pi(U) \cap \pi(\bigcup_{i\in I} U_i)$ is non-empty, then so too is $U \cap \bigcup_{i\in I} U_i$. Fix $i \in I$ such that $U \cap U_i \neq \emptyset$, as well as $x \in U \cap U_i$, and observe that the sequence with constant value x is in $\pi(U) \cap \pi(U_i)$, thus $\pi(U) \cap \bigcup_{i\in I} \pi(U_i)$ is non-empty. To see (6), observe that (4) implies that $\pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(\bigcup_{i\in I} U_i))) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\pi(\bigcup_{i\in I} U_i)))$, and (5) ensures that $\mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\pi(\bigcup_{i\in I} U_i))) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\bigcup_{i\in I} \pi(U_i)))$. □

As a corollary, we obtain the following:

Lemma 2. *The regular open algebras of (X, τ) and $(\hat{X}, \hat{\tau})$ are isomorphic.*

Proof. We will show that π induces the desired isomorphism. To see that π sends τ-regular open sets to $\hat{\tau}$-regular open sets, simply observe that if U is τ-regular open, then

$$\pi(U) = \pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(U))) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\pi(U))),$$

by (4) of Lemma 1. To see that every $\hat{\tau}$-regular open set is the image of a τ-regular open set, note that if $\bigcup_{i\in I} \pi(U_i)$ is $\hat{\tau}$-regular open, then

$$\textstyle\bigcup_{i\in I} \pi(U_i) = \mathrm{int}_{\hat{\tau}}(\mathrm{cl}_{\hat{\tau}}(\bigcup_{i\in I} \pi(U_i))) = \pi(\mathrm{int}_\tau(\mathrm{cl}_\tau(\bigcup_{i\in I} U_i))),$$

by (6) of Lemma 1. It is clear that π is injective and $\pi(\emptyset) = \emptyset$, thus the desired result follows from (2) and (6) of Lemma 1. □

While not necessary for our proof, it seems worth noting that if $\phi : X \to \hat{X}$ is the function which sends x to the sequence with constant value x, then $\phi[U]$ is dense in $\pi(U)$, for all $U \in \tau$. In particular, it follows that if U is τ-regular open, then $\pi(U) = \mathrm{int}_\tau(\mathrm{cl}_\tau(\phi[U]))$ is the smallest $\hat{\tau}$-regular open set which contains the natural copy of U sitting within $\hat{X}$.

Lemma 3. *Suppose that the axiom of dependent choice holds. Then $(\hat{X}, \hat{\tau})$ is a Baire space.*

Proof. We must show that if $U \in \tau$ and $\langle V_n\rangle \in \hat{\tau}^{\mathbb{N}}$ is a decreasing sequence of $\hat{\tau}$-dense subsets of $\hat{X}$, then $\hat{U} \cap \bigcap_{n\in\mathbb{N}} V_n \neq \emptyset$. Towards this end, appeal to the axiom of dependent choice to find a decreasing sequence $\langle U_n\rangle \in \tau^{\mathbb{N}}$ of non-empty open subsets of U such that $\hat{U}_n \subseteq V_n$, for all $n \in \mathbb{N}$. By the countable axiom of choice, there is a sequence $\langle x_n \rangle \in \prod_{n\in\mathbb{N}} U_n$, and it follows that $\langle x_n \rangle \in \hat{U} \cap \bigcap_{n\in\mathbb{N}} \hat{U}_n \subseteq \hat{U} \cap \bigcap_{n\in\mathbb{N}} V_n$. $\square$

Lemmas 2 and 3 clearly yield (2) $\Rightarrow$ (3). $\square$

As a final remark, we note that the same idea can be used to show that the axiom of dependent choice holds if and only if every forcing notion is equivalent to the Baire category forcing associated with a Baire space.

References

[1] T. Jech. *Set theory*, *Perspectives in Mathematical Logic*, second edition. Springer-Verlag, Berlin (1997)