DENSITY OF TOPOLOGICAL FULL GROUPS

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Abstract. At the request of Tsankov, we give a somewhat different proof of Medynets's result that topological full groups associated with countable groups of homeomorphisms of zero-dimensional Polish spaces are uniformly dense in their measure-theoretic counterparts.

We use $[\Gamma]_c$, $[\Gamma]_\mu$, and $[\Gamma]$ to denote the topological, measure-theoretic, and Borel full groups, respectively. Recall that $T : X \to X$ is *periodic* if $\forall x \in X \ (|[x]_T| < \aleph_0)$.

Theorem 1 (Medynets). *Suppose that X is a zero-dimensional Polish space, μ is a Borel probability measure on X, and Γ is a countable group which acts on X by homeomorphisms. Then the periodic homeomorphisms in $[\Gamma]_c$ are uniformly dense in $[\Gamma]_\mu$.*

Proof. Suppose that $T \in [\Gamma]$ and $\epsilon > 0$. We must find $S \in [\Gamma]_c$ such that $\mu(\Delta(S,T)) < \epsilon$, where $\Delta(S,T) = \{x \in X : S(x) \neq T(x)\}$. Set $\mathrm{Per}(T) = \{x \in X : |[x]_T| < \aleph_0\}$ and $\mathrm{Aper}(T) = X \setminus \mathrm{Per}(T)$. Define also

$$\mathrm{Per}_{\leq n}(T) = \{x \in X : |[x]_T| \leq n\} \text{ and } \mathrm{Per}_{>n}(T) = \{x \in X : n < |[x]_T| < \aleph_0\}.$$

Fix $N \in \mathbb{N}$ sufficiently large that $\mu(\mathrm{Per}_{>N}(T)) < \epsilon/8$. A set $B \subseteq X$ is a *complete section* if it intersects every orbit of T. By a standard argument, there is a Borel complete section $B \subseteq X$ and $N' \geq N$ such that:

1. $B \cap \mathrm{Per}_{\leq N}(T)$ is a transversal of $T|\mathrm{Per}_{\leq N}(T)$.
2. $\mu(B \cap \mathrm{Aper}(T)) < \epsilon/8$.
3. $\forall x \in \mathrm{Per}_{>N}(T) \cup \mathrm{Aper}(T) \, \exists 1 \leq n \leq N' \ (T^n(x) \in B)$.

Let F be the *gap equivalence relation* associated with B, i.e., define $R \subseteq X \times X$ by

$$R = \{(x, T^i(x)) : \forall j < i \ (T^j(x) \notin B)\},$$

and set $F = R \cup R^{-1}$.

Lemma 2. *For every Borel set $A \subseteq X$ and $\delta > 0$, there is a clopen set $C \subseteq X$ such that $\mu([A \Delta C]_F) < \delta$.*

Proof. Set $\nu = T^{-(N'-1)}{}_*\mu + \cdots + T^{N'-1}{}_*\mu$. As every finite measure on a Polish space is regular, there is a clopen set $C \subseteq X$ such that $\nu(A \Delta C) < \delta$. Noting that $\forall x \in X \ ([x]_F \subseteq \{T^n(x)\}_{-N'<n<N'})$, it follows that

$$\begin{aligned} \mu([A\Delta C]_F) &\leq \mu(T^{-(N'-1)}(A\Delta C)) + \cdots + \mu(T^{N'-1}(A\Delta C)) \\ &\leq \nu(A\Delta C) \\ &< \delta, \end{aligned}$$

which completes the proof of the lemma. □

By Lemma 2, there is a clopen set $C \subseteq X$ such that $\mu([B \Delta C]_F) < \epsilon/4$. Condition (2) then implies that

$$\begin{aligned} \mu(C \setminus \mathrm{Per}_{\leq N'}(T)) &= \mu(C \cap \mathrm{Per}_{>N'}(T)) + \mu(C \cap \mathrm{Aper}(T)) \\ &\leq \mu(\mathrm{Per}_{>N'}(T)) + \mu(B \cap \mathrm{Aper}(T)) + \mu(C \setminus B) \\ &< \epsilon/8 + \epsilon/8 + \epsilon/4 \\ &= \epsilon/2. \end{aligned}$$

Fix an enumeration $\gamma_1, \gamma_2, \ldots$ of Γ and partitions $X_1^i, X_2^i, \ldots$ of X, for $i = \pm 1$, such that $\forall n \in \mathbb{Z}^+ \ (T^i | X_n^i = \gamma_n | X_n^i)$. Fix $M \in \mathbb{N}$ such that the set

$$Y = \{x \in X \setminus (B \Delta C) : [x]_F \subseteq (X_1^{-1} \cup \cdots \cup X_M^{-1}) \cap (X_1^1 \cup \cdots \cup X_M^1)\}$$

is of μ-measure $> 1 - \epsilon/4$. For $i = \pm 1$ and $1 \leq m \leq M$, apply Lemma 2 so as to obtain a clopen set $C_m^i \subseteq X$ such that $\mu([X_m^i \Delta C_m^i]_F) < \epsilon/8M$. For $i = \pm 1$, define $Y_i \subseteq X$ by

$$Y_i = [X_1^i \Delta C_1^i]_F \cup \cdots \cup [X_M^i \Delta C_M^i]_F,$$

and observe that the set $Z = Y \setminus (Y_{-1} \cup Y_1)$ is of μ-measure at least $1 - \epsilon/2$.

For $i = \pm 1$, set $D_i = C_1^i \cup \cdots \cup C_M^i$ and define $T_i : D_i \to X$ by $T_i(x) = \gamma_m \cdot x$, where $m \geq 1$ is least such that $x \in C_m^i$. Define $n_i : X \to \mathbb{N} \cup \{\infty\}$ by

$$n_i(x) = \begin{cases} n & \text{if } 1 \leq n \leq N' \text{ is least such that } T_i^n(x) \in C \cup (X \setminus (D_{-1} \cup D_1)), \\ \infty & \text{if no such } n \text{ exists.} \end{cases}$$

Let D denote the set of $x \in X$ such that $T_{-1}^{n_{-1}(x)}(x), T_1^{n_1(x)}(x) \in D_{-1} \cup D_1$ and

$$\forall i \in \{-1, 1\} \, \forall n < n_i(x) \ (T_i^n(x) = T_{-i} \circ T_i^{n+1}(x)).$$

Finally, define $S : X \to X$ by

$$S(x) = \begin{cases} x & \text{if } x \notin D, \\ T_1(x) & \text{if } x \in D \setminus C, \text{ and} \\ T_{-1}^{n_{-1}(x)-1}(x) & \text{if } x \in C \cap D. \end{cases}$$

It is clear that $S \in [\Gamma]_c$. As $Z \subseteq D$ and $T_1 | (Z \setminus C) = T | (Z \setminus C)$, it follows that

$$\begin{aligned} \mu(\Delta(S, T)) &\leq \mu(X \setminus D) + \mu(D \setminus Z) + \mu(C \setminus \mathrm{Per}_{\leq N'}(T)) \\ &= \mu(X \setminus Z) + \mu(C \setminus \mathrm{Per}_{\leq N'}(T)) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

and this completes the proof of the theorem. □

Remark 3. Theorem 1 easily yields its Borel counterpart, where $[\Gamma]_\mu$ is replaced with the Borel full group $[\Gamma]$, and the uniform topology on $[\Gamma]_\mu$ is replaced with the uniform topology of Bezuglyi-Dooley-Kwiatkowski [1].

References

[1] S. Bezuglyĭ, A. Dooley, and J. Kwiatkowski. Topologies on the group of Borel automorphisms of a standard Borel space (2004). Preprint