AN ANTI-BASIS THEOREM FOR ANALYTIC GRAPHS OF BOREL CHROMATIC NUMBER AT LEAST THREE

BENJAMIN D. MILLER

Abstract. We show that if $\mathfrak{B}$ is a basis for the class of $\boldsymbol{\Sigma}^1_1$ directed graphs on Polish spaces which are of Borel chromatic number at least three, then the partial order $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ embeds into $(\mathfrak{B}, \leq_B)$.

0. Introduction

Suppose that X is a Polish space and $\mathcal{G}$ is a $\boldsymbol{\Sigma}^1_1$ *directed graph* on X, i.e., an irreflexive subset of $X \times X$. A *coloring* of $\mathcal{G}$ is a function $c : X \to I$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Rightarrow c(x_1) \neq c(x_2)).$$

The *chromatic number* of $\mathcal{G}$ is the least cardinal $\chi(\mathcal{G})$ of the form $|I|$, where I is a set for which there is a coloring $c : X \to I$ of $\mathcal{G}$. The *Borel chromatic number* of $\mathcal{G}$ is the least cardinal $\chi_B(\mathcal{G})$ of the form $|I|$, where I is a Polish space for which there is a Borel coloring $c : X \to I$ of $\mathcal{G}$.

A *homomorphism* from $\mathcal{G}$ to $\mathcal{H}$ (on Y) is a function $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Rightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}).$$

We use $\mathcal{G} \leq_c \mathcal{H}$ to indicate the existence of a continuous homomorphism, and we use $\mathcal{G} \leq_B \mathcal{H}$ to indicate the existence of a Borel homomorphism. As has been noted by Louveau, the following remarkable fact is a corollary of the proof of its undirected analog, which is due to Kechris-Solecki-Todorcevic [1]:

Theorem (Kechris-Solecki-Todorcevic). *Suppose that X is a Polish space and $\mathcal{G}$ is a $\boldsymbol{\Sigma}^1_1$ directed graph on X. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq \aleph_0$;
2. $\mathcal{G}_0 \leq_c \mathcal{G}$.

Here, we use $\mathcal{G}_0$ to denote the directed analog of the graph defined in §6 of Kechris-Solecki-Todorcevic [1]. It is natural to ask whether there is an analogous theorem for the class of $\boldsymbol{\Sigma}^1_1$ directed graphs $\mathcal{G}$ on Polish spaces for which $\chi_B(\mathcal{G}) \geq 3$. As the graphs of chromatic number at least three can be characterized as those whose symmetrizations contain an odd cycle, this must be understood as a question about the class $\mathfrak{C}$ of $\boldsymbol{\Sigma}^1_1$ directed graphs $\mathcal{G}$ on Polish spaces for which $\chi(\mathcal{G}) \leq 2$ and $\chi_B(\mathcal{G}) \geq 3$. More generally, we wish to understand the structure of *bases* for $\mathfrak{C}$, i.e., those classes $\mathfrak{B} \subseteq \mathfrak{C}$ with the property that, for every $\mathcal{G} \in \mathfrak{C}$, there exists $\mathcal{H} \in \mathfrak{B}$ such that $\mathcal{H} \leq_B \mathcal{G}$.

In §1, we study a particular sort of oriented graph for which Borel two-colorability can be described in terms of certain σ-ideals of $\mathbf{\Sigma}^1_1$ sets. In §2, we describe a family of locally countable oriented graphs of this form whose Borel chromatic numbers are at least three. In §3, we use such graphs which satisfy an additional growth condition to produce continuum-sized families of pairwise incompatible oriented graphs. In §4, we combine our results with a pair of dichotomy theorems (one of which is due to Louveau) to show that bases for $\mathfrak{C}$ are necessarily complicated:

Theorem. *Suppose that X is a Polish space, $\mathcal{G} \in \mathfrak{C}$, and $\mathfrak{B}$ is a basis for $\mathfrak{C}(\mathcal{G})$. Then the partial order $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ embeds into $(\mathfrak{B}, \leq_B)$.*

1. Balanced graphs

Suppose that $\mathcal{G}$ is an *oriented graph* on X, i.e, an irreflexive, asymmetric subset of $X \times X$. The *symmetrization* of $\mathcal{G}$ is given by

$$\mathcal{G}^{\pm 1} = \{(x_1, x_2) \in X \times X : (x_1, x_2) \in \mathcal{G} \text{ or } (x_2, x_1) \in \mathcal{G}\}.$$

A *$\mathcal{G}$-path* is a sequence $\langle x_0, x_1, \ldots, x_n \rangle$ such that $\forall i < n \ ((x_i, x_{i+1}) \in \mathcal{G})$. Such a path is a *$\mathcal{G}$-cycle* if $x_0 = x_n$. The *weight* of a $\mathcal{G}^{\pm 1}$-path $\gamma = \langle x_0, x_1, \ldots, x_n \rangle$ is

$$w_{\mathcal{G}}(\gamma) = \sum_{i<n} (-1)^{\mathbb{1}_{\mathcal{G}}(x_{i+1}, x_i)},$$

where $\mathbb{1}_{\mathcal{G}}$ denotes the characteristic function of $\mathcal{G}$. We say that $\mathcal{G}$ is a *balanced graph* if the weight of every $\mathcal{G}^{\pm 1}$-cycle is zero. The *weighted distance function* associated with a balanced graph $\mathcal{G}$ is given by

$$d_{\mathcal{G}}(x, y) = \begin{cases} w_{\mathcal{G}}(\gamma) & \text{if } \gamma \text{ is a } \mathcal{G}^{\pm 1}\text{-path from } x \text{ to } y, \\ \infty & \text{if there is no such path.} \end{cases}$$

Proposition 1. *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are directed graphs.*

1. *If $\mathcal{G}$ is oriented and has acyclic symmetrization, then $\mathcal{G}$ is balanced;*
2. *If $\mathcal{G}$ is balanced and $d_{\mathcal{G}}(x, y), d_{\mathcal{G}}(y, z) < \infty$, then $d_{\mathcal{G}}(x, z) = d_{\mathcal{G}}(x, y) + d_{\mathcal{G}}(y, z)$;*
3. *If π is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ and $\langle x_0, x_1, \ldots, x_n \rangle$ is a $\mathcal{G}$-path, then*
$$w_{\mathcal{H}}(\langle \pi(x_0), \pi(x_1), \ldots, \pi(x_n) \rangle) = w_{\mathcal{G}}(\langle x_0, x_1, \ldots, x_n \rangle);$$
4. *If $\mathcal{G} \leq \mathcal{H}$ and $\mathcal{H}$ is balanced, then $\mathcal{G}$ is balanced;*
5. *If $\mathcal{G}$ is balanced, then $\chi(\mathcal{G}) = 2$.*

Proof. To see (1), note that every $\mathcal{G}^{\pm 1}$-cycle passes through (x, y) the same number of times as it passes through (y, x), for all $(x, y) \in \mathcal{G}$. To see (2), fix $\mathcal{G}^{\pm 1}$-paths σ and τ from x to y and y to z, respectively, and observe that $\sigma\tau$ is a $\mathcal{G}^{\pm 1}$-path from x to z, thus

$$d_{\mathcal{G}}(x, z) = w_{\mathcal{G}}(\sigma\tau) = w_{\mathcal{G}}(\sigma) + w_{\mathcal{G}}(\tau) = d_{\mathcal{G}}(x, y) + d_{\mathcal{G}}(y, z).$$

To see (3), note that $(x_{i+1}, x_i) \in \mathcal{G} \Leftrightarrow (\pi(x_i), \pi(x_{i+1})) \in \mathcal{H}$, and appeal to the definition of weight. To see (4), note that the fact that $\mathcal{H}$ is oriented immediately implies that $\mathcal{G}$ is oriented, and since homomorphisms send $\mathcal{G}^{\pm 1}$-cycles to $\mathcal{H}^{\pm 1}$-cycles, the fact that $\mathcal{H}$ is balanced coupled with (3) ensures that $\mathcal{G}$ is balanced. To see (5), fix a set $B \subseteq \mathrm{dom}(\mathcal{G})$ which is maximal with the property that

$$\forall x, y \in B \ (d_{\mathcal{G}}(x, y) < \infty \Rightarrow d_{\mathcal{G}}(x, y) \equiv 0 \pmod 2),$$

and observe that $\mathbb{1}_B$ is a two-coloring of $\mathcal{G}$. □

Suppose that X is a Polish space and $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X. For each set $S \subseteq \mathbb{Z}$, the $(\mathcal{G}, S)$-*saturation* of a set $A \subseteq X$ is given by

$$[A]_{(\mathcal{G},S)} = \{x \in X : \exists y \in A \ (d_{\mathcal{G}}(x, y) \in S)\}.$$

The *distance set* associated with A is given by

$$\Delta_{\mathcal{G}}(A) = \{d_{\mathcal{G}}(x, y) : x, y \in A \text{ and } d_{\mathcal{G}}(x, y) < \infty\}.$$

Proposition 2. *Suppose that X is a Polish space, $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X, and $A \subseteq X$ is $\mathbf{\Sigma^1_1}$. Then there is a set $B \supseteq A$ such that:*

1. $\Delta_{\mathcal{G}}(B) = \Delta_{\mathcal{G}}(A)$;
2. $\forall S \subseteq \mathbb{Z}$ ($[B]_{(\mathcal{G},S)}$ *is Borel*).

Proof. Note first that for each $S \subseteq \mathbb{Z}$, the property of having $\Delta_{\mathcal{G}}(A) \subseteq S$ is $\mathbf{\Pi^1_1}$ on $\mathbf{\Sigma^1_1}$, thus every $\mathbf{\Sigma^1_1}$ set is contained in a Borel set with the same distance set. Now fix a sequence $\langle k_0, k_1, \dots \rangle$ of integers in which every integer appears infinitely often, and set $A_0 = A$. Given A_n, fix a Borel set $B_n \supseteq [A_n]_{(\mathcal{G},\{k_n\})}$ such that $\Delta_{\mathcal{G}}(B_n) = \Delta_{\mathcal{G}}([A_n]_{(\mathcal{G},\{k_n\})})$, and set $A_{n+1} = A_n \cup [B_n]_{(\mathcal{G},\{-k_n\})}$. We claim that the set $B = \bigcup_{n \in \mathbb{N}} A_n$ is as desired. To see (1), simply note that

$$\Delta_{\mathcal{G}}(B) = \bigcup_{n \in \mathbb{N}} \Delta_{\mathcal{G}}(A_n) = \Delta_{\mathcal{G}}(A).$$

To see (2), observe that the set $[B]_{(\mathcal{G},\{k\})} = \bigcup_{k_n = k} B_n$ is Borel, for each $k \in \mathbb{Z}$, thus the set $[B]_{(\mathcal{G},S)} = \bigcup_{k \in S} [B]_{(\mathcal{G},\{k\})}$ is Borel, for each $S \subseteq \mathbb{Z}$. □

For each set $S \subseteq \mathbb{Z}$, let $\mathcal{I}_{(\mathcal{G},S)}$ denote the σ-ideal generated by the $\mathbf{\Sigma^1_1}$ sets $A \subseteq X$ for which $\Delta_{\mathcal{G}}(A) \subseteq S$.

Proposition 3. *Suppose that X is a Polish space and $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X. Then $X \in \mathcal{I}_{(\mathcal{G},2\mathbb{Z})} \Leftrightarrow \chi_B(\mathcal{G}) \leq 2$.*

Proof. To see ($\Leftarrow$), note first that by the obvious induction, if $c : X \to \{0, 1\}$ is a two-coloring of $\mathcal{G}$ and $d_{\mathcal{G}}(x, y) \equiv 1 \pmod 2$, then $c(x) \neq c(y)$, thus $\Delta_{\mathcal{G}}(c^{-1}(\{0\}))$, $\Delta_{\mathcal{G}}(c^{-1}(\{1\})) \subseteq 2\mathbb{Z}$. It follows that if c is Borel, then $X \in \mathcal{I}$.

To see ($\Rightarrow$), note that if $X \in \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$, then Proposition 2 ensures that there are sets $B_0, B_1, \dots \subseteq X$ such that:

1. $X = \bigcup_{n \in \mathbb{N}} B_n$;

2. $\forall n \in \mathbb{N}\ (\Delta_{\mathcal{G}}(B_n) \subseteq 2\mathbb{Z})$;

3. $\forall S \subseteq \mathbb{Z}\ ([B_n]_{(\mathcal{G},S)}$ is Borel).

Define $B \subseteq X$ by

$$B = \bigcup_{n \in \mathbb{N}} [B_n]_{(\mathcal{G},2\mathbb{Z})} \setminus \bigcup_{m<n} [B_m]_{(\mathcal{G},\mathbb{Z})},$$

and observe that $\mathbb{1}_B$ is a Borel two-coloring of $\mathcal{G}$. □

Let $\mathcal{I}_{(\mathcal{G},<\mathbb{Z})}$ denote the σ-ideal generated by $\mathbf{\Sigma^1_1}$ sets $A \subseteq X$ with $|\Delta_{\mathcal{G}}(A)| < \aleph_0$.

Proposition 4. *Suppose that X is a Polish space and $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X. Then $\mathcal{I}_{(\mathcal{G},<\mathbb{Z})} = \mathcal{I}_{\{0\}}$.*

Proof. It is enough to show that if $A \subseteq X$ is $\mathbf{\Sigma^1_1}$ and $|\Delta_{\mathcal{G}}(A)| < \aleph_0$, then $A \in \mathcal{I}_{(\mathcal{G},\{0\})}$. Via the obvious induction, it is therefore enough to show that if $\Delta_{\mathcal{G}}(A)$ is non-trivial and finite, then there are Borel sets $B_1, B_2 \subseteq A$ such that $A = B_1 \cup B_2$ and $|\Delta_{\mathcal{G}}(B_1)|, |\Delta_{\mathcal{G}}(B_2)| < |\Delta_{\mathcal{G}}(A)|$. Towards this end, observe that by Proposition 2, there is a set $B \supseteq A$ such that $\Delta_{\mathcal{G}}(B) = \Delta_{\mathcal{G}}(A)$ and $\forall S \subseteq \mathbb{Z}\ ([B]_{(\mathcal{G},S)}$ is Borel). Then the sets $B_1 = [B]_{(\mathcal{G},\{0\})} \cap [B]_{(\mathcal{G},\mathbb{Z}^+)}$ and $B_2 = [B]_{(\mathcal{G},\{0\})} \setminus [B]_{(\mathcal{G},\mathbb{Z}^+)}$ are as desired. □

As a corollary, we obtain a sufficient condition for Borel two-colorability:

Proposition 5. *Suppose that X is a Polish space and $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X. If $X \in \mathcal{I}_{(\mathcal{G},<\mathbb{Z})}$, then $\chi_B(\mathcal{G}) \leq 2$.*

Proof. This follows directly from Propositions 3 and 4. □

We obtain also the following fact, which will be useful later on:

Proposition 6. *Suppose that X is a Polish space, $\mathcal{G}$ is a $\mathbf{\Sigma^1_1}$ balanced graph on X, $A \subseteq X$ is $\mathbf{\Sigma^1_1}$, and $A \notin \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$. Then, for all $k \in \mathbb{N}$, there exist $x, y \in A$ such that $d_{\mathcal{G}}(x,y) \equiv 1 \pmod 2$ and $k < d_{\mathcal{G}}(x,y) < \infty$.*

Proof. Suppose, towards a contradiction, that there exists $k \in \mathbb{N}$ such that

$$\forall x, y \in A\ (k < d_{\mathcal{G}}(x,y) < \infty \Rightarrow d_{\mathcal{G}}(x,y) \equiv 0 \pmod 2).$$

Set $B = \{x \in A : \exists y \in A\ (k < d_{\mathcal{G}}(x,y) < \infty)\}$, and observe that $\Delta_{\mathcal{G}}(B) \subseteq 2\mathbb{Z}$. Proposition 2 ensures that by enlarging B, we can assume that it is Borel. As $\Delta_{\mathcal{G}}(A \setminus B)$ is finite, it follows from Proposition 5 that $A \in \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$. □

2. Combinatorially simple examples

In this section, we describe a parameterized family of "combinatorially simple" oriented graphs with acyclic symmetrizations and Borel chromatic number at least three. For each set S, let $(S)_\emptyset = \{(s, \emptyset) : s \in S\}$, and for each set S of pairs (v, s), where $s \in 2^{<\mathbb{N}}$, we use $(S)_i$ to denote the corresponding set of pairs of the form (v, si), where $i \in \{0, 1\}$. Let $\mathbb{P}$ denote the set of sequences $p = \langle T^p_n \rangle_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, the following conditions are satisfied:

1. T^p_n is an oriented graph with connected, acyclic symmetrization;
2. T^p_n has finite domain, which we denote by D^p_n;
3. $D^p_0 = (V^p_0)_\emptyset$, where V^p_0 is a singleton whose unique element we denote by v^p;
4. D^p_{n+1} is the disjoint union of $(D^p_n)_0$, $(D^p_n)_1$, and $(V^p_{n+1})_\emptyset$;
5. $V^p_n \cap \bigcup_{m<n} V^p_m = \emptyset$;
6. $T^p_{n+1}|(D^p_n)_i = \{((v, si), (w, ti)) : ((v, s), (w, t)) \in T^p_n\}$, for each $i \in \{0, 1\}$.

We associate with each $p \in \mathbb{P}$ the set $V^p = \bigcup_{n\in\mathbb{N}} V^p_n$ and $X^p = V^p \times 2^{\mathbb{N}}$, as well as the graph $\mathcal{G}^p$ on X^p given by

$$\mathcal{G}^p = \bigcup_{n\in\mathbb{N}} \{((v, s\alpha), (w, t\alpha)) : ((v, s), (w, t)) \in T^p_n \text{ and } \alpha \in \{0, 1\}\}.$$

Proposition 7. *Each $\mathcal{G}^p$ is oriented and has acyclic symmetrization.*

Proof. It is clear that $\mathcal{G}^p$ is oriented, since each T^p_n is oriented. Similarly, if $\mathcal{G}^p$ does not have acyclic symmetrization, then there exists $n \in \mathbb{N}$ such that T^p_n does not have acyclic symmetrization, which contradicts the definition of $\mathbb{P}$. □

Corollary 8. *Each $\mathcal{G}^p$ is a balanced graph.*

Proof. This follows directly from Propositions 1 and 7. □

For each $p \in \mathbb{P}$ and infinite set $S \subseteq \mathbb{N}$, define $B^p_S \subseteq X^p$ by

$$B^p_S = \{(v^p, \alpha) \in X^p : \operatorname{supp}(\alpha) \subseteq S\},$$

where $\operatorname{supp}(\alpha) = \{n \in \mathbb{N} : \alpha(n) = 1\}$. Let $X^p_S = [B^p_S]_{(\mathcal{G}^p,\mathbb{Z})}$ and $\mathcal{G}^p_S = \mathcal{G}^p|X^p_S$. It will later be important to have a large family of graphs of this form whose Borel chromatic number is at least three. Towards this end, set

$$\mathbb{Q} = \{p \in \mathbb{P} : \forall n \in \mathbb{N}\ (d_{T^p_{n+1}}((v^p, 0^n0), (v^p, 0^n1)) \equiv 1 \pmod 2)\}.$$

Proposition 9. *Suppose that $q \in \mathbb{Q}$ and $S \subseteq \mathbb{N}$ is infinite. Then $\chi_B(\mathcal{G}^q_S) \geq 3$.*

Proof. Endow B^q_S with the topology it inherits as a closed subspace of $\{v^q\} \times 2^{\mathbb{N}}$. Proposition 3 ensures that to see that $\chi_B(\mathcal{G}^q_S) \geq 3$, it is enough to show that if $A \subseteq B^q_S$ is Baire measurable and non-meager in B^q_S, then $\Delta_{\mathcal{G}^q}(A) \nsubseteq 2\mathbb{Z}$. Towards this end, suppose that $A \subseteq B^q_S$ is Baire measurable and non-meager in B^q_S, and fix $s \in 2^{<\mathbb{N}}$ such that $\operatorname{supp}(s) \subseteq S$ and A is comeager in $B^q_S \cap (\{v^q\} \times \mathcal{N}_s)$. Fix $\alpha \in 2^{\mathbb{N}}$ such that $s0\alpha, s1\alpha \in A$, set $n = |s|$, and observe that

$$\begin{aligned} d_{\mathcal{G}^q}(s0\alpha, s1\alpha) &= d_{T^q_{n+1}}((v^q, s0), (v^q, s1)) \\ &= d_{T^q_{n+1}}((v^q, s0), (v^q, 0^n0)) + d_{T^q_{n+1}}((v^q, 0^n0), (v^q, 0^n1)) + \\ &\quad\ d_{T^q_{n+1}}(v^q, 0^n1), (v^q, s1)) \\ &= d_{T^q_n}((v^q, s), (v^q, 0^n)) + d_{T^q_{n+1}}((v^q, 0^n0), (v^q, 0^n1)) + \\ &\quad\ d_{T^q_n}((v^q, 0^n), (v^q, s)) \\ &= d_{T^q_{n+1}}((v^q, 0^n0), (v^q, 0^n1)). \end{aligned}$$

The definition of $\mathbb{Q}$ implies that the latter quantity is odd, thus $\Delta_{\mathcal{G}^q}(A) \nsubseteq 2\mathbb{Z}$. □

3. Incompatible graphs

Associated with each $p \in \mathbb{P}$ are the integers k_n^p given by

$$k_n^p = d_{T_{n+1}^p}((v^p, 0^n 0), (v^p, 0^n 1)),$$

as well as the integers i_n^p, j_n^p given by

$$i_n^p = k_n^p - \sum_{m<n} 2^{n-m} k_m^p \text{ and } j_n^p = k_n^p + \sum_{m<n} 2^{n-m} k_m^p.$$

Given a function $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$, we say that p is *f-dominating* if

$$k_0^p > f(\emptyset) \text{ and } \forall n \in \mathbb{Z}^+ \ (k_n^p > f(\langle k_0^p, \ldots, k_{n-1}^p \rangle)).$$

Define $f_0 : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ by putting $f_0(\emptyset) = 0$ and

$$f_0(\langle k_0, \ldots, k_{n-1} \rangle) = 2n + 3 \sum_{m<n} 2^{n-m} k_m,$$

for all $n \in \mathbb{Z}^+$.

Proposition 10. *Suppose that $p \in \mathbb{P}$ is f_0-dominating. Then*

$$\forall m < n \ (j_m^p < i_n^p - 2n).$$

Proof. As p is f_0-dominating, it is clear that each k_n^p is positive. Given natural numbers $m < n$, observe that

$$\begin{aligned} i_n^p - j_m^p &= k_n^p - \sum_{\ell<n} 2^{n-\ell} k_\ell^p - k_m^p - \sum_{\ell<m} 2^{m-\ell} k_\ell^p \\ &\geq k_n^p - 2 \sum_{\ell<n} 2^{n-\ell} k_\ell^p, \end{aligned}$$

and the definition of f_0 ensures that this last term is strictly greater than $2n$. □

In particular, it follows that the intervals $[i_0^p, j_0^p], [i_1^p, j_1^p], \ldots$ are pairwise disjoint.

Proposition 11. *Suppose that $p \in \mathbb{P}$ is f_0-dominating, $n \in \mathbb{N}$, and $s, t \in 2^n$. Then*

$$i_n^p \leq d_{T_{n+1}^p}((v^p, s0), (v^p, t1)) \leq j_n^p.$$

Proof. By induction on n. The case $n = 0$ is a triviality (since $i_0^p = j_0^p = k_0^p$), so suppose that we have proven the proposition below some positive integer n, and fix $s, t \in 2^n$. Then

$$\begin{aligned} d_{T_{n+1}^p}((v^p, s0), (v^p, t1)) &= d_{T_{n+1}^p}((v^p, s0), (v^p, 0^n 0)) + \\ &\quad\ d_{T_{n+1}^p}((v^p, 0^n 0), (v^p, 0^n 1)) + \\ &\quad\ d_{T_{n+1}^p}((v^p, 0^n 1), (v^p, t1)) \\ &= d_{T_n^p}((v^p, s), (v^p, 0^n)) + k_n^p + d_{T_n^p}((v^p, 0^n), (v^p, t)). \end{aligned}$$

Observe now that

$$2j^p_{n-1} = 2 \sum_{m \leq n-1} 2^{(n-1)-m} k^p_m = \sum_{m<n} 2^{n-m} k^p_m.$$

As p is f_0-dominating, the induction hypothesis implies that

$$k^p_n - 2j^p_{n-1} \leq |d_{T^p_{n+1}}((v^p, s0), (v^p, t1))| \leq k^p_n + 2j^p_{n-1}.$$

As the quantities on the left and right of this inequality are equal to i^p_n and j^p_n, respectively, the proposition follows. □

For $\alpha E_0 \beta$, let $n(\alpha, \beta) = \max\{n \in \mathbb{N} : \alpha(n) \neq \beta(n)\}$.

Proposition 12. *Suppose that $p \in \mathbb{P}$ is f_0-dominating and $\alpha E_0 \beta$. Then*

$$i^p_{n(\alpha,\beta)} \leq |d_{\mathcal{G}^p}((v^p, \alpha), (v^p, \beta))| \leq j^p_{n(\alpha,\beta)}.$$

Proof. Let $n = n(\alpha, \beta) + 1$, $s = \alpha|n$, and $t = \alpha|n$. As $d_{\mathcal{G}^p}((v^p, \alpha), (v^p, \beta)) = d_{T^p_n}((v^p, s), (v^p, t))$, the desired inequality follows from Proposition 11. □

For each $S \subseteq \mathbb{N}$, let $[S]_n = \{i \in \mathbb{N} : \exists j \in S \ (|i - j| \leq n)\}$.

Proposition 13. *Suppose that $p \in \mathbb{P}$ is f_0-dominating, $S \subseteq \mathbb{N}$, and $\ell \in \mathbb{N}$. Then*

$$\forall n \geq \ell \ (n \in S \Leftrightarrow [i^p_n - \ell, j^p_n + \ell] \cap [\Delta_{\mathcal{G}^p}(B^p_S)]_\ell \neq \emptyset).$$

Proof. To see ($\Rightarrow$), note that if $n \in S$, then $k^p_n = d_{\mathcal{G}^p}((v^p, 0^n 00^\infty), (v^p, 0^n 10^\infty))$ is in $\Delta_{\mathcal{G}^p}(B^p_S)$, and $i_n \leq k_n \leq j_n$.

To see ($\Leftarrow$), observe that if $[i^p_n - \ell, j^p_n + \ell] \cap [\Delta_{\mathcal{G}^p}(B^p_S)]_\ell \neq \emptyset$, then there exists $(\alpha, \beta) \in E_0$ such that $\mathrm{supp}(\alpha), \mathrm{supp}(\beta) \subseteq S$ and

$$i^p_n - 2\ell \leq d_{\mathcal{G}^p}((v^p, \alpha), (v^p, \beta)) \leq j^p_n + 2\ell.$$

As Proposition 12 implies that

$$i^p_{n(\alpha,\beta)} \leq d_{\mathcal{G}^p}((v^p, \alpha), (v^p, \beta)) \leq j^p_{n(\alpha,\beta)},$$

it follows that $i^p_n - 2n \leq i^p_n - 2\ell \leq j^p_{n(\alpha,\beta)}$ and $i^p_{n(\alpha,\beta)} \leq j^p_n + 2\ell \leq j^p_n + 2n$. The former inequality, in conjunction with Proposition 10, then implies that $n(\alpha, \beta) \geq n$. Coupled with the latter inequality, this implies that $i^p_{n(\alpha,\beta)} - 2n(\alpha, \beta) \leq i^p_{n(\alpha,\beta)} - 2n \leq j^p_n$, and one more application of Proposition 10 then gives that $n \geq n(\alpha, \beta)$, so $n = n(\alpha, \beta)$, thus $n \in S$. □

Recall that sets $S, T \subseteq \mathbb{N}$ are said to be *almost disjoint* if $|S \cap T| < \aleph_0$.

Proposition 14. *Suppose that $p \in \mathbb{P}$ is f_0-dominating and $S, T \subseteq \mathbb{N}$ are almost disjoint. Then $\forall n \in \mathbb{N}$ ($[\Delta(B^p_S)]_n, [\Delta(B^p_T)]_n$ are almost disjoint).*

Proof. Fix $m \geq n$ with $S \cap T \subseteq m$. It is enough to show that $[\Delta_{\mathcal{G}^p}(B^p_S)]_n \cap [\Delta_{\mathcal{G}^p}(B^p_T)]_n \subseteq i^p_m$. Towards this end, suppose that $k \geq i^p_m$ is in $[\Delta_{\mathcal{G}^p}(B^p_{\mathbb{N}})]_n$. By Proposition 12, there exists $\ell \in \mathbb{N}$ such that $i^p_\ell \leq k \leq j^p_\ell$. Then $\ell \geq m \geq n$, so Proposition 13 implies that if $k \in [\Delta_{\mathcal{G}^p}(B^p_S)]_n$, then $\ell \in S$, thus $\ell \notin T$. One more application of Proposition 13 then gives that $k \notin [\Delta_{\mathcal{G}^p}(B^p_T)]_n$. □

We are now able to construct incompatible balanced graphs:

Proposition 15. *Suppose that $p \in \mathbb{P}$ is f_0-dominating and $S, T \subseteq \mathbb{N}$ are almost disjoint. Then $\mathcal{G}^p_S$ and $\mathcal{G}^p_T$ are incompatible.*

Proof. Suppose that $\mathcal{G}$ is an analytic graph on X and there are Borel homomorphisms π_U of $\mathcal{G}$ into $\mathcal{G}^p_U$, for $U \in \{S, T\}$. Then Proposition 1 implies that $\mathcal{G}$ is balanced. For each $n \in \mathbb{N}$ and $U \in \{S, T\}$, let

$$A^U_n = \pi_U^{-1}([B^p_U]^{\{-n,\ldots,n\}}_{\mathcal{G}^p_U}),$$

and set $A_n = A^S_n \cap A^T_n$.

Lemma 16. $\forall n \in \mathbb{N}$ $(\Delta_{\mathcal{G}}(A_n)$ *is finite*$)$.

Proof. It follows from Proposition 1 that

$$\begin{array}{rcl} \Delta_{\mathcal{G}}(A_n) & \subseteq & \Delta_{\mathcal{G}}(A^S_n) \cap \Delta(A^T_n) \\ & = & \Delta_{\mathcal{G}^p}([B^p_S]^n_{\mathcal{G}^p_S}) \cap \Delta_{\mathcal{G}^p}([B^p_T]^n_{\mathcal{G}^p_T}) \\ & \subseteq & [\Delta_{\mathcal{G}^p}(B^p_S)]_{2n} \cap [\Delta_{\mathcal{G}^p}(B^p_T)]_{2n}, \end{array}$$

and Proposition 14 ensures that the latter set is finite. □

As $X = \bigcup_{n \in \mathbb{N}} A_n$, Proposition 5 implies that $\chi_B(\mathcal{G}) \leq 2$, and it follows that $\mathcal{G}^S_p, \mathcal{G}^T_p$ are incompatible. □

An *embedding* of $\mathcal{G}$ into $\mathcal{H}$ is an injection $\pi : X \to Y$ such that

$$\forall x_1, x_2 \in X \ ((x_1, x_2) \in \mathcal{G} \Rightarrow c(x_1) \neq c(x_2)).$$

We use $\mathcal{G} \sqsubseteq_c \mathcal{H}$ to indicate the existence of a continuous embedding, and we use $\mathcal{G} \sqsubseteq_B \mathcal{H}$ to indicate the existence of a Borel embedding.

Proposition 17. *Suppose that $p \in \mathbb{P}$ is f_0-dominating. Then there is a pairwise incompatible family $\langle \mathcal{G}_\alpha \rangle_{\alpha \in 2^{\mathbb{N}}}$ of Borel graphs such that $\forall \alpha \in 2^{\mathbb{N}}$ $(\mathcal{G}_\alpha \sqsubseteq_B \mathcal{G}^p)$. Moreover, if $p \in \mathbb{Q}$, then such graphs can be found with Borel chromatic number at least three.*

Proof. Fix an almost disjoint family $\langle S_\alpha \rangle_{\alpha \in 2^{\mathbb{N}}}$ of subsets of $\mathbb{N}$. Then Propositions 9 and 15 ensure that the graphs $\mathcal{G}_\alpha = \mathcal{G}^p_{S_\alpha}$ are as desired. □

4. A basis theorem and an anti-basis theorem

A result of Louveau [2] implies the following basis theorem:

Theorem 18 (Louveau). *Suppose that X is a Polish space, $\mathcal{G}$ is an analytic directed graph on X, and $\chi(\mathcal{G}) \leq 2$. Then exactly one of the following holds:*

1. $\chi_B(\mathcal{G}) \leq 2$;
2. *There exists $q \in \mathbb{Q}$ such that $\mathcal{G}_q \leq_c \mathcal{G}$.*

We will strengthen this theorem by showing that q can be taken to be f_0-dominating. In the special case that $\mathcal{G}$ is a locally countable Borel oriented graph, we actually obtain an analogous result for embeddability:

Theorem 19. *Suppose that X is a Polish space, $\mathcal{G}$ is a locally countable Borel oriented graph on X with acyclic symmetrization, and $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Then exactly one of the following holds:*

1. *$\chi_B(\mathcal{G}) \leq 2$;*
2. *There is an f-dominating $q \in \mathbb{Q}$ such that $\mathcal{G}_q \sqsubseteq_c \mathcal{G}$.*

Proof. As (1) $\Rightarrow \neg$(2) is straightforward, we shall prove only $\neg$(1) $\Rightarrow$ (2). For each $q \in \mathbb{Q}$, let E^q denote the equivalence relation on X^q given by

$$(v,\alpha)E^q(w,\beta) \Leftrightarrow ((v,\alpha),(w,\beta) \text{ are } \mathcal{G}^q\text{-connected}).$$

Similarly, define E on X by $xEy \Leftrightarrow (x,y$ are $\mathcal{G}$-connected). It is sufficient to find an f-dominating $q \in \mathbb{Q}$ and a continuous injection $\pi : X^q \to X$ such that:

1. $\forall (v,\alpha)E^q(w,\beta)\ (((v,\alpha),(w,\beta)) \in \mathcal{G}^q \Leftrightarrow (\pi(v,\alpha),\pi(w,\beta)) \in \mathcal{G})$;
2. $\forall (v,\alpha),(w,\beta) \in X^q\ (\pi(v,\alpha)E\pi(w,\beta) \Rightarrow (v,\alpha)E^q(w,\beta))$.

Towards this end, fix a countable group G of Borel automorphisms of X such that $E = \bigcup_{g\in G} \operatorname{graph}(g)$, and fix an increasing sequence of finite, symmetric sets $H_0 \subseteq H_1 \subseteq \cdots \subseteq G$ such that $G = \bigcup_{n\in\mathbb{N}} H_n$. By standard change of topology results, we can assume that X is a zero-dimensional Polish space, G acts on X by homeomorphisms, and each of the sets $\{x \in X : (x, g\cdot x) \in \mathcal{G}\}$ is clopen.

We will find clopen sets $A_n \subseteq X$, finite sets $V_n \subseteq G$, finite oriented graphs T_n whose symmetrizations are trees, group elements $g_n \in G$, and natural numbers k_n, from which we define

$$D_n = \bigcup_{m\leq n} V_m \times 2^{n-m} \text{ and } h_{(v,s)} \cdot x = vg_m^{s(0)} \cdots g_{n-1}^{s(n-1-m)} \cdot x,$$

for each $n \in \mathbb{N}$ and $(v,s) \in V_m \times 2^{n-m}$. This will be done so as to ensure that the following conditions are satisfied:

(a) $A_n \notin \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$;

(b) $D_n = \operatorname{dom}(T_n)$;

(c) $T_n^{\pm 1}$ is connected and acyclic;

(d) $k_n = d_{T_{n+1}}((1_G, 0^n 0), (1_G, 0^n 1))$;

(e) $k_n \equiv 1 \pmod 2$;

(f) $f(k_n) > f(\langle k_0, \ldots, k_{n-1}\rangle)$;

(g) $A_{n+1} \subseteq A_n \cap g_n^{-1}(A_n)$;

(h) $\forall x \in A_n\, \forall (v,s),(w,t) \in D_n\ ((h_{(v,s)} \cdot x, h_{(w,t)} \cdot x) \in \mathcal{G} \Leftrightarrow ((v,s),(w,t)) \in T_n)$;

(i) $\forall (v,s),(w,t) \in D_n \, \forall h \in H_n \; (hh_{(v,s)}(A_{n+1}) \cap h_{(w,t)} g_n(A_{n+1}) = \emptyset)$;

(j) $\forall (v,s) \in D_{n+1} \; (\mathrm{diam}(h_{(v,s)}(A_{n+1})) \leq 1/n)$;

We begin by setting $A_0 = X$, $V_0 = \{1_G\}$, and $T_0 = \emptyset$. Suppose now that we have found A_i, V_i, and T_i, for $i \leq n$, and g_i and k_i, for $i < n$. Let Λ denote the set of tuples $\lambda = \langle V, T, g, k \rangle$ such that $V \subseteq G$ is a finite set, T is an oriented graph on the set $D = (D_n)_0 \cup (D_n)_1 \cup (V)_\emptyset$ whose symmetrization is a tree, $g \in G$, $k \in \mathbb{N}$, and the following conditions are satisfied:

(b′) $V = \mathrm{dom}(T)$;

(c′) T is connected;

(d′) $k = d_T((1_G, 0^n 0), (1_G, 0^n 1))$;

(e′) $k \equiv 1 \pmod 2$;

(f′) $f(k) > f(\langle k_0, \ldots, k_{n-1} \rangle)$.

For each $\lambda \in \Lambda$ and $(v,s) \in D$, set

$$h^\lambda_{(v,s)} = \begin{cases} v & \text{if } s = \emptyset, \\ h_{(v,s|m)} g_\lambda^{s(m)} & \text{if } |s| = m+1. \end{cases}$$

Let A_λ denote the set of $x \in X$ which satisfy the following conditions:

(g′) $x \in A_n \cap g_\lambda^{-1}(A_n)$;

(h′) $\forall (v,s),(w,t) \in D \; ((h^\lambda_{(v,s)} \cdot x, h^\lambda_{(w,t)} \cdot x) \in \mathcal{G} \Leftrightarrow ((v,s),(w,t)) \in T)$;

(i′) $\forall (v,s),(w,t) \in D_n \, \forall h \in H_n \; (g_\lambda \cdot x \neq h^{-1}_{(w,t)} h h_{(v,s)} \cdot x)$.

As (g′) and (h′) are clopen and (i′) is open, it follows that each A_λ is open.

Lemma 20. *There exists $\lambda \in \Lambda$ such that $A_\lambda \notin \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$.*

Proof. Set $A = A_n \setminus \bigcup_{\lambda \in \Lambda} A_\lambda$. It is clearly sufficient to show that $A \in \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$. Suppose, towards a contradiction, that this is not the case. By a standard argument, we can write A as a union of finitely many Borel sets $B \subseteq A$ such that

$$\forall (v,s),(w,t) \in D_n \, \forall h \in H_n \; (h_{(w,t)} g_\lambda(B) \cap hh_{(v,s)}(B) = \emptyset),$$

thus there is such a set $B \subseteq A$ with $B \notin \mathcal{I}_{(\mathcal{G},2\mathbb{Z})}$. By Proposition 6, there exist $x, y \in B$ such that

$$d_\mathcal{G}(x,y) \equiv 1 \pmod 2 \text{ and } d_\mathcal{G}(x,y) > f(\langle k_0, \ldots, k_{n-1} \rangle).$$

Fix $g \in G$ such that $g \cdot x = y$, as well as a finite set $V \subseteq G$ such that $\forall v, w \in V$ ($v \neq w \Rightarrow v \cdot x \neq w \cdot x$), the sets $\{h_{(v,s)} g^i \cdot x : (v,s) \in D_n \text{ and } i \in \{0,1\}\}, \{v \cdot x : v \in V\}$ are pairwise disjoint, and the symmetrization of the restriction of $\mathcal{G}$ to their union is connected. Define T on D by

$$T = \{((v,s),(w,t)) \in D \times D : ((h^\lambda_{(v,s)}, h^\lambda_{(w,t)})) \in \mathcal{G}\},$$

and set $k = d_T((1_G, 0^n 0), (1_G, 0^n 1))$. It is now easily verified that $\lambda = \langle V, T, g, k \rangle$ is in Λ and $x \in A_\lambda$, and this contradicts the fact that $B \cap A_\lambda = \emptyset$. □

Let λ be as in Lemma 20, and set $V_{n+1} = V$, $T_{n+1} = T$, $g_n = g$, and $k_n = k$. As G acts by homeomorphisms, we can write A_λ as the union of countably many clopen sets U such that:

(d$''$) $\forall (v,s),(w,t) \in D_n \, \forall h \in H_n \; (hh_{(v,s)}(U) \cap h_{(w,t)}g_n(U) = \emptyset)$.

(e$''$) $\forall (v,s) \in D_{n+1} \; (\operatorname{diam}(h_{(v,s)}(U)) \leq 1/n)$.

Fix such a U which is not in $\mathcal{I}$, and set $A_{n+1} = U$.

This completes the recursive construction. Set $q = \langle T_n \rangle_{n\in\mathbb{N}}$, and for each $n \in \mathbb{N}$ and $(v,s) \in D_n$, put $B_{(v,s)} = h_{(v,s)}(A_n)$. Conditions (g) and (j) ensure that, for each $\alpha \in 2^{\mathbb{N}}$, the sets $B_{\alpha|0}, B_{\alpha|1}, \ldots$ are decreasing and of vanishing diameter, and since they are clopen, they have singleton intersection. Define $\pi : 2^{\mathbb{N}} \to X$ by

$$\pi(\alpha) = \text{ the unique element of } \bigcap_{n\in\mathbb{N}} B_{\alpha|n}.$$

It follows from conditions (i) and (j) that π is a continuous injection, so it only remains to check conditions (1) and (2). We note first the following lemma:

Lemma 21. *Suppose that $m \leq n$ are natural numbers, $(v,s) \in V_m \times 2^{n-m}$, and $\alpha \in 2^{\mathbb{N}}$. Then $\pi(v, s\alpha) = h_{(v,s)} \cdot \pi(v^q, 0^n\alpha)$.*

Proof. Simply observe that

$$\begin{aligned}
\{\pi(v,s\alpha)\} &= \bigcap_{i\geq n} B_{(v,s\alpha|i)} \\
&= \bigcap_{i\geq 0} h_{(v,s)} h_{(v^q,0^{n-m}(\alpha|i))}(A_{i+n}) \\
&= h_{(v,s)} \left(\bigcap_{i\geq 0} h_{(v^q,0^n(\alpha|i))}(A_{i+n}) \right) \\
&= h_{(v,s)} \left(\bigcap_{i\geq n} B_{(v^q,0^n\alpha|i)} \right) \\
&= \{h_{(v,s)} \cdot \pi(v^q, 0^n\alpha)\},
\end{aligned}$$

thus $\pi(v,s\alpha) = h_{(v,s)} \cdot \pi(v^q, 0^n\alpha)$. ◻

To see (1), suppose that $(v,\alpha)E^q(w,\beta)$, and fix $n \in \mathbb{N}$ and $s,t \in 2^{<\mathbb{N}}$ such that $(v,s),(w,t) \in D_n$ and the sequences obtained from α, β by removing s,t are identical. Let γ denote this sequence. Lemma 21 implies that $\pi(v,\alpha) = h_{(v,s)} \cdot \pi(v^q, 0^n\gamma)$ and $\pi(w,\beta) = h_{(w,t)} \cdot \pi(v^q, 0^n\gamma)$. As $x = \pi(v^q, 0^n\gamma)$ is in A_n, condition (h) ensures that

$$\begin{aligned}
((v,\alpha),(w,\beta)) \in \mathcal{G}^q &\Leftrightarrow ((v,s),(w,t)) \in T_n^q \\
&\Leftrightarrow (h_{(v,s)} \cdot x, h_{(w,t)} \cdot x) \in \mathcal{G} \\
&\Leftrightarrow (\pi(v,\alpha), \pi(w,\beta)) \in \mathcal{G}.
\end{aligned}$$

To see (2), it is enough to check that if $(v, s), (w, t) \in X_n$ and $\alpha, \beta \in 2^{\mathbb{N}}$, then there is no $h \in H_n$ such that $h \cdot \pi(v, s0\alpha) = \pi(w, t1\beta)$ (since H_n is symmetric). Suppose, towards a contradiction, that there is such an $h \in H_n$. Lemma 21 ensures that $\pi(v, s0\alpha) = h_{(v,s)} \cdot \pi(v^q, 0^{n+1}\alpha)$ and $\pi(w, t1\beta) = h_{(w,t)} g_n \cdot \pi(v^q, 0^{n+1}\beta)$. As $\pi(v^q, 0^{n+1}\alpha), \pi(v^q, 0^{n+1}\beta) \in A_{n+1}$, it follows that $\pi(w, t1\beta) \in hh_{(v,s)}(A_{n+1}) \cap h_t g_n(A_{n+1})$, which contradicts condition (i). □

As a corollary, we obtain the desired strengthening of Theorem 18:

Theorem 22. *Suppose that X is a Polish space, $\mathcal{G}$ is an analytic graph on X, $\chi(\mathcal{G}) \leq 2$, and $f : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. Then exactly one of the following holds:*

1. *$\chi_B(\mathcal{G}) \leq 2$;*
2. *There is an f-dominating $q \in \mathbb{Q}$ such that $\mathcal{G}_q \leq_c \mathcal{G}$.*

Proof. This is a direct consequence of Theorems 18 and 19. □

We are now ready to prove our main result:

Theorem 23. *Suppose that X is a Polish space, $\mathcal{G} \in \mathfrak{C}$, and $\mathfrak{B}$ is a basis for $\mathfrak{C}(\mathcal{G})$. Then the partial order $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ embeds into $(\mathfrak{B}, \leq_B)$.*

Proof. Theorem 22 ensures that for each $\mathcal{H} \in \mathfrak{C}(\mathcal{G})$, there is an f_0-dominating $q \in \mathbb{Q}$ such that $\mathcal{G}_q \leq_c \mathcal{G}$. Proposition 17 then implies that there is a pairwise incompatible family $\langle \mathcal{G}_\alpha \rangle_{\alpha \in 2^{\mathbb{N}}}$ of Borel graphs such that $\forall \alpha \in 2^{\mathbb{N}}$ $(\mathcal{G}_\alpha \sqsubseteq_B \mathcal{G}^q)$, and it follows that there is a pairwise incompatible family $\langle \mathcal{H}_\alpha \rangle_{\alpha \in 2^{\mathbb{N}}}$ of elements of $\mathfrak{B}$ such that $\forall \alpha \in 2^{\mathbb{N}}$ $(\mathcal{H}_\alpha \leq_B \mathcal{H})$. The theorem clearly follows from repeated application of this fact. □

References

[1] A. Kechris, S. Solecki, and S. Todorčević. Borel chromatic numbers. *Adv. Math.*, **141 (1)**, (1999), 1–44

[2] A. Louveau. Some dichotomy results for analytic graphs. Preprint.