Structural dichotomy theorems in descriptive set theory

Benjamin Miller
Universität Wien

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Introduction

The goal of these notes is to provide a succinct introduction to the primary structural dichotomy theorems of descriptive set theory. The only prerequisites are a rudimentary knowledge of point-set topology and set theory. Working in the base theory $\mathtt{ZF} + \mathtt{DC}$, we first discuss trees, the corresponding representations of closed, Borel, and Souslin sets, and Baire category. We then consider consequences of the open dihypergraph dichotomy and variants of the $\mathbb{G}_0$ dichotomy. While primarily focused upon Borel structures, we also note that minimal modifications of our arguments can be combined with well-known structural consequences of determinacy (which we take as a black box) to yield generalizations into the projective hierarchy and beyond.

Contents

CHAPTER 1

Preliminaries

1. Closed sets

Given a set I, define $I^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} I^n$ and $I^{\leq \mathbb{N}} = I^{<\mathbb{N}} \cup I^{\mathbb{N}}$. The *length* of a sequence $t \in I^{\leq \mathbb{N}}$ is given by $|t| = n$ if $t \in I^n$, and $|t| = \infty$ if $t \in I^{\mathbb{N}}$. Given sequences $s, t \in I^{\leq \mathbb{N}}$, we say that s is an *initial segment* of t, or t is an *extension* of s, if $|s| \leq |t|$ and $s = t \upharpoonright |s|$, in which case we write $s \sqsubseteq t$. In the special case that $s \neq t$, we say that s is a *proper initial segment* of t, or t is a *proper extension* of s, in which case we write $s \sqsubset t$. A *tree* on I is a set $T \subseteq I^{<\mathbb{N}}$ that is *closed under initial segments*, in the sense that $\forall t \in T \forall n < |t| \; t \upharpoonright n \in T$. A *subtree* of T is a tree $S \subseteq T$ on I. A *branch* through T is a sequence $x \in I^{\mathbb{N}}$ such that $\forall n \in \mathbb{N} \; x \upharpoonright n \in T$. We use $[T]$ to denote the set of all branches through T, and we say that T is *well-founded* if $[T] = \emptyset$.

Suppose now that I is a discrete topological space. For each sequence $s \in I^{<\mathbb{N}}$, let $\mathcal{N}_s$ denote the set of extensions of s in $I^{\mathbb{N}}$. These sets form a basis for the product topology on $I^{\mathbb{N}}$.

Proposition 1.1.1. *Suppose that I is discrete space and T is a tree on I. Then $[T]$ is closed.*

Proof. Observe that if $x \in \overline{[T]}$, then $\mathcal{N}_{x \upharpoonright n} \cap [T] \neq \emptyset$ for all $n \in \mathbb{N}$, so $x \upharpoonright n \in T$ for all $n \in \mathbb{N}$, thus $x \in [T]$. ⊠

Given a set $X \subseteq I^{\mathbb{N}}$, we use T_X to denote the set of proper initial segments of elements of X.

Proposition 1.1.2. *Suppose that I is a discrete space and $X \subseteq I^{\mathbb{N}}$. Then $\overline{X} = [T_X]$.*

Proof. Clearly $X \subseteq [T_X]$, so $\overline{X} \subseteq [T_X]$ by Proposition 1.1.1. Conversely, if $x \in [T_X]$, then $x \upharpoonright n \in T_X$ for all $n \in \mathbb{N}$, so $\mathcal{N}_{x \upharpoonright n} \cap X \neq \emptyset$ for all $n \in \mathbb{N}$, thus $x \in \overline{X}$. ⊠

We use (i) to denote the singleton sequence given by $s(0) = i$. The *concatenation* of sequences $s, t \in I^{<\mathbb{N}}$ is the extension $s \frown t$ of s given by $(s \frown t)(|s| + n) = t(n)$ for all $n < |t|$.

Proposition 1.1.3. *Suppose that I is a well-orderable discrete space and $C \subseteq I^{\mathbb{N}}$ is a non-empty closed set. Then there is a function $\beta \colon T_C \to C$ with the property that $\forall t \in T_C \; t \sqsubseteq \beta(t)$.*

Proof. Fix a well-ordering $\preceq$ of I, and define $\iota \colon T_C \to I$ by letting $\iota(t)$ be the $\preceq$-minimal element of I for which $t \frown (\iota(t)) \in T_C$. Define $\beta^n \colon T_C \to T_C$ by $\beta^0(t) = t$ and $\beta^{n+1}(t) = \beta^n(t) \frown ((\iota \circ \beta^n)(t))$, and set $\beta(t) = \bigcup_{n \in \mathbb{N}} \beta^n(t)$. ⊠

A *retraction* from a set X onto a subset Y is a surjection $\phi \colon X \twoheadrightarrow Y$ whose restriction to Y is the identity.

Proposition 1.1.4. *Suppose that I is a well-orderable discrete space and $C \subseteq I^{\mathbb{N}}$ is a non-empty closed set. Then there is a continuous retraction $\phi \colon I^{\mathbb{N}} \twoheadrightarrow C$.*

Proof. Proposition 1.1.2 ensures that for all sequences $x \in {\sim}C$, there is a maximal proper initial segment $\iota(x)$ of x in T_C, and Proposition 1.1.3 yields a function $\beta \colon T_C \to C$ such that $\forall t \in T_C \; t \sqsubseteq \beta(t)$. Let $\phi \colon I^{\mathbb{N}} \twoheadrightarrow C$ be the retraction agreeing with $\beta \circ \iota$ off of C. To see that ϕ is continuous, it is enough to show that if $n \in \mathbb{N}$ and $x \in I^{\mathbb{N}}$, then $\phi(\mathcal{N}_{x \upharpoonright n}) \subseteq \mathcal{N}_{\phi(x) \upharpoonright n}$. But if $x \upharpoonright n \in T_C$ then $\phi(\mathcal{N}_{x \upharpoonright n}) \subseteq \mathcal{N}_{x \upharpoonright n} = \mathcal{N}_{\phi(x) \upharpoonright n}$, and if $x \upharpoonright n \notin T_C$ then $\phi(\mathcal{N}_{x \upharpoonright n}) = \{\phi(x)\} \subseteq \mathcal{N}_{\phi(x) \upharpoonright n}$. ⊠

Now that we have explicitly proven and applied a particular instance of the axiom of choice, it should be noted that the axiom of determinacy rules out simply assuming the latter:

Theorem 1.1.5 (Solovay). *Suppose that* AD *holds. Then there is no injective ω_1-sequence of elements of $\mathbb{N}^{\mathbb{N}}$.*

2. Ranks

Suppose that R is a binary relation on X. For all $Y \subseteq X$, define $Y'_R = \{y \in Y \mid \exists x \in Y \; x \; R \; y\}$, $Y^{(0)}_R = Y$, $Y^{(\alpha+1)}_R = (Y^{(\alpha)}_R)'_R$ for all ordinals α, and $Y^{(\lambda)}_R = \bigcap_{\alpha < \lambda} Y^{(\alpha)}_R$ for all limit ordinals λ. The *rank* of R is the least ordinal $\rho(R)$ for which $X^{(\rho(R))}_R = X^{(\rho(R)+1)}_R$.

The relation R is *well-founded* if $Y \neq Y'_R$ for all non-empty sets $Y \subseteq X$. By DC, this is equivalent to the inexistence of a sequence $x \in X^{\mathbb{N}}$ with the property that $\forall n \in \mathbb{N} \; x(n+1) \; R \; x(n)$.

Proposition 1.2.1. *A binary relation R on a set X is well-founded if and only if $X^{(\rho(R))}_R = \emptyset$.*

Proof. It is clear that if R is well-founded, then $X^{(\rho(R))}_R = \emptyset$. Conversely, if there is a non-empty set $Y \subseteq X$ for which $Y = Y'$, then a straightforward transfinite induction shows that $Y \subseteq X^{(\rho(R))}_R$. ⊠

The rank of a point $x \in X$ with respect to R is the largest ordinal $\rho_R(x)$ for which $x \in X_R^{(\rho_R(x))}$, or ∞ if no such ordinal exists. We adopt the conventions that $\infty = \infty + 1$ and $\alpha < \infty$ for all ordinals α.

Proposition 1.2.2. *Suppose that R is a binary relation on a set X. Then $\forall x \in X \ \rho_R(x) = \sup\{\rho_R(w) + 1 \mid w \; R \; x\}$.*

Proof. Note that if α is an ordinal, $w \; R \; x$, and $w, x \in X_R^{(\alpha)}$, then $x \in X_R^{(\alpha+1)}$, so $\rho_R(x) \geq \rho_R(w) + 1$. But if $\alpha \geq \sup\{\rho_R(w) + 1 \mid w \; R \; x\}$ is an ordinal, then $x \notin X_R^{(\alpha+1)}$, so $\rho_R(x) \leq \alpha$. ⊠

The *horizontal sections* of a set $R \subseteq X \times Y$ are the sets of the form $R^y = \{x \in X \mid x \; R \; y\}$, where $y \in Y$. The *vertical sections* are the sets of the form $R_x = \{y \in Y \mid x \; R \; y\}$, where $x \in X$.

Proposition 1.2.3. *Suppose that X and Y are sets, R and S are binary relations on X and Y, and $\phi \colon X \to Y$ is a function.*

1. *If $\forall x \in X \ \phi(R^x) \subseteq S^{\phi(x)}$, then $\forall x \in X \ \rho_R(x) \leq \rho_S(\phi(x))$.*
2. *If $\forall x \in X \ S^{\phi(x)} \subseteq \phi(R^x)$, then $\forall x \in X \ \rho_R(x) \geq \rho_S(\phi(x))$.*

Proof. To see (1), note that if α is an ordinal for which $\rho_R(x) \leq \rho_S(\phi(x)))$ whenever $\rho_R(x) < \alpha$, then Proposition 1.2.2 ensures that

$$\begin{aligned} \rho_R(x) &= \sup\{\rho_R(w) + 1 \mid w \in R^x\} \\ &\leq \sup\{\rho_S(\phi(w)) + 1 \mid w \in R^x\} \\ &\leq \rho_S(\phi(x)) \end{aligned}$$

whenever $\rho_R(x) = \alpha$. Moreover, if $\rho_R(x) = \infty$, then $x \in X_R^{(\rho(R))}$, and since $\phi(X_R^{(\rho(R))}) \subseteq Y_S^{(\rho(S))}$, if follows that $\rho_S(\phi(x)) = \infty$.

To see (2), note that if α is an ordinal for which $\rho_R(x) \geq \rho_S(\phi(x)))$ whenever $\rho_R(x) < \alpha$, then Proposition 1.2.2 ensures that

$$\begin{aligned} \rho_R(x) &= \sup\{\rho_R(w) + 1 \mid w \in R^x\} \\ &\geq \sup\{\rho_S(\phi(w)) + 1 \mid w \in R^x\} \\ &\geq \rho_S(\phi(x)) \end{aligned}$$

whenever $\rho_R(x) = \alpha$. ⊠

3. Borel sets

Suppose that κ is an ordinal. A family of sets is a *κ-complete algebra* if it is closed under complements and unions of length strictly less than κ. An *algebra* is an $\aleph_0$-complete algebra, whereas a *σ-algebra* is an $\aleph_1$-complete algebra. A subset of a topological space is *κ-Borel* if it is in the smallest κ-complete algebra containing the open sets. A subset of a topological space is *Borel* if it is $\aleph_1$-Borel.

PROPOSITION 1.3.1. *Suppose that κ is an ordinal, X is set, and $\mathcal{X}$ is a family of subsets of X that is closed under complements. Then the closure of $\mathcal{X}$ under disjoint unions of length strictly less than κ and intersections of length strictly less than κ is a κ-complete algebra.*

PROOF. Let $\mathcal{Y}$ denote the family of sets $Y \subseteq X$ for which both Y and $\sim Y$ are in the desired closure. Clearly $\mathcal{X} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is closed under complements, so it is sufficient to show that $\mathcal{Y}$ is closed under unions of length strictly less than κ. Towards this end, suppose that $\lambda < \kappa$ and $(Y_\alpha)_{\alpha<\lambda}$ is a sequence of sets in $\mathcal{Y}$. Then the set $Z_\alpha = Y_\alpha \setminus \bigcup_{\beta<\alpha} Y_\beta = Y_\alpha \cap \bigcap_{\beta<\alpha} \sim Y_\beta$ is in the desired closure for all $\alpha < \lambda$, so the sets $\bigcup_{\alpha<\lambda} Y_\alpha = \bigcup_{\alpha<\lambda} Z_\alpha$ and $\sim\bigcup_{\alpha<\lambda} Y_\alpha = \bigcap_{\alpha<\lambda} \sim Y_\alpha$ are in the desired closure, and therefore in $\mathcal{Y}$. ⊠

A *code* for a $(\kappa+1)$-Borel subset of X is a pair (f, T), where T is a well-founded tree on $\kappa \times \kappa$ and f is a function associating to each sequence $t \in \sim T$ a subset of X that is closed or open. Given such a code, we recursively define $f^{(\alpha)}$ on $\sim T_{\sqsupset}^{(\alpha)}$ by setting $f^{(0)} = f$, letting $f^{(\alpha+1)}$ be the extension of $f^{(\alpha)}$ given by $f^{(\alpha+1)}(t) = \bigcup_{\beta<\kappa} \bigcap_{\gamma<\kappa} f^{(\alpha)}(t \frown ((\beta, \gamma)))$ whenever $\rho_{\sqsupset \restriction T}(t) = \alpha$ for all ordinals α, and defining $f^{(\lambda)} = \bigcup_{\alpha<\lambda} f^{(\alpha)}$ for all limit ordinals λ. Set $\overline{f} = f^{(\rho(\sqsupset \restriction T))}$. The $(\kappa+1)$-Borel set *coded by* (f, T) is $\overline{f}(\emptyset)$. While $\mathtt{AC}_\kappa$ and Proposition 1.3.1 ensure that every $(\kappa+1)$-Borel set is of this form, merely being $(\kappa+1)$-Borel is not a reasonable notion of definability in the absence of $\mathtt{AC}_\kappa$. Although it is easy to modify our arguments to produce sets which have $(\kappa+1)$-Borel codes, we will focus on $(\kappa+1)$-Borel sets for the sake of clarity.

4. Souslin sets

A topological space is *κ-Souslin* if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$, where κ is endowed with the discrete topology. A topological space is *analytic* if it is $\aleph_0$-Souslin.

PROPOSITION 1.4.1. *Suppose that κ is an aleph and X is non-empty and κ-Souslin. Then there is a continuous surjection $\phi\colon \kappa^{\mathbb{N}} \twoheadrightarrow X$.*

PROOF. Fix a closed set $C \subseteq \kappa^{\mathbb{N}}$ for which there is a continuous surjection $\phi'\colon C \twoheadrightarrow X$, appeal to Proposition 1.1.4 to obtain a continuous retraction $\phi''\colon \kappa^{\mathbb{N}} \twoheadrightarrow C$, and define $\phi = \phi' \circ \phi''$. ⊠

PROPOSITION 1.4.2. *Suppose that κ is an aleph, X is a κ-Souslin space, Y is a topological space, and $\phi\colon X \to Y$ is continuous. Then:*

(1) *The set $\phi(X)$ is κ-Souslin.*
(2) *If Y is Hausdorff and $A \subseteq Y$ is κ-Souslin, then $\phi^{-1}(A)$ is κ-Souslin.*

Proof. Clearly we can assume that A and X are non-empty, in which case Proposition 1.4.1 yields continuous surjections $\phi_A \colon \kappa^{\mathbb{N}} \twoheadrightarrow A$ and $\phi_X \colon \kappa^{\mathbb{N}} \twoheadrightarrow X$. To see (1), note that $(\phi \circ \phi_X)(\kappa^{\mathbb{N}}) = \phi(X)$. To see (2), let $\pi \colon \kappa^{\mathbb{N}} \times \kappa^{\mathbb{N}} \to \kappa^{\mathbb{N}}$ be the projection onto the left coordinate, and note that the set $C = \{(a, b) \in \kappa^{\mathbb{N}} \times \kappa^{\mathbb{N}} \mid (\phi \circ \phi_X)(a) = \phi_A(b)\}$ is closed and $(\phi_X \circ \pi)(C) = \phi^{-1}(A)$. ⊠

Proposition 1.4.3. *Suppose that κ is an aleph, X is a topological space, $\phi_\alpha \colon \kappa^{\mathbb{N}} \to X$ is continuous for all $\alpha < \kappa$, and $A_\alpha = \phi_\alpha(\kappa^{\mathbb{N}})$ for all $\alpha < \kappa$. Then:*

1. *The set $\bigcup_{\alpha<\kappa} A_\alpha$ is κ-Souslin.*
2. *The set $\prod_{n\in\mathbb{N}} A_n$ is κ-Souslin.*
3. *If X is Hausdorff, then $\bigcap_{n\in\mathbb{N}} A_n$ is κ-Souslin.*

Proof. To see (1), note that the function $(\alpha) \frown b \mapsto \phi_\alpha(b)$ is a continuous surjection from $\kappa^{\mathbb{N}}$ onto $\bigcup_{\alpha<\kappa} A_\alpha$.

To see (2), note that the function $(b_n)_{n\in\mathbb{N}} \mapsto (\phi_n(b_n))_{n\in\mathbb{N}}$ is a continuous surjection from $(\kappa^{\mathbb{N}})^{\mathbb{N}}$ onto $\prod_{n\in\mathbb{N}} A_n$.

To see (3), obtain a continuous surjection $\phi \colon \kappa^{\mathbb{N}} \twoheadrightarrow \prod_{n\in\mathbb{N}} A_n$ as above, let $\pi \colon X^{\mathbb{N}} \to X$ be the projection onto the 0^{th} coordinate, and note that the set $C = \phi^{-1}(\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n \mid \forall n \in \mathbb{N}\ x_m = x_n\})$ is closed and $(\pi \circ \phi)(C) = \bigcap_{n\in\mathbb{N}} A_n$. ⊠

Given a pointclass Γ of subsets of topological spaces, we say that a subset of a topological space is *co-*Γ if its complement is Γ, and *bi-*Γ if it is both Γ and co-Γ.

Proposition 1.4.4. *Suppose that κ is an aleph, X is a κ-Souslin space, and $C \subseteq X$ is closed. Then C is bi-κ-Souslin.*

Proof. By Proposition 1.4.1, we can assume that there is a continuous surjection $\phi \colon \kappa^{\mathbb{N}} \twoheadrightarrow X$. To see that C is κ-Souslin, note that the set $D = \phi^{-1}(C)$ is closed and $\phi(D) = C$. To see that C is co-κ-Souslin, note that $\sim D$ is open, set $S = \{s \in \kappa^{<\mathbb{N}} \mid \mathcal{N}_s \subseteq \sim D\}$, and observe that $\sim C = \bigcup_{s\in S} \phi(\mathcal{N}_s)$, so $\sim C$ is κ-Souslin by Proposition 1.4.3. ⊠

Proposition 1.4.5. *Suppose that κ is an aleph and X is a κ-Souslin Hausdorff space. Then every Borel subset of X is bi-κ-Souslin.*

Proof. By Propositions 1.3.1, 1.4.3, and 1.4.4. ⊠

In order to establish a natural strengthening of the converse, we will need the following simple observation:

Proposition 1.4.6. *Suppose that κ is an aleph, X is a Hausdorff space, and $\phi, \psi \colon \kappa^{\mathbb{N}} \to X$ are continuous. Then for all $c, d \in \kappa^{\mathbb{N}}$ such that $\phi(c) \neq \psi(d)$, there exists $n \in \mathbb{N}$ for which $\overline{\phi(\mathcal{N}_{c\restriction n})} \cap \psi(\mathcal{N}_{d\restriction n}) = \emptyset$.*

PROOF. As X is Hausdorff, there are disjoint open neighborhoods U and V of $\phi(c)$ and $\psi(d)$. As ϕ and ψ are continuous, there exists $n \in \mathbb{N}$ sufficiently large that $\phi(\mathcal{N}_{c\restriction n}) \subseteq U$ and $\psi(\mathcal{N}_{d\restriction n}) \subseteq V$. But then $\overline{\phi(\mathcal{N}_{c\restriction n})}$ is contained in $\sim V$, and therefore disjoint from $\psi(\mathcal{N}_{d\restriction n})$. ⊠

We say that sets A and B are *separated* by a set C if $A \subseteq C$ and $B \cap C = \emptyset$.

THEOREM 1.4.7 (Lusin). *Suppose that κ is an aleph, X is a Hausdorff space, and $A, B \subseteq X$ are disjoint κ-Souslin sets. Then there is a $(\kappa+1)$-Borel set $C \subseteq X$ separating A from B.*

PROOF. By Proposition 1.4.1, we can assume that there are continuous surjections $\phi\colon \kappa^{\mathbb{N}} \twoheadrightarrow A$ and $\psi\colon \kappa^{\mathbb{N}} \twoheadrightarrow B$. Set $A_t = \phi(\mathcal{N}_t)$ and $B_t = \psi(\mathcal{N}_t)$ for all $t \in \kappa^{<\mathbb{N}}$, define $\pi_i\colon (\kappa \times \kappa)^{<\mathbb{N}} \to \kappa^{<\mathbb{N}}$ by $\pi_i(t)(n) = t(n)(i)$ for all $i < 2$, and let T be the tree on $\kappa \times \kappa$ of all sequences $t \in (\kappa \times \kappa)^{<\mathbb{N}}$ for which $\overline{A_{\pi_0(t)}} \cap B_{\pi_1(t)} \neq \emptyset$. Proposition 1.4.6 ensures that T is well-founded. Define f on $\sim T$ by $f(t) = \overline{A_{\pi_0(t)}}$, noting that (f, T) is a code for a $(\kappa+1)$-Borel subset of X.

LEMMA 1.4.8. *Suppose that $t \in (\kappa \times \kappa)^{<\mathbb{N}}$. Then $\overline{f}(t)$ separates $A_{\pi_0(t)}$ from $B_{\pi_1(t)}$.*

PROOF. The definition of T ensures that $\overline{f}(t)$ separates $A_{\pi_0(t)}$ from $B_{\pi_1(t)}$ for all $t \in \sim T$. But if $\overline{f}(t \frown ((\alpha, \beta)))$ separates $A_{\pi_0(t)\frown(\alpha)}$ from $B_{\pi_1(t)\frown(\beta)}$ for all $\alpha, \beta < \kappa$, then $\bigcap_{\beta<\kappa} \overline{f}(t \frown ((\alpha, \beta)))$ separates $A_{\pi_0(t)\frown(\alpha)}$ from $B_{\pi_1(t)}$ for all $\alpha < \kappa$, so $\bigcup_{\alpha<\kappa} \bigcap_{\beta<\kappa} \overline{f}(t \frown ((\alpha, \beta)))$ separates $A_{\pi_0(t)}$ from $B_{\pi_1(t)}$, thus the obvious induction yields the desired result. ⊠

The special case of Lemma 1.4.8 where $t = \emptyset$ ensures that the $(\kappa+1)$-Borel set coded by (f, T) separates A from B. ⊠

THEOREM 1.4.9 (Souslin). *Suppose that X is a Hausdorff space. Then every bi-κ-Souslin subset of X is $(\kappa+1)$-Borel.*

PROOF. By the special case of Theorem 1.4.7 where $A = \sim B$. ⊠

THEOREM 1.4.10 (Souslin). *Suppose that X is an analytic Hausdorff space. Then the families of bi-analytic and Borel subsets of X coincide.*

PROOF. By Proposition 1.4.5 and Theorem 1.4.9. ⊠

PROPOSITION 1.4.11. *Suppose that X is an $\aleph_1$-Souslin Hausdorff space and $C \subseteq X$ is co-analytic. Then C is $\aleph_1$-Souslin.*

PROOF. By Proposition 1.4.1, we can assume that there is a continuous surjection $\phi\colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow \sim C$. Set $T_x = \{t \in \mathbb{N}^{<\mathbb{N}} \mid x \in \overline{\phi(\mathcal{N}_t)}\}$ for all $x \in X$, and observe that the set B of all $(f, x) \in \omega_1^{\mathbb{N}^{<\mathbb{N}}} \times X$ such that $\forall n \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$ $(t \frown (n) \notin T_x$ or $f(t \frown (n)) < f(t))$ is Borel. As Proposition 1.4.3 ensures that $\omega_1^{\mathbb{N}^{<\mathbb{N}}} \times X$ is $\aleph_1$-Souslin, Proposition 1.4.5 implies that so too is B.

LEMMA 1.4.12. *The sets C and $\mathrm{proj}_X(B)$ coincide.*

PROOF. If $x \in \sim C$, then T_x is not well-founded, so $x \notin \mathrm{proj}_X(B)$. If $x \in C$, then the special case of Proposition 1.4.6 in which ψ is constant ensures that T_x is well-founded, so Propositions 1.2.1 and 1.2.2 imply that $(f, x) \in B$ if $\forall t \in T_x\ f(t) = \rho_{\sqsupset \restriction T_x}(t)$, thus $x \in \mathrm{proj}_X(B)$. ⊠

As Proposition 1.4.2 ensures that $\mathrm{proj}_X(B)$ is $\aleph_1$-Souslin, Lemma 1.4.12 implies that so too is C. ⊠

A subset of an analytic Hausdorff space is $\mathbf{\Sigma}^1_1$ if it is analytic. More generally, for each natural number $n > 0$, a subset of an analytic Hausdorff space is $\mathbf{\Pi}^1_n$ if its complement is $\mathbf{\Sigma}^1_n$, and $\mathbf{\Sigma}^1_{n+1}$ if it is a continuous image of a $\mathbf{\Pi}^1_n$ subset of an analytic Hausdorff space. A subset of an analytic Hausdorff space is $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

A *quasi-order* on a set X is a reflexive transitive binary relation R on X. The *equivalence relation* associated with such a quasi-order is the binary relation $\equiv_R$ on X for which $x \equiv_R y$ if and only if $x \mathrel{R} y$ and $y \mathrel{R} x$. A *partial order* is a quasi-order for which the corresponding equivalence relation is equality. For all $n > 0$, let $\boldsymbol{\delta}^1_n$ denote the supremum of the lengths of well-orderings of the form $R/{\equiv_R}$, where R is a $\mathbf{\Delta}^1_n$ quasi-order on an analytic Hausdorff space.

As strict embeddability of well-orderings of $\mathbb{N}$ is an analytic binary relation on a co-analytic subset of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$, it follows that $\boldsymbol{\delta}^1_2 > \omega_1$. The following theorem ensures that $\boldsymbol{\delta}^1_1 = \omega_1$, and when combined with Propositions 1.4.3 and 1.4.11, it also implies that $\boldsymbol{\delta}^1_2 \leq \omega_2$:

THEOREM 1.4.13 (Kunen-Martin). *Suppose that κ is an aleph, X is a Hausdorff space, and R is a well-founded κ-Souslin binary relation on X. Then $\rho(R) < \kappa^+$.*

PROOF. By Proposition 1.4.1, we can assume that there is a continuous surjection $(\phi, \psi)\colon \kappa^{\mathbb{N}} \twoheadrightarrow R$. Let S be the set of non-empty sequences $s \in (\kappa^{\mathbb{N}})^{<\mathbb{N}}$ such that $\forall n < |s| - 1\ \phi(s(n)) = \psi(s(n+1))$. The well-foundedness of R yields that of $\sqsupset \restriction S$. Define $\pi\colon S \to X$ by $\pi(s) = \phi(s(|s| - 1))$, and observe that $\forall s \in S\ R^{\pi(s)} \subseteq \pi((\sqsupset \restriction S)^s)$ and

$\pi(S) = \{x \in X \mid R_x \neq \emptyset\}$, so Propositions 1.2.1 – 1.2.3 ensure that

$$\begin{aligned}\rho(\sqsupset \upharpoonright S) + 1 &= \sup\{\rho_{\sqsupset \upharpoonright S}(s) + 1 \mid s \in S\} + 1 \\ &\geq \sup\{\rho_R(\pi(s)) + 1 \mid s \in S\} + 1 \\ &\geq \sup\{\rho_R(x) \mid x \in X\} + 1 \\ &\geq \sup\{\rho_R(x) + 1 \mid x \in X\} \\ &= \rho(R),\end{aligned}$$

thus it is sufficient to show that $\rho(\sqsupset \upharpoonright S) < \kappa^+$.

Let T be the set of sequences $t \in \bigcup_{n>0}(\kappa^n)^n$ with the property that $\forall n < |t| - 1\ \phi(\mathcal{N}_{t(n)}) \cap \psi(\mathcal{N}_{t(n+1)}) \neq \emptyset$, and let $\preceq$ be the partial order on T given by $s \preceq t \iff \forall n < |s|\ s(n) \sqsubseteq t(n)$. By Proposition 1.4.6, the well-foundedness of $\sqsupset \upharpoonright S$ yields that of $\succ$. Define $\pi' \colon S \to T$ by $\pi'(s)(n) = s(n) \upharpoonright |s|$ for all $n < |s|$. As $\forall s \in S\ \pi'((\sqsupset \upharpoonright S)^s) \subseteq \succ^{\pi'(s)}$, Propositions 1.2.1 and 1.2.3 ensure that

$$\begin{aligned}\rho(\sqsupset \upharpoonright S) &= \sup\{\rho_{\sqsupset \upharpoonright S}(s) + 1 \mid s \in S\} \\ &\leq \sup\{\rho_{\succ}(\pi'(s)) + 1 \mid s \in S\} \\ &\leq \sup\{\rho_{\succ}(t) + 1 \mid t \in T\} \\ &= \rho(\succ),\end{aligned}$$

so it is sufficient to show that $\rho(\succ) < \kappa^+$. But this follows from the fact that $|T| \leq \kappa$. ⊠

The axiom of determinacy provides the primary motivation for studying κ^+-Borel and κ-Souslin sets when $\kappa > \aleph_1$:

THEOREM 1.4.14 (Kechris, Martin, Moschovakis). *Suppose that* AD *holds and $n \in \mathbb{N}$. Then there is an aleph $\boldsymbol{\kappa}^1_{2n+1}$ with the property that $\boldsymbol{\delta}^1_{2n+1} = (\boldsymbol{\kappa}^1_{2n+1})^+$. Moreover:*

1. *The $\boldsymbol{\Delta}^1_{2n+1}$ and $(\boldsymbol{\kappa}^1_{2n+1})^+$-Borel subsets of analytic Hausdorff spaces coincide.*
2. *The $\boldsymbol{\Sigma}^1_{2n+1}$ and $\boldsymbol{\kappa}^1_{2n+1}$-Souslin subsets of analytic Hausdorff spaces coincide.*
3. *The $\boldsymbol{\Sigma}^1_{2n+2}$ and $(\boldsymbol{\kappa}^1_{2n+1})^+$-Souslin subsets of analytic Hausdorff spaces coincide.*

THEOREM 1.4.15 (Woodin). *Suppose that* $\mathsf{AD}_{\mathbb{R}}$ *holds, X is an analytic Hausdorff space, and $Y \subseteq X$. Then there is an aleph κ for which Y is κ-Souslin.*

5. Baire category

A subset of a topological space is *meager* if it is a union of countably-many nowhere dense sets. A subset of a topological space is *comeager* if its complement is meager, or equivalently, if it contains an intersection of countably-many dense open sets. A *Baire space* is a topological space all of whose comeager subsets are dense.

THEOREM 1.5.1 (Baire). *Every complete metric space X is a Baire space.*

PROOF. Suppose that $C \subseteq X$ is comeager and $U \subseteq X$ is non-empty and open, and fix positive real numbers $\epsilon_n \to 0$ and dense open sets $U_n \subseteq X$ for which $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. By $\mathtt{DC}$, there is a sequence $(V_n)_{n \in \mathbb{N}}$ of non-empty open subsets of U with the property that $\operatorname{diam}(V_n) \leq \epsilon_n$, $V_n \subseteq U_n$, and $\overline{V_{n+1}} \subseteq V_n$ for all $n \in \mathbb{N}$. Then the unique point of $\bigcap_{n \in \mathbb{N}} V_n$ is in $C \cap U$. ⊠

PROPOSITION 1.5.2. *Suppose that X is a Baire space. Then every non-empty open set $U \subseteq X$ is a Baire space.*

PROOF. Suppose that $(U_n)_{n \in \mathbb{N}}$ is a sequence of dense open subsets of U, let V be the interior of $\sim U$, and observe that $U_n \cup V$ is a dense subset of X for all $n \in \mathbb{N}$, so $\bigcap_{n \in \mathbb{N}} U_n \cup V$ is also a dense subset of X, thus $\bigcap_{n \in \mathbb{N}} U_n$ is a dense subset of U. ⊠

PROPOSITION 1.5.3. *Suppose that X is a topological space, $U \subseteq X$ is a non-empty open set, and $Y \subseteq U$. Then Y is meager in U if and only if Y is meager in X.*

PROOF. It is sufficient to show that Y is nowhere dense in U if and only if Y is nowhere dense in X. As the closure of Y within U is the intersection of U with the closure of Y within X, it follows that if Y is somewhere dense in U then it is somewhere dense in X. Conversely, if Y is somewhere dense in X, then there is a non-empty open set $V \subseteq X$ contained in the closure of Y within X, and since any such set is contained in the closure of U within X, it follows that $U \cap V \neq \emptyset$, thus Y is somewhere dense in U. ⊠

The *symmetric difference* of sets X and Y is the set $X \bigtriangleup Y$ of points appearing in exactly one of them. A subset of a topological space has the *Baire property* if its symmetric difference with some open subset of the space is meager.

PROPOSITION 1.5.4. *Suppose that X is a topological space and $B \subseteq X$ has the Baire property. Then at least one of the following holds:*

(1) *The set B is meager.*
(2) *There is a non-empty open set $U \subseteq X$ with the property that $B \cap U$ is comeager in U.*

Moreover, if X is a Baire space, then exactly one of these holds.

Proof. Fix an open set $U \subseteq X$ such that $B \triangle U$ is meager. If U is empty, then B is meager. Otherwise, since $U \setminus B$ is meager in X, Proposition 1.5.3 ensures that it is meager in U, in which case $B \cap U$ is comeager in U.

To see that conditions (1) and (2) are mutually exclusive when X is a Baire space, suppose that there is a non-empty open set $U \subseteq X$ with the property that $B \cap U$ is comeager in U. If B is meager, then $B \cap U$ is meager in U by Proposition 1.5.3, so $(B \cap U) \cap (U \setminus B)$ is comeager in U, contradicting Proposition 1.5.2. ⊠

Proposition 1.5.5. *Suppose that X and Y are topological spaces, $\phi \colon X \twoheadrightarrow Y$ is a continuous open surjection, and $D \subseteq Y$. Then D is comeager if and only if the set $C = \phi^{-1}(D)$ is comeager.*

Proof. Suppose first that D is comeager. Then there are dense open sets $V_n \subseteq Y$ such that $\bigcap_{n \in \mathbb{N}} V_n \subseteq D$. The fact that ϕ is continuous ensures that the sets $U_n = \phi^{-1}(V_n)$ are open, and the fact that ϕ is open implies that they are dense, thus C is comeager.

Conversely, suppose that C is comeager. Then there are dense open sets $U_n \subseteq X$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. The fact that ϕ is open ensures that the sets $V_n = \phi(U_n)$ are open, and the fact that ϕ is a continuous surjection implies that they are dense, thus D is comeager. ⊠

Proposition 1.5.6. *Suppose that X is a second-countable Baire space and $Y \subseteq X$. Then there is a maximal open set $U \subseteq X$ for which $U \setminus Y$ is meager. Moreover, every set $B \subseteq X$ with the Baire property contained in $Y \setminus U$ is meager.*

Proof. Fix a countable basis $\mathcal{U}$ for X, and define $\mathcal{V} = \{U \in \mathcal{U} \mid U \setminus Y \text{ is meager}\}$ and $U = \bigcup \mathcal{V}$. Then $U \setminus Y = \bigcup_{V \in \mathcal{V}} V \setminus Y$ is meager. To see that U is the maximal such open set, note that if $U' \subseteq X$ is an open set not contained in U, then it contains a non-empty set $U'' \in \mathcal{U}$ not contained in U, so $U'' \notin \mathcal{V}$, thus $U'' \setminus Y$ is not meager, hence $U' \setminus Y$ is not meager.

Suppose, towards a contradiction, that there is a non-meager set $B \subseteq X$ with the Baire property contained in $Y \setminus U$. Proposition 1.5.4 yields a non-empty open set $W \subseteq X$ in which $B \cap W$ is comeager. Fix a non-empty set $V \subseteq W$ in $\mathcal{U}$. Proposition 1.5.3 ensures that $V \setminus B$ is meager, so $V \setminus Y$ is meager, thus $V \in \mathcal{V}$, hence $V \subseteq U$, in which case B is disjoint from V. But Proposition 1.5.3 implies that $B \cap V$ is comeager in V, contradicting Proposition 1.5.2. ⊠

Let BP_X denote the family of subsets of X with the Baire property, and $\mathcal{M}_X$ the family of all meager subsets of X. The *additivity* of a family $\mathcal{F}$ of sets is the least aleph $\mathtt{add}(\mathcal{F})$ with the property that there is a sequence $(F_\alpha)_{\alpha < \mathtt{add}(\mathcal{F})}$ of sets in $\mathcal{F}$ whose union is not in $\mathcal{F}$, or ∞ if no such aleph exists.

Proposition 1.5.7. *Suppose that X is a second-countable Baire space. Then BP_X contains every open subset of X and is closed under complements, and $\mathtt{add}(\mathrm{BP}_X) \geq \mathtt{add}(\mathcal{M}_X)$.*

Proof. As the empty set is meager, it follows that every open subset of X has the Baire property.

To see that BP_X is closed under complements, suppose that $B \subseteq X$ has the Baire property, fix an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager, set $C = {\sim}B$, let V be the interior of ${\sim}U$, and note that $C \bigtriangleup V \subseteq (C \bigtriangleup ({\sim}U)) \cup (({\sim}U) \bigtriangleup V) = (B \bigtriangleup U) \cup {\sim}(U \cup V)$. As $U \cup V$ is dense and open, it follows that C has the Baire property.

To see that the family of subsets of X with the Baire property is closed under unions of every length $\kappa < \mathtt{add}(\mathcal{M}_X)$, suppose that $(B_\alpha)_{\alpha<\kappa}$ is a sequence of subsets of X with the Baire property, and note that if $(U_\alpha)_{\alpha<\kappa}$ is a sequence of open subsets of X such that $B_\alpha \bigtriangleup U_\alpha$ is meager for all $\alpha < \kappa$, and $B = \bigcup_{\alpha<\kappa} B_\alpha$ and $U = \bigcup_{\alpha<\kappa} U_\alpha$, then $B \bigtriangleup U \subseteq \bigcup_{\alpha<\kappa} B_\alpha \bigtriangleup U_\alpha$ is meager, thus B has the Baire property. As the existence of such a sequence $(U_\alpha)_{\alpha<\kappa}$ is clear in the special case where $\kappa = 2$, it follows that $B_\alpha \setminus V = {\sim}({\sim}B_\alpha \cup V)$ has the Baire property for all $\alpha < \kappa$ and open sets $V \subseteq X$, so Proposition 1.5.6 yields the existence of such a sequence $(U_\alpha)_{\alpha<\kappa}$ in the general case. ⊠

Proposition 1.5.8. *Suppose that X is a second-countable Baire space and $\kappa < \mathtt{add}(\mathcal{M}_X)$ is an aleph. Then every κ^+-Borel set $B \subseteq X$ has the Baire property.*

Proof. By Proposition 1.5.7. ⊠

Theorem 1.5.9 (Lusin-Sierpiński). *Suppose that X is a second-countable Baire Hausdorff space and $\kappa < \mathtt{add}(\mathcal{M}_X)$ is an aleph. Then every κ-Souslin set $A \subseteq X$ has the Baire property.*

Proof. By Proposition 1.4.1, we can assume that there is a continuous surjection $\phi \colon \kappa^{\mathbb{N}} \twoheadrightarrow A$. For all $t \in \kappa^{<\mathbb{N}}$, set $A_t = \phi(\mathcal{N}_t)$, appeal to Proposition 1.5.6 to obtain a maximal open set $U_t \subseteq X$ for which $U_t \setminus {\sim}A_t$ is meager, and define $C_t = \overline{A_t} \setminus U_t$.

Lemma 1.5.10. *For all $t \in \kappa^{<\mathbb{N}}$, the set $A_t \setminus C_t$ is meager.*

Proof. Note that $A_t \setminus C_t = A_t \setminus (\overline{A_t} \setminus U_t) = A_t \setminus {\sim}U_t = U_t \setminus {\sim}A_t$. ⊠

As $A \setminus \bigcup_{b \in \kappa^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} C_{b \restriction n} \subseteq \bigcup_{t \in \kappa^{<\mathbb{N}}} A_t \setminus C_t$ and Lemma 1.5.10 ensures that the latter set is meager, so too is the former. As the special case of Proposition 1.4.6 where ψ is a constant function implies that $\bigcup_{b \in \kappa^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} C_{b \restriction n} \subseteq A$, it is sufficient to show that $\bigcup_{b \in \kappa^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} C_{b \restriction n}$ has the Baire property. As $C_\emptyset \setminus \bigcup_{b \in \kappa^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} C_{b \restriction n} \subseteq \bigcup_{t \in \kappa^{<\mathbb{N}}} C_t \setminus \bigcup_{\alpha < \kappa} C_{t \frown (\alpha)}$ and Proposition 1.5.7 ensures that $C_\emptyset$ has the Baire property, it is sufficient to show that $C_t \setminus \bigcup_{\alpha < \kappa} C_{t \frown (\alpha)}$ is meager for all $t \in \kappa^{<\mathbb{N}}$. As $(C_t \setminus \bigcup_{\alpha < \kappa} C_{t \frown (\alpha)}) \setminus (\bigcup_{\alpha < \kappa} A_{t \frown (\alpha)} \setminus C_{t \frown (\alpha)}) \subseteq C_t \setminus \bigcup_{\alpha < \kappa} A_{t \frown (\alpha)} = C_t \setminus A_t$, Proposition 1.5.7 ensures that $C_t \setminus \bigcup_{\alpha < \kappa} C_{t \frown (\alpha)}$ has the Baire property, and Lemma 1.5.10 implies that $\bigcup_{\alpha < \kappa} A_{t \frown (\alpha)} \setminus C_{t \frown (\alpha)}$ is meager, it only remains to note that every subset of X with the Baire property contained in $C_t \setminus A_t$ is meager, which follows from Proposition 1.5.6 and the observation that $C_t \setminus A_t = (\overline{A_t} \setminus U_t) \setminus A_t \subseteq ({\sim}U_t) \setminus A_t = ({\sim}A_t) \setminus U_t$. ⊠

Theorem 1.5.11 (Banach-Mazur). *Suppose that* $\mathtt{AD}$ *holds and X is a second-countable complete metric space. Then every set $Y \subseteq X$ has the Baire property.*

Theorem 1.5.12 (Montgomery, Novikov). *Suppose that X is a topological space, Y is a second-countable Baire space, $\kappa < \mathtt{add}(\mathcal{M}_Y)$ is an aleph, and $R \subseteq X \times Y$ is a κ^+-Borel set. Then $\{x \in X \mid R_x \cap V$ is not meager$\}$ is κ^+-Borel for all non-empty open sets $V \subseteq Y$.*

Proof. Clearly the family of κ^+-Borel sets $R \subseteq X \times Y$ with the desired property contains every κ^+-Borel rectangle. To see that it is closed under unions of length κ, suppose that $(R_\alpha)_{\alpha < \kappa}$ is a sequence of κ^+-Borel sets, set $R = \bigcup_{\alpha < \kappa} R_\alpha$, suppose that $V \subseteq Y$ is a non-empty open set, and observe that $\{x \in X \mid R_x \cap V \text{ is not meager}\} = \bigcup_{\alpha < \kappa} \{x \in X \mid (R_\alpha)_x \cap V \text{ is not meager}\}$. To see that it is closed under complements, suppose that $R \subseteq X \times Y$ is a κ^+-Borel set, set $S = {\sim}R$, suppose that $V \subseteq Y$ is a non-empty open set, fix a countable basis $\mathcal{W}$ for V consisting solely of non-empty open sets, and observe that $\{x \in X \mid S_x \cap V \text{ is not meager}\} = \bigcup_{W \in \mathcal{W}} \{x \in X \mid R_x \cap W \text{ is meager}\}$ by Propositions 1.5.3, 1.5.4, and 1.5.8. ⊠

Theorem 1.5.13 (Kuratowski-Ulam). *Suppose that X is a Baire space, Y is a second-countable Baire space, and $R \subseteq X \times Y$ has the Baire property.*

1. *The set $\{x \in X \mid R_x$ has the Baire property$\}$ is comeager.*
2. *The set R is comeager if and only if $\{x \in X \mid R_x$ is comeager$\}$ is comeager.*

PROOF. We first establish the special case of ($\Longrightarrow$) in (2) where R is dense and open. For each non-empty open set $V \subseteq Y$, define $V' = \mathrm{proj}_X(R \cap (X \times V))$. Note that if $U \subseteq X$ is a non-empty open set, then $R \cap (U \times V) \neq \emptyset$, so $U \cap V' \neq \emptyset$, thus V' is dense. Fix a countable basis $\mathcal{V}$ for Y consisting of non-empty sets, and note that the set $C = \bigcap_{V \in \mathcal{V}} V'$ is comeager, and R_x is dense and open for all $x \in C$.

We next establish ($\Longrightarrow$) in (2). Fix dense open sets $R_n \subseteq X \times Y$ with the property that $\bigcap_{n \in \mathbb{N}} R_n \subseteq R$, and observe that the sets $C_n = \{x \in X \mid (R_n)_x \text{ is dense and open}\}$ are comeager, thus so too is the set $C = \bigcap_{n \in \mathbb{N}} C_n$, and $\bigcap_{n \in \mathbb{N}} (R_n)_x$ is comeager for all $x \in C$.

To see (1), fix an open set $W \subseteq X \times Y$ for which $R \triangle W$ is meager, note that the set $C = \{x \in X \mid R_x \triangle W_x \text{ is meager}\}$ is comeager, and observe that R_x has the Baire property for all $x \in C$.

It only remains to establish ($\Longleftarrow$) in (2). Towards this end, note that $W \setminus (R \triangle W) \subseteq R$, so if W is dense, then R is comeager. But if W is not dense, then there are non-empty open sets $U \subseteq X$ and $V \subseteq Y$ with the property that $(U \times V) \cap W = \emptyset$, and if $x \in U$, then $R_x \cap V \subseteq R_x \setminus W_x \subseteq R_x \triangle W_x$, so Proposition 1.5.3 yields comeagerly many $x \in U$ for which $R_x \cap V$ is both comeager in V and meager in V, contradicting Proposition 1.5.2. ⊠

PROPOSITION 1.5.14. *Suppose that X is a second-countable Baire space. Then $\mathtt{add}(\mathrm{BP}_{X \times X}) \leq \mathtt{add}(\mathcal{M}_X)$.*

PROOF. Suppose, towards a contradiction, that $\mathtt{add}(\mathcal{M}_X)$ is strictly less than $\mathtt{add}(\mathrm{BP}_{X \times X})$, and fix a sequence $(M_\alpha)_{\alpha < \mathtt{add}(\mathcal{M}_X)}$ of meager subsets of X whose union M is not meager. Associate with each $x \in M$ the least ordinal $\alpha(x)$ for which $x \in M_{\alpha(x)}$, and let $\preceq$ be the quasi-order on M given by $x \preceq y \iff \alpha(x) \leq \alpha(y)$. As products of meager sets are meager, and $\preceq$ is a union of strictly fewer than $\mathtt{add}(\mathrm{BP}_{X \times X})$-many such products, it follows that $\preceq$ has the Baire property. As every horizontal section of $\preceq$ is meager, Theorem 1.5.13 yields a meager vertical section of $\preceq$. But M is the union of any such set with the corresponding horizontal section, and is therefore meager, a contradiction. ⊠

6. Canonical objects

A *homomorphism* from a D-ary relation R on X to a D-ary relation S on Y is a function $\phi\colon X \to Y$ for which $\phi^D(R) \subseteq S$. The *diagonal* on X is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$.

THEOREM 1.6.1 (Mycielski). *Suppose that R is a meager binary relation on $2^{\mathbb{N}}$. Then there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\sim\Delta(2^{\mathbb{N}})$ to $\sim R$.*

Proof. Fix a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of dense open subsets of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ whose intersection is disjoint from R.

Lemma 1.6.2. *Suppose that $n \in \mathbb{N}$ and $\phi\colon 2^n \to 2^{<\mathbb{N}}$. Then there is a function $\psi\colon 2^{n+1} \to 2^{<\mathbb{N}}$ such that:*

- $\forall t \in 2^{n+1}\ \phi(t \restriction n) \sqsubset \psi(t)$.
- $\forall t \in {\sim}\Delta(2^{n+1})\ \prod_{i<2} \mathcal{N}_{\psi(t(i))} \subseteq U_n$.

Proof. Fix an enumeration $(t_k)_{k<4^{n+1}-2^{n+1}}$ of ${\sim}\Delta(2^{n+1})$, as well as $\psi_0\colon 2^{n+1} \to 2^{<\mathbb{N}}$ such that $\forall t \in 2^{n+1}\ \phi(t \restriction n) \sqsubset \psi_0(t)$, and given $k < 4^{n+1} - 2^{n+1}$ and $\psi_k\colon 2^{n+1} \to 2^{<\mathbb{N}}$, fix $\psi_{k+1}\colon 2^{n+1} \to 2^{<\mathbb{N}}$ such that:

- $\forall t \in 2^{n+1}\ \psi_k(t) \sqsubseteq \psi_{k+1}(t)$.
- $\prod_{i<2} \mathcal{N}_{\psi_{k+1}(t_k(i))} \subseteq U_n$.

Clearly the function $\psi = \psi_{4^{n+1}-2^{n+1}}$ is as desired. ⊠

By Lemma 1.6.2, there are functions $\phi_n\colon 2^n \to 2^{<\mathbb{N}}$ such that:

(1) $\forall n \in \mathbb{N} \forall t \in 2^{n+1}\ \phi_n(t \restriction n) \sqsubset \phi_{n+1}(t)$.
(2) $\forall n \in \mathbb{N} \forall t \in {\sim}\Delta(2^{n+1})\ \prod_{i<2} \mathcal{N}_{\phi_{n+1}(t(i))} \subseteq U_n$.

Condition (1) ensures that we obtain a continuous function $\phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting by $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi_n(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$. To see that ϕ is a homomorphism from ${\sim}\Delta(2^{\mathbb{N}})$ to ${\sim}R$, note that if $c \in {\sim}\Delta(2^{\mathbb{N}})$, then there exists $n \in \mathbb{N}$ such that $c(0)(n) \neq c(1)(n)$, in which case condition (2) ensures that $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{m+1}(c(i)\restriction(m+1))} \subseteq U_m$ for all $m \geq n$, thus $(\phi(c(i)))_{i<2} \in {\sim}R$. ⊠

A *D-dimensional dihypergraph* on a set X is a D-ary binary relation H on X disjoint from the *D-dimensional diagonal* on X, given by $\Delta^D(X) = \{x \in X^D \mid \forall c, d \in D\ x(c) = x(d)\}$. Given a D-ary relation H on X, we say that a set $Y \subseteq X$ is *H-independent* if $H \restriction Y = \emptyset$. The *box topology* on a product $\prod_{d\in D} X_d$ of topological spaces is the topology generated by the sets of the form $\prod_{d\in D} U_d$, where $U_d \subseteq X_d$ is open for all $d \in D$.

Proposition 1.6.3. *Suppose that D is a countable set of cardinality at least two, X is a topological space, H is a box-open D-dimensional dihypergraph on X, and $Y \subseteq X$ is H-independent. Then $\overline{Y}$ is H-independent.*

Proof. If there exists $\overline{y} \in H \restriction \overline{Y}$, then there is an open neighborhood $\prod_{d\in D} U_d$ of $\overline{y}$ contained in H. Fix $y \in \prod_{d\in D} U_d \cap Y$, and observe that $y \in H \restriction Y$, a contradiction. ⊠

The *complete D-dimensional dihypergraph* on a set X is the complement of the D-dimensional diagonal on X. A *κ-coloring* of a D-dimensional dihypergraph H on X is a homomorphism $c\colon X \to \kappa$ from

H to the complete D-dimensional dihypergraph on κ. The existence of a κ-coloring of H is trivially equivalent to the existence of a covering of X by κ-many H-independent sets.

PROPOSITION 1.6.4. *Suppose that* AD *holds, D is a countable set of cardinality at least two, X is a subset of an analytic Hausdorff space, H is a box-open D-dimensional dihypergraph on X, and there is an ordinal coloring of H. Then there is an $\aleph_0$-coloring of H.*

PROOF. Fix an analytic Hausdorff space $Y \supseteq X$. Clearly we can assume that $Y \neq \emptyset$, so Proposition 1.4.1 yields a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow Y$. Fix an aleph κ for which there is a cover $(X_\alpha)_{\alpha<\kappa}$ of X by H-independent sets, and let C_α be the closure of X_α within Y for all $\alpha < \kappa$. As Theorem 1.1.5 ensures that $\{T_{\phi^{-1}(C_\alpha)} \mid \alpha < \kappa\}$ is countable, Proposition 1.1.2 ensures that $\{\phi^{-1}(C_\alpha) \mid \alpha < \kappa\}$ is countable, so the surjectivity of ϕ yields that $\{C_\alpha \mid \alpha < \kappa\}$ is countable. But Proposition 1.6.3 implies that $C_\alpha \cap X$ is H-independent for all $\alpha < \kappa$. ⊠

A subset of a topological space is F_σ if it is a union of countably-many closed sets, G_δ if it is an intersection of countably-many open sets, and $\mathbf{\Delta}^0_2$ if it is both F_σ and G_δ. A function $\phi \colon X \to Y$ is Γ-*measurable* if $\phi^{-1}(V) \in \Gamma$ for all open sets $V \subseteq Y$.

PROPOSITION 1.6.5. *Suppose that D is a countable set of cardinality at least two, X is a metric space, and H is a D-dimensional dihypergraph on X. Then the following are equivalent:*

1. *There is a cover $(C_n)_{n\in\mathbb{N}}$ of X by H-independent closed sets.*
2. *There is a $\mathbf{\Delta}^0_2$-measurable $\aleph_0$-coloring $c \colon X \to \mathbb{N}$ of H.*

PROOF. To see (2) $\Longrightarrow$ (1), observe that $c^{-1}(\{n\})$ is a union of countably-many closed sets for all $n \in \mathbb{N}$. To see (1) $\Longrightarrow$ (2), set $B_n = C_n \setminus \bigcup_{m<n} C_m$ for all $n \in \mathbb{N}$. As every closed subset of a metric space is the intersection of the ϵ-balls around it, and therefore G_δ, it follows that each of the sets B_n is F_σ, so the $\aleph_0$-coloring sending each point $x \in X$ to the unique natural number $n \in \mathbb{N}$ for which $x \in B_n$ is F_σ-measurable, and therefore $\mathbf{\Delta}^0_2$-measurable. ⊠

When D has cardinality at least two, we use $\mathbb{H}_{D^{\mathbb{N}}}$ to denote the D-dimensional dihypergraph on $D^{\mathbb{N}}$ given by $\mathbb{H}_{D^{\mathbb{N}}} = \bigcup_{t\in D^{<\mathbb{N}}} \prod_{d\in D} \mathcal{N}_{t\frown(d)}$.

PROPOSITION 1.6.6. *Suppose that D is a countable discrete space of cardinality at least two. Then every $\mathbb{H}_{D^{\mathbb{N}}}$-independent set $X \subseteq D^{\mathbb{N}}$ is meager.*

PROOF. By Proposition 1.6.3, the set $C = \overline{X}$ is $\mathbb{H}_{D^{\mathbb{N}}}$-independent. As Theorem 1.5.1 ensures that $D^{\mathbb{N}}$ is a Baire space, Proposition 1.5.8

implies that C has the Baire property, so Proposition 1.5.4 yields that if X is not meager, then there exists $t \in D^{<\mathbb{N}}$ for which $C \cap \mathcal{N}_t$ is comeager in $\mathcal{N}_t$, thus $\mathcal{N}_t \subseteq C$. But $(t \frown (d) \frown b(d))_{d \in D} \in \mathbb{H}_{D^{\mathbb{N}}} \upharpoonright C$ for all $b \in (D^{\mathbb{N}})^D$, contradicting the $\mathbb{H}_{D^{\mathbb{N}}}$-independence of C. ⊠

Proposition 1.6.7. *Suppose that D is a countable discrete space of cardinality at least two and $\kappa < \mathtt{add}(\mathcal{M}_{D^{\mathbb{N}}})$. Then there is no κ-coloring of $\mathbb{H}_{D^{\mathbb{N}}}$.*

Proof. By Theorem 1.5.1 and Proposition 1.6.6. ⊠

Proposition 1.6.8. *Suppose that* AD *holds and D is a countable discrete space of cardinality at least two. Then there is no ordinal-coloring of $\mathbb{H}_{D^{\mathbb{N}}}$.*

Proof. By Theorem 1.5.1 and Propositions 1.6.4 and 1.6.7. ⊠

A *digraph* on a set X is an irreflexive binary relation G on X. For all sequences $s \in 2^{<\mathbb{N}}$, define a homeomorphism $\iota_s \colon \mathcal{N}_{s \frown (0)} \to \mathcal{N}_{s \frown (1)}$ by setting $\iota_s(s \frown (0) \frown c) = s \frown (1) \frown c$ for all $c \in 2^{\mathbb{N}}$. For all sets $S \subseteq 2^{<\mathbb{N}}$, define $G_S = \bigcup_{s \in S} \operatorname{graph}(\iota_s)$. Following the standard abuse of language, for each infinite set $N \subseteq \mathbb{N}$, we use $\mathbb{G}_0(N)$ to denote any digraph of the form G_S, where $S \subseteq 2^{<N}$ contains an extension of every element of $2^{<\mathbb{N}}$, but only one sequence of every length in N. Define $\mathbb{G}_0 = \mathbb{G}_0(\mathbb{N})$.

Proposition 1.6.9. *Suppose that $N \subseteq \mathbb{N}$ is infinite and $B \subseteq 2^{\mathbb{N}}$ is a $\mathbb{G}_0(N)$-independent set with the Baire property. Then B is meager.*

Proof. Fix a set $S \subseteq 2^{<\mathbb{N}}$ for which $\mathbb{G}_0(N) = G_S$, and suppose, towards a contradiction, that B is not meager. By Proposition 1.5.4, there is a sequence $r \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_r$. Fix an extension $s \in S$ of r. As ι_s is a homeomorphism and Proposition 1.5.3 ensures that B is comeager in $\mathcal{N}_s$, Proposition 1.5.5 implies that the set $C = B \cap \mathcal{N}_{s \frown (0)} \cap \iota_s^{-1}(B \cap \mathcal{N}_{s \frown (1)})$ is comeager in $\mathcal{N}_{s \frown (0)}$, and therefore not empty by Theorem 1.5.1. But $(c, \iota_s(c)) \in \mathbb{G}_0(N) \upharpoonright B$ for all $c \in C$, the desired contradiction. ⊠

For all sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, define $R^{-1} = \{(y, x) \in Y \times X \mid x \mathrel{R} y\}$ and $RS = \{(x, z) \in X \times Z \mid \exists y \in Y \ x \mathrel{R} y \mathrel{S} z\}$.

Proposition 1.6.10. *Suppose that $\kappa < \mathtt{add}(\mathcal{M}_{2^{\mathbb{N}}})$, $N \subseteq \mathbb{N}$ is infinite, R is a binary relation on $2^{\mathbb{N}}$ with the Baire property, and there is a Baire-measurable κ-coloring c of $\mathbb{G}_0(N) \cap R^{-1}R$. Then R is meager.*

Proof. Suppose, towards a contradiction, that R is not meager. Theorems 1.5.1 and 1.5.13 then yield $d \in 2^{\mathbb{N}}$ for which R_d has the

Baire property and is not meager. Fix $\alpha < \kappa$ for which $c^{-1}(\{\alpha\}) \cap R_d$ is not meager. As $R_d \times R_d \subseteq R^{-1}R$, it follows that $c^{-1}(\{\alpha\}) \cap R_d$ is $\mathbb{G}_0(N)$-independent, contradicting Proposition 1.6.9. ⊠

Proposition 1.6.11. *Suppose that* AD *holds,* $N \subseteq \mathbb{N}$ *is infinite, and* R *is a binary relation on* $2^{\mathbb{N}}$ *for which there is an ordinal-coloring* c *of* $\mathbb{G}_0(N) \cap R^{-1}R$. *Then* R *is meager.*

Proof. Theorem 1.5.11 ensures that R has the Baire property and c is Baire measurable, and Theorem 1.5.11 and Proposition 1.5.14 imply that $\mathtt{add}(\mathcal{M}_{2^{\mathbb{N}}}) = \infty$, so this follows from Proposition 1.6.10. ⊠

The *concatenation* $\bigoplus_{m<n} s_m$ of a finite sequence $(s_m)_{m<n}$ of finite sequences is defined recursively by setting $\bigoplus_{m<0} s_m = \emptyset$ and letting $\bigoplus_{m<n+1} s_m$ be the concatenation of $\bigoplus_{m<n} s_m$ and s_n. The *concatenation* of an infinite sequence $(s_n)_{n\in\mathbb{N}}$ of finite sequences is given by $\bigoplus_{n\in\mathbb{N}} s_n = \bigcup_{n\in\mathbb{N}} \bigoplus_{m<n} s_m$.

Proposition 1.6.12. *Suppose that* R *is a non-meager binary relation on* $2^{\mathbb{N}}$ *with the Baire property. Then there are continuous homomorphisms* $\phi_i \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *from* $\mathbb{G}_0$ *to* $\mathbb{G}_0$ *for which* $\prod_{i<2} \phi_i(2^{\mathbb{N}}) \subseteq R$.

Proof. By Proposition 1.5.4, there exists $u \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ for which $R \cap \prod_{i<2} \mathcal{N}_{u(i)}$ is comeager in $\prod_{i<2} \mathcal{N}_{u(i)}$, in which case there are dense open sets $U_n \subseteq \prod_{i<2} \mathcal{N}_{u(i)}$ such that $\bigcap_{n\in\mathbb{N}} U_n \subseteq R$. Fix sequences $s_n \in 2^n$ with the property that $\mathbb{G}_0 = G_S$, where $S = \{s_n \mid n \in \mathbb{N}\}$.

Lemma 1.6.13. *Suppose that* $n \in \mathbb{N}$ *and* $\phi_i \colon 2^n \to 2^{<\mathbb{N}}$ *has the property that* $u(i) \sqsubseteq \phi_i(t)$ *for all* $i < 2$ *and* $t \in 2^{<\mathbb{N}}$. *Then there exists* $v \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ *such that:*

- $\forall t \in 2^n \times 2^n \ \prod_{i<2} \mathcal{N}_{\phi_i(t) \frown v(i)} \subseteq U_n$.
- $\forall i < 2 \ \phi_i(s_n) \frown v(i) \in S$.

Proof. Fix an enumeration $(t_k)_{k<4^n}$ of $2^n \times 2^n$, and $v_0 \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Given $k < 4^n$ and $v_k \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, fix $v_{k+1} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

- $\forall i < 2 \ v_k(i) \sqsubseteq v_{k+1}(i)$.
- $\prod_{i<2} \mathcal{N}_{\phi_i(t_k(i)) \frown v_{k+1}(i)} \subseteq U_n$.

Then any pair $v \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ with the property that $v_{4^n}(i) \sqsubseteq v(i)$ and $\phi_i(s_n) \frown v(i) \in S$ for all $i < 2$ is as desired. ⊠

Fix functions $\phi_{i,0} \colon 2^0 \to 2^{<\mathbb{N}}$ such that $u(i) \sqsubseteq \phi_{i,0}(\emptyset)$ for all $i < 2$, and appeal to Lemma 1.6.13 to obtain pairs $u_n \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, from which we define $\phi_{i,n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ by $\phi_{i,n+1}(t) = \phi_{i,n}(t \upharpoonright n) \frown u_n(i) \frown (t(n))$ for all $i < 2$ and $t \in 2^{n+1}$, such that:

(1) $\forall t \in 2^{n+1} \times 2^{n+1} \ \prod_{i<2} \mathcal{N}_{\phi_{i,n+1}(t(i))} \subseteq U_n$.

(2) $\forall c \in 2^{\mathbb{N}} \ (\phi_{i,n+1}(s_n \frown (j)) \frown c)_{i<2} \in \mathbb{G}_0$.

As $\forall i < 2 \forall n \in \mathbb{N} \forall t \in 2^{n+1} \ \phi_{i,n}(t \restriction n) \sqsubset \phi_{i,n+1}(t)$, we obtain continuous functions $\phi_i \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting $\phi_i(c) = \bigcup_{n \in \mathbb{N}} \phi_{i,n}(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$ and $i < 2$. To see that $\prod_{i<2} \phi_i(2^{\mathbb{N}}) \subseteq R$, note that if $c \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, then $\forall n \in \mathbb{N} \ (\phi_i(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{i,n+1}(c(i) \restriction (n+1))} \subseteq U_n$ by condition (1). To see that each ϕ_i is a homomorphism from $\mathbb{G}_0$ to $\mathbb{G}_0$, note that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $(\phi_i(s_n \frown (j) \frown c))_{j<2} = (\phi_{i,n+1}(s_n \frown (j)) \frown d)_{j<2}$, where $d = \bigoplus_{m \in \mathbb{N}} (c(m)) \frown u_{n+1+m}(i)$, and appeal to condition (2). ⊠

The equivalence relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ is given by

$$c \ \mathbb{E}_0 \ d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m).$$

Proposition 1.6.14. *The smallest equivalence relation E on $2^{\mathbb{N}}$ containing $\mathbb{G}_0$ is $\mathbb{E}_0$.*

Proof. Fix sequences $s_n \in 2^n$ such that $\mathbb{G}_0 = G_{\{s_n \mid n \in \mathbb{N}\}}$. It is enough to show that $\forall c \in 2^{\mathbb{N}} \forall u, v \in 2^n \ u \frown (0) \frown c \ E \ v \frown (1) \frown c$ for all $n \in \mathbb{N}$. But if this holds strictly below some $n \in \mathbb{N}$, then $u \frown (0) \frown c \ E \ s_n \frown (0) \frown c \ E \ s_n \frown (1) \frown c \ E \ v \frown (1) \frown c$ for all $c \in 2^{\mathbb{N}}$ and $u, v \in 2^n$, so it holds at n as well. ⊠

An equivalence relation E is *generically ergodic* if every E-invariant set with the Baire property is comeager or meager.

Proposition 1.6.15. *The relation $\mathbb{E}_0$ is generically ergodic.*

Proof. Suppose that $B \subseteq 2^{\mathbb{N}}$ is an $\mathbb{E}_0$-invariant non-meager set with the Baire property. By Proposition 1.5.4 and the obvious induction, it is sufficient to show that if $i < 2$, $s \in 2^{<\mathbb{N}}$, and $B \cap \mathcal{N}_{s \frown (i)}$ is comeager in $\mathcal{N}_{s \frown (i)}$, then $B \cap \mathcal{N}_{s \frown (1-i)}$ is comeager in $\mathcal{N}_{s \frown (1-i)}$. As ι_s is a homeomorphism, this follows from Proposition 1.5.5. ⊠

Proposition 1.6.16. *Suppose that X is a Baire space, Y is a second-countable T_0 space, E is a generically ergodic equivalence relation on X, and $\phi \colon X \to Y$ is a Baire-measurable homomorphism from E to $\Delta(Y)$. Then there exists $y \in Y$ for which $\phi^{-1}(\{y\})$ is comeager.*

Proof. Fix a countable basis $\mathcal{V}$ for Y, let $\mathcal{W}$ be the set of all $V \in \mathcal{V}$ with the property that $\phi^{-1}(V)$ is comeager, and observe that the set $C = (\bigcap_{W \in \mathcal{W}} \phi^{-1}(W)) \setminus (\bigcup_{V \in \mathcal{V} \setminus \mathcal{W}} \phi^{-1}(V))$ is comeager. As Y is T_0, it follows that $\phi \restriction C$ is constant. ⊠

Given a topological space X, we say that a set $Y \subseteq X$ is $\aleph_0$-*universally Baire* if for every continuous function $\phi \colon 2^{\mathbb{N}} \to X$, the set $\phi^{-1}(Y)$ has the Baire property. The *incomparability relation* associated with a quasi-order R on a set X is the binary relation $\perp_R$ on X for which $x \perp_R y$ if and only if neither $x \mathrel{R} y$ nor $y \mathrel{R} x$.

PROPOSITION 1.6.17 (M-Vidnyánszky). *Suppose that X is a topological space and R is an $\aleph_0$-universally-Baire quasi-order on X for which there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $\perp_R$. Then there are continuous homomorphisms $\phi_i \colon 2^{\mathbb{N}} \to \phi(2^{\mathbb{N}})$ from $\mathbb{G}_0$ to $\perp_R \restriction \phi(2^{\mathbb{N}})$ such that $\prod_{i<2} \phi_i(2^{\mathbb{N}}) \subseteq \perp_R$.*

PROOF. As the quasi-order $R_0 = (\phi \times \phi)^{-1}(R)$ has the Baire property, so too does $\perp_{R_0}$, as does every horizontal and vertical section of either relation.

LEMMA 1.6.18. *The relation $\perp_{R_0}$ is not meager.*

PROOF. Suppose, towards a contradiction, that $\perp_{R_0}$ is meager. Then the set $C = \{c \in 2^{\mathbb{N}} \mid (\perp_{R_0})_c \text{ is meager}\}$ is comeager, by Theorem 1.5.13. The binary relation R_0' on $2^{\mathbb{N}}$ given by $c \mathrel{R_0'} d \iff \forall^* b \in 2^{\mathbb{N}} \ (b \mathrel{R_0} c \implies b \mathrel{R_0} d)$ is clearly a quasi-order. Note that if $(d, c) \in (2^{\mathbb{N}} \times C) \setminus R_0'$, then $(c, d]_{R_0}$ is not meager, so $c <_{R_0} d$. As $\mathbb{G}_0 \subseteq \perp_{R_0}$, it follows that $\mathbb{G}_0 \restriction C \subseteq \equiv_{R_0'}$. As Proposition 1.5.5 ensures that every comeager subset of $2^{\mathbb{N}}$ has an $\mathbb{E}_0$-invariant comeager subset, Proposition 1.6.14 yields an $\mathbb{E}_0$-invariant comeager set $C' \subseteq C$ for which $\mathbb{E}_0 \restriction C' \subseteq \equiv_{R_0'}$. Observe that for all $s \in 2^{\mathbb{N}}$, the set $B_s = \{c \in 2^{\mathbb{N}} \mid \forall^* b \in \mathcal{N}_s \ b \mathrel{R_0} c\}$ has the Baire property, by Theorems 1.5.12 and 1.5.13. As Proposition 1.5.4 implies that $c \equiv_{R_0'} d \iff \forall s \in 2^{<\mathbb{N}} \ (c \in B_s \iff d \in B_s)$ for all $c, d \in C$, Proposition 1.6.15 ensures that $\equiv_{R_0'}$ has a comeager equivalence class. Fixing $s, t \in 2^{<\mathbb{N}}$ for which $R_0 \cap (\mathcal{N}_s \times \mathcal{N}_t)$ is comeager in $\mathcal{N}_s \times \mathcal{N}_t$, Theorem 1.5.13 implies that $\forall^* c \in \mathcal{N}_t \forall^* b \in \mathcal{N}_s \ b \mathrel{R_0} c$, so $\forall^* b, c \in \mathcal{N}_s \ b \mathrel{R_0} c$, thus $\equiv_{R_0}$ has an equivalence class that is comeager in $\mathcal{N}_s$. But Proposition 1.6.9 then ensures that $\equiv_{R_0} \cap \mathbb{G}_0 \neq \emptyset$, the desired contradiction. ⊠

By Proposition 1.6.12 and Lemma 1.6.18, there are continuous homomorphisms $\phi_i' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\mathbb{G}_0$ to $\mathbb{G}_0$ for which $\prod_{i<2} \phi_i'(2^{\mathbb{N}}) \subseteq \perp_{R_0}$, in which case the functions $\phi_i = \phi \circ \phi_i'$ are as desired. ⊠

PROPOSITION 1.6.19 (M-Vidnyánszky). *Suppose that X is an analytic Hausdorff space and R is an $\aleph_0$-universally-Baire quasi-order on X for which there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $\perp_R$. Then there is a continuous homomorphism $\pi \colon 2^{\mathbb{N}} \to X$ from $\sim\Delta(2^{\mathbb{N}})$ to $\perp_R$.*

Proof. Proposition 1.4.1 yields a continuous surjection $\psi \colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow X$. We will recursively construct functions $\psi_n \colon 2^n \to \mathbb{N}^n$ and continuous homomorphisms $\phi_s \colon 2^{\mathbb{N}} \to \psi(\mathcal{N}_{\psi_n(s)})$ from $\mathbb{G}_0$ to $\perp_R \restriction \psi(\mathcal{N}_{\psi_n(s)})$ such that:

(1) $\forall i < 2 \forall n \in \mathbb{N} \forall s \in 2^n \ \psi_n(s) \sqsubseteq \psi_{n+1}(s \frown (i))$.
(2) $\forall i < 2 \forall n \in \mathbb{N} \forall s \in 2^n \ \phi_{s \frown (i)}(2^{\mathbb{N}}) \subseteq \phi_s(2^{\mathbb{N}})$.
(3) $\forall n \in \mathbb{N} \forall s \in 2^n \ \prod_{i<2} \phi_{s \frown (i)}(2^{\mathbb{N}}) \subseteq \perp_R$.

We begin by setting $\phi_0 = \phi$ and $\psi_0(\emptyset) = \emptyset$. Suppose that $n \in \mathbb{N}$ and we have found $(\phi_s)_{s \in 2^n}$ and ψ_n. For all $s \in 2^n$, Proposition 1.6.17 yields continuous homomorphisms $\phi_{s,i} \colon 2^{\mathbb{N}} \to \phi_s(2^{\mathbb{N}})$ from $\mathbb{G}_0$ to $\perp_R \restriction \phi_s(2^{\mathbb{N}})$ for which $\prod_{i<2} \phi_{s,i}(2^{\mathbb{N}}) \subseteq \perp_R$. Fix extensions $\psi_{n+1}(s \frown (i)) \in \mathbb{N}^{n+1}$ of $\psi_n(s)$ such that $\phi_{s,i}^{-1}(\psi(\mathcal{N}_{\psi_{n+1}(s \frown (i))}))$ is not meager for all $i < 2$. As Proposition 1.4.2 ensures that the latter sets are analytic, Proposition 1.5.9 implies that they have the Baire property, so the special case of Proposition 1.6.12 where $R = \prod_{i<2} \phi_{s,i}^{-1}(\psi(\mathcal{N}_{\psi_{n+1}(s \frown (i))}))$ yields continuous homomorphisms $\phi'_{s,i} \colon 2^{\mathbb{N}} \to \phi_{s,i}^{-1}(\psi(\mathcal{N}_{\psi_{n+1}(s \frown (i))}))$ from $\mathbb{G}_0$ to $\mathbb{G}_0 \restriction \phi_{s,i}^{-1}(\psi(\mathcal{N}_{\psi_{n+1}(s \frown (i))}))$. Define $\phi_{s \frown (i)} = \phi_{s,i} \circ \phi'_{s,i}$.

Condition (1) ensures that we obtain a continuous map $\psi_\infty \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by setting $\psi_\infty(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$. Define $\pi = \psi \circ \psi_\infty$, and note that for all $c \in 2^{\mathbb{N}}$, Proposition 1.4.6 ensures that $\pi(c)$ is the unique element of $\bigcap_{n \in \mathbb{N}} \psi(\mathcal{N}_{\psi_n(c \restriction n)})$, and since $\phi_{c \restriction n}(2^{\mathbb{N}}) \subseteq \psi(\mathcal{N}_{\psi_n(c \restriction n)})$ for all $n \in \mathbb{N}$ and the former sets have non-empty intersection by condition (2), it follows that $\pi(c)$ is also the unique element of $\bigcap_{n \in \mathbb{N}} \phi_{c \restriction n}(2^{\mathbb{N}})$. To see that π is a homomorphism from $\sim\Delta(2^{\mathbb{N}})$ to $\perp_R$, observe that if $c, d \in 2^{\mathbb{N}}$ are distinct, then there is a maximal natural number $n \in \mathbb{N}$ for which $c \restriction n = d \restriction n$, and since $\pi(c) \in \phi_{s \frown (c(n))}(2^{\mathbb{N}})$ and $\pi(d) \in \phi_{s \frown (d(n))}(2^{\mathbb{N}})$, where $s = c \restriction n = d \restriction n$, condition (3) ensures that $\pi(c) \perp_R \pi(d)$. ⊠

Let $\mathbb{F}_0$ denote the subequivalence relation of $\mathbb{E}_0$ given by

$$c \mathrel{\mathbb{F}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \ \textstyle\sum_{k<m} c(k) \equiv \sum_{k<m} d(k) \pmod 2.$$

Proposition 1.6.20. *Suppose that E is an equivalence relation on $2^{\mathbb{N}}$ and F is an index-two subequivalence relation of E with the property that $\mathbb{G}_0 \subseteq E \setminus F$. Then $\mathbb{F}_0 \subseteq F$ and $\mathbb{E}_0 \setminus \mathbb{F}_0 \subseteq E \setminus F$.*

Proof. Note that if $c \mathrel{\mathbb{E}_0} d$, then Proposition 1.6.14 yields a $\mathbb{G}_0^{\pm 1}$-path γ from c to d, so the fact that $\mathbb{G}_0 \subseteq E$ ensures that $c \mathrel{E} d$. Moreover, the fact that $\mathbb{G}_0 \subseteq \mathbb{E}_0 \setminus \mathbb{F}_0$ and $\mathbb{F}_0$ has index two below $\mathbb{E}_0$ ensures that $c \mathrel{\mathbb{F}_0} d \iff \gamma$ has evenly-many edges, whereas the fact that $\mathbb{G}_0 \subseteq E \setminus F$ and F has index two below E implies that $c \mathrel{F} d \iff \gamma$ has evenly-many edges, thus $c \mathrel{\mathbb{F}_0} d \iff c \mathrel{F} d$. ⊠

A *partial transversal* of an equivalence relation E on X over a subequivalence relation F of E is a set $Y \subseteq X$ for which $E \restriction Y = F \restriction Y$.

PROPOSITION 1.6.21. *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a partial transversal of $\mathbb{E}_0$ over $\mathbb{F}_0$ with the Baire property. Then B is meager.*

PROOF. As $\mathbb{G}_0 \subseteq \mathbb{E}_0 \setminus \mathbb{F}_0$, it follows that B is $\mathbb{G}_0$-independent, so Proposition 1.6.9 ensures that it is meager. ⊠

A *homomorphism* from a sequence $(R_i)_{i \in I}$ of D-ary relations on X to a sequence $(S_i)_{i \in I}$ of D-ary relations on Y is a function $\phi \colon X \to Y$ that is a homomorphism from R_i to S_i for all $i \in I$.

PROPOSITION 1.6.22. *Suppose that N is a nowhere dense binary relation on $2^{\mathbb{N}}$ and R is a meager binary relation on $2^{\mathbb{N}}$. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\sim\Delta(2^{\mathbb{N}}), \mathbb{F}_0, \mathbb{E}_0 \setminus \mathbb{F}_0, \sim\mathbb{E}_0)$ to $(\sim N, \mathbb{F}_0, \mathbb{E}_0 \setminus \mathbb{F}_0, \sim R)$.*

PROOF. Fix a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of dense open symmetric subsets of $\sim\overline{N}$ whose intersection is disjoint from R.

LEMMA 1.6.23. *Suppose that $n \in \mathbb{N}$ and $\phi \colon 2^n \to 2^{<\mathbb{N}}$. Then there exist $\ell > 0$ and $u \in 2^\ell \times 2^\ell$ such that:*

- $\forall t \in 2^n \times 2^n \ \prod_{i<2} \mathcal{N}_{\phi(t(i)) \frown u(i)} \subseteq U_n$.
- $\sum_{k<\ell} u(0)(k) \not\equiv \sum_{k<\ell} u(1)(k) \pmod 2$.

PROOF. Fix an enumeration $(t_k)_{k<4^n}$ of $2^n \times 2^n$, and $u_0 \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Given $k < 4^n$ and $u_k \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, fix $u_{k+1} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

- $\forall i < 2 \ u_k(i) \sqsubseteq u_{k+1}(i)$.
- $\prod_{i<2} \mathcal{N}_{\phi(t_k(i)) \frown u_{k+1}(i)} \subseteq U_n$.

Then any $\ell > 0$ and pair $u \in 2^\ell \times 2^\ell$ such that $u_{4^n}(i) \sqsubseteq u(i)$ for all $i < 2$ and $\sum_{k<\ell} u(0)(k) \not\equiv \sum_{k<\ell} u(1)(k) \pmod 2$ is as desired. ⊠

Fix $\phi_0 \colon 2^0 \to 2^0$, and appeal to Lemma 1.6.23 to obtain $\ell_n > 0$ and pairs $u_n \in 2^{\ell_n} \times 2^{\ell_n}$, from which we define $\phi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ by $\phi_{n+1}(t) = \phi_n(t \restriction n) \frown u_n(t(n))$ for all $t \in 2^{n+1}$, such that:

1. $\forall t \in 2^n \times 2^n \ \prod_{i<2} \mathcal{N}_{\phi_{n+1}(t(i) \frown (i))} \subseteq U_n$.
2. $\sum_{k<\ell_n} u_n(0)(k) \not\equiv \sum_{k<\ell_n} u_n(1)(k) \pmod 2$.

As $\forall n \in \mathbb{N} \forall t \in 2^{n+1} \ \phi_n(t \restriction n) \sqsubset \phi_{n+1}(t)$, we obtain a continuous function $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$. To see that ϕ is a homomorphism from $\sim\Delta(2^{\mathbb{N}})$ to $\sim N$, note that if $c \in \sim\Delta(2^{\mathbb{N}})$, then there exists $n \in \mathbb{N}$ for which $c(0)(n) \neq c(1)(n)$, so $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i) \restriction (n+1))} \subseteq U_n \subseteq \sim N$ by condition (1). To see that ϕ is a homomorphism from $(\mathbb{F}_0, \mathbb{E}_0 \setminus \mathbb{F}_0)$ to $(\mathbb{F}_0, \mathbb{E}_0 \setminus \mathbb{F}_0)$, note that if $c \in \mathbb{E}_0$, then there exists $n \in \mathbb{N}$ such that $c(0)(m) = c(1)(m)$ for all $m \geq n$, and condition (2) ensures that if $\ell = \sum_{m<n} \ell_m$, then

$$\begin{aligned} &c(0) \mathbb{F}_0 c(1) \\ &\quad\iff \textstyle\sum_{m<n} c(0)(m) \equiv \sum_{m<n} c(1)(m) \pmod 2 \\ &\quad\iff |\{m < n \mid c(0)(m) \neq c(1)(m)\}| \text{ is even} \\ &\quad\iff |\{m < n \mid u_m(c(0)(m)) \not\equiv u_m(c(1)(m)) \pmod 2\}| \text{ is even} \\ &\quad\iff \textstyle\sum_{m<n} u_m(c(0)(m)) \equiv \sum_{m<n} u_m(c(1)(m)) \pmod 2 \\ &\quad\iff \textstyle\sum_{m<\ell} \phi_n(c(0) \upharpoonright n)(m) \equiv \sum_{m<\ell} \phi_n(c(1) \upharpoonright n)(m) \pmod 2 \\ &\quad\iff \phi(c(0)) \mathbb{F}_0 \phi(c(1)). \end{aligned}$$

To see that ϕ is a homomorphism from $\sim\mathbb{E}_0$ to $\sim R$, note that if $c \in \sim\mathbb{E}_0$, then there is an infinite set $N \subseteq \mathbb{N}$ such that $c(0)(n) \neq c(1)(n)$ for all $n \in N$, so $\forall n \in N\ (\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i)\upharpoonright(n+1))} \subseteq U_n$ by condition (1), thus $(\phi(c(i)))_{i<2} \in \sim R$. ⊠

For all sequences $t \in \bigcup_{n\in\mathbb{N}} 2^n \times 2^n$, define $\iota_t \colon \mathcal{N}_{t(0)\frown(0)} \to \mathcal{N}_{t(1)\frown(1)}$ by setting $\iota_t(t(0) \frown (0) \frown c) = t(1) \frown (1) \frown c$ for all $c \in 2^{\mathbb{N}}$. For all sets $T \subseteq \bigcup_{n\in\mathbb{N}} 2^n \times 2^n$, define $G_T = \bigcup_{t\in T} \operatorname{graph}(\iota_t)$.

Proposition 1.6.24. *Suppose that $T \subseteq \bigcup_{n\in\mathbb{N}} 2^n \times 2^n$ contains an extension of every element of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, and R is a transitive binary relation on $2^{\mathbb{N}}$ with the Baire property containing G_T. Then R is comeager or meager.*

Proof. Suppose, towards a contradiction, that R is neither comeager nor meager. By Proposition 1.5.4, there exist pairs $u, v \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ with the property that $R \cap (\mathcal{N}_{u(1)} \times \mathcal{N}_{v(0)})$ is comeager in $\mathcal{N}_{u(1)} \times \mathcal{N}_{v(0)}$ and $R \cap (\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)})$ is meager in $\mathcal{N}_{u(0)} \times \mathcal{N}_{v(1)}$. Fix $s, t \in T$ with the property that $u(i) \sqsubseteq s(i)$ and $v(i) \sqsubseteq t(i)$ for all $i < 2$. As $(\iota_s^{-1} \times \iota_t)(R \cap (\mathcal{N}_{s(1)\frown(1)} \times \mathcal{N}_{t(0)\frown(0)})) \subseteq R$, Proposition 1.5.5 ensures that $R \cap (\mathcal{N}_{s(0)\frown(0)} \times \mathcal{N}_{t(1)\frown(1)})$ is comeager in $\mathcal{N}_{s(0)\frown(0)} \times \mathcal{N}_{t(1)\frown(1)}$. But $R \cap (\mathcal{N}_{s(0)\frown(0)} \times \mathcal{N}_{t(1)\frown(1)})$ is also meager in $\mathcal{N}_{s(0)\frown(0)} \times \mathcal{N}_{t(1)\frown(1)}$ by Proposition 1.5.3, contradicting Theorem 1.5.1 and Proposition 1.5.2. ⊠

For each infinite set $N \subseteq \mathbb{N}$, we use $\mathbb{H}_0(N)$ to denote any digraph of the form G_T, where $T \subseteq \bigcup_{n\in N} \mathbb{N}^n \times \mathbb{N}^n$ contains an extension of every element of $2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, but only one pair corresponding to each length in N.

Proposition 1.6.25. *Suppose that $\kappa < \mathtt{add}(\mathcal{M}_{2^{\mathbb{N}}})$ and R is a linear quasi-order on $2^{\mathbb{N}}$ with the Baire property containing $\mathbb{H}_0(2\mathbb{N}+1)$. Then there is no Baire-measurable κ-coloring c of $\equiv_R \cap \mathbb{G}_0(2\mathbb{N})$.*

Proof. As Theorem 1.5.1 and Proposition 1.6.24 ensure that R is comeager, so too is $\equiv_R$, contradicting Proposition 1.6.10. ⊠

Proposition 1.6.26. *Suppose that* AD *holds and R is a linear quasi-order on $2^{\mathbb{N}}$ containing $\mathbb{H}_0(2\mathbb{N}+1)$. Then there is no ordinal-coloring c of $\equiv_R \cap \mathbb{G}_0(2\mathbb{N})$.*

Proof. As Theorem 1.5.11 ensures that R has the Baire property and c is Baire measurable, and Theorem 1.5.11 and Proposition 1.5.14 imply that $\mathtt{add}(\mathcal{M}_{2^{\mathbb{N}}}) = \infty$, this follows from Proposition 1.6.25. ⊠

The *strict quasi-order* associated with a quasi-order R on a set X is the binary relation $<_R$ on X for which $x <_R y$ if and only if $x \mathrel{R} y$ but $\neg y \mathrel{R} x$. The partial order $\mathbb{R}_0$ on $2^{\mathbb{N}}$ is given by

$$c <_{\mathbb{R}_0} d \iff \exists n \in \mathbb{N}\ (c(n) < d(n) \text{ and } \forall m > n\ c(m) = d(m)).$$

The *odometer* is the homeomorphism of $2^{\mathbb{N}}$ given by

$$\sigma((1)^n \frown (0) \frown c) = (0)^n \frown (1) \frown c.$$

Proposition 1.6.27. *The transitive closure R of* $\operatorname{graph}(\sigma) \setminus \{((1)^\infty, (0)^\infty)\}$ *is* $<_{\mathbb{R}_0}$.

Proof. It is enough to show that $\forall c \in 2^{\mathbb{N}} \forall u, v \in 2^n\ u \frown (0) \frown c \mathrel{R} v \frown (1) \frown c$ for all $n \in \mathbb{N}$. But if this holds strictly below some $n \in \mathbb{N}$, then $u \frown (0) \frown c \mathrel{S} (1)^n \frown (0) \frown c \mathrel{R} (0)^n \frown (1) \frown c \mathrel{S} v \frown (1) \frown c$ for all $c \in 2^{\mathbb{N}}$ and $u, v \in 2^n$, where $S = \Delta(2^{\mathbb{N}}) \cup R$, so it holds at n. ⊠

A *reduction* of a D-ary relation R on X to a D-ary relation S on Y is a homomorphism from $(R, {\sim}R)$ to $(S, {\sim}S)$. An *embedding* of R into S is an injective reduction of R to S.

Proposition 1.6.28. *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a non-meager set with the Baire property. Then there is a continuous embedding of $\mathbb{R}_0$ into $\mathbb{R}_0 \restriction B$.*

Proof. By Proposition 1.5.4, there is a sequence $u \in 2^{<\mathbb{N}}$ for which $B \cap \mathcal{N}_u$ is comeager in $\mathcal{N}_u$, in which case there are dense open sets $U_n \subseteq \mathcal{N}_u$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq B$.

Lemma 1.6.29. *Suppose that $n \in \mathbb{N}$ and $\phi \colon 2^n \to 2^{<\mathbb{N}}$ has the property that $u \sqsubseteq \phi(s)$ for all $s \in 2^n$. Then there exists $v \in 2^{<\mathbb{N}}$ such that $\forall s \in 2^n\ \mathcal{N}_{\phi(s) \frown v} \subseteq U_n$.*

Proof. Fix an enumeration $(s_k)_{k<2^n}$ of 2^n, set $v_0 = \emptyset$, and given $k < 2^n$ and $v_k \in 2^{<\mathbb{N}}$, fix $v_{k+1} \in 2^{<\mathbb{N}}$ such that $v_k \sqsubseteq v_{k+1}$ and $\mathcal{N}_{\phi(s_k) \frown v_{k+1}} \subseteq U_n$. Then any $v \in 2^{<\mathbb{N}}$ such that $v_{2^n} \sqsubseteq v$ is as desired. ⊠

Fix $\phi_0 \colon 2^0 \to 2^{<\mathbb{N}}$ such that $u \sqsubseteq \phi_0(\emptyset)$, and appeal to Lemma 1.6.29 to obtain sequences $u_n \in 2^{<\mathbb{N}}$, from which we define $\phi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ by $\phi_{n+1}(s) = \phi_n(s \upharpoonright n) \frown u_n \frown s(n)$ for all $s \in 2^{n+1}$, such that $\forall s \in 2^{n+1}\ \mathcal{N}_{\phi_{n+1}(s)} \subseteq U_n$. As $\forall n \in \mathbb{N} \forall s \in 2^{n+1}\ \phi_n(s \upharpoonright n) \sqsubset \phi_{n+1}(s)$, we obtain a continuous embedding $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{R}_0$ into $\mathbb{R}_0$ by setting $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi_n(c \upharpoonright n)$ for all $c \in 2^{\mathbb{N}}$. To see that $\phi(2^{\mathbb{N}}) \subseteq B$, observe that if $c \in 2^{\mathbb{N}}$, then $\forall n \in \mathbb{N}\ \phi(c) \in \mathcal{N}_{\phi_{n+1}(c \upharpoonright (n+1))} \subseteq U_n$. ⊠

Proposition 1.6.30. *Suppose that R is an $\aleph_0$-universally Baire quasi-order on $2^{\mathbb{N}}$ for which $\mathbb{R}_0 \subseteq R \subseteq \mathbb{E}_0$. Then there is a continuous embedding of $\mathbb{R}_0$ or $\mathbb{E}_0$ into R.*

Proof. Note that the set $X = \{c \in 2^{\mathbb{N}} \mid c <_R \sigma(c)\}$ has the Baire property, and Proposition 1.6.27 ensures that $R \upharpoonright X = \mathbb{R}_0 \upharpoonright X$. If X is not meager, then Proposition 1.6.28 therefore yields a continuous embedding of $\mathbb{R}_0$ into $R \upharpoonright X$. If X is meager, then Proposition 1.5.5 ensures that $\bigcup_{n\in\mathbb{Z}} \sigma^n(X)$ is meager, so Proposition 1.6.27 implies that $[X]_{\mathbb{E}_0}$ is meager, and since Proposition 1.6.27 also ensures that $R \upharpoonright \sim[X]_{\mathbb{E}_0} = \mathbb{E}_0 \upharpoonright \sim[X]_{\mathbb{E}_0}$, Proposition 1.6.28 yields a continuous embedding of $\mathbb{E}_0$ into $R \upharpoonright \sim[X]_{\mathbb{E}_0}$. ⊠

Proposition 1.6.31. *Suppose that N is a nowhere dense binary relation on $2^{\mathbb{N}}$ and R is a meager binary relation on $2^{\mathbb{N}}$. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\sim\Delta(2^{\mathbb{N}}), \mathrm{graph}(\sigma) \setminus \{((1)^\infty, (0)^\infty)\}, \sim\mathbb{E}_0)$ to $(\sim N, \mathbb{H}_0(2\mathbb{N}+1), \sim R)$.*

Proof. Fix a set $T \subseteq \bigcup_{n\in\mathbb{N}} 2^n \times 2^n$ for which $\mathbb{H}_0(2\mathbb{N}+1) = G_T$, as well as a decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of dense open symmetric subsets of $\sim\overline{N}$ whose intersection is disjoint from R.

Lemma 1.6.32. *Suppose that $n \in \mathbb{N}$ and $\phi \colon 2^n \to 2^{<\mathbb{N}}$. Then there exist $\ell \in \mathbb{N}$ and $u \in 2^\ell \times 2^\ell$ such that:*

- $\forall s \in 2^n \times 2^n\ \prod_{i<2} \mathcal{N}_{\phi(s(i)) \frown u(i)} \subseteq U_n$.
- $(\phi((1-i)^n) \frown u(i))_{i<2} \in T$.

Proof. Fix an enumeration $(s_k)_{k<4^n}$ of $2^n \times 2^n$, and $u_0 \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Given $k < 4^n$ and $u_k \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, fix $u_{k+1} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

- $\forall i < 2\ u_k(i) \sqsubseteq u_{k+1}(i)$.
- $\prod_{i<2} \mathcal{N}_{\phi(s_k(i)) \frown u_{k+1}(i)} \subseteq U_n$.

Then any $\ell \in \mathbb{N}$ and $u \in 2^\ell \times 2^\ell$ with the property that $u_{4^n}(i) \sqsubseteq u(i)$ and $(\phi((1-i)^n) \frown u(i))_{i<2} \in T$ are as desired. ⊠

Fix $\phi_0 \colon 2^0 \to 2^{<\mathbb{N}}$, and appeal to Lemma 1.6.32 to obtain $\ell_n \in \mathbb{N}$ and pairs $u_n \in 2^{\ell_n} \times 2^{\ell_n}$, from which we define $\phi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ by $\phi_{n+1}(s) = \phi_n(s \upharpoonright n) \frown u_n(s(n)) \frown (s(n))$ for all $s \in 2^{n+1}$, such that:

(1) $\forall s \in 2^n \times 2^n \; \prod_{i<2} \mathcal{N}_{\phi_{n+1}(s(i) \frown (i))} \subseteq U_n$.

(2) $\forall c \in 2^{\mathbb{N}} \; (\phi_{n+1}((1-i)^n \frown (i)) \frown c)_{i<2} \in \mathbb{H}_0(2\mathbb{N}+1)$.

As $\forall n \in \mathbb{N} \forall s \in 2^{n+1} \; \phi_n(s \restriction n) \sqsubset \phi_{n+1}(s)$, we obtain a continuous function $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$. To see that ϕ is a homomorphism from $\sim\Delta(2^{\mathbb{N}})$ to $\sim N$, note that if $c \in \sim\Delta(2^{\mathbb{N}})$, then there exists $n \in \mathbb{N}$ with the property that $c(0)(n) \neq c(1)(n)$, so $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i) \restriction (n+1))} \subseteq U_n \subseteq \sim N$ by condition (1). To see that ϕ is a homomorphism from $\operatorname{graph}(\sigma) \setminus \{((1)^\infty, (0)^\infty)\}$ to $\mathbb{H}_0(2\mathbb{N}+1)$, note that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $(\phi((1-i)^n \frown (i) \frown c))_{i<2} = (\phi_{n+1}((1-i)^n \frown (i)) \frown d)_{i<2}$, where $d = \bigoplus_{m \in \mathbb{N}} u_{n+1+m}(c(m)) \frown (c(m))$, and appeal to condition (2). To see that ϕ is a homomorphism from $\sim\mathbb{E}_0$ to $\sim R$, note that if $c \in \sim\mathbb{E}_0$, then there is an infinite set $N \subseteq \mathbb{N}$ such that $c(0)(n) \neq c(1)(n)$ for all $n \in N$, so $\forall n \in N \; (\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i) \restriction (n+1))} \subseteq U_n$ by condition (1), thus $(\phi(c(i)))_{i<2} \in \sim R$. ⊠

CHAPTER 2

The box-open dihypergraph dichotomy

1. Colorings of box-open dihypergraphs

Here we consider the circumstances under which a box-open countable-dimensional dihypergraph admits an ordinal coloring.

THEOREM 2.1.1 (Feng, Carroy-M-Soukup). *Suppose that D is a countable discrete space of cardinality at least two, κ is an aleph, X is a κ-Souslin Hausdorff space, and H is a box-open D-dimensional dihypergraph on X. Then at least one of the following holds:*

(1) *There is a κ-coloring of H.*
(2) *There is a continuous homomorphism $\phi \colon D^{\mathbb{N}} \to X$ from $\mathbb{H}_{D^{\mathbb{N}}}$ to H.*

PROOF. We can clearly assume that $X \neq \emptyset$, in which case Proposition 1.4.1 yields a continuous surjection $\phi_X \colon \kappa^{\mathbb{N}} \twoheadrightarrow X$. Recursively define an increasing sequence $(T^\alpha)_{\alpha<\kappa^+}$ of subsets of $\kappa^{<\mathbb{N}}$, as well as a decreasing sequence $(X^\alpha)_{\alpha<\kappa^+}$ of subsets of X, by setting $X^0 = X$, $T^\alpha = \{t \in \kappa^{<\mathbb{N}} \mid \phi_X(\mathcal{N}_t) \cap X^\alpha \text{ is } H\text{-independent}\}$ and $X^{\alpha+1} = \sim\bigcup_{t \in T^\alpha} \phi_X(\mathcal{N}_t)$ for all $\alpha < \kappa^+$, and $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ for all limit ordinals $\lambda < \kappa^+$.

LEMMA 2.1.2. *Suppose that $\alpha < \kappa^+$ and $t \in \sim T^{\alpha+1}$. Then there is a sequence $(t_d)_{d \in D}$ of proper extensions of t in $\sim T^\alpha$ with the property that $\prod_{d \in D} \phi_X(\mathcal{N}_{t_d}) \subseteq H$.*

PROOF. As $t \notin T^{\alpha+1}$, there exists $x \in H \upharpoonright (\phi_X(\mathcal{N}_t) \cap X^{\alpha+1})$. As H is box open, there is an open neighborhood $\prod_{d \in D} U_d$ of x contained in H. Fix a sequence $b \in \mathcal{N}_t^D$ such that $\phi_X^D(b) = x$, and for all $d \in D$, appeal to the continuity of ϕ_X to obtain a natural number $n_d > |t|$ such that $\phi_X(\mathcal{N}_{b(d) \upharpoonright n_d}) \subseteq U_d$, noting that the sequence $t_d = b(d) \upharpoonright n_d$ is in $\sim T^\alpha$, since $x(d) \in X^{\alpha+1}$. ⊠

As $(T^\alpha)_{\alpha<\kappa^+}$ is increasing, there is an ordinal $\alpha < \kappa^+$ with the property that $T^\alpha = T^{\alpha+1}$.

LEMMA 2.1.3. *If $\emptyset \in T^\alpha$, then there is a κ-coloring of H.*

PROOF. As the sets of the form $\phi_X(\mathcal{N}_t) \cap X^\beta$, where $\beta \leq \alpha$ and $t \in T^\beta$, are H-independent, it is sufficient to show that they cover X.

But if $x \in X$, then there is a least ordinal $\gamma \leq \alpha + 1$ such that $x \notin X^\gamma$, and since γ is necessarily the successor of some ordinal $\beta \leq \alpha$, there exists $t \in T^\beta$ such that $x \in \phi_X(\mathcal{N}_t)$, so $x \in \phi_X(\mathcal{N}_t) \cap X^\beta$. ⊠

By Lemma 2.1.3, we can assume that $\emptyset \notin T^\alpha$. Lemma 2.1.2 and DC then yield a sequence of functions $\phi_n \colon D^n \to \sim T^\alpha$ such that:

(a) $\forall d \in D \forall t \in D^n \; \phi_n(t) \sqsubset \phi_{n+1}(t \frown (d))$.
(b) $\forall t \in D^n \; \prod_{d \in D} \phi_X(\mathcal{N}_{\phi_{n+1}(t \frown (d))}) \subseteq H$.

Condition (a) ensures that we obtain a continuous map $\phi_\infty \colon D^\mathbb{N} \to \kappa^\mathbb{N}$ by setting $\phi_\infty(b) = \bigcup_{n \in \mathbb{N}} \phi_n(b \upharpoonright n)$ for all $b \in D^\mathbb{N}$. To see that the map $\phi = \phi_X \circ \phi_\infty$ is a homomorphism from $\mathbb{H}_{D^\mathbb{N}}$ to H, note that if $n \in \mathbb{N}$ and $t \in D^n$, then $\phi^D(\prod_{d \in D} \mathcal{N}_{t \frown (d)}) \subseteq \prod_{d \in D} \phi_X(\mathcal{N}_{\phi_{n+1}(t \frown (d))}) \subseteq H$ by condition (b). ⊠

Theorem 2.1.4 (Feng, Carroy-M-Soukup). *Suppose that D is a countable discrete space of cardinality at least two, X is an analytic Hausdorff space, and H is a box-open D-dimensional dihypergraph on X. Then exactly one of the following holds:*

(1) *There is an $\aleph_0$-coloring of H.*
(2) *There is a continuous homomorphism $\phi \colon D^\mathbb{N} \to X$ from $\mathbb{H}_{D^\mathbb{N}}$ to H.*

Proof. Proposition 1.6.7 ensures that the two conditions are mutually exclusive, and the special case of Theorem 2.1.1 where $\kappa = \aleph_0$ implies that at least one of them holds. ⊠

Theorem 2.1.5 (Feng, Carroy-M-Soukup). *Suppose that* $\mathtt{AD}_\mathbb{R}$ *holds, D is a countable discrete space of cardinality at least two, X is a subset of an analytic Hausdorff space, and H is a box-open D-dimensional dihypergraph on X. Then exactly one of the following holds:*

(1) *There is an $\aleph_0$-coloring of H.*
(2) *There is a continuous homomorphism $\phi \colon D^\mathbb{N} \to X$ from $\mathbb{H}_{D^\mathbb{N}}$ to H.*

Proof. Proposition 1.6.7 ensures that the two conditions are mutually exclusive. Theorem 1.4.15 yields an aleph κ for which X is κ-Souslin, so Theorem 2.1.1 ensures that there is a κ-coloring of H or a continuous homomorphism $\phi \colon D^\mathbb{N} \to X$ from $\mathbb{H}_{D^\mathbb{N}}$ to H, thus Proposition 1.6.4 implies that at least one of the two conditions holds. ⊠

Remark 2.1.6. Theorem 2.1.5 continues to hold under the weaker hypothesis that AD holds (see [**CMS**]), yielding analogous generalizations of the other consequences of $\mathtt{AD}_\mathbb{R}$ established in this chapter.

The following observation often ensures that the homomorphisms given by the above results are injective:

Proposition 2.1.7. *Suppose that D is a set of cardinality at least two, X is a set, H is a D-dimensional dihypergraph on X consisting solely of injective sequences, and $\phi \colon D^{\mathbb{N}} \to X$ is a homomorphism from $\mathbb{H}_{D^{\mathbb{N}}}$ to H. Then ϕ is injective.*

Proof. Suppose that $a, b \in D^{\mathbb{N}}$ are distinct, fix $c \in \mathbb{H}_{D^{\mathbb{N}}}$ for which $a, b \in c(D)$, and note that $\phi \circ c \in H$, so $\phi(a) \neq \phi(b)$. ⊠

A *digraph* on X is an irreflexive binary relation on X, and a *graph* is a symmetric digraph. Note that every homomorphism $\phi \colon X \to Y$ from a digraph G on X to a graph H on Y is a homomorphism from $G^{\pm 1}$ to H. The *complete graph* on X is given by $K_X = {\sim}\Delta(X)$. As $K_X = \mathbb{H}_{2^{\mathbb{N}}}^{\pm 1}$, it follows that a map $\phi \colon 2^{\mathbb{N}} \to X$ is a homomorphism from $\mathbb{H}_{2^{\mathbb{N}}}$ to a graph G if and only if it is a homomorphism from K_X to G.

We next consider the circumstances under which a set can be well-ordered:

Theorem 2.1.8 (Souslin). *Suppose that κ is an aleph and X is a κ-Souslin Hausdorff space. Then at least one of the following holds:*

(1) *The cardinality of X is at most κ.*
(2) *There is a continuous injection $\phi \colon 2^{\mathbb{N}} \hookrightarrow X$.*

Proof. As every K_X-independent set $Y \subseteq X$ contains at most one point, this follows from the special cases of Theorem 2.1.1 and Proposition 2.1.7 where $D = 2$ and $H = K_X$. ⊠

Theorem 2.1.9 (Souslin). *Suppose that X is an analytic Hausdorff space. Then exactly one of the following holds:*

(1) *The set X is countable.*
(2) *There is a continuous injection $\phi \colon 2^{\mathbb{N}} \hookrightarrow X$.*

Proof. As $\mathfrak{c} \not\leq \aleph_0$, this follows from the special case of Theorem 2.1.8 where $\kappa = \aleph_0$. ⊠

Theorem 2.1.10 (Davis). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds and X is a subset of an analytic Hausdorff space. Then exactly one of the following holds:*

(1) *The set X is countable.*
(2) *There is a continuous injection $\phi \colon 2^{\mathbb{N}} \hookrightarrow X$.*

Proof. As $\mathfrak{c} \not\leq \aleph_0$ and every K_X-independent set $Y \subseteq X$ contains at most one point, this follows from the analog of the proof of Theorem 2.1.8 in which one replaces the use of Theorem 2.1.1 with that of Theorem 2.1.5. ⊠

Finally, we consider the circumstances under which a space can be covered by a well-orderable family of compact sets.

THEOREM 2.1.11 (Carroy-M-Soukup). *Suppose that κ is an aleph, X is a metric space, and $Y \subseteq X$ is κ-Souslin. Then at least one of the following holds:*

(1) *There is a cover of Y by at most κ-many compact subsets of X.*
(2) *There is a closed continuous injection $\phi \colon \mathbb{N}^{\mathbb{N}} \hookrightarrow X$ with the property that $\phi(\mathbb{N}^{\mathbb{N}}) \subseteq Y$.*

PROOF. Let H be the $\mathbb{N}$-dimensional dihypergraph on X consisting of all injective sequences $x \in X^{\mathbb{N}}$ with no convergent subsequence. Note that if $x \in H$, $\epsilon_n \leq \inf_{m \in \mathbb{N} \setminus \{n\}} d(x(m), x(n))$ for all $n \in \mathbb{N}$, and $\epsilon_n \to 0$, then $\prod_{n \in \mathbb{N}} \mathcal{B}(x(n), \epsilon_n/2) \subseteq H$, so H is box open. As every closed H-independent set is compact, Proposition 1.6.3 ensures that if there is a κ-coloring of $H \restriction Y$, then condition (1) holds. Otherwise, Theorem 2.1.1 yields a continuous homomorphism $\phi \colon \mathbb{N}^{\mathbb{N}} \to Y$ from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H, and Proposition 2.1.7 ensures that ϕ is injective. To see that ϕ sends closed subsets of $\mathbb{N}^{\mathbb{N}}$ to closed subsets of X, it is sufficient to show that every sequence $a \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ for which $\phi \circ a$ converges in X has a convergent subsequence. If there exists $b \in \mathbb{N}^{\mathbb{N}}$ such that $a(n)(i) < b(i)$ for all $i, n \in \mathbb{N}$, then the compactness of $\prod_{i \in \mathbb{N}} b(i)$ yields the desired subsequence. So suppose, towards a contradiction, that there does not exist such a b. Then there is a least $i \in \mathbb{N}$ for which $\{a(n)(i) \mid n \in \mathbb{N}\}$ is infinite. By passing to a subsequence, we can assume that for all distinct $m, n \in \mathbb{N}$, the sequences $a(m)$ and $a(n)$ differ from one another for the first time on their i^{th} coordinates. Fix $b \in \mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ for which $a(\mathbb{N}) \subseteq b(\mathbb{N})$, and observe that $\phi \circ b \in H$, contradicting the fact that $\phi \circ a$ converges. ⊠

A subset of a topological space is K_σ if it is a union of countably-many compact sets.

THEOREM 2.1.12 (Hurewicz, Kechris–Saint Raymond). *Suppose that X is a metric space and $Y \subseteq X$ is analytic. Then exactly one of the following holds:*

(1) *There is a K_σ subset of X containing Y.*
(2) *There is a closed continuous injection $\phi \colon \mathbb{N}^{\mathbb{N}} \hookrightarrow X$ with the property that $\phi(\mathbb{N}^{\mathbb{N}}) \subseteq Y$.*

PROOF. As $\mathbb{N}^{\mathbb{N}}$ is not K_σ and preimages of compact sets under continuous closed injections are compact, this follows from the special case of Theorem 2.1.11 where $\kappa = \aleph_0$. ⊠

THEOREM 2.1.13 (Kechris–Saint Raymond). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, X is an analytic metric space, and $Y \subseteq X$. Then exactly one of the following holds:*

(1) *There is a K_σ subset of X containing Y.*
(2) *There is a closed continuous injection $\phi\colon \mathbb{N}^\mathbb{N} \hookrightarrow X$ with the property that $\phi(\mathbb{N}^\mathbb{N}) \subseteq Y$.*

Proof. As $\mathbb{N}^\mathbb{N}$ is not K_σ and preimages of compact sets under continuous closed injections are compact, this follows from the analog of the proof of Theorem 2.1.11 in which one replaces the use of Theorem 2.1.1 with that of Theorem 2.1.5. ⊠

2. Partial compactifications

Given sets $Y \subseteq X^{<\mathbb{N}}$ and $Z \subseteq X^{\leq\mathbb{N}}$, let $Y \frown Z$ denote the set of sequences of the form $y \frown z$, where $y \in Y$ and $z \in Z$. Given a sequence $(X_n)_{n\in\mathbb{N}}$ of topological spaces, the *product topology* on $(\prod_{n\in\mathbb{N}} X_n) \cup \bigcup_{n\in\mathbb{N}} \prod_{m<n} X_m$ is the topology generated by the basic open sets of the form $(\prod_{m<n} U_m) \frown ((\prod_{m\in\mathbb{N}} X_{m+n}) \cup \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_{\ell+n})$, where $n \in \mathbb{N}$ and $U_m \subseteq X_m$ is open for all $m < n$.

Proposition 2.2.1. *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a sequence of compact spaces and U_n is a proper open subset of X_n for all $n \in \mathbb{N}$. Then $(\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}}(\prod_{m<n} U_m) \frown (\sim U_n)^1$ is compact.*

Proof. Suppose that $\mathcal{V}$ is a family of open subsets of $(\prod_{n\in\mathbb{N}} X_n) \cup \bigcup_{n\in\mathbb{N}} \prod_{m<n} X_m$ covering the space in question. For all $n \in \mathbb{N}$, let $\mathcal{V}_n$ be the family of all open hyperrectangles $\prod_{m<n} V_m \subseteq \prod_{m<n} X_m$ such that $(\prod_{m<n} V_m) \frown \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_{\ell+n}$ is contained in a set in $\mathcal{V}$.

Lemma 2.2.2. *Suppose that $n \in \mathbb{N}$ and $K_m \subseteq U_m$ is a non-empty compact set for all $m < n$. Then there is a compact set $K_n \subseteq U_n$ for which there is a finite set $\mathcal{F}_{n+1} \subseteq \mathcal{V}_{n+1}$ covering $(\prod_{m<n} K_m) \frown (\sim K_n)^1$.*

Proof. As $(\prod_{m<n} K_m) \frown (\sim U_n)^1$ is compact, there is a finite subcover $\mathcal{F}_{n+1} \subseteq \mathcal{V}_{n+1}$ of $(\prod_{m<n} K_m) \frown (\sim U_n)^1$. Let $\boldsymbol{\mathcal{F}}_{n+1}$ be the family of sets $\mathcal{F} \subseteq \mathcal{F}_{n+1}$ for which $\{V_n \mid \prod_{m<n+1} V_m \in \mathcal{F}\}$ covers $\sim U_n$, and define $K_n = \sim\bigcap_{\mathcal{F}\in\boldsymbol{\mathcal{F}}_{n+1}} \bigcup_{\prod_{m<n+1} V_m\in\mathcal{F}} V_n$. As $\sim U_n \subseteq \sim K_n$, it follows that $K_n \subseteq U_n$. To see that $\mathcal{F}_{n+1}$ covers $(\prod_{m<n} K_m) \frown (\sim K_n)^1$, suppose that $x \in (\prod_{m<n} K_m) \frown (\sim K_n)^1$, and observe that the corresponding family $\mathcal{F} = \{\prod_{m<n+1} V_m \in \mathcal{F}_{n+1} \mid x \restriction n \in \prod_{m<n} V_m\}$ is in $\boldsymbol{\mathcal{F}}_{n+1}$, so the definition of K_n yields a hyperrectangle $\prod_{m<n+1} V_m \in \mathcal{F}$ for which $x_n \in V_n$, thus $x \in \prod_{m<n+1} V_m \in \mathcal{F}_{n+1}$. ⊠

Observe that if $n \in \mathbb{N}$, $K_m \subseteq U_m$ is a non-empty compact set for all $m < n$, $\mathcal{F}_{m+1} \subseteq \mathcal{V}_{m+1}$ is a cover of $(\prod_{\ell<m} K_\ell) \frown (\sim K_m)^1$ for all $m < n$, and $\mathcal{F}_{n+1} \subseteq \mathcal{V}_{n+1}$ is a cover of $(\prod_{m<n} K_m) \frown X^1$, then the basic open subsets of $(\prod_{m\in\mathbb{N}} X_m) \cup \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_m$ associated with

the sets in $\bigcup_{m<n+1} \mathcal{F}_{m+1}$ cover $(\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$. By Lemma 2.2.2 and DC, we can therefore assume that there are non-empty compact sets $K_n \subseteq U_n$ and finite subcovers $\mathcal{F}_{n+1} \subseteq \mathcal{V}_{n+1}$ of $(\prod_{m<n} K_m) \frown (\sim K_n)^1$ for all $n \in \mathbb{N}$. Let $\mathcal{V}_\infty$ be the family of all basic open subsets of $(\prod_{n\in\mathbb{N}} X_n) \cup \bigcup_{n\in\mathbb{N}} \prod_{m<n} X_m$ contained in a set in $\mathcal{V}$. As $\prod_{n\in\mathbb{N}} K_n$ is compact, there is a finite subcover $\mathcal{F}_\infty \subseteq \mathcal{V}_\infty$ of $\prod_{n\in\mathbb{N}} K_n$. Fix $n \in \mathbb{N}$ for which every set in $\mathcal{F}_\infty$ is of the form $(\prod_{\ell<m} V_\ell) \frown ((\prod_{\ell\in\mathbb{N}} X_{\ell+m}) \cup \bigcup_{\ell\in\mathbb{N}} \prod_{k<\ell} X_{k+m})$, where $m < n+1$ and $\prod_{\ell<m} V_\ell \subseteq \prod_{\ell<m} X_\ell$ is an open hyperrectangle. Then $\mathcal{F}_\infty$ is a cover of $(\prod_{m<n} K_m) \frown \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_{\ell+n}$, so the sets in $\mathcal{F}_\infty$ along with the basic open subsets of $(\prod_{m\in\mathbb{N}} X_m) \cup \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_m$ associated with the sets in $\bigcup_{m<n} \mathcal{F}_{m+1}$ cover $(\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$. ⊠

An *ultrametric* on X is a function $\rho\colon X \times X \to [0, \infty)$ such that $\rho(x, y) = 0 \iff x = y$ and $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, and $\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$ for all $x, y, z \in X$. Given a point $x \in X$ and a set $Y \subseteq X$, define $\rho(x, Y) = \inf_{y\in Y} \rho(x, y)$.

PROPOSITION 2.2.3. *Suppose that X is an ultrametric space, $x, y \in X$, $Z \subseteq Y$, and $\rho_X(x, Z) > \rho_X(y, Z)$. Then $\rho_X(x, Z) \leq \rho(x, y)$.*

PROOF. Fix $z \in Z$ with the property that $\rho_X(x, Z) > \rho_X(y, z)$. As $\rho_X(x, Z) \leq \rho_X(x, z) \leq \max\{\rho_X(x, y), \rho_X(y, z)\}$, it follows that $\rho_X(x, Z) \leq \rho_X(x, y)$. ⊠

PROPOSITION 2.2.4. *Suppose that $(X_n)_{n\in\mathbb{N}}$ is a sequence of (complete) ultrametric spaces and $U_n \subseteq X_n$ is open for all $n \in \mathbb{N}$. Then $(\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$ admits a compatible (complete) ultrametric.*

PROOF. Fix compatible (complete) ultrametrics ρ_n on X_n such that $\mathrm{diam}_{\rho_n}(X_n) < 1$ for all $n \in \mathbb{N}$ and $\mathrm{diam}_{\rho_n}(X_n) \to 0$, and define $\rho\colon ((\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1)^2 \to [0, \infty)$ by $\rho(x, y) = \max_{n<|x|,|y|} \rho_n(x(n), y(n)) \prod_{m<n} \max\{\rho_m(x(m), \sim U_m), \rho_m(y(m), \sim U_m)\}$.

To see that ρ is an ultrametric, suppose that $x, y, z \in (\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$, and fix $n \in \mathbb{N}$ with the property that $\rho(x, z) = \rho_n(x(n), z(n)) \prod_{m<n} \max\{\rho_m(x(m), \sim U_m), \rho_m(z(m), \sim U_m)\}$.

LEMMA 2.2.5. *If $m < n$, then $\rho_m(x(m), \sim U_m) = \rho_m(z(m), \sim U_m)$.*

PROOF. Observe that if $\rho_m(x(m), \sim U_m) \neq \rho_m(z(m), \sim U_m)$, then Proposition 2.2.3 ensures that $\max\{\rho_m(x(m), \sim U_m), \rho_m(z(m), \sim U_m)\} \leq$

$\rho_m(x(m), z(m))$, so

$$\begin{aligned}
&\rho_n(x(n), z(n)) \textstyle\prod_{\ell<n} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(z(\ell), \sim U_\ell)\} \\
&\quad < \textstyle\prod_{\ell<m+1} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(z(\ell), \sim U_\ell)\} \\
&\quad \leq \rho_m(x(m), z(m)) \textstyle\prod_{\ell<m} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(z(\ell), \sim U_\ell)\} \\
&\quad \leq \rho(x, z),
\end{aligned}$$

contradicting the definition of n. ⊠

Observe now that if $\rho_m(x(m), \sim U_m) = \rho_m(y(m), \sim U_m)$ for all $m < n$, then the fact that $\rho_n(x(n), z(n)) \leq \max\{\rho_n(x(n), y(n)), \rho_n(y(n), z(n))\}$ ensures that $\rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$. Otherwise, there is a least natural number $m < n$ for which $\rho_m(x(m), \sim U_m) \neq \rho_m(y(m), \sim U_m)$, in which case one more application of Proposition 2.2.3 ensures that $\rho_m(x(m), \sim U_m) \leq \rho_m(x(m), y(m))$, so

$$\begin{aligned}
\rho(x, z) &= \rho_n(x(n), z(n)) \textstyle\prod_{\ell<n} \rho_\ell(x(\ell), \sim U_\ell) \\
&< \textstyle\prod_{\ell<m+1} \rho_\ell(x(\ell), \sim U_\ell) \\
&\leq \rho_m(x(m), y(m)) \textstyle\prod_{\ell<m} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(y(\ell), \sim U_\ell)\} \\
&\leq \rho(x, y).
\end{aligned}$$

To see that the topology generated by ρ is coarser than that inherited from the product topology, suppose that $\epsilon > 0$ and $x \in (\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$, and fix $n \in \mathbb{N}$ such that $\operatorname{diam}_{\rho_m}(X_m) < \epsilon$ for all $m \geq n$. Then the intersection of $(\prod_{m<\min\{n,|x|\}} \mathcal{B}_{\rho_m}(x(m), \epsilon)) \frown ((\prod_{m\in\mathbb{N}} X_{m+\min\{n,|x|\}}) \cup \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_{\ell+\min\{n,|x|\}})$ with the space in question is contained in $\mathcal{B}_\rho(x, \epsilon)$, for if y is in the aforementioned intersection and $m < \min\{|x|, |y|\}$, then either $m < \min\{n, |x|\}$ or $m \geq n$, in which case $\rho_m(x(m), y(m)) \prod_{\ell<m} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(y(\ell), \sim U_\ell)\} \leq \rho_m(x(m), y(m)) < \epsilon$, so $y \in \mathcal{B}_\rho(x, \epsilon)$.

To see that the topology generated by ρ is finer than that inherited from the product topology, observe that if $0 < \epsilon < 1$, $x \in (\prod_{n\in\mathbb{N}} U_n) \cup \bigcup_{n\in\mathbb{N}} (\prod_{m<n} U_m) \frown (\sim U_n)^1$, and $1 \leq n \leq |x|$ is a natural number, then $\mathcal{B}_\rho(x, \epsilon \prod_{m<n-1} \rho_m(x(m), \sim U_m))$ is contained in $(\prod_{m<n} \mathcal{B}_{\rho_m}(x(m), \epsilon)) \frown ((\prod_{m\in\mathbb{N}} X_{m+n}) \cup \bigcup_{m\in\mathbb{N}} \prod_{\ell<m} X_{\ell+n})$, for if y is in the former set and $m < \min\{n-1, |y|\}$, then

$$\begin{aligned}
&\rho_m(x(m), y(m)) \textstyle\prod_{\ell<m} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(y(\ell), \sim U_\ell)\} \\
&\quad < \textstyle\prod_{\ell<n-1} \rho_\ell(x(\ell), \sim U_\ell) \\
&\quad \leq \rho_m(x(m), \sim U_m) \textstyle\prod_{\ell<m} \max\{\rho_\ell(x(\ell), \sim U_\ell), \rho_\ell(y(\ell), \sim U_\ell)\},
\end{aligned}$$

so $\rho_m(x(m), y(m)) < \rho_m(x(m), \sim U_m)$, and it follows that $y(m) \in U_m$, hence $m + 1 < |y|$, in which case the obvious induction ensures that

$n-1<|y|$, so if $m<n$, then

$$\begin{aligned}\rho_m(x(m),y(m))\textstyle\prod_{\ell<m}\max\{\rho_\ell(x(\ell),{\sim}U_\ell),\rho_\ell(y(\ell),{\sim}U_\ell)\}\\ <\epsilon\textstyle\prod_{\ell<n-1}\rho_\ell(x(\ell),{\sim}U_\ell)\\ \le\epsilon\textstyle\prod_{\ell<m}\max\{\rho_\ell(x(\ell),{\sim}U_\ell),\rho_\ell(y(\ell),{\sim}U_\ell)\},\end{aligned}$$

thus $\rho_m(x(m),y(m))<\epsilon$, and therefore $y\in(\prod_{m<n}\mathcal{B}_{\rho_m}(x(m),\epsilon))\frown((\prod_{m\in\mathbb{N}}X_{m+n})\cup\bigcup_{m\in\mathbb{N}}\prod_{\ell<m}X_{\ell+n})$.

To see that the completeness of each ρ_n yields that of ρ, suppose that $(x_k)_{k\in\mathbb{N}}$ is a ρ-Cauchy sequence, and note that if $n\in\mathbb{N}$ has the property that $|x_k|>n$ for all but finitely many $k\in\mathbb{N}$ and $(x_k(m))_{k\in\mathbb{N}}$ converges to a point of U_m for all $m<n$, then there exists $\epsilon_m>0$ such that $\forall^\infty k\in\mathbb{N}\ \rho(x_k(m),{\sim}U_m)\ge\epsilon_m$ for all $m<n$, so $(x_k(n))_{k\in\mathbb{N}}$ is a ρ_n-Cauchy sequence, thus the completeness of ρ_n ensures that it converges. A straightforward recursive construction therefore yields a sequence $x\in(\prod_{n\in\mathbb{N}}U_n)\cup\bigcup_{n\in\mathbb{N}}(\prod_{m<n}U_m)\frown({\sim}U_n)^1$ with the property that $x_k(n)\to x(n)$ for all $n<|x|$, in which case $x_k\to x$. ⊠

Let $\mathrm{Cnvg}(X)$ denote the set of sequences $(x_n)_{n\in\mathbb{N}}$ of elements of X that converge to an element of X.

Proposition 2.2.6. *Suppose that X and Y are metric spaces, $D\subseteq X$ is dense, and $\phi\colon D\to Y$ is a continuous homomorphism from $\mathrm{Cnvg}(X)\restriction D$ to $\mathrm{Cnvg}(Y)$. Then there is a continuous extension $\psi\colon X\to Y$ of ϕ.*

Proof. Note first that if $(w_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are sequences of elements of D that converge to the same point of X, then the sequence $(v_n)_{n\in\mathbb{N}}$, given by $v_{2n}=w_n$ and $v_{2n+1}=x_n$, is also convergent, thus so too is $(\phi(v_n))_{n\in\mathbb{N}}$, hence $(\phi(w_n))_{n\in\mathbb{N}}$ and $(\phi(x_n))_{n\in\mathbb{N}}$ converge to the same point of Y. It follows that there is a unique extension $\psi\colon X\to Y$ of ϕ such that $x_n\to x\implies\psi(x_n)\to\psi(x)$ for all $(x_n)_{n\in\mathbb{N}}\in D^{\mathbb{N}}$ and $x\in X$. To see that ψ is continuous, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of elements of X converging to some $x\in X$, fix sequences $(x_{m,n})_{m\in\mathbb{N}}$ of points of D converging to x_n for all $n\in\mathbb{N}$, fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to zero, fix a function $f\colon\mathbb{N}\to\mathbb{N}$ such that $\rho_X(x_{f(n),n},x_n),\rho_Y(\psi(x_{f(n),n}),\psi(x_n))<\epsilon_n$ for all $n\in\mathbb{N}$, and observe that $x_{f(n),n}\to x$, so $\psi(x_{f(n),n})\to\psi(x)$, thus $\psi(x_n)\to\psi(x)$. ⊠

3. Separation by unions of closed hyperrectangles

A *hyperrectangular homomorphism* from a pair (R_X,S_X) of subsets of $\prod_{d\in D}X_d$ to a pair (R_Y,S_Y) of subsets of $\prod_{d\in D}Y_d$ is a function $\phi\colon\prod_{d\in D}\mathrm{proj}_d(R_X\cup S_X)\to\prod_{d\in D}Y_d$ of the form $\phi(x)(d)=(\phi_d\circ x)(d)$, where $\phi_d \colon \mathrm{proj}_d(R_X \cup S_X) \to Y_d$ for all $d \in D$, with the property that $\phi(R_X) \subseteq R_Y$ and $\phi(S_X) \subseteq S_Y$.

We use $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$ to denote the one-point compactification of $\mathbb{N}$, and $\mathbb{H}_{(D\times\mathbb{N})^\mathbb{N},\infty}$ to denote the D-ary relation on the subspace $(D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1)$ of $(D \times \mathbb{N}_*)^{\leq\mathbb{N}}$ consisting of all sequences of the form $(t \frown ((d, \infty)))_{d\in D}$, where $t \in (D \times \mathbb{N})^{<\mathbb{N}}$.

Proposition 2.3.1. *Suppose that D is a non-empty countable discrete space and $C \subseteq (D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1)$ is an $\mathbb{H}_{(D\times\mathbb{N})^\mathbb{N},\infty}$-independent closed set. Then C is meager.*

Proof. As $C \cap (D \times \mathbb{N})^\mathbb{N}$ is $\mathbb{H}_{(D\times\mathbb{N})^\mathbb{N}}$-independent, $(D \times \mathbb{N})^\mathbb{N}$ is comeager in $(D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1)$, and Theorem 1.5.1 and Proposition 2.2.4 ensure that the latter is a Baire space, Proposition 1.6.6 ensures that C is meager. ⊠

Theorem 2.3.2 (Carroy-M-Soukup). *Suppose that D is a non-empty countable discrete space, κ is an aleph, $(X_d)_{d\in D}$ is a sequence of metric spaces, $R \subseteq \prod_{d\in D} X_d$ is κ-Souslin, and $S \subseteq {\sim}R$. Then at least one of the following holds:*

1. *There is a union of at most κ-many closed hyperrectangles separating R from S.*
2. *There exists a continuous hyperrectangular homomorphism ϕ: $\prod_{d\in D}(D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown \{((d, \infty))\}) \to \prod_{d\in D} X_d$ from $(\Delta^D((D \times \mathbb{N})^\mathbb{N}), \mathbb{H}_{(D\times\mathbb{N})^\mathbb{N},\infty})$ to (R, S).*

Proof. Let H be the $(D\times\mathbb{N})$-dimensional dihypergraph on R consisting of all sequences $(x_{d,n})_{(d,n)\in D\times\mathbb{N}}$ of elements of R for which there exists $y \in S$ with the property that $\forall d \in D\ y(d) = \lim_{n\to\infty} x_{d,n}(d)$. Observe that if $(x_{d,n})_{(d,n)\in D\times\mathbb{N}} \in H$, $\epsilon_n \to 0$, and $U_{d,n} = \{x \in R \mid \rho_{X_d}(x(d), x_{d,n}(d)) < \epsilon_n\}$ for all $(d, n) \in D \times \mathbb{N}$, then $\prod_{(d,n)\in D\times\mathbb{N}} U_{d,n} \subseteq H$, so H is box open. Moreover, if $Q \subseteq R$ is H-independent, then there does not exist $y \in (\prod_{d\in D} \overline{\mathrm{proj}_d(Q)}) \cap S$, since otherwise there are sequences $(x_{d,n})_{n\in\mathbb{N}}$ of elements of $\mathrm{proj}_d(Q)$ such that $x_{d,n} \to y(d)$ for all $d \in D$, as well as $x'_{d,n} \in Q$ such that $x_{d,n} = x'_{d,n}(d)$ for all $(d, n) \in D \times \mathbb{N}$, thus $x'_{d,n}(d) \to y(d)$ for all $d \in D$. It follows that if there is a κ-coloring $c \colon R \to \kappa$ of H, then $\bigcup_{\alpha<\kappa} \prod_{d\in D} \overline{\mathrm{proj}_d(c^{-1}(\{\alpha\}))}$ separates R from S. Otherwise, Theorem 2.1.1 yields a continuous homomorphism $\phi' \colon (D \times \mathbb{N})^\mathbb{N} \to R$ from $\mathbb{H}_{(D\times\mathbb{N})^\mathbb{N}}$ to H. Note that for all $d \in D$, the function $\mathrm{proj}_d \circ \phi'$ is a continuous homomorphism from $\mathrm{Cnvg}((D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown \{((d, \infty))\})) \restriction (D \times \mathbb{N})^\mathbb{N}$ to $\mathrm{Cnvg}(X_d)$, so Proposition 2.2.6 ensures the existence of a continuous extension $\phi_d \colon (D \times \mathbb{N})^\mathbb{N} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown \{((d, \infty))\}) \to X_d$ of $\mathrm{proj}_d \circ \phi'$, in which case the function $\phi = \prod_{d \in D} \phi_d$ is a hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S). ⊠

THEOREM 2.3.3 (Lecomte-Zeleny, Carroy-M-Soukup). *Suppose that D is a non-empty countable discrete space, $(X_d)_{d \in D}$ is a sequence of metric spaces, $R \subseteq \prod_{d \in D} X_d$ is analytic, and $S \subseteq {\sim}R$. Then exactly one of the following holds:*

1. *There is a union of countably-many closed hyperrectangles separating R from S.*
2. *There exists a continuous hyperrectangular homomorphism ϕ: $\prod_{d \in D}(D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown \{((d, \infty))\}) \to \prod_{d \in D} X_d$ from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S).*

PROOF. To see that conditions (1) and (2) are mutually exclusive, note that if $(\prod_{d \in D} C_{d,n})_{n \in \mathbb{N}}$ is a sequence of hyperrectangles whose union separates $\Delta^D((D \times \mathbb{N})^{\mathbb{N}})$ from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$, then $(\bigcap_{d \in D} C_{d,n})_{n \in \mathbb{N}}$ is a cover of $(D \times \mathbb{N})^{\mathbb{N}}$ by $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$-independent sets, and appeal to Proposition 2.3.1, noting that $(D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1)$ is a Baire space in which $(D \times \mathbb{N})^{\mathbb{N}}$ is comeager, by Theorem 1.5.1 and Proposition 2.2.4. To see that at least one of the two conditions holds, appeal to the special case of Theorem 2.3.2 where $\kappa = \aleph_0$. ⊠

THEOREM 2.3.4 (Carroy-M-Soukup). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, D is a non-empty countable discrete space, $(X_d)_{d \in D}$ is a sequence of analytic metric spaces, $R \subseteq \prod_{d \in D} X_d$, and $S \subseteq {\sim}R$. Then exactly one of the following holds:*

1. *There is a union of countably-many closed hyperrectangles separating R from S.*
2. *There exists a continuous hyperrectangular homomorphism ϕ: $\prod_{d \in D}(D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown \{((d, \infty))\}) \to \prod_{d \in D} X_d$ from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S).*

PROOF. The proof that conditions (1) and (2) are mutually exclusive is exactly the same as in Theorem 2.3.3. The proof that at least one of the two conditions holds is analogous to that of Theorem 2.3.2, replacing the use of Theorem 2.1.1 with that of Theorem 2.1.5. ⊠

In particular, we obtain a characterization of the circumstances under which two disjoint sets can be separated by a well-ordered union of closed sets:

THEOREM 2.3.5 (Carroy-M-Soukup). *Suppose that κ is an aleph, X is a metric space, $A \subseteq X$ is κ-Souslin, and $Y \subseteq {\sim}A$. Then at least one of the following holds:*

(1) *There is a union of at most κ-many closed sets separating A from Y.*
(2) *There is a continuous reduction $\pi \colon \mathbb{N}^{\mathbb{N}} \cup (\mathbb{N}^{<\mathbb{N}} \frown \{(\infty)\}) \to A \cup Y$ of $\mathbb{N}^{\mathbb{N}}$ to A.*

Proof. This is the special case of Theorem 2.3.2 where $D = 1$. ⊠

Theorem 2.3.6 (Hurewicz, Kechris-Louveau-Woodin). *Suppose that X is a metric space, $A \subseteq X$ is analytic, and $Y \subseteq \sim A$. Then exactly one of the following holds:*

(1) *There is an F_σ subset of X separating A from Y.*
(2) *There is a continuous reduction $\pi \colon \mathbb{N}^{\mathbb{N}} \cup (\mathbb{N}^{<\mathbb{N}} \frown \{(\infty)\}) \to A \cup Y$ of $\mathbb{N}^{\mathbb{N}}$ to A.*

Proof. This is the special case of Theorem 2.3.3 where $D = 1$. ⊠

Theorem 2.3.7 (Kechris-Louveau-Woodin). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, X is an analytic metric space, $A \subseteq X$, and $Y \subseteq \sim A$. Then exactly one of the following holds:*

(1) *There is an F_σ subset of X separating A from Y.*
(2) *There is a continuous reduction $\pi \colon \mathbb{N}^{\mathbb{N}} \cup (\mathbb{N}^{<\mathbb{N}} \frown \{(\infty)\}) \to A \cup Y$ of $\mathbb{N}^{\mathbb{N}}$ to A.*

Proof. This is the special case of Theorem 2.3.4 where $D = 1$. ⊠

We next generalize Theorems 2.1.1, 2.1.4, and 2.1.5 beyond box-open dihypergraphs:

Theorem 2.3.8 (Carroy-M-Soukup). *Suppose that D is a countable discrete space of cardinality at least two, κ is an aleph, X is a κ-Souslin metric space, and H is a D-dimensional dihypergraph on X. Then at least one of the following holds:*

(1) *There is a cover of X by at most κ-many H-independent closed sets.*
(2) *There is a continuous homomorphism $\phi \colon (D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1) \to X$ from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H.*

Proof. Observe that if $(\prod_{d \in D} C_{\alpha,d})_{\alpha < \kappa}$ is a sequence of hyperrectangles whose union separates $\Delta^D(X)$ from H, then $(\bigcap_{d \in D} C_{\alpha,d})_{\alpha<\kappa}$ is a cover of X by H-independent sets. By Theorem 2.3.2, we can therefore assume that there is a continuous hyperrectangular homomorphism $\prod_{d \in D} \phi_d$ from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to $(\Delta^D(X), H)$. But then the function $\phi = \bigcup_{d \in D} \phi_d$ is a homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H, and Proposition 2.2.6 ensures that it is continuous. ⊠

THEOREM 2.3.9 (Lecomte-Zeleny, Carroy-M-Soukup). *Suppose that D is a countable discrete space of cardinality at least two, X is an analytic metric space, and H is a D-dimensional dihypergraph on X. Then exactly one of the following holds:*

(1) *There is a $\mathbf{\Delta}^0_2$-measurable $\aleph_0$-coloring of H.*
(2) *There is a continuous homomorphism $\phi\colon (D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1) \to X$ from $\mathbb{H}_{(D\times\mathbb{N})^{\mathbb{N}},\infty}$ to H.*

PROOF. To see that conditions (1) and (2) are mutually exclusive, note that $(D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1)$ is a Baire space by Theorem 1.5.1 and Proposition 2.2.4, and appeal to Propositions 1.6.5 and 2.3.1. To see that at least one of them holds, appeal to the special case of Theorem 2.3.8 where $\kappa = \aleph_0$. ⊠

THEOREM 2.3.10 (Carroy-M-Soukup). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, D is a countable discrete space of cardinality at least two, X is a subset of an analytic metric space, and H is a D-dimensional dihypergraph on X. Then exactly one of the following holds:*

(1) *There is a $\mathbf{\Delta}^0_2$-measurable $\aleph_0$-coloring of H.*
(2) *There is a continuous homomorphism $\phi\colon (D \times \mathbb{N})^{\mathbb{N}} \cup ((D \times \mathbb{N})^{<\mathbb{N}} \frown (D \times \{\infty\})^1) \to X$ from $\mathbb{H}_{(D\times\mathbb{N})^{\mathbb{N}},\infty}$ to H.*

PROOF. The proof that conditions (1) and (2) are mutually exclusive is exactly the same as in Theorem 2.3.9. The proof that at least one of the two conditions holds is analogous to that of Theorem 2.3.8, replacing the use of Theorem 2.3.2 with that of Theorem 2.3.4. ⊠

CHAPTER 3

The $\mathbb{G}_0$ dichotomy, I: Abstract colorings

1. Colorings within cliques

Given a binary relation R on X, we say that a set $Y \subseteq X$ is an *R-clique* if $y \mathrel{R} z$ for all distinct $y, z \in Y$.

THEOREM 3.1.1 (Geschke). *Suppose that κ is an aleph, X is a Hausdorff space, G is a κ-Souslin digraph on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, there is a κ-coloring of $G \restriction Y$.*
(2) *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique.*

PROOF. Suppose that condition (1) fails, and fix an R-clique $Y \subseteq X$ for which there is no κ-coloring of $G \restriction Y$. By Proposition 1.4.1, we can assume that there is a continuous surjection $\phi_G\colon \kappa^{\mathbb{N}} \twoheadrightarrow G$. By Propositions 1.4.1, 1.4.2, and 1.4.3, we can assume that there is a continuous function $\phi_X\colon \kappa^{\mathbb{N}} \to X$ for which $\phi_X(\kappa^{\mathbb{N}})$ is the union of the left and right projections of G onto X. Fix a decreasing sequence $(R_n)_{n\in\mathbb{N}}$ of open subsets of $X \times X$ whose intersection is R, as well as sequences $s_n \in 2^n$ for which $\mathbb{G}_0 = G_{\{s_n \mid n\in\mathbb{N}\}}$.

We will define a decreasing sequence $(Y^\alpha)_{\alpha<\kappa^+}$ of subsets of Y, off of which there are κ-colorings of $G \restriction Y$. We begin by setting $Y^0 = Y$. For all limit ordinals $\lambda < \kappa^+$, we set $Y^\lambda = \bigcap_{\alpha<\lambda} Y^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a\colon 2^{n^a} \to \kappa^{<\mathbb{N}}$, $\psi^a_n\colon 2^{n^a-(n+1)} \to \kappa^{n^a}$ for all $n < n^a$, and $\phi_X(\mathcal{N}_{\phi^a(s)}) \times \phi_X(\mathcal{N}_{\phi^a(t)}) \subseteq R_{n^a}$ for all distinct $s, t \in 2^{n^a}$. A *one-step extension* of an approximation a is an approximation b such that:

(a) $n^b = n^a + 1$.
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}\ (s \sqsubset t \Longrightarrow \phi^a(s) \sqsubset \phi^b(t))$.
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}\ (s \sqsubset t \Longrightarrow \psi^a_n(s) \sqsubset \psi^b_n(t))$.

Similarly, a *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma\colon 2^{n^\gamma} \to \kappa^{\mathbb{N}}$, $\psi^\gamma_n\colon 2^{n^\gamma-(n+1)} \to \kappa^{\mathbb{N}}$ for all $n < n^\gamma$, and $(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown t))$ for all $n < n^\gamma$ and $t \in 2^{n^\gamma - (n+1)}$. A configuration γ is *compatible* with an approximation a if the following conditions hold:

(i) $n^a = n^\gamma$.
(ii) $\forall t \in 2^{n^a}\ \phi^a(t) \sqsubseteq \phi^\gamma(t)$.
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}\ \psi_n^a(t) \sqsubseteq \psi_n^\gamma(t)$.

A configuration γ is *compatible* with a set $Y' \subseteq Y$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq Y'$. An approximation a is Y'*-terminal* if no configuration is compatible with a one-step extension of a and Y'. Let $A(a, Y')$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(s_{n^a})$, where γ varies over all configurations compatible with a and Y'.

LEMMA 3.1.2. *Suppose that $Y' \subseteq Y$ and a is a Y'-terminal approximation. Then $A(a, Y')$ is G-independent.*

PROOF. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and Y', with the property that $((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a})) \in G$. Fix a sequence $d \in \kappa^{\mathbb{N}}$ such that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a}))$, and let γ denote the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi_n^\gamma(t \frown (i)) = \psi_n^{\gamma_i}(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi_{n^a}^\gamma(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is Y'-terminal. ⊠

Define $Y^{\alpha+1}$ to be the difference of Y^α and the union of the sets of the form $A(a, Y^\alpha)$, where a varies over all Y^α-terminal approximations.

LEMMA 3.1.3. *Suppose that $\alpha < \kappa^+$ and a is a non-$Y^{\alpha+1}$-terminal approximation. Then a has a non-Y^α-terminal one-step extension.*

PROOF. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $Y^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in Y^{\alpha+1}$, so b is not Y^α-terminal. ⊠

Fix $\alpha < \kappa^+$ such that the families of Y^α-terminal approximations and $Y^{\alpha+1}$-terminal approximations are one and the same, and let a_0 denote the unique approximation for which $n^{a_0} = 0$ and $\phi^{a_0}(\emptyset) = \emptyset$. As $A(a_0, Y') = Y'$ for all $Y' \subseteq Y$, we can assume that a_0 is not Y^α-terminal, since otherwise $Y^{\alpha+1} = \emptyset$, so there is a κ-coloring of $G \restriction Y$.

By recursively applying Lemma 3.1.3, we obtain non-Y^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi', \psi_n \colon 2^{\mathbb{N}} \to \kappa^{\mathbb{N}}$ by $\phi'(c) = \bigcup_{n \in \mathbb{N}} \phi^{a_n}(c \restriction n)$ and $\psi_n(c) = \bigcup_{m > n} \psi_n^{a_m}(c \restriction (m - (n+1)))$ for all $n \in \mathbb{N}$. Clearly these functions are continuous.

To see that the function $\phi = \phi_X \circ \phi'$ is a homomorphism from $\mathbb{G}_0$ to G, we will show the stronger fact that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then

$$(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi')(s_n \frown (0) \frown c), (\phi_X \circ \phi')(s_n \frown (1) \frown c)).$$

And for this, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi')(s_n \frown (0) \frown c), (\phi_X \circ \phi')(s_n \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (0) \frown s)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (1) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \upharpoonright (m - (n + 1))$. The fact that a_m is not Y^α-terminal yields a configuration γ compatible with a_m, in which case $((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown s)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$.

To see that $\phi(2^{\mathbb{N}})$ is an R-clique, observe that if $c, d \in 2^{\mathbb{N}}$ are distinct and $n \in \mathbb{N}$ is sufficiently large that $c \upharpoonright n \neq d \upharpoonright n$, then $\phi(c) \in \phi_X(\mathcal{N}_{\phi^{a_n}(c \upharpoonright n)})$ and $\phi(d) \in \phi_X(\mathcal{N}_{\phi^{a_n}(d \upharpoonright n)})$, so $\phi(c) \mathrel{R_n} \phi(d)$. ⊠

THEOREM 3.1.4 (Geschke). *Suppose that X is a Hausdorff space, G is an analytic digraph on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, there is an $\aleph_0$-coloring of $G \upharpoonright Y$.*
(2) *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique.*

PROOF. This is the special case of Theorem 3.1.1 where $\kappa = \aleph_0$. ⊠

THEOREM 3.1.5 (Geschke). *Suppose that X is an analytic Hausdorff space, G is a $\boldsymbol{\Sigma}^1_2$ digraph on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, there is an $\aleph_1$-coloring of $G \upharpoonright Y$.*
(2) *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique.*

PROOF. Note that G is $\aleph_1$-Souslin by Propositions 1.4.2 and 1.4.10, and appeal to the special case of Theorem 3.1.1 where $\kappa = \aleph_1$. ⊠

THEOREM 3.1.6 (Geschke). *Suppose that* `AD` *holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, G is a $\boldsymbol{\Sigma}^1_{2n+1}$ digraph on X, and R is a reflexive G_δ binary relation on X. Then exactly one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, there is a $\boldsymbol{\kappa}^1_{2n+1}$-coloring of $G \upharpoonright Y$.*
(2) *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique.*

PROOF. Proposition 1.6.11 ensures that the two conditions are mutually exclusive. To see that at least one of them holds, note that G is $\boldsymbol{\kappa}^1_{2n+1}$-Souslin by Theorem 1.4.14, and appeal to the special case of Theorem 3.1.1 where $\kappa = \boldsymbol{\kappa}^1_{2n+1}$. ⊠

THEOREM 3.1.7 (Geschke). *Suppose that* AD *holds,* $n \in \mathbb{N}$, X *is an analytic Hausdorff space,* G *is a* $\boldsymbol{\Sigma}^1_{2n+2}$ *digraph on* X, *and* R *is a reflexive* G_δ *binary relation on* X. *Then exactly one of the following holds:*

(1) *For every* R*-clique* $Y \subseteq X$, *there is a* $(\boldsymbol{\kappa}^1_{2n+1})^+$*-coloring of* $G \upharpoonright Y$.
(2) *There is a continuous homomorphism* $\phi \colon 2^{\mathbb{N}} \to X$ *from* $\mathbb{G}_0$ *to* G *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-clique.*

PROOF. Proposition 1.6.11 ensures that the two conditions are mutually exclusive. To see that at least one of them holds, note that G is $(\boldsymbol{\kappa}^1_{2n+1})^+$-Souslin by Theorem 1.4.14, and appeal to the special case of Theorem 3.1.1 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$. ⊠

REMARK 3.1.8. For all $n \in \mathbb{N}$, the weakenings of the corresponding special cases of Theorems 3.1.6 and 3.1.7 in which conditions (1) and (2) are not required to be mutually exclusive are consequences of $\mathtt{Det}(\boldsymbol{\Delta}^1_n)$, yielding analogous generalizations of the other consequences of AD established in this chapter.

THEOREM 3.1.9 (Geschke). *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds,* X *is an analytic Hausdorff space,* G *is a digraph on* X, *and* R *is a reflexive* G_δ *binary relation on* X. *Then exactly one of the following holds:*

(1) *For every* R*-clique* $Y \subseteq X$, *there is an ordinal-coloring of* $G \upharpoonright Y$.
(2) *There is a continuous homomorphism* $\phi \colon 2^{\mathbb{N}} \to X$ *from* $\mathbb{G}_0$ *to* G *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-clique.*

PROOF. Proposition 1.6.11 ensures that the two conditions are mutually exclusive. To see that at least one of them holds, note that there is an aleph κ for which G is κ-Souslin by Theorem 1.4.14, and appeal to Theorem 3.1.1. ⊠

2. Discrete perfect sets within cliques

An *extended-valued quasi-metric* on X is a map $\rho \colon X \times X \to [0, \infty]$ such that $\rho(x, x) = 0$ for all $x \in X$, $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$, and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$. Given $\epsilon \geq 0$, we say that (X, ρ) is ϵ*-discrete* if $\rho(x, y) > \epsilon$ for all distinct $x, y \in X$.

THEOREM 3.2.1 (Geschke). *Suppose that* $\delta \geq 0$, $\epsilon \geq 2\delta$, κ *is an aleph,* X *is a Hausdorff space,* ρ *is an extended-valued quasi-metric on* X *for which* $\rho^{-1}([0, \delta])$ *is* $\aleph_0$*-universally Baire and* $\rho^{-1}([0, \epsilon])$ *is co-*κ*-Souslin, and* R *is a reflexive* G_δ *binary relation on* X. *Then at least one of the following holds:*

(1) *Every R-clique $Y \subseteq X$ is a union of at most κ-many sets of ρ-diameter at most ϵ.*
(2) *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), \rho \restriction \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. Suppose that condition (1) fails, fix an R-clique $Y \subseteq X$ for which there is no cover of Y by at most κ-many sets of ρ-diameter at most ϵ, set $G = \rho^{-1}((\epsilon, \infty])$, and observe that Theorem 3.1.1 yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique. Define $G' = (\rho \circ (\phi \times \phi))^{-1}([0, \delta])$, and observe that $\mathbb{G}_0 \cap (G')^{-1}G' = \emptyset$, so Proposition 1.6.10 ensures that G' is meager, thus Theorem 1.6.1 yields a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\sim\Delta(2^{\mathbb{N}})$ to $\sim G'$. Define $\pi = \phi \circ \psi$. ⊠

Theorem 3.2.2 (Geschke). *Suppose that κ is an aleph, X is a Hausdorff space, ρ is an extended-valued quasi-metric on X for which there are arbitrarily small $\delta, \epsilon > 0$ such that $\rho^{-1}([0, \delta])$ is $\aleph_0$-universally Baire and $\rho^{-1}([0, \epsilon])$ is co-κ-Souslin, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, the space $(Y, \rho \restriction Y)$ has a basis of cardinality at most κ.*
(2) *There exist $\delta > 0$ and a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), \rho \restriction \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. By Theorem 3.2.1, it is enough to note that if $\epsilon_n \to 0$, $\mathcal{Y}_n$ is a cover of Y by sets of ρ-diameter at most ϵ_n for all $n \in \mathbb{N}$, and $\mathcal{U}_n = \{\mathcal{B}_\rho(Y', \epsilon_n) \cap Y \mid Y' \in \mathcal{Y}_n\}$ for all $n \in \mathbb{N}$, then the set $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is a basis for (Y, ρ). ⊠

The special case of either of the above theorems, where ρ is the characteristic function of the complement of an equivalence relation and $R = X \times X$, is a version of Harrington-Shelah's perfect set theorem for co-κ-Souslin equivalence relations. The analogous special cases of the following results are Silver's perfect set theorem for co-analytic equivalence relations, Burgess's perfect set theorem for analytic equivalence relations, and their generalizations under determinacy.

Theorem 3.2.3 (Harrington-Friedman-Kechris, Geschke). *Suppose that X is an analytic Hausdorff space, ρ is an extended-valued quasi-metric on X for which there are arbitrarily small $\epsilon > 0$ such that $\rho^{-1}([0, \epsilon])$ is co-analytic, and R is a reflexive G_δ binary relation on X. Then exactly one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, the space $(Y, \rho \restriction Y)$ is separable.*

(2) *There exist* $\delta > 0$ *and a continuous injection* $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\pi(2^{\mathbb{N}})$ *is an* R*-clique and* $(\pi(2^{\mathbb{N}}), \rho \upharpoonright \pi(2^{\mathbb{N}}))$ *is* δ*-discrete.*

PROOF. As Proposition 1.5.9 ensures that there are arbitrarily small $\delta > 0$ for which $\rho^{-1}([0,\delta])$ is $\aleph_0$-universally Baire, the special case of Theorem 3.2.2 where $\kappa = \aleph_0$ yields $\neg(1) \implies (2)$. To see that the two conditions are mutually exclusive, note that condition (2) ensures that the cardinality of any basis for (X, ρ) is at least $\mathfrak{c}$. ⊠

THEOREM 3.2.4 (Geschke). *Suppose that* X *is an analytic Hausdorff space,* ρ *is an extended-valued quasi-metric on* X *for which there are arbitrarily small* $\delta, \epsilon > 0$ *such that* $\rho^{-1}([0,\delta])$ *is* $\aleph_0$*-universally Baire and* $\rho^{-1}([0,\epsilon])$ *is* $\mathbf{\Pi}^1_2$*, and* R *is a reflexive* G_δ *binary relation on* X*. Then at least one of the following holds:*

(1) *For every* R*-clique* $Y \subseteq X$*, the space* $(Y, \rho \upharpoonright Y)$ *has a basis of cardinality at most* $\aleph_1$*.*
(2) *There exist* $\delta > 0$ *and a continuous injection* $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\pi(2^{\mathbb{N}})$ *is an* R*-clique and* $(\pi(2^{\mathbb{N}}), \rho \upharpoonright \pi(2^{\mathbb{N}}))$ *is* δ*-discrete.*

PROOF. As Propositions 1.4.2 and 1.4.10 ensure that there are arbitrarily small $\epsilon > 0$ for which $\rho^{-1}([0,\epsilon])$ is co-$\aleph_1$-Souslin, this follows from the special case of Theorem 3.2.2 where $\kappa = \aleph_1$. ⊠

THEOREM 3.2.5 (Geschke). *Suppose that* AD *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space,* ρ *is an extended-valued quasi-metric on* X *for which there are arbitrarily small* $\epsilon > 0$ *such that* $\rho^{-1}([0,\epsilon])$ *is* $\mathbf{\Pi}^1_{2n+1}$*, and* R *is a reflexive* G_δ *binary relation on* X*. Then exactly one of the following holds:*

(1) *For every* R*-clique* $Y \subseteq X$*, the space* $(Y, \rho \upharpoonright Y)$ *has a basis of cardinality at most* $\boldsymbol{\kappa}^1_{2n+1}$*.*
(2) *There exist* $\delta > 0$ *and a continuous injection* $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\pi(2^{\mathbb{N}})$ *is an* R*-clique and* $(\pi(2^{\mathbb{N}}), \rho \upharpoonright \pi(2^{\mathbb{N}}))$ *is* δ*-discrete.*

PROOF. As Theorem 1.1.5 ensures that $2^{\mathbb{N}}$ cannot be well-ordered, Theorem 1.5.11 implies that $\rho^{-1}([0,\delta])$ is $\aleph_0$-universally Baire for all $\delta > 0$, and Theorem 1.4.14 yields arbitrarily small $\epsilon > 0$ for which $\rho^{-1}([0,\epsilon])$ is co-$\boldsymbol{\kappa}^1_{2n+1}$-Souslin, this follows from the special case of Theorem 3.2.2 where $\kappa = \boldsymbol{\kappa}^1_{2n+1}$. ⊠

THEOREM 3.2.6 (Geschke). *Suppose that* AD *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* ρ *is an extended-valued quasi-metric on* X *for which there are arbitrarily small* $\epsilon > 0$ *such that* $\rho^{-1}([0,\epsilon])$ *is* $\mathbf{\Pi}^1_{2n+2}$*, and* R *is a reflexive* G_δ *binary relation on* X*. Then exactly one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, the space $(Y, \rho \upharpoonright Y)$ has a basis of cardinality at most $(\boldsymbol{\kappa}^1_{2n+1})^+$.*
(2) *There exist $\delta > 0$ and a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), \rho \upharpoonright \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. As Theorem 1.1.5 ensures that $2^{\mathbb{N}}$ cannot be well-ordered, Theorem 1.5.11 implies that $\rho^{-1}([0, \delta])$ is $\aleph_0$-universally Baire for all $\delta > 0$, and Theorem 1.4.14 yields arbitrarily small $\epsilon > 0$ for which $\rho^{-1}([0, \epsilon])$ is co-$(\boldsymbol{\kappa}^1_{2n+1})^+$-Souslin, this follows from the special case of Theorem 3.2.2 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$. ⊠

Theorem 3.2.7 (Geschke). *Suppose that $\mathsf{AD}_{\mathbb{R}}$ holds, X is an analytic Hausdorff space, and ρ is an extended-valued quasi-metric on X. Then exactly one of the following holds:*

(1) *For every R-clique $Y \subseteq X$, the space $(Y, \rho \upharpoonright Y)$ has a well-orderable basis.*
(2) *There exist $\delta > 0$ and a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), \rho \upharpoonright \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. As Theorem 1.1.5 ensures that $2^{\mathbb{N}}$ cannot be well-ordered, Theorem 1.5.11 implies that $\rho^{-1}([0, \delta])$ is $\aleph_0$-universally Baire for all $\delta > 0$, and Theorem 1.4.15 yields an aleph κ for which there are arbitrarily small $\epsilon > 0$ such that $\rho^{-1}([0, \epsilon])$ is co-κ-Souslin, this follows from Theorem 3.2.2. ⊠

3. Scrambled sets

Note that if X is a metric space, $\phi \colon X \to \mathbb{R}$, and $y \in X$, then $\liminf_{\rho_X(x,y) \to \infty} \phi(x)$ and $\limsup_{\rho_X(x,y) \to \infty} \phi(x)$ do not depend on y. We denote them by $\liminf_{||x|| \to \infty} \phi(x)$ and $\limsup_{||x|| \to \infty} \phi(x)$.

Suppose that $S \curvearrowright X$ is an action of a metric semigroup on a metric space. We say that two points x and y of X are *proximal* if $\liminf_{||s|| \to \infty} \rho_X(s \cdot x, s \cdot y) = 0$, we use P^X_S to denote the set of all such pairs, and we say that a set $Y \subseteq X$ is *proximal* if it is a P^X_S-clique.

Proposition 3.3.1. *Suppose that $S \curvearrowright X$ is an action of a metric semigroup by continuous functions on a metric space. Then P^X_S is G_δ.*

Proof. The desired result follows from the fact that if $r \in S$, then $P^X_S = \bigcap_{\epsilon > 0} \bigcap_{n \in \mathbb{N}} \bigcup_{\rho_S(r,s) \geq n} \{(x, y) \in X \times X \mid \rho_X(s \cdot x, s \cdot y) < \epsilon\}$. ⊠

Associated with $S \curvearrowright X$ is the function $\rho^X_S \colon X \times X \to [0, \infty]$ given by $\rho^X_S(x, y) = \limsup_{||s|| \to \infty} \rho_X(s \cdot x, s \cdot y)$.

PROPOSITION 3.3.2. *Suppose that $S \curvearrowright X$ is an action of a metric semigroup by continuous functions on a metric space. Then ρ_S^X is Borel.*

PROOF. It is sufficient to observe that if $r \in S$ and $\delta > 0$, then $(\rho_S^X)^{-1}([\delta, \infty]) = \bigcap_{\epsilon<\delta} \bigcap_{n\in\mathbb{N}} \bigcup_{\rho_S(r,s)\geq n} \{(x, y) \mid \rho_X(s \cdot x, s \cdot y) > \epsilon\}$. ⊠

PROPOSITION 3.3.3. *Suppose that $S \curvearrowright X$ is an action of a metric semigroup on a metric space. Then ρ_S^X is an extended-valued quasi-metric.*

PROOF. It is sufficient to show that if $x, y, z \in X$ and $\epsilon > 0$, then $\rho_S^X(x, z) \leq \rho_S^X(x, y) + \rho_S^X(y, z) + \epsilon$. Towards this end, suppose that $r \in S$, fix $n \in \mathbb{N}$ such that $\sup_{\rho_S(r,s)\geq n} \rho_X(s \cdot x, s \cdot y) \leq \rho_S^X(x, y) + \epsilon/3$ and $\sup_{\rho_S(r,s)\geq n} \rho_X(s\cdot y, s\cdot z) \leq \rho_S^X(y, z)+\epsilon/3$, as well as $s \in S$ with the property that $\rho_S(r, s) \geq n$ and $\rho_X(s \cdot x, s \cdot z) \geq \rho_S^X(x, z) - \epsilon/3$. Then $\rho_S^X(x, z) \leq \epsilon/3 + \rho_X(s \cdot x, s \cdot z) \leq \epsilon/3 + \rho_X(s \cdot x, s \cdot y) + \rho_X(s \cdot y, s \cdot z) \leq \epsilon/3 + \rho_S^X(x, y) + \epsilon/3 + \rho_S^X(y, z) + \epsilon/3$. ⊠

We say that a set $Y \subseteq X$ is *scrambled* if it is proximal but 0-discrete, and we say that $S \curvearrowright X$ is *Li-Yorke chaotic* if there is a scrambled uncountable set $Y \subseteq X$.

THEOREM 3.3.4 (Geschke). *Suppose that $S \curvearrowright X$ is a Li-Yorke chaotic action of a metric semigroup by continuous functions on an analytic metric space. Then there is a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is scrambled.*

PROOF. This is the special case of Theorem 3.2.1 where $\delta = \epsilon = 0$, $\kappa = \aleph_0$, $\rho = \rho_S^X$, and $R = P_S^X$. ⊠

We say that a set $Y \subseteq X$ is *uniformly scrambled* if it is proximal but there exists $\epsilon > 0$ for which $(Y, \rho_S^X \restriction Y)$ is ϵ-discrete, and we say that $S \curvearrowright X$ is *uniformly Li-Yorke chaotic* if there is a uniformly-scrambled uncountable set $Y \subseteq X$. As this rules out the separability of $(Y, \rho_S^X \restriction Y)$, the following fact ensures that it yields a uniformly-scrambled non-empty perfect set:

THEOREM 3.3.5 (Geschke). *Suppose that $S \curvearrowright X$ is an action of a metric semigroup by continuous functions on an analytic metric space. Then exactly one of the following holds:*

(1) *For every proximal set $Y \subseteq X$, the space $(Y, \rho_S^X \restriction Y)$ is separable.*
(2) *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is scrambled.*

PROOF. This is the special case of Theorem 3.2.3 where $\rho = \rho_S^X$ and $R = P_S^X$. ⊠

CHAPTER 4

The $\mathbb{G}_0$ dichotomy, II: Borel colorings

1. Borel colorings

Given a set $R \subseteq X \times Y$, we say that a pair (A, B) of sets is *R-independent* if $R \cap (A \times B) = \emptyset$.

Proposition 4.1.1. *Suppose that κ is an aleph for which every $(\kappa + 1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X and Y are analytic Hausdorff spaces, $R \subseteq X \times Y$ is κ-Souslin, and (A, B) is an R-independent pair of κ-Souslin sets. Then there is an R-independent pair (A', B') of $(\kappa+1)$-Borel sets for which $A \subseteq A'$ and $B \subseteq B'$.*

Proof. As A is disjoint from $\mathrm{proj}_X(R \cap (X \times B))$, and Propositions 1.4.2 and 1.4.3 ensure that the latter set is κ-Souslin, Theorem 1.4.7 yields a $(\kappa+1)$-Borel set $A' \subseteq X$ separating the former from the latter. As B is disjoint from $\mathrm{proj}_Y(R \cap (A' \times Y))$, and Propositions 1.4.2 and 1.4.3 ensure that the latter set is κ-Souslin, Theorem 1.4.7 yields a $(\kappa + 1)$-Borel set $B' \subseteq X$ separating the former from the latter. ⊠

Proposition 4.1.2. *Suppose that κ is an aleph for which every $(\kappa+1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X is an analytic Hausdorff space, G is a κ-Souslin digraph on X, and $A \subseteq X$ is a G-independent κ-Souslin set. Then there is a G-independent $(\kappa+1)$-Borel set $B \supseteq A$.*

Proof. The fact that A is G-independent ensures that (A, A) is a G-independent pair, so Proposition 4.1.1 yields a G-independent pair (C, D) of $(\kappa + 1)$-Borel supersets of A. Set $B = C \cap D$. ⊠

Theorem 4.1.3 (Kanovei). *Suppose that κ is an aleph for which κ^+-$\mathtt{DC}$ holds, $\lambda \geq \kappa$ is an aleph for which every $(\lambda + 1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, X is an analytic Hausdorff space, and G is a κ-Souslin digraph on X. Then at least one of the following holds:*

1. *There is a $(\lambda + 1)$-Borel κ-coloring of G.*
2. *There is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

PROOF. By Proposition 1.4.1, we can assume that there are continuous surjections $\phi_G \colon \kappa^{\mathbb{N}} \to G$ and $\phi_X \colon \kappa^{\mathbb{N}} \to X$. Fix sequences $s_n \in 2^n$ for which $\mathbb{G}_0 = G_{\{s_n | n \in \mathbb{N}\}}$.

We will recursively a decreasing sequence $(B^\alpha)_{\alpha < \kappa^+}$ of $(\lambda+1)$-Borel subsets of X, off of which there are $(\lambda+1)$-Borel κ-colorings of G. We begin setting $B^0 = X$. For all limit ordinals $\mu < \kappa^+$, we set $B^\mu = \bigcap_{\alpha<\mu} B^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \kappa^{n^a}$, and $\psi^a_n \colon 2^{n^a-(n+1)} \to \kappa^{n^a}$ for all $n < n^a$. A *one-step extension* of such an a is an approximation b for which:

(a) $n^b = n^a + 1$.

(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}\ (s \sqsubset t \Longrightarrow \phi^a(s) \sqsubset \phi^b(t))$.

(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}\ (s \sqsubset t \Longrightarrow \psi^a_n(s) \sqsubset \psi^b_n(t))$.

Similarly, a *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \kappa^{\mathbb{N}}$, $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \kappa^{\mathbb{N}}$ for all $n < n^\gamma$, and $(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown t))$ for all $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. A configuration γ is *compatible* with an approximation a if the following conditions hold:

(i) $n^a = n^\gamma$.

(ii) $\forall t \in 2^{n^a}\ \phi^a(t) \sqsubseteq \phi^\gamma(t)$.

(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}\ \psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

A configuration γ is *compatible* with a set $X' \subseteq X$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq X'$. An approximation a is *X'-terminal* if no configuration is compatible with a one-step extension of a and X'. Let $A(a, X')$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(s_{n^a})$, where γ varies over all configurations compatible with a and X'.

LEMMA 4.1.4. *Suppose that $X' \subseteq X$ and a is a Y-terminal approximation. Then $A(a, X')$ is G-independent.*

PROOF. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and X', with the property that $((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a})) \in G$. Fix a sequence $d \in \kappa^{\mathbb{N}}$ such that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a}))$, and let γ denote the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi^\gamma_n(t \frown (i)) = \psi^{\gamma_i}_n(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi^\gamma_{n^a}(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is X'-terminal. ⊠

For all B^α-terminal approximations a, Proposition 4.1.2 yields a G-independent $(\lambda+1)$-Borel set $B(a, B^\alpha) \supseteq A(a, B^\alpha)$. Let $B^{\alpha+1}$ be the set obtained from B^α by subtracting the union of the sets of the form $B(a, B^\alpha)$, where a varies over all B^α-terminal approximations.

Lemma 4.1.5. *Suppose that $\alpha < \kappa^+$ and a is a non-$B^{\alpha+1}$-terminal approximation. Then a has a non-B^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $B^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in B^{\alpha+1}$, so b is not B^α-terminal. ⊠

Fix $\alpha < \kappa^+$ such that the families of B^α-terminal approximations and $B^{\alpha+1}$-terminal approximations are one and the same, and let a_0 denote the unique approximation for which $n^{a_0} = 0$. As $A(a_0, X') = X'$ for all $X' \subseteq X$, we can assume that a_0 is not B^α-terminal, since otherwise $B^{\alpha+1} = \emptyset$, so there is a $(\lambda+1)$-Borel κ-coloring of G.

By recursively applying Lemma 4.1.5, we obtain non-B^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi, \psi_n \colon 2^{\mathbb{N}} \to \kappa^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi^{a_n}(c \restriction n)$ and $\psi_n(c) = \bigcup_{m > n} \psi_n^{a_m}(c \restriction (m - (n+1)))$ for all $n \in \mathbb{N}$. Clearly these functions are continuous.

To establish that the function $\pi = \phi_X \circ \phi$ is a homomorphism from G_S to G, we will show the stronger fact that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then

$$(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c)).$$

And for this, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (0) \frown s)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (1) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \restriction (m - (n+1))$. The fact that a_m is not B^α-terminal yields a configuration γ compatible with a_m. Then $((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown s)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$. ⊠

Remark 4.1.6. The assumption of κ^+-DC can be reduced to κ-DC by first running the above argument without Proposition 4.1.2 (i.e., by setting $B(a, B^\alpha) = A(a, B^\alpha)$ as in the proof of Theorem 3.1.1) to obtain an upper bound $\alpha' < \kappa^+$ for the least ordinal $\alpha < \kappa^+$ such that the families of B^α-terminal and $B^{\alpha+1}$-terminal approximations coincide.

Remark 4.1.7. Under the stronger assumption that there is a function sending each code for a $(\lambda+1)$-Borel subset of an analytic Hausdorff space to a witness that the set is λ-Souslin, the assumption of κ-DC can be removed by working with codes for the $(\lambda+1)$-Borel sets B^α. Under AD, the existence of such a function follows from Theorem 1.4.14 and other well-known consequences of determinacy (i.e., the coding lemma and projective uniformization) when $\lambda = \boldsymbol{\kappa}^1_{2n+1}$.

Remark 4.1.8. Kanovei has shown that both κ-$\mathtt{DC}$ and the assumption that every $(\lambda+1)$-Borel subset of an analytic Hausdorff space is λ-Souslin can be removed (see [**Kan97**]), and the ideas underlying his argument can be used to obtain analogous generalizations of the corollaries established in this chapter.

Theorem 4.1.9 (Kechris-Solecki-Todorcevic). *Suppose that X is an analytic Hausdorff space and G is an analytic digraph on X. Then exactly one of the following holds:*

1. *There is a Borel $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

Proof. This follows from Theorem 1.4.10, Proposition 1.6.10, and the special case of Remark 4.1.6 where $\kappa = \lambda = \aleph_0$. ⊠

Theorem 4.1.10 (Kanovei). *Suppose that $\mathtt{AD}$ holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, and G is a $\mathbf{\Sigma}^1_{2n+1}$ digraph on X. Then exactly one of the following holds:*

1. *There is a $\mathbf{\Delta}^1_{2n+1}$-measurable $\boldsymbol{\kappa}^1_{2n+1}$-coloring of G.*
2. *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

Proof. This follows from Theorem 1.4.14, Proposition 1.6.11, and the special case of Remark 4.1.7 where $\kappa = \lambda = \boldsymbol{\kappa}^1_{2n+1}$. ⊠

Theorem 4.1.11 (Kanovei). *Suppose that $\mathtt{AD}$ holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, and G is a $\mathbf{\Sigma}^1_{2n+2}$ digraph on X. Then exactly one of the following holds:*

1. *There is a $\mathbf{\Delta}^1_{2n+3}$-measurable $(\boldsymbol{\kappa}^1_{2n+1})^+$-coloring of G.*
2. *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

Proof. By Theorem 1.4.14, Proposition 1.6.11, and the special case of Remark 4.1.7 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$ and $\lambda = \boldsymbol{\kappa}^1_{2n+3}$. ⊠

Theorem 4.1.12 (Kechris-Solecki-Todorcevic). *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, X is an analytic Hausdorff space, and G is a digraph on X. Then exactly one of the following holds:*

1. *There is an ordinal-valued-coloring of G.*
2. *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

Proof. By the special case of Theorem 3.1.9 where $R = X \times X$. ⊠

2. Index two subequivalence relations

Given an equivalence relation E on X, we say that a digraph G on X is *E-invariant* if $G = EGE$.

PROPOSITION 4.2.1. *Suppose that κ is an aleph for which every $(\kappa+1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X is an analytic Hausdorff space, E is a κ-Souslin equivalence relation on X, G is an E-invariant κ-Souslin digraph on X, and $B \subseteq X$ is a G-independent $(\kappa+1)$-Borel set. Then B is contained in an E-invariant G-independent $(\kappa+1)$-Borel set.*

PROOF. Set $B_0 = B$, and given $n \in \mathbb{N}$ and a G-independent $(\kappa+1)$-Borel set $B_n \subseteq X$, define $A_{n+1} = [B_n]_E$, and appeal to Proposition 4.1.2 to obtain a G-independent $(\kappa+1)$-Borel set $B_{n+1} \supseteq A_{n+1}$. It only remains to note that $\bigcup_{n\in\mathbb{N}} B_n$ is E-invariant and G-independent. ⊠

A *transversal* of an equivalence relation E on X over a subequivalence relation F is a maximal set $Y \subseteq X$ for which $E \restriction Y = F \restriction Y$.

THEOREM 4.2.2. *Suppose that κ is an aleph for which κ-*`DC`* holds and every $(\kappa+1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X is an analytic Hausdorff space, E is a κ-Souslin equivalence relation on X, and F is a $\aleph_0$-universally-Baire co-κ-Souslin index-two subequivalence relation of E. Then at least one of the following holds:*

1. *There is a $(\kappa+1)$-Borel transversal of E over F.*
2. *There exists a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, {\sim}\mathbb{E}_0)$ to $(F \setminus \Delta(X), E \setminus F, {\sim}E)$.*

PROOF. Define $G = E \setminus F$. If there is a $(\kappa+1)$-Borel κ-coloring c of G, then each of the sets $c^{-1}(\{\alpha\})$ is a $(\kappa+1)$-Borel partial transversal of E over F. As $x \mathrel{F} y \iff (E \setminus F)_x \cap (E \setminus F)_y \neq \emptyset$ for all $x, y \in X$, it follows that F is κ-Souslin, so Proposition 4.2.1 yields F-invariant $(\kappa+1)$-Borel partial transversals $B_\alpha \subseteq X$ of E over F containing $c^{-1}(\{\alpha\})$. As $[B_\alpha]_E$ can be expressed as $\{x \in X \mid \exists y \in [x]_E \ y \in B_\alpha\}$ and $B_\alpha \cup \{x \in X \mid \forall y \in X \ (y \notin (E \setminus F)_x \text{ or } y \in B_\alpha)\}$, Theorem 1.4.9 ensures that it is $(\kappa+1)$-Borel, thus so too is the transversal of E over F given by $\bigcup_{\alpha<\kappa} B_\alpha \setminus \bigcup_{\beta<\alpha} [B_\beta]_E$. By Remark 4.1.6, we can therefore assume that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G. Let D', E', and F' be the pullbacks of the diagonal on X, E, and F through $\phi \times \phi$. As $\mathbb{G}_0 \cap (F')^{-1}F' = \emptyset$, Proposition 1.6.10 ensures that F' is meager. As $E' = F' \cup (\mathrm{id} \times (\iota_\emptyset \cup \iota_\emptyset^{-1}))(F')$, Proposition 1.5.5 implies that E' has the Baire property. As $\mathbb{G}_0 \subseteq E' \setminus F'$, Proposition 1.6.20 ensures that $\mathbb{F}_0 \subseteq F'$ and $\mathbb{E}_0 \setminus \mathbb{F}_0 \subseteq E' \setminus F'$, in which case Proposition 1.6.22 yields a continuous homomorphism $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, {\sim}\mathbb{E}_0)$ to $(F' \setminus D', E' \setminus F', {\sim}E')$. Set $\pi = \phi \circ \psi$. ⊠

THEOREM 4.2.3 (Louveau). *Suppose that X is an analytic Hausdorff space, E is an analytic equivalence relation on X, and F is a* *co-analytic index-two subequivalence relation of E. Then exactly one of the following holds:*

1. *There is a Borel transversal of E over F.*
2. *There exists a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, \sim\mathbb{E}_0)$ to $(F \setminus \Delta(X), E \setminus F, \sim E)$.*

Proof. This follows from Theorems 1.4.10 and 1.5.9, Proposition 1.6.22, and the special case of Theorem 4.2.2 where $\kappa = \aleph_0$. ⊠

Theorem 4.2.4. *Suppose that* AD *holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, E is a $\mathbf{\Sigma}^1_{2n+1}$ equivalence relation on X, and F is a $\mathbf{\Pi}^1_{2n+1}$ index-two subequivalence relation of E. Then exactly one of the following holds:*

1. *There is a $\mathbf{\Delta}^1_{2n+1}$ transversal of E over F.*
2. *There exists a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, \sim\mathbb{E}_0)$ to $(F \setminus \Delta(X), E \setminus F, \sim E)$.*

Proof. This follows from Remark 4.1.7, Theorem 1.5.9, Proposition 1.6.22, and the proof of the special case of Theorem 4.2.2 where $\kappa = \boldsymbol{\kappa}^1_{2n+1}$. ⊠

Theorem 4.2.5. *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds, X is an analytic Hausdorff space, E is an equivalence relation on X, and F is an index-two subequivalence relation of E. Then exactly one of the following holds:*

1. *There is a transversal of E over F.*
2. *There exists a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, \sim\mathbb{E}_0)$ to $(F \setminus \Delta(X), E \setminus F, \sim E)$.*

Proof. This follows from Theorem 1.5.9, Proposition 1.6.22, and the analog of the proof of the special case of Theorem 4.2.2 for $F \subseteq E$ where the use of Proposition 4.2.1 is removed and the use of Theorem 4.1.3 is replaced with that of Theorem 4.1.12. ⊠

3. Perfect antichains

We say that a set $Y \subseteq X$ is an *R-antichain* if it is an $\perp_R$-clique, and an *R-chain* if it is $\perp_R$-independent.

Theorem 4.3.1 (M-Vidnyánszky). *Suppose that κ is an aleph for which κ-*DC *holds, $\lambda \geq \kappa$ is an aleph for which every $(\lambda+1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, X is an analytic Hausdorff space, and R is an $\aleph_0$-universally Baire quasi-order on X for which $\perp_R$ is κ-Souslin. Then at least one of the following holds:*

1. *There is a cover of X by at most κ-many $(\lambda + 1)$-Borel R-chains.*

(2) *There is a continuous injection* $\phi\colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-antichain.*

Proof. If condition (1) fails, then Remark 4.1.6 yields a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $\perp_R$, in which case Proposition 1.6.19 ensures that condition (2) holds. ⊠

Theorem 4.3.2 (M-Vidnyánszky). *Suppose that* X *is an analytic Hausdorff space and* R *is an* $\aleph_0$*-universally Baire quasi-order on* X *for which* $\perp_R$ *is analytic. Then exactly one of the following holds:*

(1) *There is a cover of* X *by countably-many Borel* R*-chains.*
(2) *There is a continuous injection* $\phi\colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-antichain.*

Proof. The special case of Theorem 4.3.1 where $\kappa = \lambda = \aleph_0$ ensures that at least one of the two conditions holds, and the fact that $\mathfrak{c} \not\leq \aleph_0$ implies that they are mutually exclusive. ⊠

Theorem 4.3.3 (M-Vidnyánszky). *Suppose that* AD *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* R *is a quasi-order on* X *for which* $\perp_R$ *is* $\mathbf{\Sigma}^1_{2n+1}$*. Then exactly one of the following holds:*

(1) *There is a cover of* X *by at most* $\boldsymbol{\kappa}^1_{2n+1}$*-many* $\mathbf{\Delta}^1_{2n+1}$ R*-chains.*
(2) *There is a continuous injection* $\phi\colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-antichain.*

Proof. As Theorem 1.5.11 ensures that R is $\aleph_0$-universally Baire, Remark 4.1.7 and the proof of the special case of Theorem 4.3.1 where $\kappa = \lambda = \boldsymbol{\kappa}^1_{2n+1}$ ensure that (1) or (2) holds, and Theorem 1.1.5 and the fact that $\mathfrak{c} \not\leq \aleph_0$ imply that they are mutually exclusive. ⊠

Theorem 4.3.4 (M-Vidnyánszky). *Suppose that* AD *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* R *is a quasi-order on* X *for which* $\perp_R$ *is* $\mathbf{\Sigma}^1_{2n+2}$*. Then exactly one of the following holds:*

(1) *There is a cover of* X *by at most* $(\boldsymbol{\kappa}^1_{2n+1})^+$*-many* $\mathbf{\Delta}^1_{2n+3}$ R*-chains.*
(2) *There is a continuous injection* $\phi\colon 2^{\mathbb{N}} \hookrightarrow X$ *for which* $\phi(2^{\mathbb{N}})$ *is an* R*-antichain.*

Proof. As Theorem 1.5.11 ensures that R is $\aleph_0$-universally Baire, Remark 4.1.7 and the proof of the special case of Theorem 4.3.1 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$ and $\lambda = \boldsymbol{\kappa}^1_{2n+3}$ ensure that (1) or (2) holds, and they are mutually exclusive by Theorem 1.1.5 and the fact that $\mathfrak{c} \not\leq \aleph_0$. ⊠

Theorem 4.3.5 (Foreman). *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds,* X *is an analytic Hausdorff space, and* R *is a quasi-order on* X*. Then exactly one of the following holds:*

(1) *There is a cover of X by a well-ordered sequence of R-chains.*
(2) *There is a continuous injection $\phi\colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\phi(2^{\mathbb{N}})$ is an R-antichain.*

Proof. As Theorem 1.5.11 ensures that R is $\aleph_0$-universally Baire, the analog of the proof of Theorem 4.3.1, where the use of Theorem 4.1.3 is replaced with that of Theorem 4.1.12, ensures that at least one of the two conditions holds, and Theorem 1.1.5 and the fact that $\mathfrak{c} \not\leq \aleph_0$ imply that they are mutually exclusive. ⊠

4. Parametrization and uniformization

Theorem 4.4.1 (M-Vidnyánszky). *Suppose that κ is an aleph for which κ-DC holds, $\lambda \geq \kappa$ is an aleph for which every $(\lambda+1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, X and Y are analytic Hausdorff spaces, R is an $\aleph_0$-universally Baire quasi-order on Y for which $\perp_R$ is κ-Souslin, and $S \subseteq X \times Y$ is κ-Souslin. Then at least one of the following holds:*

(1) *The set S is a union of at most κ-many $(\lambda+1)$-Borel-in-S sets whose vertical sections are R-chains.*
(2) *There exist $x \in X$ and a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain.*

Proof. Suppose that condition (1) fails, let G be the graph on $X \times Y$ with respect to which (x, y) and (x', y') are neighbors if and only if they are both in S, $x = x'$, and $y \perp_R y'$, and observe that if a set $T \subseteq S$ is G-independent, then its vertical sections are R-chains, so by Remark 4.1.6, we can assume that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X \times Y$ from $\mathbb{G}_0$ to G. Then $\mathrm{proj}_X \circ \phi$ is a homomorphism from $\mathbb{G}_0$ to equality, so Proposition 1.6.14 ensures that it is a homomorphism from $\mathbb{E}_0$ to equality, hence Propositions 1.6.15 and 1.6.16 imply that it is constant. Let x denote its constant value, and observe that $\mathrm{proj}_Y \circ \phi$ is a homomorphism from $\mathbb{G}_0$ to $\perp_R$. As $\bigcup_{i<2} \mathrm{proj}_i(\mathbb{G}_0) = 2^{\mathbb{N}}$, it follows that $(\mathrm{proj}_Y \circ \phi)(2^{\mathbb{N}}) \subseteq S_x$, so the proof of Proposition 1.6.19 yields a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain. ⊠

Theorem 4.4.2 (M-Vidnyánszky). *Suppose that X and Y are analytic Hausdorff spaces, R is an $\aleph_0$-universally Baire quasi-order on Y for which $\perp_R$ is analytic, and $S \subseteq X \times Y$ is an analytic set whose vertical sections are unions of countably-many R-chains. Then S is a union of countably-many Borel-in-S sets whose vertical sections are R-chains.*

Proof. The special case of Theorem 4.4.1 where $\kappa = \lambda = \aleph_0$ ensures that if the conclusion fails, then there exist $x \in X$ and a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain. As $\mathfrak{c} \not\leq \aleph_0$, this contradicts the fact that S_x is a union of countably-many R-chains. ⊠

Theorem 4.4.3 (M-Vidnyánszky). *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, R is a quasi-order on Y for which $\perp_R$ is $\boldsymbol{\Sigma}^1_{2n+1}$, and $S \subseteq X \times Y$ is a $\boldsymbol{\Sigma}^1_{2n+1}$ set whose vertical sections are unions of at most $\boldsymbol{\kappa}^1_{2n+1}$-many R-chains. Then S is a union of at most $\boldsymbol{\kappa}^1_{2n+1}$-many $\boldsymbol{\Delta}^1_{2n+1}$-in-$S$ sets whose vertical sections are R-chains.*

Proof. As Theorem 1.5.11 ensures that R is $\aleph_0$-universally Baire, Remark 4.1.7 and the proof of the special case of Theorem 4.4.1 where $\kappa = \lambda = \boldsymbol{\kappa}^1_{2n+1}$ ensure that if the conclusion fails, then there exist $x \in X$ and a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain. As Theorem 1.1.5 ensures that $\mathfrak{c} \not\leq \boldsymbol{\kappa}^1_{2n+1}$, this contradicts the fact that S_x is a union of at most $\boldsymbol{\kappa}^1_{2n+1}$-many R-chains. ⊠

Theorem 4.4.4 (M-Vidnyánszky). *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, R is a quasi-order on Y for which $\perp_R$ is $\boldsymbol{\Sigma}^1_{2n+2}$, and $S \subseteq X \times Y$ is a $\boldsymbol{\Sigma}^1_{2n+2}$ set whose vertical sections are unions of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many R-chains. Then S is a union of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many $\boldsymbol{\Delta}^1_{2n+3}$-in-$S$ sets whose vertical sections are R-chains.*

Proof. As Theorem 1.5.11 ensures that R is $\aleph_0$-universally Baire, Remark 4.1.7 and the proof of the special case of Theorem 4.4.1 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$ and $\lambda = \boldsymbol{\kappa}^1_{2n+3}$ ensure that if the conclusion fails, then there exist $x \in X$ and a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain. As Theorem 1.1.5 ensures that $\mathfrak{c} \not\leq (\boldsymbol{\kappa}^1_{2n+1})^+$, this contradicts the fact that S_x is a union of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many R-chains. ⊠

Theorem 4.4.5. *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds, X and Y are analytic Hausdorff spaces, R is a quasi-order on Y, and $S \subseteq X \times Y$ is a set whose vertical sections are well-ordered unions of R-chains. Then S is a well-ordered union of sets whose vertical sections are R-chains.*

Proof. The analog of the proof of the special case of Theorem 4.4.1 in which the use of Theorem 4.1.3 is replaced with that of Theorem 4.1.12 ensures that if the conclusion fails, then there exist $x \in X$ and a continuous injection $\pi\colon 2^{\mathbb{N}} \hookrightarrow S_x$ for which $\pi(2^{\mathbb{N}})$ is an R-antichain. As Theorem 1.1.5 ensures that there is no well-ordering of $\mathfrak{c}$, this contradicts the fact that S_x is a well-ordered union of R-chains. ⊠

In particular, we obtain generalizations of the Lusin-Novikov uniformization theorem for sets with countable vertical sections:

THEOREM 4.4.6 (Lusin-Novikov, Conley-M). *Suppose that X and Y are analytic Hausdorff spaces, F is a Borel equivalence relation on Y, and $S \subseteq X \times Y$ is an analytic set whose vertical sections are unions of countably-many F-classes. Then S is a union of countably-many Borel-in-S sets whose non-empty vertical sections are F-classes.*

PROOF. By the special case of Theorem 4.4.2 where $R = F$, it is enough to show that every Borel-in-S subset of S whose vertical sections are contained in F-classes is contained in a Borel-in-S subset of S whose non-empty vertical sections are F-classes. But this follows from Proposition 4.2.1. ⊠

THEOREM 4.4.7 (Conley-M). *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, F is a $\boldsymbol{\Delta}^1_{2n+1}$ equivalence relation on Y, and $S \subseteq X \times Y$ is a $\boldsymbol{\Sigma}^1_{2n+1}$ set whose vertical sections are unions of at most $\boldsymbol{\kappa}^1_{2n+1}$-many F-classes. Then S is a union of at most $\boldsymbol{\kappa}^1_{2n+1}$-many $\boldsymbol{\Delta}^1_{2n+1}$-in-$S$ sets whose non-empty vertical sections are F-classes.*

PROOF. By the proof of the special case of Theorem 4.4.3 where $R = F$, it is enough to show that there is a function sending each code for a $(\boldsymbol{\kappa}^1_{2n+1} + 1)$-Borel-in-$S$ subset of S whose vertical sections are contained in F-classes to a code for a $(\boldsymbol{\kappa}^1_{2n+1} + 1)$-Borel-in-$S$ superset contained in S whose non-empty vertical sections are F-classes. But this follows Remark 4.1.7 and the proof of Proposition 4.2.1. ⊠

THEOREM 4.4.8 (Conley-M). *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, F is a $\boldsymbol{\Delta}^1_{2n+2}$ equivalence relation on Y, and $S \subseteq X \times Y$ is a $\boldsymbol{\Sigma}^1_{2n+2}$ set whose vertical sections are unions of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many F-classes. Then S is a union of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many $\boldsymbol{\Delta}^1_{2n+3}$-in-$S$ sets whose non-empty vertical sections are F-classes.*

PROOF. By the proof of the special case of Theorem 4.4.4 where $R = F$, it is enough to show that there is a function sending each code for a $(\boldsymbol{\kappa}^1_{2n+3} + 1)$-Borel-in-$S$ subset of S whose vertical sections are contained in F-classes to a code for a $(\boldsymbol{\kappa}^1_{2n+3} + 1)$-Borel-in-$S$ superset contained in S whose vertical sections are F-classes. But this follows Remark 4.1.7 and the proof of Proposition 4.2.1. ⊠

THEOREM 4.4.9 (Conley-M). *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds, X and Y are analytic Hausdorff spaces, F is an equivalence relation on Y, and $S \subseteq X \times Y$ is a set whose vertical sections are well-ordered unions of*

F-classes. Then S is a well-ordered union of sets whose non-empty vertical sections are F-classes.

PROOF. This is a trivial consequence of Theorem 4.4.5. ⊠

Given an equivalence relation E on X, we say that a set $S \subseteq X \times Y$ is *E-invariant* if the vertical sections of E-related points coincide. Note that if $X = Y = 2^{\mathbb{N}}$ and $E = \mathbb{E}_0$, then the non-empty vertical sections of the E-invariant set $S = \mathbb{E}_0$ are countable, but Proposition 1.6.9 ensures that S is not a union of countably-many E-invariant Borel sets whose non-empty vertical sections are singletons.

THEOREM 4.4.10. *Suppose that X and Y are analytic Hausdorff spaces, E and F are Borel equivalence relations on X and Y, and $S \subseteq X \times Y$ is an E-invariant analytic set whose vertical sections are unions of countably-many F-classes. Then exactly one of the following holds:*

1. *The set S is a union of countably-many E-invariant Borel-in-S sets whose vertical sections are unions of finitely-many F-classes.*
2. *There are continuous embeddings $\phi\colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\psi\colon 2^{\mathbb{N}} \to Y$ of $\Delta(2^{\mathbb{N}})$ into F for which $(\phi \times \psi)(\mathbb{E}_0) \subseteq S$.*

PROOF. This is a straightforward corollary of Theorem 4.4.6 and [**CCM16**, Theorem 1]. ⊠

THEOREM 4.4.11. *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, E and F are $\boldsymbol{\Delta}^1_{2n+1}$ equivalence relations on X and Y, and $S \subseteq X \times Y$ is an E-invariant $\boldsymbol{\Sigma}^1_{2n+1}$ whose vertical sections are unions of at most $\boldsymbol{\kappa}^1_{2n+1}$-many F-classes. Then exactly one of the following holds:*

1. *The set S is a union of at most $\boldsymbol{\kappa}^1_{2n+1}$-many E-invariant $\boldsymbol{\Delta}^1_{2n+1}$-in-$S$ sets whose vertical sections are unions of finitely-many F-classes.*
2. *There are continuous embeddings $\phi\colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\psi\colon 2^{\mathbb{N}} \to Y$ of $\Delta(2^{\mathbb{N}})$ into F for which $(\phi \times \psi)(\mathbb{E}_0) \subseteq S$.*

PROOF. This is a straightforward corollary of Theorem 4.4.7 and the analog of [**CCM16**, Theorem 1] for $\boldsymbol{\Delta}^1_{2n+1}$ equivalence relations. ⊠

THEOREM 4.4.12. *Suppose that* AD *holds, $n \in \mathbb{N}$, X and Y are analytic Hausdorff spaces, E and F are $\boldsymbol{\Delta}^1_{2n+2}$ equivalence relations on X and Y, and $S \subseteq X \times Y$ is an E-invariant $\boldsymbol{\Sigma}^1_{2n+2}$ whose vertical sections are unions of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many F-classes. Then exactly one of the following holds:*

(1) *The set S is a union of at most $(\boldsymbol{\kappa}^1_{2n+1})^+$-many E-invariant $\boldsymbol{\Delta}^1_{2n+3}$-in-$S$ sets whose vertical sections are unions of finitely-many F-classes.*
(2) *There are continuous embeddings $\phi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\psi \colon 2^{\mathbb{N}} \to Y$ of $\Delta(2^{\mathbb{N}})$ into F for which $(\phi \times \psi)(\mathbb{E}_0) \subseteq S$.*

Proof. This is a straightforward corollary of Theorem 4.4.7 and the analog of [**CCM16**, Theorem 1] for $\boldsymbol{\Delta}^1_{2n+2}$ equivalence relations. ⊠

Theorem 4.4.13. *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, X and Y are analytic Hausdorff spaces, E and F are equivalence relations on X and Y, and $S \subseteq X \times Y$ is an E-invariant set whose vertical sections are well-ordered unions of F-classes. Then exactly one of the following holds:*

(1) *The set S is a well-ordered union of E-invariant sets whose vertical sections are unions of finitely-many F-classes.*
(2) *There are continuous embeddings $\phi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E and $\psi \colon 2^{\mathbb{N}} \to Y$ of $\Delta(2^{\mathbb{N}})$ into F for which $(\phi \times \psi)(\mathbb{E}_0) \subseteq S$.*

Proof. This is a straightforward corollary of Theorem 4.4.9 and the analog of [**CCM16**, Theorem 1] under $\mathtt{AD}_{\mathbb{R}}$. ⊠

CHAPTER 5

The $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy

1. Borel local colorings

Given a binary relation R on X, the *downward R-saturation* of a set $Y \subseteq X$ is given by $[Y]_R = \{x \in X \mid \exists y \in Y \; x \; R \; y\}$, and the *upward R-saturation* of a set $Y \subseteq X$ is given by $[Y]^R = \{x \in X \mid \exists y \in Y \; y \; R \; x\}$. We say that Y is *downward R-invariant* if $Y = [Y]_R$, and *upward R-invariant* if $Y = [Y]^R$.

Proposition 5.1.1. *Suppose that κ is an aleph for which every $(\kappa+1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X is an analytic Hausdorff space, R is a κ-Souslin quasi-order on X, and (A_0, A_1) is an R-independent pair of κ-Souslin sets. Then there is an R-independent pair (B_0, B_1) of $(\kappa+1)$-Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is upward R-invariant, and B_1 is downward R-invariant.*

Proof. Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$. Given $n \in \mathbb{N}$ and an R-independent pair $(A_{0,n}, A_{1,n})$ of κ-Souslin sets, appeal to Proposition 4.1.1 to obtain an R-independent pair $(B_{0,n}, B_{1,n})$ of $(\kappa+1)$-Borel sets such that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$, and set $A_{0,n+1} = [B_{0,n}]^R$ and $A_{1,n+1} = [B_{1,n}]_R$. Define $B_0 = \bigcup_{n \in \mathbb{N}} B_{0,n}$ and $B_1 = \bigcup_{n \in \mathbb{N}} B_{1,n}$. ⊠

The *lexicographical ordering* of 2^α is the partial order given by $c <_{R_{\text{lex}} \upharpoonright 2^\alpha} d \iff \exists \beta < \alpha \; (c \upharpoonright [0, \beta) = d \upharpoonright [0, \beta) \text{ and } c(\beta) < d(\beta))$.

Theorem 5.1.2. *Suppose that κ is an aleph for which κ^+-*`DC`* holds, $\lambda \geq \kappa$ is an aleph for which every $(\lambda+1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, X is an analytic Hausdorff space, G is a κ-Souslin digraph on X, and R is a κ-Souslin quasi-order on X. Then at least one of the following holds:*

1. *There exist a quasi-order $R' \supseteq R$ that admits a $(\lambda+1)$-Borel reduction to $R_{\text{lex}} \upharpoonright 2^\alpha$ for some $\alpha < \kappa^+$, and for which there is a $(\lambda+1)$-Borel κ-coloring of $\equiv_{R'} \cap \, G$.*
2. *There exists a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ to (G, R).*

Proof. By Proposition 1.4.1, we can assume that there are continuous surjections $\phi_G \colon \kappa^{\mathbb{N}} \twoheadrightarrow G$, $\phi_R \colon \kappa^{\mathbb{N}} \twoheadrightarrow R$, and $\phi_X \colon \kappa^{\mathbb{N}} \twoheadrightarrow X$. Fix

$s_{2n} \in 2^{2n}$ and $t_{2n+1} \in 2^{2n+1} \times 2^{2n+1}$ for which $\mathbb{G}_0(2\mathbb{N}) = G_{\{s_{2n} \mid n \in \mathbb{N}\}}$ and $\mathbb{H}_0(2\mathbb{N}+1) = G_{\{t_{2n+1} \mid n \in \mathbb{N}\}}$.

We will construct decreasing sequences $(B^\alpha)_{\alpha<\kappa^+}$ of $(\lambda+1)$-Borel subsets of X and $(R^\alpha)_{\alpha<\kappa^+}$ of quasi-orders containing R such that for all $\alpha < \kappa^+$, there exist $\beta < \kappa^+$ for which there is a $(\lambda+1)$-Borel reduction of R^α to $R_{\text{lex}} \restriction 2^\beta$, and a $(\lambda+1)$-Borel κ-coloring of $\equiv_{R^\alpha} \cap G$ off of B^α. We begin by setting $B^0 = X$ and $R^0 = X \times X$. For all limit ordinals $\mu < \kappa^+$, we set $B^\mu = \bigcap_{\alpha<\mu} B^\alpha$ and $R^\mu = \bigcap_{\alpha<\mu} R^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \kappa^{n^a}$, and $\psi^a_n \colon 2^{n^a-(n+1)} \to \kappa^{n^a}$ for all $n < n^a$. A *one-step extension* of such an a is an approximation b for which:

(a) $n^b = n^a + 1$.
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}$ $(s \sqsubset t \Longrightarrow \phi^a(s) \sqsubset \phi^b(t))$.
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}$ $(s \sqsubset t \Longrightarrow \psi^a_n(s) \sqsubset \psi^b_n(t))$.

An approximation a is *even* if n^a is even, and *odd* if n^a is odd.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, such that $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \kappa^{\mathbb{N}}$, $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \kappa^{\mathbb{N}}$ for all $n < n^\gamma$, $(\phi_G \circ \psi^\gamma_n)(s) = ((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown s))$ for all even $n < n^\gamma$ and $s \in 2^{n^\gamma-(n+1)}$, and along similar lines, $(\phi_R \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(t_n(0) \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(t_n(1) \frown (1) \frown t))$ for all odd $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. A configuration γ is *compatible* with an approximation a if the following conditions hold:

(i) $n^a = n^\gamma$.
(ii) $\forall t \in 2^{n^a}$ $\phi^a(t) \sqsubseteq \phi^\gamma(t)$.
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}$ $\psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

A configuration γ is *compatible* with a set $X' \subseteq X$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq X'$, and *compatible* with a quasi-order $R' \supseteq R$ on X if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma})$ is contained in a single $\equiv_{R'}$-class. An approximation a is (X', R')-*terminal* if no configuration is compatible with a one-step extension of a, X', and R'. For all even approximations a, let $A(a, X', R')$ be the set of points of the form $(\phi_X \circ \phi^\gamma)(s_{n^a})$, where γ varies over configurations compatible with a, X', and R'. For all odd approximations a and $i < 2$, let $A_i(a, X', R')$ be the set of points of the form $(\phi_X \circ \phi^\gamma)(t_{n^a}(i))$, where γ varies over all configurations compatible with a, X', and R'.

LEMMA 5.1.3. *Suppose that $X' \subseteq X$, $R' \supseteq R$ is a quasi-order on X, and a is an (X', R')-terminal even approximation. Then $A(a, X', R')$ is $(\equiv_{R'} \cap G)$-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a, X', and R', with the property that $((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a})) \in G$. Fix a sequence $d \in \kappa^{\mathbb{N}}$ such that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a}))$, and let γ denote the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi_n^\gamma(t \frown (i)) = \psi_n^{\gamma_i}(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi_{n^a}^\gamma(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is (X', R')-terminal. ⊠

Lemma 5.1.4. *Suppose that $X' \subseteq X$, $R' \supseteq R$ is a quasi-order on X, and a is an (X', R')-terminal odd approximation. Then $(A_0(a, X', R'), A_1(a, X', R'))$ is an $(\equiv_{R'} \cap R)$-independent pair.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a, X', and R', with the property that $((\phi_X \circ \phi^{\gamma_0})(t_{n^a}(0)), (\phi_X \circ \phi^{\gamma_1})(t_{n^a}(1))) \in \equiv_{R'} \cap R$. Fix a sequence $d \in \kappa^{\mathbb{N}}$ such that $\phi_R(d) = ((\phi_X \circ \phi^{\gamma_0})(t_{n^a}(0)), (\phi_X \circ \phi^{\gamma_1})(t_{n^a}(1)))$, and let γ denote the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi_n^\gamma(t \frown (i)) = \psi_n^{\gamma_i}(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi_{n^a}^\gamma(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is (X', R')-terminal. ⊠

For all (B^α, R^α)-terminal even approximations a, Proposition 4.1.2 gives rise to a $(\equiv_{R^\alpha} \cap G)$-independent $(\lambda + 1)$-Borel set $B(a, B^\alpha, R^\alpha) \supseteq A(a, B^\alpha, R^\alpha)$. Let $B^{\alpha+1}$ be the difference of B^α and the union of the sets of the form $B(a, B^\alpha, R^\alpha)$, where a varies over all (B^α, R^α)-terminal even approximations.

For all (B^α, R^α)-terminal odd approximations a and $i < 2$, another application of Proposition 5.1.1 yields an $(\equiv_{R'} \cap R)$-independent pair $(B_0(a, B^\alpha, R^\alpha), B_1(a, B^\alpha, R^\alpha))$ of $(\lambda + 1)$-Borel sets with the property that $A_i(a, B^\alpha, R^\alpha) \subseteq B_i(a, B^\alpha, R^\alpha)$ for all $i < 2$. Fix an injective enumeration $(a_\beta^\alpha)_{\beta < \beta_\alpha}$ of the family of all (B^α, R^α)-terminal odd approximations, define $\pi^\alpha \colon X \to 2^{\beta_\alpha}$ by $\pi^\alpha(x)(\beta) = \chi_{B_0(a^\beta, B^\alpha, R^\alpha)}(x)$ for all $\beta < \beta_\alpha$, and let $R^{\alpha+1}$ be the subquasiorder of R^α with respect to which $x \mathrel{R^{\alpha+1}} y \iff (x <_{R^\alpha} y \text{ or } (x \equiv_{R^\alpha} y \text{ and } \pi^\alpha(x) \mathrel{R_{\text{lex}}} \pi^\alpha(y)))$.

Lemma 5.1.5. *Suppose that $\alpha < \kappa^+$ and a is a non-$(B^{\alpha+1}, R^{\alpha+1})$-terminal approximation. Then a has a non-(B^α, R^α)-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b, $B^{\alpha+1}$, and $R^{\alpha+1}$. If a is odd, then $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in B^{\alpha+1}$, so b is not (B^α, R^α)-terminal. If a is even, then $(\phi_X \circ \phi^\gamma)(t_{n^b}(0)) \equiv_{R^{\alpha+1}} (\phi_X \circ \phi^\gamma)(t_{n^b}(1))$, so b is not (B^α, R^α)-terminal. ⊠

Fix $\alpha < \kappa^+$ such that the families of (B^α, R^α)-terminal approximations and $(B^{\alpha+1}, R^{\alpha+1})$-terminal approximations are one and the same, and let a_0 denote the unique approximation for which $n^{a_0} = 0$. Then $A(a_0, X', R') = X'$ for all $X' \subseteq X$ and quasi-orders $R' \supseteq R$ on X, so we can assume that a_0 is not (B^α, R^α)-terminal, since otherwise $B^{\alpha+1} = \emptyset$, in which case there is a $(\lambda+1)$-Borel κ-coloring of $\equiv_{R^\alpha} \cap G$.

By recursively applying Lemma 5.1.5, we obtain non-(B^α, R^α)-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi, \psi_n \colon 2^\mathbb{N} \to \kappa^\mathbb{N}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi^{a_n}(c \restriction n)$ and $\psi_n(c) = \bigcup_{m > n} \psi_n^{a_m}(c \restriction (m - (n+1)))$ for all $n \in \mathbb{N}$. Clearly these functions are continuous.

To establish that the function $\pi = \phi_X \circ \phi$ is a homomorphism from $\mathbb{G}_0(2\mathbb{N})$ to G, we will show that if $c \in 2^\mathbb{N}$ and $n \in 2\mathbb{N}$, then

$$(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c)).$$

And for this, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (0) \frown s)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (1) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \restriction (m - (n+1))$. The fact that a_m is not (B^α, R^α)-terminal yields a configuration γ compatible with a_m. Then $((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown s)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$.

To establish that the function $\pi = \phi_X \circ \phi$ is a homomorphism from $\mathbb{H}_0(2\mathbb{N}+1)$ to R, we will show that if $c \in 2^\mathbb{N}$ and $n \in 2\mathbb{N}+1$, then

$$(\phi_R \circ \psi_n)(c) = ((\phi_X \circ \phi)(t_n(0) \frown (0) \frown c), (\phi_X \circ \phi)(t_n(1) \frown (1) \frown c)).$$

And for this, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(t_n(0) \frown (0) \frown c), (\phi_X \circ \phi)(t_n(1) \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(t_n(0) \frown (0) \frown t)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(t_n(1) \frown (1) \frown t)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(t)}) \subseteq V$, where $t = c \restriction (m - (n+1))$. The fact that a_m is not (B^α, R^α)-terminal yields a configuration γ compatible with a_m. Then $((\phi_X \circ \phi^\gamma)(t_n(0) \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(t_n(1) \frown (1) \frown t)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(t) \in V$, thus $U \cap V \neq \emptyset$. ⊠

REMARK 5.1.6. The assumption of κ^+-DC can again be reduced to κ-DC by first running the argument without Proposition 5.1.1.

REMARK 5.1.7. Under the stronger assumption that there is a function sending each code for a $(\lambda+1)$-Borel subset of an analytic Hausdorff space to a witness that the set is λ-Souslin, the assumption of κ-DC can again be removed by working with codes for $(\lambda+1)$-Borel sets.

Remark 5.1.8. The ideas behind [**Kan97**] can again be used to eliminate both κ-$\mathtt{DC}$ and the assumption that every $(\lambda+1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, and to obtain analogous generalizations of the corollaries established in this chapter.

Theorem 5.1.9. *Suppose that X is an analytic Hausdorff space, G is an analytic digraph on X, and R is an analytic quasi-order on X. Then exactly one of the following holds:*

1. *There exists a quasi-order $R' \supseteq R$ that admits a Borel reduction to $R_{\mathrm{lex}} \upharpoonright 2^\alpha$ for some $\alpha < \omega_1$, and for which there is a Borel $\aleph_0$-coloring of $\equiv_{R'} \cap G$.*
2. *There exists a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ to (G, R).*

Proof. This follows from Theorem 1.4.10, Proposition 1.6.25, and the special case of Remark 5.1.6 where $\kappa = \lambda = \aleph_0$. ⊠

Theorem 5.1.10. *Suppose that $\mathtt{AD}$ holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, G is a $\mathbf{\Sigma}^1_{2n+1}$ digraph on X, and R is a $\mathbf{\Sigma}^1_{2n+1}$ quasi-order on X. Then exactly one of the following holds:*

1. *There exists a quasi-order $R' \supseteq R$ that admits a $\mathbf{\Delta}^1_{2n+1}$-measurable reduction to $R_{\mathrm{lex}} \upharpoonright 2^\kappa$ for some $\kappa < \boldsymbol{\delta}^1_{2n+1}$, and for which there is a $\mathbf{\Delta}^1_{2n+1}$ $\boldsymbol{\kappa}^1_{2n+1}$-coloring of $\equiv_{R'} \cap G$.*
2. *There exists a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ to (G, R).*

Proof. This follows from Proposition 1.6.26 and the special case of Remark 5.1.7 where $\kappa = \lambda = \boldsymbol{\kappa}^1_{2n+1}$. ⊠

Theorem 5.1.11. *Suppose that $\mathtt{AD}$ holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, G is a $\mathbf{\Sigma}^1_{2n+2}$ digraph on X, and R is a $\mathbf{\Sigma}^1_{2n+2}$ quasi-order on X. Then exactly one of the following holds:*

1. *There exists a quasi-order $R' \supseteq R$ for which there are a $\mathbf{\Delta}^1_{2n+3}$-measurable reduction of R' to $R_{\mathrm{lex}} \upharpoonright 2^\alpha$ for some $\alpha < (\boldsymbol{\kappa}^1_{2n+1})^{++}$ and a $\mathbf{\Delta}^1_{2n+3}$ $(\boldsymbol{\kappa}^1_{2n+1})^+$-coloring of $\equiv_{R'} \cap G$.*
2. *There exists a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ to (G, R).*

Proof. This follows from Proposition 1.6.26 and the special case of Remark 5.1.7 where $\kappa = (\boldsymbol{\kappa}^1_{2n+1})^+$ and $\lambda = \boldsymbol{\kappa}^1_{2n+3}$. ⊠

Theorem 5.1.12. *Suppose that $\mathtt{AD}_{\mathbb{R}}$ holds, X is an analytic Hausdorff space, G is a digraph on X, and R is a quasi-order on X. Then exactly one of the following holds:*

(1) *There exists a quasi-order* $R' \supseteq R$ *that admits a reduction to* $R_{\text{lex}} \upharpoonright 2^\alpha$ *for some ordinal* α*, and for which there is an ordinal-valued coloring of* $\equiv_{R'} \cap G$*.*
(2) *There exists a continuous homomorphism* $\phi \colon 2^{\mathbb{N}} \to X$ *from* $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ *to* (G, R)*.*

PROOF. This follows from Theorem 1.4.15, Proposition 1.6.26, and the simplification of the proof of Theorem 5.1.2 in which the use of Proposition 5.1.1 is eliminated. ⊠

2. Linearizability of quasi-orders

PROPOSITION 5.2.1. *Suppose that κ is an aleph for which every $(\kappa+1)$-Borel subset of an analytic Hausdorff space is κ-Souslin, X is an analytic Hausdorff space, E is a κ-Souslin equivalence relation on X, R is a co-κ-Souslin quasi-order on X, and (A_0, A_1) is an $(E \setminus R)$-independent pair of κ-Souslin sets. Then there is an $(E \setminus R)$-independent pair (B_0, B_1) of $(\kappa + 1)$-Borel sets such that $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, B_0 is downward $(E \cap R)$-invariant, and B_1 is upward $(E \cap R)$-invariant.*

PROOF. Set $A_{0,0} = A_0$ and $A_{1,0} = A_1$. Given $n \in \mathbb{N}$ and an $(E \setminus R)$-independent pair $(A_{0,n}, A_{1,n})$ of κ-Souslin sets, appeal to Proposition 4.1.1 to obtain an $(E \setminus R)$-independent pair $(B_{0,n}, B_{1,n})$ of $(\kappa+1)$-Borel sets with the property that $A_{0,n} \subseteq B_{0,n}$ and $A_{1,n} \subseteq B_{1,n}$, and set $A_{0,n+1} = [B_{0,n}]_{E\cap R}$ and $A_{1,n+1} = [B_{1,n}]^{E\cap R}$. Define $B_0 = \bigcup_{n\in\mathbb{N}} B_{0,n}$ and $B_1 = \bigcup_{n\in\mathbb{N}} B_{1,n}$. ⊠

THEOREM 5.2.2. *Suppose that κ is an aleph for which κ-`DC` holds, $\lambda \geq \kappa$ is an aleph for which every $(\lambda + 1)$-Borel subset of an analytic Hausdorff space is λ-Souslin, X is an analytic Hausdorff space, and R is an $\aleph_0$-universally-Baire bi-κ-Souslin quasi-order on X. Then at least one of the following holds:*

1. *There is a quasi-order $S \supseteq R$ that admits a $(\lambda + 1)$-Borel reduction to $R_{\text{lex}} \restriction 2^\alpha$ for some $\alpha < \kappa^+$ and for which $\equiv_R$ and $\equiv_S$ coincide.*
2. *There is a continuous embedding of $\mathbb{E}_0$ or $\mathbb{R}_0$ into R.*

PROOF. Define $G = {\sim}R$. Suppose first that there is a quasi-order $R' \supseteq R$ that admits a $(\lambda + 1)$-Borel reduction to $R_{\text{lex}} \restriction 2^\alpha$ for some $\alpha < \kappa^+$, and for which there is a $(\lambda + 1)$-Borel κ-coloring c of $\equiv_{R'} \cap G$. Then Proposition 5.2.1 yields $(\equiv_{R'} \setminus R)$-independent pairs $(B_{0,\alpha}, B_{1,\alpha})$ of $(\lambda + 1)$-Borel sets such that $B_{0,\alpha}$ is downward $(\equiv_{R'} \cap R)$-invariant, $B_{1,\alpha}$ is upward $(\equiv_{R'} \cap R)$-invariant, and $c^{-1}(\{\alpha\}) \subseteq B_{0,\alpha} \cap B_{1,\alpha}$ for all $\alpha < \kappa$. Define $\phi \colon X \to (2 \times 2)^\kappa$ by $\phi(x)(\alpha) = (1 - \chi_{B_{0,\alpha}}(x), \chi_{B_{1,\alpha}}(x))$, let S' be the quasi-order on X given by $x \; S' \; y \iff x <_{R'} y$ or ($x \equiv_{R'} y$ and $\phi(x) \; R_{\text{lex}} \; \phi(y)$), and note that $R \subseteq S'$ and $\equiv_{S'} \cap R \subseteq \equiv_R$, thus $E \cap R \subseteq \equiv_R$, where E is the smallest equivalence relation containing $\equiv_{S'} \cap R$. By Proposition 4.2.1, there are E-invariant $(\equiv_{S'} \setminus R)$-independent Borel sets $B_\alpha \supseteq c^{-1}(\{\alpha\})$. Define $\psi \colon X \to 2^\kappa$ by $\psi(x)(\alpha) = \chi_{B_\alpha}(x)$, let S be the quasi-order on X given by $x \; S \; y \iff x <_{S'} y$ or ($x \equiv_{S'} y$ and $\psi(x) \; R_{\text{lex}} \; \psi(y)$), and observe that $R \subseteq S$ and $\equiv_R$ and $\equiv_S$ coincide.

By Theorem 5.1.2, we can therefore assume that there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_0(2\mathbb{N}), \mathbb{H}_0(2\mathbb{N}+1))$ to (G, R).

Let D_0 and R_0 be the pullbacks of $\Delta(X)$ and R through $\phi \times \phi$. As $\equiv_{R_0} \cap \mathbb{G}_0(2\mathbb{N}) = \emptyset$, Proposition 1.6.25 ensures that $\equiv_{R_0}$ is meager, so R_0 is not comeager. As $\mathbb{H}_0(2\mathbb{N}+1) \subseteq R_0$, Proposition 1.6.24 therefore implies that R_0 is meager. By Proposition 1.6.31, there is a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\sim\Delta(2^{\mathbb{N}}), \mathrm{graph}(\sigma) \setminus \{((1)^\infty, (0)^\infty)\}, \sim\mathbb{E}_0)$ to $(\sim D_0, R_0, \sim R_0)$, in which case $\phi \circ \psi$ is a homomorphism from $(\sim\Delta(2^{\mathbb{N}}), \mathbb{R}_0, \sim\mathbb{E}_0)$ to $(\sim\Delta(X), R, \sim R)$. As the pullback R_0' of R through $\phi \circ \psi$ is $\aleph_0$-universally Baire and $\mathbb{R}_0 \subseteq R_0' \subseteq \mathbb{E}_0$, Proposition 1.6.30 yields a continuous embedding $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ or $\mathbb{R}_0$ into R_0', in which case $\phi \circ \psi \circ \pi$ is a continuous embedding of $\mathbb{E}_0$ or $\mathbb{R}_0$ into R. ⊠

Theorem 5.2.3 (Kanovei, Louveau). *Suppose that X is an analytic Hausdorff space and R is a Borel quasi-order on X. Then exactly one of the following holds:*

(1) *There is a quasi-order $S \supseteq R$ that admits a Borel reduction $\phi \colon X \to 2^\alpha$ to $R_{\mathrm{lex}} \restriction 2^\alpha$ for some $\alpha < \omega_1$, and for which $\equiv_R$ and $\equiv_S$ coincide.*
(2) *There is a continuous embedding $\psi \colon 2^{\mathbb{N}} \hookrightarrow X$ of $\mathbb{E}_0$ or $\mathbb{R}_0$ into R.*

Proof. The special case of Theorem 5.2.2, where $\kappa = \lambda = \aleph_0$, ensures that at least one of the conditions holds. To see that they are mutually exclusive, note that otherwise, the pullback S_0 of $R_{\mathrm{lex}} \restriction 2^\alpha$ through $(\phi \circ \psi) \times (\phi \circ \psi)$ has the Baire property and is not meager, since it is linear. As $\mathbb{H}_0 \subseteq \mathbb{R}_0 \subseteq S_0$, Proposition 1.6.24 ensures that S_0 is comeager, so $\equiv_{S_0}$ is comeager. Let R_0 be the pullback of R through ϕ, and observe that $\equiv_{R_0}$ and $\equiv_{S_0}$ coincide, thus the former is comeager, as well. But $R_0 \subseteq \mathbb{E}_0$, contradicting the fact that $\mathbb{E}_0$ is meager. ⊠

Theorem 5.2.4 (Kanovei, Louveau). *Suppose that* AD *holds, $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a $\boldsymbol{\Delta}^1_{2n+1}$ quasi-order on X. Then exactly one of the following holds:*

(1) *There is a quasi-order $S \supseteq R$ that admits a $\boldsymbol{\Delta}^1_{2n+1}$ reduction to $R_{\mathrm{lex}} \restriction 2^\kappa$ for some $\kappa < (\boldsymbol{\kappa}^1_{2n+1})^+$, and for which $\equiv_R$ and $\equiv_S$ coincide.*
(2) *There is a continuous embedding of $\mathbb{E}_0$ or $\mathbb{R}_0$ into R.*

Proof. Theorem 1.4.14 and Remark 5.1.7 ensure that at least one of the following conditions hold. To see that they are mutually exclusive, appeal to Theorem 1.5.11 and the second half of the proof of Theorem 5.2.3. ⊠

THEOREM 5.2.5 (Kanovei, Louveau). *Suppose that* $\mathtt{AD}$ *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* R *is a* $\boldsymbol{\Delta}^1_{2n+2}$ *quasi-order on* X*. Then exactly one of the following holds:*

(1) *There is a quasi-order* $S \supseteq R$ *that admits a* $\boldsymbol{\Delta}^1_{2n+3}$ *reduction to* $R_{\mathrm{lex}} \upharpoonright 2^\kappa$ *for some* $\kappa < (\boldsymbol{\kappa}^1_{2n+1})^{++}$*, and for which* $\equiv_R$ *and* $\equiv_S$ *coincide.*
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *or* $\mathbb{R}_0$ *into* R*.*

PROOF. As in the proof of Theorem 5.2.4. ⊠

THEOREM 5.2.6 (Kanovei, Louveau). *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds,* X *is an analytic Hausdorff space, and* R *is a quasi-order on* X*. Then exactly one of the following holds:*

(1) *There is a quasi-order* $S \supseteq R$ *that admits a reduction to* $R_{\mathrm{lex}} \upharpoonright 2^\alpha$ *for some ordinal* α*, and for which* $\equiv_R$ *and* $\equiv_S$ *coincide.*
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *or* $\mathbb{R}_0$ *into* R*.*

PROOF. Theorem 1.4.15 and the analog of the proof of Theorem 5.2.2, where the use of Theorem 5.1.2 is replaced with that of Theorem 5.1.12, ensures that at least one of the two conditions holds. The proof of Theorem 5.2.4 ensures that they are mutually exclusive. ⊠

THEOREM 5.2.7 (Harrington-Kechris-Louveau). *Suppose that* X *is an analytic Hausdorff space and* E *is a Borel equivalence relation on* X*. Then exactly one of the following holds:*

(1) *There is a Borel reduction of* E *to equality on* $2^{\mathbb{N}}$*.*
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *into* E*.*

PROOF. As the class of equivalence relations is closed downward under embeddability, this follows from the special case of Theorem 5.2.3 in which R is an equivalence relation. ⊠

THEOREM 5.2.8 (Harrington-Kechris-Louveau). *Suppose that* $\mathtt{AD}$ *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* E *is a* $\boldsymbol{\Delta}^1_{2n+1}$ *equivalence relation on* X*. Then exactly one of the following holds:*

(1) *There is a* $\boldsymbol{\Delta}^1_{2n+1}$ *reduction of* E *to equality on* $2^{\boldsymbol{\kappa}^1_{2n+1}}$*.*
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *into* E*.*

PROOF. As the class of equivalence relations is closed downward under embeddability, this follows from the special case of Theorem 5.2.4 in which R is an equivalence relation. ⊠

THEOREM 5.2.9 (Harrington-Kechris-Louveau). *Suppose that* $\mathtt{AD}$ *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* E *is a* $\boldsymbol{\Delta}^1_{2n+2}$ *equivalence relation on* X*. Then exactly one of the following holds:*

(1) *There is a* $\boldsymbol{\Delta}^1_{2n+3}$ *reduction of* E *to equality on* $2^{(\boldsymbol{\kappa}^1_{2n+1})^+}$.
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *into* E.

PROOF. As the class of equivalence relations is closed downward under embeddability, this follows from the special case of Theorem 5.2.5 in which R is an equivalence relation. ⊠

THEOREM 5.2.10 (Harrington-Kechris-Louveau). *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds,* X *is an analytic Hausdorff space, and* E *is an equivalence relation on* X*. Then exactly one of the following holds:*

(1) *There is a reduction of* E *to equality on* 2^κ *for some aleph* κ.
(2) *There is a continuous embedding of* $\mathbb{E}_0$ *into* E.

PROOF. As the class of equivalence relations is closed downward under embeddability, this follows from the special case of Theorem 5.2.6 in which R is an equivalence relation. ⊠

A quasi-order R on a set X is *linear* if X is an R-chain.

THEOREM 5.2.11 (Harrington-Marker-Shelah). *Suppose that* X *is an analytic Hausdorff space and* R *is a linear Borel quasi-order on* X*. Then there is a Borel reduction of* R *to* $R_{\text{lex}} \upharpoonright 2^\alpha$ *for some* $\alpha < \omega_1$.

PROOF. Otherwise, Theorem 5.2.3 yields a Borel reduction of $\mathbb{E}_0$ or $\mathbb{R}_0$ to a linear Borel quasi-order, which the second part of the proof of Theorem 5.2.3 rules out. ⊠

THEOREM 5.2.12. *Suppose that* $\mathtt{AD}$ *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* R *is a linear* $\boldsymbol{\Delta}^1_{2n+1}$ *quasi-order on* X*. Then there is a* $\boldsymbol{\Delta}^1_{2n+1}$ *reduction of* R *to* $R_{\text{lex}} \upharpoonright 2^\alpha$ *for some* $\alpha < (\boldsymbol{\kappa}^1_{2n+1})^+$.

PROOF. Otherwise, Theorem 5.2.4 yields a reduction of $\mathbb{E}_0$ or $\mathbb{R}_0$ to a linear quasi-order, which the proof of Theorem 5.2.4 rules out. ⊠

THEOREM 5.2.13. *Suppose that* $\mathtt{AD}$ *holds,* $n \in \mathbb{N}$*,* X *is an analytic Hausdorff space, and* R *is a linear* $\boldsymbol{\Delta}^1_{2n+2}$ *quasi-order on* X*. Then there is a* $\boldsymbol{\Delta}^1_{2n+3}$ *reduction of* R *to* $R_{\text{lex}} \upharpoonright 2^\alpha$ *for some* $\alpha < (\boldsymbol{\kappa}^1_{2n+1})^{++}$.

PROOF. Otherwise, Theorem 5.2.5 yields a reduction of $\mathbb{E}_0$ or $\mathbb{R}_0$ to a linear quasi-order, which the proof of Theorem 5.2.5 rules out. ⊠

THEOREM 5.2.14. *Suppose that* $\mathtt{AD}_{\mathbb{R}}$ *holds,* X *is an analytic Hausdorff space, and* R *is a linear quasi-order on* X*. Then there is a reduction of* R *to* $R_{\text{lex}} \upharpoonright 2^\alpha$ *for some ordinal* α.

PROOF. Otherwise, Theorem 5.2.6 yields a reduction of $\mathbb{E}_0$ or $\mathbb{R}_0$ to a linear quasi-order, which the proof of Theorem 5.2.6 rules out. ⊠

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