Reducibility of countable equivalence relations

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Introduction

These are the notes accompanying a course on Borel reducibility of countable Borel equivalence relations at the University of Vienna in Fall 2018. I am grateful to all of the participants for their interest and participation.

Contents

1. The perfect set theorem for measures

When D is a discrete space, we endow $D^{\mathbb{N}}$ with the complete ultrametric given by $d_{D^{\mathbb{N}}}(a,b) = 1/2^{n(a,b)}$ for all distinct $a, b \in D^{\mathbb{N}}$, where $n(a,b)$ is the least coordinate at which a and b differ. The underlying topology is generated by the sets of the form $\mathcal{N}_s = \{c \in D^{\mathbb{N}} \mid s \sqsubseteq c\}$, where $s \in D^{<\mathbb{N}}$.

A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, *Polish* if it is second countable and completely metrizable, K_σ if it is a countable union of compact sets, and *zero-dimensional* if it has a clopen basis. A subset of a metric space is *δ-bounded* if it can be covered by finitely-many balls of radius strictly less than δ, and *totally bounded* if it is δ-bounded for all $\delta > 0$.

A *Borel space* is a set X equipped with a σ-algebra of *Borel sets*. A *Borel measure* on such a space is a measure defined on the Borel sets. Two such Borel measures μ and ν are *orthogonal* if there is a μ-conull Borel set that is also ν-null. When X is a zero-dimensional Polish space, we use $P(X)$ to denote the set of Borel probability measures on X, equipped with the (Polish) topology generated by the sets of the form $\{\mu \mid \mu(U) \in V\}$, where $U \subseteq X$ is clopen and $V \subseteq [0,1]$ is open.

We will slightly abuse language by saying that a sequence $(B_i)_{i \in I}$ of subsets of a space X is *in* a pointclass Γ if the corresponding set $\{(i,x) \in I \times X \mid x \in B_i\}$ is in Γ.

THEOREM 1.1 (Burgess-Mauldin). *Suppose that X is a zero-dimensional Polish space and $A \subseteq P(X)$ is an analytic set of pairwise orthogonal measures. Then exactly one of the following holds:*

1. *The set A is countable.*
2. *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \to A$ for which there is a K_σ sequence $(K_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint subsets of X such that $\pi(c)(K_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

PROOF. Fix a compatible complete metric d_X on X. By the perfect set theorem for analytic Hausdorff spaces, it is sufficient to show that if there is a continuous injection $\phi \colon 2^{\mathbb{N}} \to A$, then condition (2) holds. Towards this end, fix real numbers $\delta_n, \epsilon_n > 0$ such that $\delta_n \to 0$ and $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. We will recursively construct $k_n \in \mathbb{N}$, $\psi_n \colon 2^n \to 2^{k_n}$, and sequences $(U_s)_{s \in 2^n}$ of open subsets of X such that:

(a) $\forall i < 2 \forall n \in \mathbb{N} \forall s \in 2^n \ \psi_n(s) \frown (i) \sqsubseteq \psi_{n+1}(s \frown (i))$.
(b) $\forall n \in \mathbb{N} \forall s \in 2^{n+1} \ U_s$ is δ_n-bounded.
(c) $\forall n \in \mathbb{N} \forall s \in 2^{n+1} \forall \mu \in \phi(\mathcal{N}_{\psi_{n+1}(s)}) \ \mu(U_s) > 1 - \epsilon_n$.
(d) $\forall n \in \mathbb{N} \forall s, t \in 2^{n+1} \ (s \neq t \Longrightarrow \overline{U_s} \cap \overline{U_t} = \emptyset)$.

We begin by setting $k_0 = 0$, $\psi_0(\emptyset) = \emptyset$, and $U_\emptyset = X$. Suppose now that $n \in \mathbb{N}$ and we have already found k_n and ψ_n. For all $i < 2$ and $s \in 2^n$, set $\mu_{s \frown (i)} = \phi(\psi_n(s) \frown (i) \frown (0)^\infty)$. For all distinct $s, t \in 2^{n+1}$, fix a Borel set $B_{s,t} \subseteq X$ that is μ_s-conull and μ_t-null. Then the sets of the form $B_s = \bigcap_{t \in 2^{n+1} \setminus \{s\}} B_{s,t} \setminus B_{t,s}$ are pairwise disjoint, and $\mu_s(B_s) = 1$ for all $s \in 2^{n+1}$. By the tightness of Borel probability measures on Polish spaces, there are compact sets $K_s \subseteq B_s$ with the property that $\mu_s(K_s) > 1 - \epsilon_n$ for all $s \in 2^{n+1}$. By compactness, there exists $0 < \delta'_n < \delta_n$ such that $d(x, y) > 2\delta'_n$ for all distinct $s, t \in 2^{n+1}$ and $(x, y) \in K_s \times K_t$. Compactness also ensures that for all $s \in 2^{n+1}$, there is a finite set $F_s \subseteq K_s$ for which K_s is contained in the δ_n-bounded open set $U_s = \mathcal{B}(F_s, \delta'_n)$. Note that $\overline{U_s} \cap \overline{U_t} = \emptyset$ for all distinct $s, t \in 2^{n+1}$. By the regularity of Borel probability measures on Polish spaces and the fact that X is second countable and zero-dimensional, there are clopen sets $V_s \subseteq U_s$ such that $\mu_s(V_s) > 1 - \epsilon_n$ for all $s \in 2^{n+1}$. As ϕ is continuous, there exists $k_{n+1} > k_n$ such that $\mu(V_{s \frown (i)}) > 1 - \epsilon_n$ for all $i < 2$, $s \in 2^n$, and $\mu \in \phi(\mathcal{N}_{\psi_n(s) \frown (i) \frown (0)^{k_{n+1} - (k_n + 1)}})$. For all $i < 2$ and $s \in 2^n$, define $\psi_{n+1}(s \frown (i)) = \psi_n(s) \frown (i) \frown (0)^{k_{n+1} - (k_n + 1)}$.

Condition (a) ensures that we obtain a continuous injection $\psi \colon 2^\mathbb{N} \to 2^\mathbb{N}$ by setting $\psi(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \upharpoonright n)$ for all $c \in 2^\mathbb{N}$, in which case the function $\pi = \phi \circ \psi$ is also a continuous injection. Condition (b) and the fact that $\delta_n \to 0$ ensure that the sets $K_n = \bigcap_{m \geq n} \bigcup_{s \in 2^m} \mathcal{N}_s \times \overline{U_s}$ are totally bounded, and therefore compact, in which case the set $K = \bigcup_{n \in \mathbb{N}} K_n$ is K_σ. For all $c \in 2^\mathbb{N}$, condition (c) and the fact that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$ ensures that $\mu_c(\bigcap_{m \geq n} U_{c \upharpoonright m}) \to 1$, so the fact that $K_c = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \overline{U_{c \upharpoonright m}}$ implies that $\mu_c(K_c) = 1$. Finally, for all distinct $c, d \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, condition (d) ensures that $\bigcap_{m \geq n} \overline{U_{c \upharpoonright m}}$ and $\bigcap_{m \geq n} \overline{U_{d \upharpoonright m}}$ are disjoint for all $n \in \mathbb{N}$, thus so too are K_c and K_d. ⊠

2. Compressibility

Given an equivalence relation E on X, we say that a set $Y \subseteq X$ is *E-complete* if it intersects every E-class. A *compression* of E is an injection $\phi \colon X \to X$ such that $\text{graph}(\phi) \subseteq E$ and $X \setminus \phi(X)$ is E-complete. A Borel space is *standard* if its Borel sets coincide with those of a Polish topology. We say that a Borel equivalence relation on a standard Borel space is *compressible* if it admits a Borel compression. Following the usual abuse of language, we say that an equivalence relation is *countable* if all of its classes are countable.

Proposition 2.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $B \subseteq X$ is an E-complete Borel set for which $E \upharpoonright B$ is compressible. Then there is a Borel injection $\pi \colon X \to B$ whose graph is contained in E.*

Proof. Fix a Borel compression $\phi \colon B \to B$ of $E \upharpoonright B$, and appeal to the Lusin-Novikov uniformization theorem to obtain a Borel function $\psi \colon X \to B \setminus \phi(B)$ whose graph is contained in E, as well as a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel sets on which ψ is injective. Then the function $\pi = \bigcup_{n \in \mathbb{N}} (\phi^n \circ \psi) \upharpoonright B_n$ is as desired. ⊠

Given a group G, we say that a function $\rho \colon E \to G$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ for all $x \mathrel{E} y \mathrel{E} z$. When $G = (0, \infty)$, we set $|S|^\rho_x = \sum_{y \in S} \rho(y, x)$ for all $x \in X$ and $S \subseteq [x]_E$. We say that a function $\phi \colon X \to X$ whose graph is contained in E is *ρ-increasing* at S if $|\phi^{-1}(S)|^\rho_x \leq |S|^\rho_x$, and *strictly ρ-increasing* at S if $|\phi^{-1}(S)|^\rho_x < |S|^\rho_x$. A *compression* of ρ over a subequivalence relation F of E is a function $\phi \colon X \to X$, whose graph is contained in E, that is ρ-increasing at every F-class, and for which the set of F-classes at which it is strictly ρ-increasing is (E/F)-complete. Again following the usual abuse of language, we say that an equivalence relation is *finite* if all of its classes are finite. We say that a Borel cocycle $\rho \colon E \to (0, \infty)$ is *compressible* over a finite Borel subequivalence relation of E if there is a Borel compression of ρ over a finite Borel subequivalence relation of E. We say that a Borel cocycle $\rho \colon E \to (0, \infty)$ is *μ-nowhere compressible* over a finite Borel subequivalence relation of E if there is no μ-positive Borel set $B \subseteq X$ for which $\rho \upharpoonright (E \upharpoonright B)$ is compressible over a finite Borel subequivalence relation of $E \upharpoonright B$.

A Borel measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-conull or μ-null, *E-quasi-invariant* if the family of μ-null sets is closed under E-saturation, *ρ-invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T \colon X \to X$ whose graphs are contained in E, and *E-invariant* if it is invariant with respect to the constant cocycle.

Theorem 2.2 (Hopf). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-quasi-invariant Borel probability measure on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle that is μ-nowhere compressible over a finite Borel subequivalence relation of E. Then there is a ρ-invariant Borel probability measure $\nu \sim \mu$.*

Proof. As there is no Borel compression of ρ over a finite Borel subequivalence relation of E, the generalization of Nadkarni's characterization of the existence of invariant Borel probability measures to Borel cocycles ensures the existence of a ρ-invariant Borel probability measure on X. Ditzen's generalization of the Farrell-Varadarajan uniform ergodic decomposition theorem therefore yields an E-invariant Borel function $\phi \colon X \to P(X)$ that is a *decomposition* of the set of all ρ-invariant Borel probability measures into E-ergodic ρ-invariant Borel probability measures, in the sense that $\phi(x)$ is E-ergodic and ρ-invariant for all $x \in X$, $\phi^{-1}(\{\mu\})$ is μ-conull for all E-ergodic ρ-invariant Borel probability measures μ on X, and $\nu(B) = \int \phi(x)(B) \ d\nu(x)$ for all ρ-invariant Borel probability measures ν on X and Borel sets $B \subseteq X$. Let ν' be the Borel probability measure on X given by $\nu'(B) = \int \phi(x)(B) \ d\mu(x)$.

Lemma 2.3. *The measure ν' is ρ-invariant.*

Proof. Note that if $\psi \colon X \to (0, \infty)$ is a Borel function, then $\int \psi(x) \ d\nu'(x) = \int \int \psi(y) \ d\phi(x)(y) \ d\mu(x)$ by countable additivity. So if $B \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in E, then $\nu'(T(B)) = \int \phi(x)(T(B)) \ d\mu(x) = \int \int \rho(T(y), y) \ d\phi(x)(y) \ d\mu(x) = \int \rho(T(x), x) \ d\nu'(x)$. ⊠

Lemma 2.4. *The measure μ is absolutely continuous with respect to the measure ν'.*

Proof. Suppose that $B \subseteq X$ is a μ-positive Borel set, and define $N = \{x \in X \mid \phi(x)(B) = 0\}$. Observe now that if $x \in {\sim}N$, then $\phi(x) \neq \phi(y)$ for all $y \in N$, in which case $\phi(x)(N) = 0$. In particular, it follows that if ν is a ρ-invariant Borel probability measure on X, then $\nu(B \cap N) \leq \int_N \phi(x)(B) \ d\nu(x) + \int_{\sim N} \phi(x)(N) \ d\nu(x) = 0$, thus $[B \cap N]_E$ is ν-null. One more application of the generalization of Nadkarni's theorem to Borel cocycles therefore ensures that $\rho \restriction (E \restriction [B \cap N]_E)$ is compressible over a finite Borel subequivalence relation of $E \restriction [B \cap N]_E$, so $[B \cap N]_E$ is μ-null, thus $B \setminus N$ is μ-positive, and it follows that $\nu'(B) \geq \int_{B \setminus N} \phi(x)(B) \ d\mu(x) > 0$. ⊠

Fix an E-invariant μ-null Borel set $N \subseteq X$ of maximal ν'-measure, and observe that the normalization of the ρ-invariant Borel measure ν on X given by $\nu(B) = \nu'(B \setminus N)$ is as desired. ⊠

3. Increasing unions

Given a class $\mathcal{E}$ of equivalence relations, we use *hyper-$\mathcal{E}$* to denote the class of equivalence relations of the form $\bigcup_{n \in \mathbb{N}} E_n$, where $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of equivalence relations in $\mathcal{E}$.

QUESTION 3.1. Is every hyperhyperfinite Borel equivalence relation on a standard Borel space hyperfinite?

Given a Borel measure μ on a standard Borel space X, we say that a Borel equivalence relation E on X is μ-$\mathcal{E}$ if its restriction to some μ-conull Borel set is in $\mathcal{E}$.

PROPOSITION 3.2. *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces that is closed under Borel restrictions and countable intersections, X is a standard Borel space, E is a countable Borel equivalence relation on X, Φ is a countable set of Borel partial functions from X to X such that $E = \bigcup_{\phi \in \Phi} \operatorname{graph}(\phi)$, and μ is an E-quasi-invariant finite Borel measure on X. Then the following are equivalent:*

(1) *The equivalence relation E is μ-hyper-$\mathcal{E}$.*
(2) *For all $\epsilon > 0$ and Borel sets $R \subseteq E$ with finite vertical sections, there exists $E' \subseteq E$ in $\mathcal{E}$ with $\mu(\{x \in X \mid R_x \nsubseteq [x]_{E'}\}) < \epsilon$.*
(3) *For all $\epsilon > 0$ and finite sets $\Phi' \subseteq \Phi$, there exists $E' \subseteq E$ in $\mathcal{E}$ such that $\mu(\bigcup_{\phi' \in \Phi'} \{x \in \operatorname{dom}(\phi') \mid \neg x \mathrel{E'} \phi'(x)\}) < \epsilon$.*

PROOF. To see (1) $\implies$ (2), fix a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is hyper-$\mathcal{E}$, as well as an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of equivalence relations in $\mathcal{E}$ such that $E \restriction C = \bigcup_{n \in \mathbb{N}} E_n$. As μ is E-quasi-invariant, the set $N = [\sim C]_E$ is μ-null. But if $\epsilon > 0$, $R \subseteq E$ is a Borel set with finite vertical sections, and $B_n = \{x \in X \mid R_x \nsubseteq [x]_{E_n}\}$ for all $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} B_n \subseteq N$, so $\mu(B_n) < \epsilon$ for some $n \in \mathbb{N}$.

To see (2) $\implies$ (3), note that if $E' \subseteq E$ is an equivalence relation and $\Phi' \subseteq \Phi$ is finite, then $R = \bigcup_{\phi' \in \Phi'} \operatorname{graph}(\phi')$ has finite vertical sections and $\{x \in X \mid R_x \nsubseteq [x]_{E'}\} = \bigcup_{\phi' \in \Phi'} \{x \in \operatorname{dom}(\phi') \mid \neg x \mathrel{E'} \phi'(x)\}$.

To see (3) $\implies$ (1), fix real numbers $\epsilon_m > 0$ with $\sum_{m \in \mathbb{N}} \epsilon_m < \infty$, an enumeration $(\phi_k)_{k \in \mathbb{N}}$ of Φ, and equivalence relations $E_m \subseteq E$ in $\mathcal{E}$ such that the set $A_m = \bigcup_{k < m} \{x \in \operatorname{dom}(\phi_k) \mid \neg x \mathrel{E_m} \phi_k(x)\}$ has μ-measure at most ϵ_m for all $m \in \mathbb{N}$. Then the set $B_n = \bigcup_{m \geq n} A_m$ has μ-measure at most $\sum_{m \geq n} \epsilon_m$ for all $n \in \mathbb{N}$, so the set $N = \bigcap_{n \in \mathbb{N}} B_n$ is μ-null. Note that if $x \mathrel{E} y$, then there exists $k \in \mathbb{N}$ such that $\phi_k(x) = y$, and if $x \notin N$, then there exists $n > k$ for which $x \notin B_n$, so $x \mathrel{(\bigcap_{m \geq n} E_m)} y$, thus $(\bigcap_{m \geq n} E_m \restriction \sim N)_{n \in \mathbb{N}}$ is an increasing sequence of equivalence relations in $\mathcal{E}$ whose union is $E \restriction \sim N$, hence E is μ-hyper-$\mathcal{E}$. ⊠

We say that μ is E-$\mathcal{E}$ if E is μ-$\mathcal{E}$.

PROPOSITION 3.3 (Dye-Krieger). *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces that is closed under Borel restrictions and countable intersections, X is a standard*

Borel space, E is a countable Borel equivalence relation on X, and μ is an E-hyper-hyper-$\mathcal{E}$ E-quasi-invariant finite Borel measure on X. Then μ is E-hyper-$\mathcal{E}$.

Proof. Suppose that $\epsilon > 0$ and $R \subseteq E$ is a Borel set with finite vertical sections. By Proposition 3.2, there is a hyper-$\mathcal{E}$ equivalence relation $E' \subseteq E$ for which the set $B = \{x \in X \mid R_x \nsubseteq [x]_{E'}\}$ has μ-measure at most $\epsilon/2$. Set $R' = R \cap (\sim B \times X)$, and appeal again to Proposition 3.2 to obtain an equivalence relation $E'' \subseteq E'$ in $\mathcal{E}$ with $\mu(\{x \in X \mid R'_x \nsubseteq [x]_{E''}\}) < \epsilon/2$, so $\mu(\{x \in X \mid R_x \nsubseteq [x]_{E''}\}) < \epsilon$. One last application of Proposition 3.2 then ensures that E is μ-hyper-$\mathcal{E}$. ⊠

In the special case that $\mathcal{E}$ is the class of finite Borel equivalence relations on standard Borel spaces, we obtain the following.

Theorem 3.4 (Segal). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X. Then the set of E-hyperfinite E-quasi-invariant Borel probability measures is Borel.*

Proof. We can assume, without loss of generality, that X is a Polish space. Fix a countable basis $\mathcal{B}$ for X that is closed under finite unions, appeal to the Lusin-Novikov uniformization theorem to obtain a countable set Φ of Borel functions from X to X with the property that $E = \bigcup_{\phi \in \Phi} \text{graph}(\phi)$, and set $\Psi = \{\phi \upharpoonright U \mid \phi \in \Phi \text{ and } U \in \mathcal{B}\}$. For each finite set $\Psi' \subseteq \Psi$, let $B_{\Psi'}$ be the Borel set of $x \in X$ such that:

(1) $\exists \psi' \in \Psi' \; x = \psi'(x)$.
(2) $\forall \psi' \in \Psi' \; (x \in \text{dom}(\psi') \implies \exists \psi'' \in \Psi' \; x = (\psi'' \circ \psi')(x))$.
(3) $\forall \psi', \psi'' \in \Psi' \; (x \in \text{dom}(\psi') \cap (\psi')^{-1}(\text{dom}(\psi'')) \implies$
$\exists \psi''' \in \Psi' \; \psi'''(x) = (\psi'' \circ \psi')(x))$.

Then the restriction $F_{\Psi'}$ of $\bigcup_{\psi' \in \Psi'} \text{graph}(\psi')$ to $B_{\Psi'}$ is a finite Borel equivalence relation.

Lemma 3.5. *Suppose that $E' \subseteq E$ is a finite Borel partial equivalence relation on X, μ is a finite Borel measure on X, and $\epsilon > 0$. Then there is a finite set $\Psi' \subseteq \Psi$ for which $\mu(\{x \in X \mid [x]_{E'} \neq [x]_{F_{\Psi'}}\}) < \epsilon$.*

Proof. Fix an enumeration $(\phi_k)_{k \in \mathbb{N}}$ of Φ, as well as a natural number n sufficiently large that the μ-measure of the complement of the set $A = \{x \in X \mid \forall y, z \in [x]_{E'} \exists k < n \; \phi_k(y) = z\}$ is at most $\epsilon/2$. Set $B_m = \{x \in X \mid x \; E' \; \phi_m(x)\}$ and appeal to the regularity of finite Borel measures on Polish spaces to obtain sets $U_m \in \mathcal{B}$ such that $\sum_{k<n} (\phi_k)_* \mu(B_m \bigtriangleup U_m) < \epsilon/2n$ for all $m < n$. To see that the set $\Psi' = \{\phi_k \upharpoonright U_k \mid k < n\}$ is as desired, set $B = A \setminus \bigcup_{m<n} [B_m \bigtriangleup U_m]_{E'}$, and note that if $x \in B$, then $[x]_{E'} = \{\psi'(x) \mid \psi' \in \Psi'\}$, so the fact

that B is E'-invariant ensures that $[y]_{E'} = \{\psi'(y) \mid \psi' \in \Psi'\}$ for all $y \in [x]_{E'}$, thus $[x]_{E'} \subseteq B_{\Psi'}$, hence $[x]_{E'} = [x]_{F_{\Psi'}}$, so it only remains to observe that $\mu(\sim B) \leq \mu(\sim A) + \sum_{m<n} \mu(A \cap [B_m \triangle U_m]_{E'}) \leq \epsilon/2 + \sum_{k,m<n} (\phi_k)_* \mu(B_m \triangle U_m) < \epsilon$. ⊠

Proposition 3.2 and Lemma 3.5 ensure that an E-quasi-invariant finite Borel measure μ on X is E-hyperfinite if and only if for all $\epsilon > 0$ and finite sets $\Phi' \subseteq \Phi$, there is a finite set $\Psi' \subseteq \Psi$ such that $\mu(\bigcup_{\phi' \in \Phi'} \{x \in \mathrm{dom}(\phi') \mid \neg x \ F_{\Psi'} \ \phi'(x)\}) < \epsilon$. The desired result is therefore a consequence of the fact that the set of E-quasi-invariant Borel probability measures on X is Borel. ⊠

4. Smooth-to-one homomorphisms

The *diagonal* on X is given by $\Delta(X) = \{(x,x) \mid x \in X\}$, and we use $\mathbb{E}_0$ to denote the equivalence relation on $2^{\mathbb{N}}$ with respect to which $c \ \mathbb{E}_0 \ d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m)$. We identify the product of equivalence relations E on X and F on Y with the equivalence relation on $X \times Y$ for which two pairs (x,y) and (x',y') are equivalent if and only if $x \ E \ x'$ and $y \ F \ y'$. A *homomorphism* from a binary relation R on X to a binary relation S on Y is a function $\phi \colon X \to Y$ such that $(\phi \times \phi)(R) \subseteq S$, a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, and an *embedding* of R into S is an injective reduction of R to S. We say that a Borel equivalence relation E on a standard Borel space X is *smooth* if there is a Borel reduction of E to equality on a standard Borel space. A *partial transversal* of E is a set $Y \subseteq X$ whose intersection with each E-class consists of at most one point. The Lusin-Novikov uniformization theorem ensures that when E is countable, the smoothness of E is equivalent to the existence of cover of X by countably-many Borel partial transversals of E. Given a class $\mathcal{E}$ of countable Borel equivalence relations on standard Borel spaces, a standard Borel space X, and a countable Borel equivalence relation E on X, we say that a Borel set $B \subseteq X$ is E-$\mathcal{E}$ if $E \restriction B \in \mathcal{E}$.

Proposition 4.1. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X, and $\phi \colon X \to Y$ is Borel. Then the following are equivalent:*

1. *The function ϕ is E-smooth-to-one.*
2. *The graph of ϕ is $(E \times \Delta(Y))$-smooth.*
3. *There is a cover $(B_n)_{n \in \mathbb{N}}$ of X by Borel sets with the property that ϕ is injective on each $(E \restriction B_n)$-class for all $n \in \mathbb{N}$.*

Proof. To see $\neg(2) \implies \neg(1)$, note that if the graph of ϕ is not $(E \times \Delta(Y))$-smooth, then the $\mathbb{E}_0$ dichotomy yields a continuous embedding $\psi\colon 2^{\mathbb{N}} \to \mathrm{graph}(\phi)$ of $\mathbb{E}_0$ into $E \times \Delta(Y)$. Then $\mathrm{proj}_Y \circ \psi$ is a continuous homomorphism from $\mathbb{E}_0$ to equality, and is therefore constant. Let $y \in Y$ be its constant value, and observe that $\mathrm{proj}_X \circ \psi$ is an embedding of $\mathbb{E}_0$ into $E \upharpoonright \phi^{-1}(\{y\})$, thus the latter is non-smooth.

To see $(2) \implies (3)$, fix Borel partial transversals R_n of $E \times \Delta(Y)$ with the property that $\mathrm{graph}(\phi) = \bigcup_{n \in \mathbb{N}} R_n$, and observe that the Borel sets of the form $B_n = \mathrm{proj}_X(R_n)$ cover X and ϕ is injective on each $(E \upharpoonright B_n)$-class for all $n \in \mathbb{N}$.

To see $(3) \implies (1)$, note that for all $y \in Y$, the sets of the form $B_n \cap \phi^{-1}(\{y\})$ are partial transversals of E and cover $\phi^{-1}(\{y\})$, so $\phi^{-1}(\{y\})$ is E-smooth. ⊠

Proposition 4.2. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relation on X and Y, and $\phi\colon X \to Y$ is a Borel homomorphism from E to F. Then ϕ is E-smooth-to-one if and only if there is an E-complete Borel set $B \subseteq X$ such that ϕ is injective on each $(E \upharpoonright B)$-class.*

Proof. If ϕ is smooth-to-one, then Proposition 4.1 yields a cover $(B_n)_{n \in \mathbb{N}}$ of X by Borel sets such that ϕ is injective on each $(E \upharpoonright B_n)$-class for all $n \in \mathbb{N}$, so the Borel set $B = \bigcup_{n \in \mathbb{N}} B_n \setminus \bigcup_{m<n} [B_m]_E$ is E-complete and ϕ is injective on each $(E \upharpoonright B)$-class. Conversely, if $B \subseteq X$ is an E-complete Borel set such that ϕ is injective on each $(E \upharpoonright B)$-class and $y \in Y$, then $\phi^{-1}(\{y\}) \subseteq \bigcup_{z \in [y]_F} [B \cap \phi^{-1}(\{z\})]_E$. As each $B \cap \phi^{-1}(\{z\})$ is a partial transversal of E, the fact that the family of Borel sets on which E is smooth is closed under countable unions and E-saturations yields that $\phi^{-1}(\{y\})$ is E-smooth. ⊠

5. Structurability

Suppose that N is a countable set, $L = (R_n)_{n \in N}$ is a relational language, and k_n is the arity of R_n for all $n \in \mathbb{N}$. An *L-structuring* of an equivalence relation E on X is an E-invariant function assigning an L-structure $M^x = ([x]_E, (R^x_n)_{n \in N})$ to each $x \in X$. We say that such an assignment is *Borel* if $\{(x, (x_i)_{i<k_n}) \in X \times X^{k_n} \mid (x_i)_{i<k_n} \in R^x_n\}$ is Borel for all $n \in N$. Given a class $\mathcal{M}$ of L-structures, an *$\mathcal{M}$-structuring* of E is an L-structuring of E such that $M^x \in \mathcal{M}$ for all $x \in X$. We say that a Borel equivalence relation on a standard Borel space is *$\mathcal{M}$-structurable* if it admits a Borel $\mathcal{M}$-structuring. In particular, the following observation ensures that the classes of smooth and hyperfinite countable Borel equivalence relations on standard Borel spaces are closed downward under smooth-to-one Borel homomorphisms.

PROPOSITION 5.1. *Suppose that L is a countable relational language and $\mathcal{M}$ is an isomorphism-invariant class of countable L-structures for which the class of $\mathcal{M}$-structurable countable Borel equivalence relations on standard Borel spaces is closed under Borel restrictions and saturations. Then it is downward closed under smooth-to-one Borel homomorphisms.*

PROOF. Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, $\phi\colon X \to Y$ is an E-smooth-to-one Borel homomorphism from E to F, and F is $\mathcal{M}$-structurable. By Proposition 4.2, there is an E-complete Borel set $B \subseteq X$ such that $\phi \restriction B$ is injective on $(E \restriction B)$-classes.

LEMMA 5.2. *There is a Borel partial function $\psi\colon X \times \mathbb{N} \rightharpoonup Y$ bijectively sending* $\mathrm{dom}(\psi) \cap ([x]_E \times \mathbb{N})$ *to* $[\phi(x)]_F$ *for all $x \in X$.*

PROOF. Appeal to the Feldman-Moore theorem to obtain a countable group $G = \{g_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of Y such that $F = E_G^Y$, set $\phi_n = g_n \circ \phi$ and $B_n = B \cap \phi_n^{-1}(\phi_n(B) \setminus \bigcup_{m<n} \phi_m(B))$ for all $n \in \mathbb{N}$, define $A = \bigcup_{n\in\mathbb{N}} B_n \times \{n\}$, and observe that the function $\psi\colon A \to Y$ given by $\psi(x, n) = \phi_n(x)$ is as desired. ⊠

For each set N, let $I(N)$ denote the equivalence relation $N \times N$. As F is $\mathcal{M}$-structurable, so too is $(E \times I(\mathbb{N})) \restriction \mathrm{dom}(\psi)$. The closure of $\mathcal{M}$-structurability under saturations therefore ensures that $E \times I(\mathbb{N})$ is $\mathcal{M}$-structurable, so the closure of $\mathcal{M}$-structurability under Borel restrictions implies that E is $\mathcal{M}$-structurable. ⊠

We say that an element F of a class $\mathcal{E}$ is *universal* for $\mathcal{E}$ under a quasi-order $\leq$ if $E \leq F$ for all $E \in \mathcal{E}$. We say that a class $\mathcal{M}$ of countable L-structures is *Borel-on-Borel* if for all standard Borel spaces X, countable Borel equivalence relations E on X, and Borel L-structurings $x \mapsto M^x$ of E, the set $\{x \in X \mid M^x \in \mathcal{M}\}$ is Borel. An *invariant embedding* of an equivalence relation E on X into an equivalence relation F on Y is an embedding $\pi\colon X \to Y$ of E into F with the property that $\pi([x]_E) = [\pi(x)]_F$ for all $x \in X$.

PROPOSITION 5.3. *Suppose that L is a countable relational language and $\mathcal{M}$ is an isomorphism-invariant Borel-on-Borel class of countable L-structures. Then there is a universal $\mathcal{M}$-structurable countable Borel equivalence relation on a standard Borel space under Borel invariant embeddability.*

PROOF. The Feldman-Moore theorem ensures that every countable Borel equivalence relation on a standard Borel space is generated by a

Borel action of the free group $G = \mathbb{F}_{\aleph_0}$. Fix a countable set N disjoint from $\mathbb{N}$ for which there is an injective enumeration $(R_n)_{n \in N}$ of the relation symbols of L, and let k_n be the arity of R_n for all $n \in N$.

The *right Bernoulli shift* of G on $\prod_{n \in N} 2^{G^{k_n}}$ is the map from $G \times \prod_{n \in N} 2^{G^{k_n}}$ to $\prod_{n \in N} 2^{G^{k_n}}$ given by $(g \cdot x)(n)((g_i)_{i<k_n}) = x(n)((g_i g)_{i<k_n})$. Note that if $x \in X$, then $(1_G \cdot x)(n)((g_i)_{i<k_n}) = x(n)((g_i)_{i<k_n})$ for all $n \in N$ and $(g_i)_{i<k_n} \in G^{k_n}$, thus $1_G \cdot x = x$. Similarly, if $g, h \in G$ and $x \in X$, then

$$\begin{aligned}(g \cdot (h \cdot x))(n)((g_i)_{i<k_n}) &= (h \cdot x)(n)((g_i g)_{i<k_n}) \\ &= x(n)((g_i g h)_{i<k_n}) \\ &= ((gh) \cdot x)(n)((g_i)_{i<k_n})\end{aligned}$$

for all $n \in N$ and $(g_i)_{i<k_n} \in G^{k_n}$, thus $g \cdot (h \cdot x) = (gh) \cdot x$.

Let X_L be the set of all $x \in \prod_{n \in N} 2^{G^{k_n}}$ with the property that $(g_i \cdot x)_{i<k_n} = (h_i \cdot x)_{i<k_n} \implies x(n)((g_i)_{i<k_n}) = x(n)((h_i)_{i<k_n})$ for all $n \in N$ and $(g_i)_{i<k_n}, (h_i)_{i<k_n} \in G^{k_n}$. Observe that if $g \in G$ and $x \in X_L$, then $(g_i \cdot (g \cdot x))_{i<k_n} = (h_i \cdot (g \cdot x))_{i<k_n} \implies x(n)((g_i g)_{i<k_n}) = x(n)((h_i g)_{i<k_n}) \implies (g \cdot x)(n)((g_i)_{i<k_n}) = (g \cdot x)(n)((h_i)_{i<k_n})$ for all $n \in N$ and $(g_i)_{i<k_n}, (h_i)_{i<k_n} \in G^{k_n}$, so $g \cdot x \in X_L$.

The definition of X_L ensures that for all $n \in N$ and $x \in X_L$, we obtain a k_n-ary relation R_n^x on Gx by setting $(g_i \cdot x)_{i<k_n} \in R_n^x \iff x(n)((g_i)_{i<k_n}) = 1$ for all $(g_i)_{i<k_n} \in G^{k_n}$. Note that if $g \in G$, $n \in N$, $(g_i)_{i<k_n} \in G^{k_n}$, and $x \in X$ then

$$\begin{aligned}(g_i \cdot x)_{i<k_n} \in R_n^{g \cdot x} &\iff (g_i g^{-1} \cdot (g \cdot x))_{i<k_n} \in R_n^{g \cdot x} \\ &\iff (g \cdot x)(n)((g_i g^{-1})_{i<k_n}) = 1 \\ &\iff x(n)((g_i)_{i<k_n}) = 1 \\ &\iff (g_i \cdot x)_{i<k_n} \in R_n^x.\end{aligned}$$

It follows that the assignment $x \mapsto M^x = (Gx, (R_n^x)_{n \in N})$ is an L-structuring of $E_G^{X_L}$, in which case the restriction of this assignment to the set $X_{\mathcal{M}} = \{x \in X_L \mid M^x \in \mathcal{M}\}$ is an $\mathcal{M}$-structuring of $E_G^{X_{\mathcal{M}}}$.

A *homomorphism* from an action $G \curvearrowright X$ to an action $G \curvearrowright Y$ is a function $\phi \colon X \to Y$ such that $\phi(g \cdot x) = g \cdot \phi(x)$ for all $x \in X$. Given a standard Borel space X, a Borel action $G \curvearrowright X$, and a Borel L-structuring $x \mapsto M^x = (Gx, (R_n^x)_{n \in N})$ of E_G^X, define a function $\phi \colon X \to \prod_{n \in N} 2^{G^{k_n}}$ by $\phi(x)(n)((g_i)_{i<k_n}) = 1 \iff (g_i \cdot x)_{i<k_n} \in R_n^x$ for all $n \in N$, $(g_i)_{i<k_n} \in G^{k_n}$, and $x \in X$, and observe that if $g \in G$ and

$x \in X$, then

$$\begin{aligned}\phi(g \cdot x)(n)((g_i)_{i<k_n}) = 1 &\iff (g_i g \cdot x)_{i<k_n} \in R^x_n \\ &\iff \phi(x)(n)((g_i g)_{i<k_n}) = 1 \\ &\iff (g \cdot \phi(x))(n)((g_i)_{i<k_n}) = 1,\end{aligned}$$

so $\phi(g \cdot x) = g \cdot \phi(x)$, thus ϕ is a homomorphism of G-actions.

An *embedding* of an action $G \curvearrowright X$ into an action $G \curvearrowright Y$ is an injective homomorphism from $G \curvearrowright X$ to $G \curvearrowright Y$. Let L' be the language obtained from L by adding unary function symbols S_n for all $n \in \mathbb{N}$. Let $\mathcal{M}'$ be the class of L'-structures whose L-reducts are in $\mathcal{M}$.

Suppose now that X is a standard Borel space, $G \curvearrowright X$ is a Borel action, and $x \mapsto M^x = (Gx, (R^x_n)_{n \in N})$ is a Borel $\mathcal{M}$-structuring of E^X_G, fix a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel subsets of X separating points, and let $x \mapsto (M')^x = (Gx, (R^x_n)_{n \in N} \cup (S^x_n)_{n \in \mathbb{N}})$ be the $\mathcal{M}'$-structuring of E^X_G with respect to which $(M')^x$ is the expansion of M^x such that $y \in S^x_n \iff y \in B_n$ for all $n \in \mathbb{N}$, $x \in X$, and $y \in Gx$. Let ϕ be the homomorphism from $G \curvearrowright X$ to $G \curvearrowright \prod_{n \in N} 2^{G^{k_n}} \times (2^G)^{\mathbb{N}}$ from the previous paragraph.

To see that ϕ is injective, note that if $x, y \in X$ are distinct, then there exists $n \in \mathbb{N}$ such that $x \in S^x_n$ but $y \notin S^y_n$, so $\phi(x)(n)(1_G) \neq \phi(y)(n)(1_G)$, thus $\phi(x) \neq \phi(y)$.

To see that $\phi(X) \subseteq X_{L'}$, note that if $n \in N$, $(g_i)_{i<k_n}, (h_i)_{i<k_n} \in G^{k_n}$, and $x \in X$ has the property that $(g_i \cdot \phi(x))_{i<k_n} = (h_i \cdot \phi(x))_{i<k_n}$, then the fact that ϕ is a homomorphism ensures that $(\phi(g_i \cdot x))_{i<k_n} = (\phi(h_i \cdot x))_{i<k_n}$, so the fact that ϕ is injective implies that $(g_i \cdot x)_{i<k_n} = (h_i \cdot x)_{i<k_n}$, thus $\phi(x)(n)((g_i)_{i<k_n}) = 1 \iff (g_i \cdot x)_{i<k_n} \in R^x_n \iff (h_i \cdot x)_{i<k_n} \in R^x_n \iff \phi(x)(n)((h_i)_{i<k_n}) = 1$. Of course, the same argument shows that if $n \in \mathbb{N}$, $g, h \in G$, and $x \in X$ has the property that $g \cdot \phi(x) = h \cdot \phi(x)$, then $\phi(x)(n)(g) = \phi(x)(n)(h)$.

The fact that $x \mapsto (M')^x$ is an $\mathcal{M}'$-structuring of E now implies that $\phi(X) \subseteq X_{\mathcal{M}'}$, thus $G \curvearrowright X_{\mathcal{M}'}$ is a universal Borel G-action on a standard Borel space whose orbit equivalence relation is $\mathcal{M}$-structurable under Borel embeddability. As every embedding of G-actions is an invariant embedding of orbit equivalence relations, it follows that $E^{X_{\mathcal{M}'}}_G$ is a universal $\mathcal{M}$-structurable countable Borel equivalence relation on a standard Borel space under Borel invariant embeddability. ⊠

6. Treeability

A *graphing* of an equivalence relation is a graph whose connected components coincide with the equivalence classes. A *treeing* of an equivalence relation is an acyclic graphing. We say that a countable

Borel equivalence relation E on a standard Borel space is *treeable* if there is a Borel treeing of E.

Proposition 6.1 (Jackson-Kechris-Louveau). *The class of treeable countable Borel equivalence relations on standard Borel spaces is downward closed under smooth-to-one Borel homomorphisms.*

Proof. By Proposition 5.1, we need only establish closure under saturations and Borel restrictions.

To establish closure under saturations, suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $B \subseteq X$ is Borel, and T is a Borel treeing of $E \upharpoonright B$, appeal to the Lusin-Novikov uniformization theorem to obtain a Borel function $\phi \colon [B]_E \setminus B \to B$ whose graph is contained in E, and observe that $\mathrm{graph}(\phi)^{\pm 1} \cup T$ is a Borel treeing of $E \upharpoonright [B]_E$.

To establish closure under Borel restrictions, suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, T is a Borel treeing of E, and $B \subseteq X$ is Borel. For all $x \in [B]_E$, let $d_T(x, B)$ be the minimal number of edges along a T-path from x to B. By the Lusin-Novikov uniformization theorem, there is a Borel function $\phi \colon [B]_E \setminus B \to B$ such that $d_T(\phi(x), B) < d_T(x, B)$ for all $x \in [B]_E \setminus B$. Define $\psi \colon [B]_E \to B$ by $\psi(x) = \phi^{d_T(x,B)}(x)$, let F be the subequivalence relation of $E \upharpoonright [B]_E$ given by $x \mathrel{F} y \iff \psi(x) = \psi(y)$, and observe that $(\psi \times \psi)(T \setminus F)$ is a treeing of $E \upharpoonright B$. ⊠

7. Cost

We begin this section with a basic fact concerning integration.

Proposition 7.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $R \subseteq E$ is Borel, and μ is an E-invariant Borel measure. Then $\int |R_x| \ d\mu(x) = \int |R^y| \ d\mu(y)$.*

Proof. By the Lusin-Novikov uniformization theorem, there are Borel partial injections $\phi_n \colon X \rightharpoonup X$ whose graphs partition R. Then

$$\begin{aligned}
\int |R^y| \ d\mu(y) &= \int \textstyle\sum_{n \in \mathbb{N}} \chi_{\phi_n(\mathrm{dom}(\phi_n))}(y) \ d\mu(y) \\
&= \textstyle\sum_{n \in \mathbb{N}} \mu(\phi_n(\mathrm{dom}(\phi_n))) \\
&= \textstyle\sum_{n \in \mathbb{N}} \mu(\mathrm{dom}(\phi_n)) \\
&= \int \textstyle\sum_{n \in \mathbb{N}} \chi_{\mathrm{dom}(\phi_n)}(x) \ d\mu(x) \\
&= \int |R_x| \ d\mu(x),
\end{aligned}$$

which completes the proof. ⊠

Suppose that X is a standard Borel space, G is a Borel graph on X, and μ is a Borel measure on X. The *cost* of G with respect to μ is given by $C_\mu(G) = \frac{1}{2} \int |G_x| \ d\mu(x)$.

Proposition 7.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\phi \colon X \rightharpoonup X$ is a Borel partial function whose graph is contained in E with the property that $x \notin \{f(x), f^2(x)\}$ for all $x \in X$, and μ is an E-invariant Borel measure. Then $C_\mu(\mathrm{graph}(\phi)^{\pm 1}) = \mu(\mathrm{dom}(\phi))$.*

Proof. As $\mathrm{graph}(\phi) \cap \mathrm{graph}(\phi)^{-1} = \emptyset$ and Proposition 7.1 ensures that $\int |\mathrm{graph}(\phi)_x| \ d\mu(x) = \int |\mathrm{graph}(\phi)^y| \ d\mu(y) = \int |\mathrm{graph}(\phi)_x^{-1}| \ d\mu(x)$, it follows that $C_\mu(\mathrm{graph}(\phi)^{\pm 1}) = \int |\mathrm{graph}(\phi)_x| \ d\mu(x) = \mu(\mathrm{dom}(\phi))$. ⊠

Proposition 7.3 (Levitt). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $B \subseteq X$ is a Borel transversal of E, T is a Borel treeing of E, and μ is an E-invariant Borel measure on X. Then $C_\mu(T) = \mu(\sim B)$.*

Proof. For all $x \in X$, let $d_T(x, B)$ denote the number of edges along the unique injective T-path from x to a point of B, and define $\phi \colon \sim B \to X$ by $\phi(x) =$ the unique T-neighbor of x with the property that $d_T(\phi(x), B) < d_T(x, B)$. Then $T = \mathrm{graph}(\phi)^{\pm 1}$, so Proposition 7.2 ensures that $C_\mu(T) = \mu(\mathrm{dom}(\phi)) = \mu(\sim B)$. ⊠

We say that a set $Y \subseteq X$ is *G-connected* if $G \upharpoonright Y$ has a single connected component.

Proposition 7.4. *Suppose that X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, and G is a Borel graphing of E. Then E is the union of an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations whose classes are G-connected.*

Proof. Fix an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations whose union is E, and define $x \ E_n \ y$ if and only if $x \ E \ y$ and there is a G-path from x to y that lies within a single F_n-class. ⊠

An equivalence relation is *aperiodic* if all of its classes are infinite.

Proposition 7.5 (Levitt). *Suppose that X is a standard Borel space, E is an aperiodic hyperfinite Borel equivalence relation on X, T is a Borel treeing of E, and μ is an E-invariant finite Borel measure on X. Then $C_\mu(T) = \mu(X)$.*

Proof. By Proposition 7.4, there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E such that $E = \bigcup_{n \in \mathbb{N}} E_n$ and each equivalence class of each E_n is T-connected. Fix a decreasing sequence of Borel transversals $B_n \subseteq X$ of E_n. Proposition 7.3 ensures that $C_\mu(E_n \cap T) = \mu(\sim B_n)$ for all $n \in \mathbb{N}$. As the set $B = \bigcap_{n\in\mathbb{N}} B_n$ is a partial transversal of E, E is aperiodic, and μ is E-invariant, it follows that B is μ-null, so $\mu(B_n) \to 0$, thus the fact that $C_\mu(E_n \cap T) \to C_\mu(T)$ implies that $C_\mu(T) = \mu(X)$. ⊠

A graph G is *n-regular* if $|G_x| = n$ for all $x \in X$.

Proposition 7.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and there is a two-regular Borel graphing G of E. Then E is hyperfinite.*

Proof. We can clearly assume that every equivalence class of E is infinite, and therefore that G is acyclic. By the Lusin-Novikov uniformization theorem, there is a Borel function $\phi\colon X \to X$ whose graph is contained in G. Let d_G denote the (extended-valued) graph metric on X induced by G, and let F be the subequivalence relation of E consisting of all $(x, y) \in E$ for which $d_G(x, y) = d_G(\phi(x), \phi(y))$. As every E-class is the union of two F-classes, it only remains to show that F is hyperfinite. Define $T\colon X \rightharpoonup X$ by $T(x) =$ the first point of $[x]_F \setminus \{x\}$ along the injective G-ray $(x, \phi(x), \ldots)$. By throwing out an F-invariant Borel set on which F is smooth, we can assume that T is a Borel automorphism. But then F is the orbit equivalence relation induced by T, and is therefore hyperfinite. ⊠

We say that G is *μ-acyclic* if there is a μ-conull Borel set $C \subseteq X$ for which $G \restriction C$ is acyclic.

Proposition 7.7 (Levitt). *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, G is a Borel graphing of E, and μ is an E-invariant finite Borel measure on X. Then $C_\mu(G) \geq \mu(X)$, and if equality holds, then E is μ-hyperfinite and G is μ-acyclic.*

Proof. As $C_\mu(G) < \infty$ and μ is E-quasi-invariant, by throwing out an E-invariant μ-null Borel set, we can assume that G is locally finite. We say that a set $Y \subseteq X$ is *G-convex* if every injective G-path between elements of Y lies entirely within Y. The *pruning derivative* on the family of all G-convex sets $Y \subseteq X$ is the function given by $Y' = \{y \in Y \mid |G_y \cap Y| \geq 2\}$. The G-convexity of Y yields that of Y'. Note that if every $(E \restriction Y)$-class has at least two elements, then every point of $Y \setminus Y'$ has a unique $(G \restriction Y)$-neighbor, and if every $(E \restriction Y)$-class has at least three elements, then this $(G \restriction Y)$-neighbor is necessarily in Y'. Letting $\phi\colon Y \setminus Y' \to Y'$ be the function sending each point of $Y \setminus Y'$ to this $(G \restriction Y)$-neighbor, it follows that $G \restriction Y$ is the disjoint union of $G \restriction Y'$ with $\mathrm{graph}(\phi)^{\pm 1}$. The fact that G is locally finite ensures that if $E \restriction Y$ is aperiodic, then so too is $E \restriction Y'$.

By starting with $Y = X$ and recursively applying the pruning derivative, we obtain a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of G-convex Borel subsets of X and Borel functions $\phi_n \colon B_n \setminus B_{n+1} \to B_{n+1}$ such that $B_0 = X$ and $G \restriction B_n$ is the disjoint union of $G \restriction B_{n+1}$ with $\mathrm{graph}(\phi_n)^{\pm 1}$ for all $n \in \mathbb{N}$. Then the set $B = \bigcap_{n \in \mathbb{N}} B_n$ is G-convex, and G is the disjoint union of $G \restriction B$ with $\mathrm{graph}(\psi)^{\pm 1}$, where $\psi \colon {\sim}B \to X$ is given by $\psi = \bigcup_{n \in \mathbb{N}} \phi_n$. As G is locally finite, the pruning derivative terminates after ω-many steps, that is, every point of B has at least two $(G \restriction B)$-neighbors.

Proposition 7.2 ensures that $C_\mu(G) = \mu({\sim}B) + C_\mu(G \restriction B) \geq \mu(X)$, so it only remains to show that if $C_\mu(G \restriction B) = \mu(B)$, then E is μ-hyperfinite and G is μ-acyclic. The fact that ψ sends points of ${\sim}B$ to points of strictly larger pruning rank ensures that every simple G-cycle lies entirely within B (since it would otherwise contain a point of minimal pruning rank). It follows that the restriction of G to the set $A = \{x \in X \mid B \cap [x]_E = \emptyset\}$ is acyclic, and since $E \restriction A = E_t(\psi \restriction A)$, it follows that $E \restriction A$ is hypersmooth, and therefore hyperfinite. So we can assume that $\mu(A) < \mu(X)$. As μ is E-quasi-invariant, it follows that $\mu(B) > 0$. As the family of Borel subsets of X on which E is hyperfinite is closed under E-saturations, it only remains to show that $E \restriction B$ is $(\mu \restriction B)$-hyperfinite and $G \restriction B$ is $(\mu \restriction B)$-acyclic. By throwing out an $(E \restriction B)$-invariant $(\mu \restriction B)$-null Borel subset of B, we can assume that $G \restriction B$ is a two-regular Borel graph, and therefore generates a hyperfinite equivalence relation by Proposition 7.6. To see that $G \restriction B$ is acyclic, note that otherwise there exists $x \in B$ for which $[x]_{E \restriction B}$ is finite, and the fact that ψ is finite-to-one yields $n \in \mathbb{N}$ for which $B_n \cap \psi^{-1}([x]_{E \restriction B}) = \emptyset$, thus $[x]_E = \bigcup_{m \leq n} \psi^{-m}([x]_{E \restriction B})$ is finite, contradicting the aperiodicity of E. ⊠

The *cost* of a countable Borel equivalence relation E on a standard Borel space X with respect to an E-invariant finite Borel measure μ on X is given by $C_\mu(E) = \inf\{C_\mu(G) \mid G \text{ is a Borel graphing of } E\}$.

Proposition 7.8 (Gaboriau). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $B \subseteq X$ is an E-complete Borel set, and μ is an E-invariant finite Borel measure on X. Then $C_\mu(E) - \mu(X) = C_{\mu \restriction B}(E \restriction B) - \mu(B)$.*

Proof. To see that $C_\mu(E) - \mu(X) \leq C_{\mu \restriction B}(E \restriction B) - \mu(B)$, note that if $\epsilon > 0$, then there is a Borel graphing H of $E \restriction B$ with the property that $C_\mu(H) \leq C_{\mu \restriction B}(E \restriction B) + \epsilon$, and the Lusin-Novikov uniformization theorem yields a Borel function $\phi\colon {\sim}B \to B$ whose graph is contained in E. As the graph $G = \mathrm{graph}(\phi)^{\pm 1} \cup H$ generates E, and Proposition 7.2 ensures that $C_\mu(G) = \mu({\sim}B) + C_\mu(H)$, it follows that $C_\mu(E) - \mu(X) \leq C_\mu(G) - \mu(X) = C_\mu(H) - \mu(B) \leq C_{\mu \restriction B}(E \restriction B) - \mu(B) + \epsilon$.

To see that $C_{\mu \restriction B}(E \restriction B) - \mu(B) \leq C_\mu(E) - \mu(X)$, note that if $\epsilon > 0$, then there is a Borel graphing G of E with the property that $C_\mu(G) \leq C_\mu(E) + \epsilon$, and the Lusin-Novikov uniformization theorem yields a Borel function $\phi\colon {\sim}B \to X$ whose graph is contained in G and has the property that $d_G(\phi(x), B) < d_G(x, B)$ for all $x \in {\sim}B$. Define $\psi\colon X \to B$ by $\psi(x) = \phi^{d_G(x,B)}(x)$, and let F be the subequivalence relation of E given by $x \mathrel{F} y \iff \psi(x) = \psi(y)$. Then the graph $H = (\psi \times \psi)(G \setminus F)$ generates $E \restriction B$ and

$$\begin{aligned} C_\mu(H) &= \tfrac{1}{2} \int |H_x| \; d\mu(x) \\ &\leq \tfrac{1}{2} \int_B \textstyle\sum_{y \in [x]_F} |(G \setminus F)_y| \; d\mu(x) \\ &= \tfrac{1}{2} \int |(G \setminus F)_x| \; d\mu(x) \\ &= C_\mu(G \setminus F). \end{aligned}$$

As $\mathrm{graph}(\phi)^{\pm 1} \subseteq F \cap G$, it follows from Proposition 7.2 that $C_\mu(H) \leq C_\mu(G) - \mu({\sim}B)$, in which case $C_{\mu \restriction B}(E \restriction B) - \mu(B) \leq C_\mu(H) - \mu(B) \leq C_\mu(G) - \mu(X) \leq C_\mu(E) - \mu(X) + \epsilon$. ⊠

Remark 7.9. Proposition 7.8 ensures that if $C_\mu(E) > \mu(X)$, then $C_{\mu/\mu(X)}(E) \leq C_{(\mu \restriction B)/\mu(B)}(E \restriction B)$, with equality holding if and only if B is μ-conull.

Given sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, let RS denote the set of pairs $(x, z) \in X \times Z$ for which there exists $y \in Y$ such that $x \mathrel{R} y \mathrel{S} z$.

Proposition 7.10 (Gaboriau). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, F is a Borel subequivalence relation of E whose classes have bounded finite size, $B \subseteq X$ is a Borel transversal of F, G is a Borel graphing of E disjoint from F for which $FGF \restriction B$ is acyclic, and μ is an E-invariant finite Borel measure on X. Then $C_\mu(FGF \restriction B) - \mu(B) \leq C_\mu(G) - \mu(X)$.*

Proof. Let $(X)^3_E$ denote the space of injective triples of pairwise E-related points of X, and fix a Borel coloring $c\colon (X)^3_E \to \mathbb{N}$ of the graph on $(X)^3_E$ in which two triples are related if and only if their images intersect, as well as an infinite-to-one function $d\colon \mathbb{N} \to \mathbb{N}$. We will define an increasing sequence of finite Borel subequivalence relations F_n of F and a decreasing sequence of Borel transversals $B_n \supseteq B$ of F_n such that $C_\mu(F_{n+1} G F_{n+1} \restriction B_{n+1}) - \mu(B_{n+1}) \leq C_\mu(F_n G F_n \restriction B_n) - \mu(B_n)$ for all $n \in \mathbb{N}$. We begin by setting $B_0 = X$ and $F_0 = \Delta(X)$, so that $C_\mu(G) - \mu(X) = C_\mu(F_0GF_0 \restriction B_0) - \mu(B_0)$. Given $n \in \mathbb{N}$ for which we have already found B_n and F_n, let R_n be the set of triples $(x, y, z) \in (B_n \setminus B) \times B_n \times B_n$ with the property that $c(x, y, z) = d(n)$, $x\ F_nGF_n\ y\ F_nGF_n\ z$, and $x\ (F \setminus F_n)\ z$, define $\phi_n \colon B_n \setminus B \rightharpoonup B_n$ by $\phi_n(x) = z \iff \exists y \in B_n\ (x, y, z) \in R_n$, let F_{n+1} be the equivalence relation generated by F_n and $\mathrm{graph}(\phi_n)$, set $B_{n+1} = B_n \setminus \mathrm{dom}(\phi_n)$, and define $\psi_n \colon \mathrm{dom}(\phi_n) \to B_n$ by $\psi_n(x) = y \iff (x, y, \phi_n(x)) \in R_n$. Proposition 7.2 then ensures that

$$\begin{aligned}
&C_\mu(F_{n+1}GF_{n+1} \restriction B_{n+1}) \\
&\quad = \tfrac{1}{2} \int_{B_{n+1}} |B_{n+1} \cap (F_{n+1}GF_{n+1})_x| \ d\mu(x) \\
&\quad \leq \tfrac{1}{2} \int_{B_{n+1} \setminus \phi_n(\mathrm{dom}(\phi_n))} |B_n \cap (F_nGF_n)_x| \ d\mu(x) + \\
&\qquad \tfrac{1}{2} \int_{\phi_n(\mathrm{dom}(\phi_n))} |B_n \cap (F_nGF_n)_x| \ d\mu(x) + \\
&\qquad \tfrac{1}{2} \int_{\phi_n(\mathrm{dom}(\phi_n))} |B_n \cap (F_nGF_n)_{\phi_n^{-1}(x)}| \ d\mu(x) - \\
&\qquad C_\mu(\mathrm{graph}(\psi_n)^{\pm 1}) \\
&\quad = \tfrac{1}{2} \int_{B_n} |B_n \cap (F_nGF_n)_x| \ d\mu(x) - \mu(\mathrm{dom}(\psi_n)) \\
&\quad = C_\mu(F_nGF_n \restriction B_n) - (\mu(B_n) - \mu(B_{n+1})),
\end{aligned}$$

thus $C_\mu(F_{n+1}GF_{n+1} \restriction B_{n+1}) - \mu(B_{n+1}) \leq C_\mu(F_nGF_n \restriction B_n) - \mu(B_n)$. This completes the recursive construction.

Define $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$ and $F_\infty = \bigcup_{n \in \mathbb{N}} F_n$. The fact that F is finite ensures that for all $x \in X$, there exists $n \in \mathbb{N}$ such that $[x]_{F_\infty} = [x]_{F_n}$, so $B_\infty \cap [x]_{F_\infty} = B_n \cap [x]_{F_n}$, thus B_∞ is a transversal of F_∞.

Lemma 7.11. *The relations F and F_∞ coincide on B_∞.*

Proof. Suppose, towards a contradiction, that $F \restriction B_\infty \nsubseteq F_\infty$, and let k be the minimal natural number with the property that there is an $(F_\infty GF_\infty \restriction B_\infty)$-path $(x_i)_{i \leq k}$ such that $x_0 \notin B$ and $x_0\ (F \setminus F_\infty)\ x_k$. Define $\phi \colon X \to B$ by $\phi(x) =$ the unique element of $B \cap [x]_F$, and note that $(\phi(x_i))_{i \leq k}$ is an $(FGF \restriction B)$-path whose initial and terminal points coincide, so the acyclicity of $FGF \restriction B$ yields $0 < i < k$ with the property that $\phi(x_{i-1}) = \phi(x_{i+1})$. As the minimality of k ensures that $x_{i-1}\ (F \setminus F_\infty)\ x_{i+1}$, it follows that $k = 2$. Fix $m \in \mathbb{N}$ for which $x_0\ F_mGF_m\ x_1\ F_mGF_m\ x_2$, as well as $n > m$ with the property that $c(x_0, x_1, x_2) = d(n)$, and observe that $x_0\ F_{n+1}\ x_2$, a contradiction. ⊠

Lemma 7.11 ensures that $B = B_\infty$, thus $F = F_\infty$, in which case $FGF \restriction B = \bigcup_{n \in \mathbb{N}} F_nGF_n \restriction B$. Set $k = \max_{x \in X} |[x]_F|$, and observe that if $H \subseteq E$ is a Borel graph, then Proposition 7.1 ensures that

$$\begin{aligned} C_\mu(FH \cup HF) &\leq \int \textstyle\sum_{y \in [x]_F} |H_y| \; d\mu(x) \\ &\leq k \int \textstyle\sum_{y \in [x]_F} |H_y|/|[x]_F| \; d\mu(x) \\ &= k \int |H_x| \; d\mu(x) \\ &= 2kC_\mu(H). \end{aligned}$$

As $F(FG \cup GF) \cup (FG \cup GF)F = FGF$, it follows that $C_\mu(FGF) \leq 2kC_\mu(FG \cup GF) \leq 4k^2 C_\mu(G)$. In particular, as we can clearly assume that $C_\mu(G) < \infty$, it follows that $C_\mu(FGF) < \infty$. Then the measure ν on X given by $\nu(A) = \int_A |(FGF)_x| \; d\mu(x)$ is finite, so the fact that $\bigcap_{n \in \mathbb{N}} B_n \setminus B = \emptyset$ ensures that $\nu(B_n \setminus B) \to 0$. As one more application of Proposition 7.1 yields that

$$\begin{aligned} C_\mu((F_n G F_n \restriction B_n) \setminus (F_n G F_n \restriction B)) &= C_\mu(F_n G F_n \cap ((B_n \setminus B) \times B)^{\pm 1}) \\ &\leq \int_{B_n \setminus B} |(F_n G F_n)_x| \; d\mu(x) \\ &\leq \nu(B_n \setminus B), \end{aligned}$$

the fact that $C_\mu(F_n G F_n \restriction B) \to C_\mu(FGF \restriction B)$ therefore implies that $C_\mu(F_n G F_n \restriction B_n) - \mu(B_n) \to C_\mu(FGF \restriction B) - \mu(B)$, and it follows that $C_\mu(FGF \restriction B) - \mu(B) \leq C_\mu(G) - \mu(X)$. ⊠

Theorem 7.12 (Gaboriau). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, T is a Borel treeing of E, and μ is an E-invariant finite Borel measure on X for which $C_\mu(T) < \infty$. Then $C_\mu(E) = C_\mu(T)$.*

Proof. It is sufficient to show that if $\epsilon > 0$ and G is a Borel graphing of E, then $C_\mu(T) \leq C_\mu(G) + \epsilon$. By the Lusin-Novikov uniformization theorem, there are countable sets Φ_G and Φ_T of Borel partial injections of X into X such that $(\mathrm{graph}(\phi)^i)_{(i,\phi) \in \{\pm 1\} \times \Phi_H}$ partitions H for all $H \in \{G, T\}$. By replacing each $\phi \in \Phi_G$ with countably-many restrictions, we can assume that for all $\phi \in \Phi_G$, there is a Φ_T-word w_ϕ such that $\phi = w_\phi \restriction \mathrm{dom}(\phi)$. The fact that $C_\mu(T) < \infty$ ensures the existence of a finite set W of Φ_G-words such that $C_\mu(T \setminus \bigcup_{w \in W} \mathrm{graph}(w)^{\pm 1}) \leq \epsilon$. Let $\Phi_G \restriction W$ be the set of $\phi \in \Phi_G$ appearing in some $w \in W$, set $\Phi_H = \{\phi \in \Phi_G \restriction W \mid |w_\phi| \geq 2\}$, define $H = \bigcup_{\phi \in \Phi_H} \mathrm{graph}(\phi)^{\pm 1}$ and $U = \bigcup_{\phi \in (\Phi_G \restriction W) \setminus \Phi_H} \mathrm{graph}(\phi)^{\pm 1} \cup (T \setminus \bigcup_{w \in W} \mathrm{graph}(w)^{\pm 1})$, and observe that $H \cup U$ is a graphing of E and $C_\mu(H \cup U) \leq C_\mu(G) + \epsilon$.

For all $\phi \in \Phi_H$, set $X_\phi = \{1, \ldots, |w_\phi| - 1\} \times \{\phi\} \times \mathrm{dom}(\phi)$ and define $\overline{\phi} \colon \mathrm{dom}(\phi) \cup X_\phi \to X_\phi \cup \phi(\mathrm{dom}(\phi))$ by $\overline{\phi}(x) = (1, \phi, x)$ for all $x \in \mathrm{dom}(\phi)$, $\overline{\phi}(i, \phi, x) = (i+1, \phi, x)$ for all $1 \leq i \leq |w_\phi| - 2$ and $x \in \mathrm{dom}(\phi)$, and $\overline{\phi}(|w_\phi| - 1, \phi, x) = \phi(x)$ for all $x \in \mathrm{dom}(\phi)$.

Define $\overline{X} = X \cup \bigcup_{\phi \in \Phi_H} X_\phi$, let $\pi \colon \overline{X} \to X$ be the extension of the identity function on X given by $\pi(i, \phi, x) = (w_\phi \restriction i)(x)$ for all $\phi \in \Phi_H$, $1 \leq i \leq |w_\phi| - 1$, and $x \in \mathrm{dom}(\phi)$, let $\overline{E}$ be the pullback of E through π, set $\overline{H} = \bigcup_{\phi \in \Phi_H} \mathrm{graph}(\overline{\phi})^{\pm 1}$, and let $\overline{\mu}$ be the extension of μ to an $\overline{E}$-invariant finite Borel measure on $\overline{X}$ given by $\overline{\mu}(\{i\} \times \{\phi\} \times B) = \mu(B)$ for all $\phi \in \Phi_H$, $1 \leq i \leq |w_\phi| - 1$, and Borel sets $B \subseteq \mathrm{dom}(\phi)$.

Let $\overline{F}$ be the pullback of equality on X through π. As π is injective on $\{i\} \times \{\phi\} \times \mathrm{dom}(\phi)$ for all $\phi \in \Phi_H$ and $1 \leq i \leq |w_\phi| - 1$, it follows that the classes of $\overline{F}$ have bounded finite cardinality.

Lemma 7.13. *The graphs $\overline{F}(\overline{H} \cup U)\overline{F} \restriction X$ and T coincide.*

Proof. As $\overline{FHF} \restriction X = (\pi \times \pi)(\overline{H})$ and $\overline{FUF} \restriction X = U$, their union is contained in T. To see that $T \subseteq \overline{F}(\overline{H} \cup U)\overline{F}$, suppose that $x \mathrel{T} y$. If $(x, y) \notin \bigcup_{v \in W} \mathrm{graph}(v)^{\pm 1}$, then $x \mathrel{U} y$. Otherwise, fix $v \in W$ for which $(x, y) \in \mathrm{graph}(v)^{\pm 1}$. As T is acyclic, there exist $i < |v|$ and $j < |w_{v(i)}|$ with the property that $(x, y) \in \mathrm{graph}(w_{v(i)}(j))^{\pm 1}$, in which case $|w_{v(i)}| = 1 \Longrightarrow x \mathrel{U} y$ and $|w_{v(i)}| \geq 2 \Longrightarrow x \mathrel{\overline{FHF}} y$. ⊠

As $\overline{H} \cup U$ is clearly a graphing of $\overline{E}$, Proposition 7.10 ensures that $C_\mu(T) - \mu(X) \leq C_{\overline{\mu}}(\overline{H} \cup U) - \overline{\mu}(\overline{X})$. As the fact that

$$\begin{aligned} C_{\overline{\mu}}(\overline{H}) &= \textstyle\sum_{\phi \in \Phi_H} C_{\overline{\mu}}(\mathrm{graph}(\overline{\phi})^{\pm 1}) \\ &= \textstyle\sum_{\phi \in \Phi_H} \overline{\mu}(\mathrm{dom}(\overline{\phi})) \\ &= \textstyle\sum_{\phi \in \Phi_H} \mu(\mathrm{dom}(\phi))|w_\phi| \\ &= C_\mu(H) + \overline{\mu}(\overline{X}) - \mu(X) \end{aligned}$$

implies that $C_\mu(H \cup U) - \mu(X) = C_{\overline{\mu}}(\overline{H} \cup U) - \overline{\mu}(\overline{X})$, it follows that $C_\mu(T) \leq C_\mu(H \cup U) \leq C_\mu(G) + \epsilon$. ⊠

Remark 7.14 (Gaboriau). Conversely, if G is a non-μ-acyclic Borel graphing of E for which $C_\mu(G) < \infty$, then $C_\mu(E) < C_\mu(G)$. To see this, let C_G be the standard Borel space of simple G-cycles, fix a Borel coloring $c \colon C_G \to \mathbb{N}$ of the graph on C_G in which two simple G-cycles are related if and only if they pass through a common point, and define $\phi_n \colon X \rightharpoonup X$ by $\phi_n(x) = y \iff \exists \gamma \in c^{-1}(\{n\})\ (x, y) \sqsubseteq \gamma$ for all $n \in \mathbb{N}$. As μ is E-quasi-invariant, the fact that G is not μ-acyclic yields $n \in \mathbb{N}$ for which the domain of ϕ_n is μ-positive. Then the graph $H = G \setminus \mathrm{graph}(\phi_n)^{\pm 1}$ also generates E, and since Proposition 7.2 ensures that $C_\mu(H) < C_\mu(G)$, it follows that $C_\mu(E) < C_\mu(G)$.

Remark 7.15 (Gaboriau). Theorem 7.12 implies its generalization in which the hypothesis that $C_\mu(T) < \infty$ is removed. To see this, it is sufficient to show that if G is a Borel graphing of E, $r \in \mathbb{R}$, and $C_\mu(T) > r$, then $C_\mu(G) > r$. Towards this end, again fix countable sets Φ_G and Φ_T of Borel partial injections of X into X such that $(\mathrm{graph}(\phi)^i)_{(i,\phi)\in\{\pm 1\}\times\Phi_H}$ partitions H for all $H \in \{G, T\}$, and note once more that by replacing each $\phi \in \Phi_G$ with countably-many restrictions, we can assume that for all $\phi \in \Phi_G$, there is a Φ_T-word w_ϕ such that $\phi = w_\phi \restriction \mathrm{dom}(\phi)$. Fix a finite set $\Psi_T \subseteq \Phi_T$ such that $C_\mu(H) > r$, where $H = \bigcup_{\psi\in\Psi_T} \mathrm{graph}(\psi)^{\pm 1}$, as well as a finite set $\Psi_G \subseteq \Phi_G$ such that $C_\mu(H) - C_\mu(H \setminus F) > r$, where F is the equivalence relation generated by $\bigcup_{\psi\in\Psi_G} \mathrm{graph}(\psi)^{\pm 1}$. Define $\Psi'_T = \Psi_T \cup \{\phi \in \Phi_T \mid \exists \psi \in \Psi_G\ \phi \text{ appears in } w_\psi\}$, and observe that $\bigcup_{\psi\in\Psi'_T} \mathrm{graph}(\psi)^{\pm 1}$ and $\bigcup_{\psi\in\Psi_G\cup(\Psi'_T\setminus\Psi_T)} \mathrm{graph}(\psi)^{\pm 1} \cup (H \setminus F)$ generate the same equivalence relation, so Theorem 7.12 ensures that the cost of the former is at most that of the latter, thus $C_\mu(\bigcup_{\psi\in\Psi_T} \mathrm{graph}(\psi)^{\pm 1}) \leq C_\mu(\bigcup_{\psi\in\Psi_G} \mathrm{graph}(\psi)^{\pm 1}) + C_\mu(H \setminus F)$, hence $C_\mu(\bigcup_{\psi\in\Psi_G} \mathrm{graph}(\psi)^{\pm 1}) > r$.

8. Codes

Given a compact space X and a metric space Y, let $C(X, Y)$ denote the space of continuous functions from X to Y, equipped with the metric $d_{C(X,Y)}(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$.

Proposition 8.1. *Suppose that X is a compact Polish space and Y is a Polish metric space. Then $C(X, Y)$ is Polish.*

Proof. To see that $C(X, Y)$ is separable, fix a countable basis $\mathcal{U}$ for X and a countable dense set $D \subseteq Y$. For all rational $\epsilon > 0$, finite covers $\mathcal{V} \subseteq \mathcal{U}$ of X, and functions $\phi \colon \mathcal{V} \to D$ for which it is possible, fix a continuous function $f_{\epsilon,\mathcal{V},\phi} \colon X \to Y$ such that $d_Y(\phi(V), f_{\epsilon,\mathcal{V},\phi}(x)) < \epsilon$ for all $V \in \mathcal{V}$ and $x \in V$. To see that the set of all $f_{\epsilon,\mathcal{V},\phi}$ is dense, note that if $\epsilon > 0$ and $f \colon X \to Y$ is continuous, then there is a finite cover $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathrm{diam}(f(V)) < \epsilon$ for all $V \in \mathcal{V}$, as well as a function $\phi \colon \mathcal{V} \to D$ such that $d_Y(\phi(V), f(x)) < 2\epsilon$ for all $V \in \mathcal{V}$ and $x \in V$. But then $f_{2\epsilon,\mathcal{V},\phi}$ exists and $d_{C(X,Y)}(f, f_{2\epsilon,\mathcal{V},\phi}) < 4\epsilon$.

To see that $C(X, Y)$ is complete, note that if $(f_n)_{n\in\mathbb{N}}$ is Cauchy, then we obtain a function $f \colon X \to Y$ by setting $f(x) = \lim_{n\to\infty} f_n(x)$. To see that f is continuous, observe that if $\epsilon > 0$ and $x \in X$, then there exists $n \in \mathbb{N}$ such that $d_{C(X,Y)}(f_m, f_n) < \epsilon$ for all $m \geq n$, thus $d_Y(f_n(x), f(x)) \leq \epsilon$ for all $x \in X$, so if U is an open neighborhood of x such that $f_n(U) \subseteq \mathcal{B}(f_n(x), \epsilon)$, then $f(U) \subseteq \mathcal{B}(f_n(x), 2\epsilon) \subseteq \mathcal{B}(f(x), 3\epsilon)$. To see that $f_n \to f$, note that if $\epsilon > 0$ and $n \in \mathbb{N}$ is sufficiently large that $d_{C(X,Y)}(f_m, f_n) < \epsilon$ for all $m \geq n$, then $d_{C(X,Y)}(f_n, f) \leq \epsilon$. ⊠

Proposition 8.2. *Suppose that X is a compact space and Y is a metric space. Then the function $\phi\colon C(X,Y) \times X \to Y$ given by $\phi(f,x) = f(x)$ is continuous.*

Proof. Given $\epsilon > 0$, $f \in C(X,Y)$, and $x \in X$, fix $0 < \delta < \epsilon$ and an open neighborhood $U \subseteq X$ of x such that $f(U) \subseteq \mathcal{B}(f(x), \delta)$, and observe that $\phi(\mathcal{B}(f, \epsilon - \delta) \times U) \subseteq \mathcal{B}(f(x), \epsilon)$. ⊠

A *code* for a partial function is a sequence $c \in C(X,Y)^{\mathbb{N}}$. The partial function $\pi_c\colon X \rightharpoonup Y$ *coded* by such a sequence is given by $\pi_c(x) = y \iff \forall^\infty n \in \mathbb{N}\ c(n)(x) = y$. We identify each partial function $\pi\colon X \rightharpoonup Y$ with the extension $\overline{\pi}\colon X \to Y \sqcup \{\emptyset\}$ given by $\overline{\pi}(x) = \emptyset$ for all $x \in {\sim}\mathrm{dom}(\phi)$.

Proposition 8.3. *Suppose that X is a zero-dimensional Polish space, Y is a metric space of cardinality at least two, μ is a finite Borel measure on X, and $\pi\colon X \rightharpoonup Y$ is a μ-measurable partial function. Then there is a code c for a partial function such that $\overline{\pi}(x) = \overline{\pi_c}(x)$ for μ-almost all $x \in X$.*

Proof. Fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive real numbers for which $\sum_{n\in\mathbb{N}} \epsilon_n < \infty$, as well as closed sets $C_n \subseteq \mathrm{dom}(\pi)$ on which π is continuous and clopen sets $U_n \subseteq X$ such that $\mu(\mathrm{dom}(\pi) \setminus C_n) \leq \epsilon_n$ and $\mu(\mathrm{dom}(\pi) \mathbin{\triangle} U_n) \leq \epsilon_n$ for all $n \in \mathbb{N}$, in which case the corresponding set $N = (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \mathrm{dom}(\pi) \setminus C_m) \cup (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \mathrm{dom}(\pi) \mathbin{\triangle} U_m)$ is μ-null. Fix continuous retractions $\pi_n\colon X \to C_n$, as well as points $y_n \in Y$ with the property that $(y_n)_{n\in\mathbb{N}}$ is not eventually constant, and let c be the code for a partial function given by $c(n) \restriction U_n = (\pi \circ \pi_n) \restriction U_n$ and $c(n) \restriction {\sim}U_n = y_n$ for all $n \in \mathbb{N}$. It only remains to observe that if $x \in {\sim}N$, then $x \in \mathrm{dom}(\pi) \implies \exists n \in \mathbb{N} \forall m \geq n\ x \in C_m \cap U_m \implies \exists n \in \mathbb{N} \forall m \geq n\ c(m)(x) = (\pi \circ \pi_m)(x) = \pi(x) \implies \overline{\pi}(x) = \overline{\pi_c}(x)$, and $x \notin \mathrm{dom}(\pi) \implies \exists n \in \mathbb{N} \forall m \geq n\ x \notin U_m \implies \exists n \in \mathbb{N} \forall m \geq n\ c(m)(x) = y_m \implies \overline{\pi}(x) = \overline{\pi_c}(x)$. ⊠

A subset of a topological space is F_σ if it is a union of countably-many closed sets.

Proposition 8.4. *Suppose that X is a compact Polish space and Y is a Polish metric space. Then the partial function $\phi\colon C(X,Y)^{\mathbb{N}} \times X \rightharpoonup Y$ given by $\phi(c,x) = \pi_c(x)$ is Borel.*

Proof. The domain of ϕ is the set of $(c,x) \in C(X,Y)^{\mathbb{N}} \times X$ for which $c(n)(x)$ is eventually constant, which is F_σ by Proposition 8.2. Similarly, the graph of ϕ is the set of $((c,x),y) \in (C(X,Y)^{\mathbb{N}} \times X) \times Y$ for which $c(n)(x)$ is eventually constant with value y, which is also F_σ by Proposition 8.2. ⊠

Proposition 8.5. *Suppose that X is a compact Polish space and Y is a Polish metric space. Then the partial function $\phi\colon C(X,Y)^{\mathbb{N}} \times P(X) \rightharpoonup P(Y)$ given by $\phi(c,\mu) = (\pi_c)_*\mu$ is Borel.*

Proof. Suppose that $B \subseteq Y$ and $C \subseteq \mathbb{R}$ are Borel. As Proposition 8.4 ensures that the set of $(c,x) \in C(X,Y)^{\mathbb{N}} \times X$ for which $x \in \pi_c^{-1}(B)$ is Borel, it follows that so too is the set of $(c,\mu) \in C(X,Y)^{\mathbb{N}} \times P(X)$ for which $\mu(\pi_c^{-1}(B)) \in C$ and $\mu(\mathrm{dom}(\pi_c)) = 1$. ⊠

A *code* for a subset of X is a code c for a partial function $\pi_c\colon X \to 2$. The set $B_c \subseteq X$ *coded* by such a sequence is the support of π_c.

9. Measure-hyper-$\mathcal{E}$-to-one homomorphisms

Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a Polish metric space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms. A *code* for a partial witness to the hyper-$\mathcal{E}$-ness of a partial equivalence relation E on a compact Polish space X is a pair $(c,d) \in (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$. The E-*scope* of such a code is the set of $x \in \mathrm{dom}(E)$ for which the partial equivalence relations $E_n = (\pi_{c(n)} \times \pi_{c(n)})^{-1}(E_{\mathcal{E}}) \upharpoonright [x]_E$ are increasing and their union is $[x]_E \times [x]_E$, the sets $B_n = B_{d(n)} \cap \mathrm{dom}(E_n)$ are E_n-complete, and each $\pi_{c(n)}$ is injective on each $(E_n \upharpoonright B_n)$-class.

Proposition 9.1. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a Polish metric space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a compact Polish space, and E is a countable Borel partial equivalence relation on X for which there is a Borel homomorphism $\phi\colon \mathrm{dom}(E) \to (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$ from E to equality such that x is in the E-scope of $\phi(x)$ for all $x \in \mathrm{dom}(E)$. Then E is hyper-$\mathcal{E}$.*

Proof. Define $(c_x,d_x) = \phi(x)$ for all $x \in \mathrm{dom}(E)$, as well as $\pi_n\colon \mathrm{dom}(E) \rightharpoonup X_{\mathcal{E}}$ by $\pi_n(x) = \pi_{c_x(n)}(x)$, $E_n = E \cap (\pi_n \times \pi_n)^{-1}(E_{\mathcal{E}})$, and $B_n = \{x \in \mathrm{dom}(E) \mid x \in B_{d_x(n)}\}$ for all $n \in \mathbb{N}$. Then $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of Borel equivalence relations whose union is E, and each π_n is a Borel homomorphism from E_n to $E_{\mathcal{E}}$. As each B_n is E_n-complete and each π_n is injective on each $(E_n \upharpoonright B_n)$-class, Proposition 4.2 ensures that each π_n is E-smooth-to-one, so each E_n is in $\mathcal{E}$, thus E is hyper-$\mathcal{E}$. ⊠

Proposition 9.2. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a Polish metric space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a compact zero-dimensional Polish space, E is a countable Borel partial equivalence relation* *on X, and μ is an E-hyper-$\mathcal{E}$ finite Borel measure on X. Then there is a code for a partial witness to the hyper-$\mathcal{E}$-ness of E whose E-scope is μ-conull.*

Proof. Fix a μ-conull Borel set $C \subseteq X$ such that $E \restriction C$ is hyper-$\mathcal{E}$, an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of equivalence relations in $\mathcal{E}$ whose union is $E \restriction C$, and smooth-to-one Borel homomorphisms $\pi_n \colon \mathrm{dom}(E_n) \to X_{\mathcal{E}}$ from E_n to $E_{\mathcal{E}}$ for all $n \in \mathbb{N}$. By the Lusin-Novikov uniformization theorem, there is a Borel function $\pi \colon [C]_E \to C$ whose graph is contained in E. By replacing C with $[C]_E$, E_n with $(\pi\times\pi)^{-1}(E_n)$, and π_n with $\pi_n\circ\pi$, we can assume that C is E-invariant. Fix an E-quasi-invariant finite Borel measure ν such that $\mu \ll \nu$ and the two measures agree on every E-invariant Borel set. By Proposition 4.2, there are E_n-complete Borel sets $B_n \subseteq \mathrm{dom}(E_n)$ such that π_n is injective on each $(E_n \restriction B_n)$-class for all $n \in \mathbb{N}$, and by Proposition 8.3, there exists $(c,d) \in (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$ for which the set $D = \{x \in C \mid \forall n \in \mathbb{N}\ (\overline{\pi_n}(x) = \overline{\pi_{c(n)}}(x) \text{ and } (x \in B_n \iff x \in B_{d(n)}))\}$ is ν-conull. As ν is E-quasi-invariant, the set $\sim[\sim D]_E$ is ν-conull, thus μ-conull. But $\sim[\sim D]_E$ is contained in the E-scope of (c,d). ⊠

Proposition 9.3. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a standard Borel space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a standard Borel space, and E is a countable Borel equivalence relation on X. Then the set of E-hyper-$\mathcal{E}$ Borel probability measures is analytic.*

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that X is a compact zero-dimensional Polish space. We can clearly assume that $X_{\mathcal{E}}$ is a Polish metric space. As the set R of $((c,d),x) \in ((C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}) \times X$ for which x is in the E-scope of (c,d) is Borel, so too is the set S of $(\mu,(c,d)) \in P(X) \times ((C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}})$ for which $\mu(R_{(c,d)}) = 1$. But if μ is a finite Borel measure on X, then the special case of Proposition 9.1 for constant homomorphisms ensures that if $\mu \in \mathrm{proj}_{P(X)}(S)$ then E is μ-hyper-$\mathcal{E}$, and conversely, Proposition 9.2 implies that if E is μ-hyper-$\mathcal{E}$ then $\mu \in \mathrm{proj}_{P(X)}(S)$. ⊠

A *partial witness* to the E-hyper-$\mathcal{E}$-to-one-ness of a partial function $\phi \colon X \rightharpoonup Y$ is a partial function $\pi \colon Y \rightharpoonup (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$. The *scope* of such a partial witness is the set of $x \in \mathrm{dom}(\phi)$ for which $\phi(x) \in \mathrm{dom}(\pi)$ and x is in the $(E \restriction \phi^{-1}(\{\phi(x)\}))$-scope of $(\pi\circ\phi)(x)$.

A *disintegration* of a Borel probability measure μ on X through a Borel function $\phi \colon X \to Y$ is a function $\psi \colon Y \to P(X)$ with the property that $\phi^{-1}(\{y\})$ is $\psi(y)$-conull for $(\phi_*\mu)$-almost all $y \in Y$, and $\mu(B) = \int \psi(y)(B) \; d\phi_*\mu(y)$ for all Borel sets $B \subseteq X$.

Proposition 9.4. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a Polish metric space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a compact zero-dimensional Polish space, Y is a standard Borel space, E is a countable Borel equivalence relation on X, μ is a Borel probability measure on X, $\phi\colon X \rightharpoonup Y$ is a Borel partial function whose domain is μ-conull, and there is a Borel disintegration $\psi\colon Y \to P(X)$ of μ through ϕ such that $E \restriction \phi^{-1}(\{y\})$ is $\psi(y)$-hyper-$\mathcal{E}$ for $(\phi_*\mu)$-almost all $y \in Y$. Then there is a Borel partial witness to the E-hyper-$\mathcal{E}$-to-one-ness of ϕ whose scope is μ-conull.*

Proof. As the set R of $((c,d),x) \in ((C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}) \times \operatorname{dom}(\phi)$ for which x is in the $(E \restriction \phi^{-1}(\{\phi(x)\}))$-scope of (c,d) is Borel, so too is the set S of $(y,(c,d)) \in Y \times ((C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}})$ for which $\psi(y)(R_{(c,d)}) = 1$, thus the Jankov-von Neumann uniformization theorem yields a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization $\pi\colon \operatorname{proj}_Y(S) \to (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$ of S. As Proposition 9.2 ensures that $\operatorname{proj}_Y(S)$ is $(\phi_*\mu)$-conull, there is a $(\phi_*\mu)$-conull Borel set $D \subseteq \operatorname{dom}(\pi)$ on which π is Borel. Let C be the set of $x \in \phi^{-1}(D)$ in the $E \restriction \phi^{-1}(\{\phi(x)\})$-scope of $(\pi \circ \phi)(x)$. Then $\mu(C) = \int \psi(y)(C) \; d\phi_*\mu(y) = 1$, so $\pi \restriction D$ is a Borel partial witness to the E-hyper-$\mathcal{E}$-to-one-ness of ϕ whose scope is μ-conull. ⊠

Proposition 9.5. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a standard Borel space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0,\infty)$ is a Borel cocycle for which every E-ergodic ρ-invariant Borel probability measure is E-hyper-$\mathcal{E}$. Then so too is every ρ-invariant Borel probability measure.*

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that X is a compact zero-dimensional Polish space. We can clearly assume that $X_{\mathcal{E}}$ is a Polish metric space. Given a ρ-invariant Borel probability measure μ, fix an E-invariant Borel function $\phi\colon X \to P(X)$ that is a *decomposition* of μ into E-ergodic ρ-invariant Borel probability measures, in the sense that $\phi(x)$ is E-ergodic and ρ-invariant for all $x \in X$, $\phi^{-1}(\{\nu\})$ is ν-conull for all $\nu \in \phi(X)$, and $\mu(B) = \int \phi(x)(B) \; d\mu(x)$ for all Borel sets $B \subseteq X$. As the identity function on $P(X)$ is a disintegration of μ through ϕ, Proposition 9.4 yields a Borel partial witness $\pi\colon P(X) \rightharpoonup (C(X,X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X,2)^{\mathbb{N}})^{\mathbb{N}}$

to the E-hyper-$\mathcal{E}$-to-one-ness of ϕ whose scope $C \subseteq X$ is μ-conull, and since $(\pi \circ \phi) \restriction C$ is a Borel homomorphism from $E \restriction C$ to equality such that x is in the E-scope of $(\pi \circ \phi)(x)$ for all $x \in C$, Proposition 9.1 ensures that $E \restriction C$ is hyper-$\mathcal{E}$, thus μ is E-hyper-$\mathcal{E}$. ⊠

Given any class $\mathcal{E}$ of countable Borel equivalence relations on standard Borel spaces, we say that a countable Borel equivalence relation on a standard Borel space X is *measure-$\mathcal{E}$* if it is μ-$\mathcal{E}$ for all Borel probability measures μ on X.

QUESTION 9.6. Is a countable Borel equivalence relation hyperfinite if and only if it is measure hyperfinite?

PROPOSITION 9.7. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a standard Borel space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X is a standard Borel space, E is a countable Borel equivalence relation on X, and there is an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from E to a measure-hyperfinite countable Borel equivalence relation on a standard Borel space. Then E is measure-hyper-$\mathcal{E}$.*

PROOF. We will first show that if there is an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism $\phi\colon X \to Y$ from E to equality on a standard Borel space, then E is measure-hyper-$\mathcal{E}$. By the isomorphism theorem for standard Borel spaces, we can assume that X and Y are compact zero-dimensional Polish spaces. Clearly we can assume that $X_{\mathcal{E}}$ is a Polish metric space. But given any Borel probability measure μ on X, Proposition 9.4 yields a Borel partial witness π to the E-hyper-$\mathcal{E}$-to-one-ness of ϕ whose scope $C \subseteq X$ is μ-conull, in which case $(\pi \circ \phi) \restriction C$ is a Borel homomorphism from $E \restriction C$ to equality with the property that x is in the E-scope of $(\pi \circ \phi)(x)$ for all $x \in C$, thus Proposition 9.1 ensures that $E \restriction C$ is hyper-$\mathcal{E}$.

Suppose now that Y is a standard Borel space, F is a measure-hyperfinite countable Borel equivalence relation on Y, and $\phi\colon X \to Y$ is an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from E to F. Given a Borel probability measure μ on X, fix a $(\phi_*\mu)$-conull Borel set $D \subseteq Y$ on which F is hyperfinite, as well as an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite Borel equivalence relations whose union is $F \restriction D$. Then the Borel set $C = \phi^{-1}(D)$ is μ-conull, and for all $n \in \mathbb{N}$, the function $\phi \restriction C$ is an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from the equivalence relation $E_n = (E \cap (\phi \times \phi)^{-1}(F_n)) \restriction C$ to F_n, so the previous paragraph ensures that E_n is μ-hyper-$\mathcal{E}$. As $E \restriction C = \bigcup_{n\in\mathbb{N}} E_n$, Proposition 3.3 implies that E is μ-hyper-$\mathcal{E}$. ⊠

A *code* for an E-hyper-$\mathcal{E}$-to-one partial homomorphism from an equivalence relation E on X to a partial equivalence relation F on Y is a pair $(c, d) \in C(X, Y)^{\mathbb{N}} \times C(Y, (C(X, X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X, 2)^{\mathbb{N}})^{\mathbb{N}})^{\mathbb{N}}$. The *scope* of such a code (c, d) is the set of all $x \in X$ with the property that $[x]_E \subseteq \mathrm{dom}(\pi_c)$, $\pi_c([x]_E) \subseteq \mathrm{dom}(\pi_d) \cap \mathrm{dom}(F) \cap [\pi_c(x)]_F$, and y is in the $E \restriction \pi_c^{-1}(\{\pi_c(y)\})$-scope of $(\pi_d \circ \pi_c)(y)$ for all $y \in [x]_E$.

Proposition 9.8. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a Polish metric space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, X and Y are compact zero-dimensional Polish spaces, $D \subseteq Y$ is a Borel set, E and F are countable Borel equivalence relations on X and Y, and μ is a finite Borel measure on X. Then the following are equivalent:*

1. *There exists a code (c, d) for an E-hyper-$\mathcal{E}$-to-one partial homomorphism from E to $F \restriction D$ whose scope is μ-conull.*
2. *There exist a μ-conull Borel set $C \subseteq X$ and an E-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D$.*
3. *There exist a μ-conull Borel set $C \subseteq X$ and an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D$.*

Proof. To see (1) $\implies$ (2), note that if (c, d) is a code for an E-hyper-$\mathcal{E}$-to-one partial homomorphism from E to $F \restriction D$ with scope $C \subseteq X$, then $\pi_c \restriction C$ is an E-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D$. As (2) $\implies$ (3) is clear, it only remains to establish (3) $\implies$ (1). Towards this end, suppose that there is a μ-conull Borel set $C \subseteq X$ for which there is an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism $\phi \colon C \to D$ from $E \restriction C$ to $F \restriction D$. By the Lusin-Novikov uniformization theorem, there is a Borel function $\psi \colon [C]_E \to C$ whose graph is contained in E. By replacing C with $[C]_E$ and ϕ with $\phi \circ \psi$, we can assume that C is E-invariant. Fix an E-quasi-invariant finite Borel measure ν such that $\mu \ll \nu$ and the two measures agree on every E-invariant Borel set. By Proposition 9.4, there is a Borel partial witness $\pi \colon Y \rightharpoonup (C(X, X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X, 2)^{\mathbb{N}})^{\mathbb{N}}$ to the E-hyper-$\mathcal{E}$-to-one-ness of ϕ whose scope is ν-conull. By Proposition 8.3, there are codes c and d for partial functions $\pi_c \colon X \rightharpoonup Y$ and $\pi_d \colon Y \rightharpoonup (C(X, X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X, 2)^{\mathbb{N}})^{\mathbb{N}}$ such that $\phi(x) = \pi_c(x)$ and $(\pi \circ \phi)(x) = (\pi_d \circ \phi)(x)$ for ν-almost all $x \in X$. Then the E-quasi-invariance of ν ensures that (c, d) is a code for an E-hyper-$\mathcal{E}$-to-one partial homomorphism from E to $F \restriction D$ whose scope is ν-conull, and therefore μ-conull. ⊠

Proposition 9.9. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation $E_{\mathcal{E}}$ on a standard Borel space $X_{\mathcal{E}}$ under smooth-to-one Borel homomorphisms, I, X, and Y are standard*

Borel spaces, $(D_i)_{i \in I}$ is a Borel sequence of subsets of Y, and E and F are countable Borel equivalence relations on X and Y. Then the set of $(\mu, i) \in P(X) \times I$ for which there exist a μ-conull Borel set $C \subseteq X$ and an E-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D_i$ is analytic and coincides with the set of $(\mu, i) \in P(X) \times I$ for which there exist a μ-conull Borel set $C \subseteq X$ and an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D_i$.

Proof. By the isomorphism theorem for standard Borel spaces, we can assume that X and Y are compact zero-dimensional Polish spaces. Clearly we can assume that $X_{\mathcal{E}}$ and Y are Polish metric spaces. As the set R of $((c, d, i), x) \in (C(X, Y)^{\mathbb{N}} \times C(Y, (C(X, X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X, 2)^{\mathbb{N}})^{\mathbb{N}})^{\mathbb{N}} \times I) \times X$ for which x is in the D_i-scope of (c, d) is Borel, so too is the set S of $((\mu, i), (c, d)) \in (P(X) \times I) \times (C(X, Y)^{\mathbb{N}} \times C(Y, (C(X, X_{\mathcal{E}})^{\mathbb{N}})^{\mathbb{N}} \times (C(X, 2)^{\mathbb{N}})^{\mathbb{N}})^{\mathbb{N}})$ for which $\mu(R_{(c,d,i)}) = 1$. But Proposition 9.8 ensures that $(\mu, i) \in \mathrm{proj}_{P(X)}(S)$ if and only if there exist a μ-conull Borel set $C \subseteq X$ and an E-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D_i$ if and only if there exist a μ-conull Borel set $C \subseteq X$ and an E-measure-hyper-$\mathcal{E}$-to-one Borel homomorphism from $E \restriction C$ to $F \restriction D_i$. ⊠

10. Productive hyperfiniteness

Suppose that Γ is a countable group. We say that a Borel action of Γ on a standard Borel space is *hyperfinite* if the induced orbit equivalence relation is hyperfinite. We say that Γ is *hyperfinite* if every Borel action of Γ on a standard Borel space is hyperfinite.

The *diagonal product* of actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ is the action $\Gamma \curvearrowright X \times Y$ given by $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$. We say that a Borel action of Γ on a standard Borel space is *productively hyperfinite* if its diagonal product with every Borel action of Γ on a standard Borel space is hyperfinite.

Proposition 10.1. *Suppose that Γ is a countable group, X is a standard Borel space, and $\Gamma \curvearrowright X$ is a hyperfinite Borel action such that the stabilizer of every point is hyperfinite and only countably-many points have infinite stabilizers. Then $\Gamma \curvearrowright X$ is productively hyperfinite.*

Proof. Let C be the set of $x \in X$ whose stabilizers are infinite, fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations whose union is E_Γ^X, and suppose that Y is a standard Borel space and $\Gamma \curvearrowright Y$ is a Borel action. As each $E_\Gamma^{X \times Y} \restriction (\{x\} \times Y)$ is generated by the stabilizer of x, and therefore hyperfinite, we need only show that $E_\Gamma^{(\sim C) \times Y}$ is hyperfinite. But if F_n is the subequivalence relation with respect to which two $E_\Gamma^{(\sim C)\times Y}$-equivalent pairs (x,y) and (x',y') are related exactly when $x \mathrel{E_n} x'$ for all $n \in \mathbb{N}$, then each F_n is finite and their union is $E_\Gamma^{(\sim C)\times Y}$. ⊠

11. Actions of $\mathrm{SL}_2(\mathbb{Z})$

Define $\sim$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ by $v \sim w \iff \exists r > 0 \; rv = w$, set $\mathbb{T} = (\mathbb{R}^2 \setminus \{(0,0)\})/{\sim}$, and define $\mathrm{proj}_\mathbb{T} \colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{T}$ by setting $\mathrm{proj}_\mathbb{T}(v) = [v]_\sim$. Note that if $A \in \mathrm{GL}_2(\mathbb{Z})$, $r > 0$, and $v \in \mathbb{R}^2 \setminus \{(0,0)\}$, then $A(rv) = r(Av)$, so the usual action $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2 \setminus \{(0,0)\}$ by matrix multiplication factors over $\sim$ to an action $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Proposition 11.1 (Jackson-Kechris-Louveau). *The action* $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ *is hyperfinite.*

Proof. Define an action $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{R} \cup \{\infty\}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot x = \frac{ax+b}{cx+d}$ (where $\frac{a\infty+b}{c\infty+d} = \frac{a}{c}$), let $\phi \colon \mathbb{R} \setminus \mathbb{Q} \to \mathbb{Z}^\mathbb{N}$ be the function sending each irrational number to its continued fraction expansion, and recall that the *unilateral shift* on $\mathbb{Z}^\mathbb{N}$ is the function $s \colon \mathbb{Z}^\mathbb{N} \to \mathbb{Z}^\mathbb{N}$ given by $s(x)(n) = x(n+1)$. It is well-known that if $x, y \in \mathbb{R} \setminus \mathbb{Q}$, then $x \mathrel{E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\cup\{\infty\}}} y \iff \phi(x) \mathrel{E_t(s)} \phi(y)$ (see, for example, Theorem 175 of *The Theory of Numbers* by Hardy-Wright). As $E_t(s)$ is hyperfinite, so too is $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\cup\{\infty\}}$.

As the set $X = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0 \text{ and } (y = 0 \implies x > 0)\}$ is $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}^2\setminus\{(0,0)\}}$-complete, we need only show that $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{T}} \restriction \mathrm{proj}_\mathbb{T}(X)$ is hyperfinite. Define $\pi \colon X \to \mathbb{R} \cup \{\infty\}$ by $\pi(x,y) = x/y$, and note that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \pi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \frac{a(x/y)+b}{c(x/y)+d} = \frac{ax+by}{cx+dy} = \pi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right))$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $(x,y) \in X$, thus π induces an embedding of $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{T}} \restriction \mathrm{proj}_\mathbb{T}(X)$ into $E_{\mathrm{GL}_2(\mathbb{Z})}^{\mathbb{R}\cup\{\infty\}}$. ⊠

Proposition 11.2 (Conley-Miller). *The action* $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ *is productively hyperfinite.*

Proof. Note that if $\theta \in \mathbb{T}$ has a non-trivial stabilizer, then it is the equivalence class of an eigenvector of a non-trivial matrix in $\mathrm{SL}_2(\mathbb{Z})$ whose corresponding eigenvector is positive. As $\mathrm{SL}_2(\mathbb{Z})$ is countable and every such matrix admits at most two such classes of eigenvectors, there are only countably-many such θ. By Propositions 10.1 and 11.1, it only remains to show that the stabilizer of each $\theta \in \mathbb{T}$ is cyclic.

We first consider the case that $\theta \cap \mathbb{Z}^2 \neq \emptyset$. Let v denote the unique element of $\theta \cap \mathbb{Z}^2$ of minimal length. Note that the stabilizers of θ and v coincide, for if A is in the stabilizer of θ, then v is an eigenvector of A, so minimality ensures that $Av = v$. Minimality also ensures that the coordinates of v are relatively prime, so there exists $a \in \mathbb{Z}^2$ such that $a \cdot v = 1$, in which case the matrix $B = \left(\begin{smallmatrix} a_0 & a_1 \\ -v_1 & v_0 \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$ and $Bv = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, thus conjugation by B yields an isomorphism of the stabilizer of v with that of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. But if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \iff \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $n \in \mathbb{Z}$, thus the group of such matrices is cyclic.

It remains to consider the case that $\theta \cap \mathbb{Z}^2 = \emptyset$.

Lemma 11.3. *The stabilizer of each $v = (x, y)$ in θ is trivial.*

Proof. Suppose, towards a contradiction, that there is a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z}) \setminus \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ such that such that $Av = v$. Then $(a-1)x + by = cx + (d-1)y = 0$, so there exists $(a', b') \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $a'x + b'y = 0$. As $\theta \cap \mathbb{Z}^2 \neq \emptyset$, it follows that neither x nor y is zero, so neither a' nor b' is zero, thus $y = -(a'/b')x$, in which case there exist $i, j \in \{\pm 1\}$ for which $(ib', ja') \in \theta$, the desired contradiction. ⊠

Note that the set Λ of eigenvalues of matrices in the stabilizer of θ is a group under multiplication.

Lemma 11.4. *The group Λ is cyclic.*

Proof. It is sufficient to show that 1 is isolated in $\Lambda \cap [1, \infty)$. Towards this end, suppose that A is in the stabilizer of θ and v is an eigenvector of A with eigenvalue $\lambda > 1$. If μ is the other eigenvalue of A, then $\lambda\mu = \det(A) = 1$, so $\mathrm{tr}(A) = \lambda + \mu = \lambda + 1/\lambda$. As $\mathrm{tr}(A) \in \mathbb{Z}$, it follows that $\lambda + 1/\lambda = n$ for some $n \geq 2$, in which case $\lambda = (n + \sqrt{n^2 - 4})/2$. The fact that $\lambda > 1$ therefore ensures that $n \neq 2$, thus $\lambda \geq (3 + \sqrt{5})/2$. ⊠

By Lemma 11.4, there is a matrix A in the stabilizer of θ which has an eigenvalue λ generating Λ. If B is any matrix in the stabilizer of θ, then there exists $n \in \mathbb{Z}$ for which v is an eigenvector of B with eigenvalue λ^n, in which case $A^n B^{-1}$ is in the stabilizer of v, so $B = A^n$, thus A generates the stabilizer of θ, hence the latter is cyclic. ⊠

Let $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ denote the group of all functions $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = Ax + b$ (under composition), where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$, and define $\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \colon \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ by $\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}(Ax + b) = A$. Set $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, let $\mathrm{proj}_{\mathbb{T}^2}$ denote the projection from $\mathbb{R}^2$ to $\mathbb{T}^2$, and let $\mathbb{m}^2$ denote the usual Lebesgue probability measure on $\mathbb{T}^2$. Note that if $A \in \mathrm{SL}_2(\mathbb{Z})$, $b \in \mathbb{Z}^2$, $v \in \mathbb{R}^2$, and $w \in \mathbb{Z}^2$, then $A(v + w) + b = Av + (Aw + b)$, so $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ factors to an action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$.

Proposition 11.5. *There is an $\mathbb{m}^2$-treeable Borel subequivalence relation E of $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}^2}$ that is not $\mathbb{m}^2$-hyperfinite.*

Proof. We first note that the free part of the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ is $\mathbb{m}^2$-conull.

Lemma 11.6. *The non-free part of* $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ *is contained in the* $E^{\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$*-saturation of* $\mathrm{proj}_{\mathbb{T}^2}(\mathbb{Q} \times \mathbb{R})$.

Proof. If $\mathrm{proj}_{\mathbb{T}^2}(x, y)$ is in the non-free part, then there exists $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \setminus \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ for which $((a-1)x + by, cx + (d-1)y) \in \mathbb{Z}^2$, so there exists $(a', b') \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $a'x + b'y \in \mathbb{Z}$. If either a' or b' is zero, then y or x is rational, so $\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is in $\mathbb{Q} \times \mathbb{R}$. Otherwise, there are relatively prime $a'', b'' \in \mathbb{Z}$ such that $a''x + b''y \in \mathbb{Q}$, in which case there are $c'', d'' \in \mathbb{Z}$ such that $a''d'' - b''c'' = 1$, thus $\left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\left(\begin{smallmatrix} a'' & b'' \\ c'' & d'' \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{Q} \times \mathbb{R}$. ⊠

We next observe that $\mathrm{SL}_2(\mathbb{Z})$ contains a copy F_2 of the free group on two generators.

Lemma 11.7. *The group generated by the matrices* $A = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $B = \left(\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$ *is free.*

Proof. Note that if $n \neq 0$, $x, y \in \mathbb{R}$, $\left(\begin{smallmatrix} x_A \\ y_A \end{smallmatrix}\right) = A^n \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+3ny \\ y \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} x_B \\ y_B \end{smallmatrix}\right) = B^n \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ 3nx+y \end{smallmatrix}\right)$, then

$$|x| < |y| \Longrightarrow |x_A| > (3|n|-1)|y| \geq 2|y| \Longrightarrow |x_A| - |y_A| > |y| - |x|$$

and

$$|y| < |x| \Longrightarrow |y_B| > (3|n|-1)|x| \geq 2|x| \Longrightarrow |y_B| - |x_B| > |x| - |y|.$$

A straightforward induction therefore ensures that if W is a non-trivial reduced word in A and B, $|x| < |y|$ if and only if the rightmost entry of W is a power of A, and $\left(\begin{smallmatrix} x_W \\ y_W \end{smallmatrix}\right) = W \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, then $||x_W| - |y_W|| > ||x| - |y||$, so $\left(\begin{smallmatrix} x_W \\ y_W \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, thus $W \neq \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. ⊠

Note that the push-forward G of the Cayley graph of F_2 through $F_2 \curvearrowright \mathbb{T}^2$ is acyclic on the free part $B \subseteq X$ of $F_2 \curvearrowright \mathbb{T}^2$, so $E^B_{F_2}$ is treeable. Moreover, as $C_{\mathbb{m}^2}(G) = 2$, Proposition 7.5 ensures that $E^{\mathbb{T}^2}_{F_2}$ is not $\mathbb{m}^2$-hyperfinite. ⊠

Remark 11.8. Jackson-Kechris-Louveau have shown that $E^{\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$ is itself treeable, but we will not need this stronger result.

12. Projective rigidity

Given sets X and Y, a binary relation R on X, a countable group Δ, an action $\Delta \curvearrowright Y$, and a function $\rho \colon R \to \Delta$, we say that a function $\phi \colon X \to Y$ is *ρ-invariant* if $x_1 \mathrel{R} x_2 \Longrightarrow \phi(x_1) = \rho(x_1, x_2) \cdot \phi(x_2)$ for all $x_1, x_2 \in X$. Given a class $\mathcal{E}$ of countable Borel equivalence relations on standard Borel spaces, we say that a Borel action $\Delta \curvearrowright Y$ is *projectively $\mathcal{E}$-rigid* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to \Delta$ is a Borel function, $\phi, \psi \colon X \to Y$ are ρ-invariant Borel functions, and ϕ is E-$\mathcal{E}$-to-one, the *difference set* $D(\phi, \psi) = \{x \in X \mid \phi(x) \neq \psi(x)\}$ is E-$\mathcal{E}$.

Theorem 12.1 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms. Then $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is projectively measure-hyper-$\mathcal{E}$ rigid.*

Proof. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, $\phi, \psi \colon X \to \mathbb{R}^2$ are ρ-invariant Borel functions, and ϕ is E-measure-hyper-$\mathcal{E}$-to-one, and define functions $\pi \colon D(\phi, \psi) \to \mathbb{T}$ and $\sigma \colon E \restriction D(\phi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\phi(x) - \psi(x))$ and $\sigma(x_1, x_2) = (\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \circ \rho)(x_1, x_2)$.

Lemma 12.2. *The function π is σ-invariant.*

Proof. Simply observe that if $x_1, x_2 \in D(\phi, \psi)$ are E-related, then

$$\begin{aligned}
\pi(x_1) &= \mathrm{proj}_{\mathbb{T}}(\phi(x_1) - \psi(x_1)) \\
&= \mathrm{proj}_{\mathbb{T}}(\rho(x_1, x_2) \cdot \phi(x_2) - \rho(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot \phi(x_2) - \sigma(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot (\phi(x_2) - \psi(x_2))) \\
&= \sigma(x_1, x_2) \cdot \mathrm{proj}_{\mathbb{T}}(\phi(x_2) - \psi(x_2)) \\
&= \sigma(x_1, x_2) \cdot \pi(x_2),
\end{aligned}$$

thus π is σ-invariant. ⊠

As $(\mathrm{proj}_{\mathbb{T}^2} \circ \phi) \restriction D(\phi, \psi)$ is also σ-invariant, it follows that $\pi \times (\mathrm{proj}_{\mathbb{T}^2} \circ \phi) \restriction D(\phi, \psi)$ is a measure-hyper-$\mathcal{E}$-to-one homomorphism from $E \restriction D(\phi, \psi)$ to the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T} \times \mathbb{T}^2$. As Proposition 11.2 ensures that the latter relation is hyperfinite, Proposition 9.7 implies that the former is measure-hyper-$\mathcal{E}$. ⊠

Question 12.3. Is there a more combinatorial way of producing projectively-measure-hyper-$\mathcal{E}$-rigid Borel actions?

13. Projective separability and products

Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces. A *μ-homomorphism* from E to F is a Borel homomorphism from $E \restriction C$ to F, where $C \subseteq X$ is a μ-conull Borel set.

We say that a countable Borel equivalence relation F on a standard Borel space is *projectively $\mathcal{E}$-separable* if for every standard Borel space X, countable Borel equivalence relation E on X, and E-quasi-invariant non-E-$\mathcal{E}$ finite Borel measure μ on X, there is a countable set Φ of E-$\mathcal{E}$-to-one μ-homomorphisms from E to F such that every E-$\mathcal{E}$-to-one μ-homomorphism from E to F agrees with a function in Φ on a set of positive μ-measure.

THEOREM 13.1 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, Δ is a countable group, Y is a standard Borel space, and $\Delta \curvearrowright Y$ is a projectively-measure-hyper-$\mathcal{E}$-rigid Borel action. Then E_Δ^Y is projectively measure-hyper-$\mathcal{E}$-separable.*

PROOF. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant non-E-hyper-$\mathcal{E}$ finite Borel measure on X. Clearly we can assume that X is a Polish space. Fix a countable basis $\mathcal{U}$ for X closed under finite unions, as well as a countable group Γ of Borel automorphisms of X generating E. By Proposition 3.3, there is a finite set $S \subseteq \Gamma$ for which the equivalence relation $E' = E_{\langle S \rangle}^X$ is non-μ-hyper-$\mathcal{E}$, and therefore non-μ-hyper-hyper-$\mathcal{E}$. For each Borel set $B \subseteq X$, let E_B denote the equivalence relation on X generated by the set $R_B = \bigcup_{\gamma \in S} \operatorname{graph}(\gamma \restriction B)$.

LEMMA 13.2. *There exists $\epsilon > 0$ such that E_B is non-μ-hyper-$\mathcal{E}$ for all Borel sets $B \subseteq X$ of μ-measure at least $\mu(X) - \epsilon$.*

PROOF. Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and suppose, towards a contradiction, that there are Borel sets $B_n \subseteq X$ of μ-measure at least $\mu(X) - \epsilon_n$ with the property that E_{B_n} is μ-hyper-$\mathcal{E}$ for all $n \in \mathbb{N}$. Setting $C_n = \bigcap_{m \geq n} B_m$ for all $n \in \mathbb{N}$, it follows that $\mu(C_n) \to \mu(X)$. As μ is E'-quasi-invariant, the E'-invariant Borel set $C = \sim[\sim\bigcup_{n \in \mathbb{N}} C_n]_{E'}$ is μ-conull. But $(E_{C_n} \restriction C)_{n \in \mathbb{N}}$ is an increasing sequence of μ-hyper-$\mathcal{E}$ countable Borel equivalence relations whose union is $E' \restriction C$, contradicting the fact that E' is non-μ-hyper-hyper-$\mathcal{E}$. ⊠

Observe that if $\phi \colon X \rightharpoonup Y$ is a μ-homomorphism from E to E_Δ^Y, then there is a finite set $T \subseteq \Delta$ for which the set $B_{\phi,T}$ of all $x \in \bigcap_{\gamma \in \langle S \rangle} \gamma^{-1}(\operatorname{dom}(\phi))$ such that $\forall \gamma \in S \exists \delta \in T \ \phi(x) = \delta \cdot \phi(\gamma \cdot x)$ has μ-measure strictly greater than $\mu(X) - \epsilon/2$, as well as a function $U \colon T^S \to \mathcal{U}$ for which the set $B_{\phi,T,U}$ of all $x \in B_{\phi,T}$ such that $x \in U(f) \iff \forall \gamma \in S \ \phi(x) = f(\gamma) \cdot \phi(\gamma \cdot x)$ for all $f \in T^S$ has μ-measure at least $\mu(X) - \epsilon/2$. Now suppose that $\psi \colon X \rightharpoonup Y$ is another μ-homomorphism from E to E_Δ^Y for which the corresponding set $B_{\psi,T,U}$ has μ-measure at least $\mu(X) - \epsilon/2$, so that the set $B = B_{\phi,T,U} \cap B_{\psi,T,U}$ has μ-measure at least $\mu(X) - \epsilon$. Fix linear orderings of S and T^S, and observe that both ϕ and ψ are invariant with respect to the function $\sigma \colon R_B \to \Delta$ given by $\sigma(x, y) = f(\gamma)$, where f is the least element of T^S such that $x \in U(f)$, and γ is the least element of S such that $\gamma \cdot x = y$. Let $\overline{\sigma}$ be the extension of σ to $R_B^{\pm 1}$ given by $\overline{\sigma}(x, y) = \sigma(x, y)^{-1}$ for all $(x, y) \in R_B^{-1} \setminus R_B$, appeal to the Lusin-Novikov uniformization theorem to obtain a Borel function $\theta \colon E_B \to X^{<\mathbb{N}}$ sending each pair $(x, y) \in E_B$ to an R_B-path from x to y, and observe that both ϕ and ψ are invariant with respect to the function $\rho \colon E_B \to \Delta$ given by $\rho(x, y) = \prod_{n<|\gamma(x,y)|-1} \overline{\sigma}(\theta_n(x, y), \theta_{n+1}(x, y))$, so if ϕ is E-measure-hyper-$\mathcal{E}$-to-one, then $D(\phi \restriction B, \psi \restriction B)$ is not $(\mu \restriction B)$-conull. But there are only countably-many possibilities for T and U. ⊠

Proposition 13.3 (Conley-Miller). *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces such that the family of Borel sets on which any equivalence relation is in $\mathcal{E}$ is closed under countable unions. Then the projectively $\mathcal{E}$-separable countable Borel equivalence relations on standard Borel spaces are closed downward under countable-to-one Borel homomorphisms.*

Proof. Suppose that Y and Y' are standard Borel spaces, F and F' are countable Borel equivalence relations on Y and Y', F' is projectively $\mathcal{E}$-separable, and there is a countable-to-one Borel homomorphism $\psi \colon Y \to Y'$ from F to F'. By the Lusin-Novikov uniformization theorem, there is a countable set Φ of Borel functions $\phi \colon \psi(Y) \to Y$ such that $\mathrm{graph}(\psi)^{-1} = \bigcup_{\phi \in \Phi} \mathrm{graph}(\phi)$. Given a standard Borel space X, a countable Borel equivalence relation E on X, and an E-quasi-invariant non-E-$\mathcal{E}$ finite Borel measure μ on X, fix a countable set Φ' of E-$\mathcal{E}$-to-one μ-homomorphisms from E to F' such that every E-$\mathcal{E}$-to-one μ-homomorphism from E to F' agrees with a function in Φ' on a set of positive μ-measure. Then every E-$\mathcal{E}$-to-one μ-homomorphism from E to F agrees with a function of the form $\phi \circ \phi'$, where $\phi \in \Phi$ and $\phi' \in \Phi'$, on a set of positive μ-measure. ⊠

Remark 13.4 (Conley-Miller). If E is a non-measure-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space, then $E \times \Delta(\mathbb{R})$ is not projectively $\mathcal{E}$-separable. It follows that if E is projectively measure-$\mathcal{E}$-separable, then there is no countable-to-one Borel homomorphism from $E \times \Delta(\mathbb{R})$ to E.

Remark 13.5 (Conley-Miller). We say that E is *$\mathcal{E}$-to-one measure homomorphible* to F if there is an $\mathcal{E}$-to-one μ-homomorphism from E to F for every Borel probability measure μ on X. Under the above assumptions, it is not difficult to see that if ν is a continuous finite Borel measure on $\mathbb{R}$ and $B \subseteq X \times \mathbb{R}$ is a $(\mu \times \nu)$-positive Borel set, then $(E \times \Delta(\mathbb{R})) \restriction B$ is not projectively $\mathcal{E}$-separable, so there is no countable-to-one Borel homomorphism from $(E \times \Delta(\mathbb{R})) \restriction B$ to E, thus $E \times \Delta(\mathbb{R})$ is not countable-to-one measure homomorphible to F.

Remark 13.6 (Conley-Miller). If $\mathcal{F}$ is a class of countable Borel equivalence relations on standard Borel spaces that is closed downward under smooth-to-one Borel homomorphisms, then again under the above assumptions, E cannot be a maximal element of $\mathcal{F}$ under any quasi-order between countable-to-one measure homomorphibility and continuous embeddability.

14. Measures and products

Let $\ll^{\mathcal{E}}_{E,F}$ denote the set of $(\mu, \nu) \in P(X) \times P(Y)$ for which μ is E-ergodic and E-quasi-invariant, ν is F-ergodic and F-quasi-invariant, and there is an E-$\mathcal{E}$-to-one μ-homomorphism $\phi \colon X \rightharpoonup Y$ from E to F such that $\phi_* \mu \ll \nu$.

Proposition 14.1 (Conley-Miller). *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces, X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, μ is an E-ergodic E-quasi-invariant non-E-$\mathcal{E}$ Borel probability measure on X, and F is projectively $\mathcal{E}$-separable. Then the μ^{th} vertical section of $\ll^{\mathcal{E}}_{E,F}$ is a union of countably-many measure-equivalence classes.*

Proof. As any two F-ergodic F-quasi-invariant Borel measures are either equivalent or orthogonal, it follows that any non-zero Borel measure on Y is absolutely continuous with respect to at most one such measure. As F is projectively $\mathcal{E}$-separable, it is therefore sufficient to show that if $C \subseteq X$ is a μ-conull Borel set, $\phi, \psi \colon C \to Y$ are Borel homomorphisms from $E \restriction C$ to F for which $\sim D(\phi, \psi)$ is μ-positive, and ν is an F-quasi-invariant Borel measure on Y for which $\phi_* \mu \ll \nu$, then $\psi_* \mu \ll \nu$. Towards this end, suppose that $B \subseteq Y$ is a $(\psi_* \mu)$-positive Borel set. The E-ergodicity of μ then ensures that $[\psi^{-1}(B)]_E$ is μ-conull. As the fact that ψ is a homomorphism from $E \restriction C$ to F implies that $[\psi^{-1}(B)]_E \cap C$ is contained in $\psi^{-1}([B]_F)$, the latter set is also μ-conull. In particular, it follows that $\psi^{-1}([B]_F) \setminus D(\phi, \psi)$ is μ-positive, thus so too is $\phi^{-1}([B]_F)$. The fact that $\phi_* \mu \ll \nu$ therefore ensures that $[B]_F$ is ν-positive, in which case the F-quasi-invariance of ν implies that B is ν-positive. ⊠

A *μ-reduction* of E to F is a Borel reduction of $E \upharpoonright C$ to F, where $C \subseteq X$ is a μ-conull Borel set. A *μ-embedding* is an injective μ-reduction. We say that E is *measure reducible* to F if there is a μ-reduction of E to F for every Borel probability measure μ on X. We say that E is *measure embeddable* into F if there is a μ-embedding of E into F for every Borel probability measure μ on X.

We say that $\mathcal{E}$ is *dichotomous* if it is strictly contained in hyper-$\mathcal{E}$ but every hyper-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space is measure embeddable into every non-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space. Given such an $\mathcal{E}$, we use $E_{\mathcal{E}}^{+}$ to denote any hyper-$\mathcal{E}$ non-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space.

Question 14.2. Is there a dichotomous class containing the hyperfinite Borel equivalence relations on standard Borel spaces?

We say that a Borel measure μ on X is *(E, F)-ergodic* if for every Borel homomorphism $\phi \colon X \to Y$ from E to F, there exists $y \in Y$ for which $\phi^{-1}([y]_F)$ is μ-conull.

Question 14.3. Is the measure hyper-$\mathcal{E}$-ness of E equivalent to the inexistence of an $(E, E_{\mathcal{E}}^{+})$-ergodic Borel probability measure?

Proposition 14.4 (Conley-Miller). *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces containing all equivalence relations on countable standard Borel spaces, X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, μ is an E-ergodic E-quasi-invariant non-E-$\mathcal{E}$ Borel probability measure on X, and ν is an F-ergodic F-quasi-invariant F-projectively-$\mathcal{E}$-separable Borel probability measure on Y. Then there is a ν-conull Borel set $D \subseteq Y$ with the property that whenever X' and Y' are standard Borel spaces, E' and F' are countable Borel equivalence relations on X' and Y', μ is (E, F')-ergodic, and μ' is a Borel probability measure on X' for which there is a $(\mu \times \mu')$-reduction of $E \times E'$ to $(F \upharpoonright D) \times F'$, then there is also a μ'-reduction of E' to F'.*

Proof. By Proposition 14.1, there is an F-invariant F-projectively-$\mathcal{E}$-separable ν-conull Borel set $D \subseteq Y$ with the property that the μ^{th} vertical section of $\ll_{E,F \upharpoonright D}^{\mathcal{E}}$ is contained in the measure-equivalence class of $\nu \upharpoonright D$. To see that this set is desired, suppose that $C \subseteq X \times X'$ is a $(\mu \times \mu')$-conull Borel set and $\pi \colon C \to D \times Y'$ is a Borel reduction of $(E \times E') \upharpoonright C$ to $(F \upharpoonright D) \times F'$. Then the set $R = \{(x, (x', y')) \in X \times (X' \times Y') \mid (\mathrm{proj}_{Y'} \circ \pi)(x, x') \; F' \; y'\}$ is Borel, thus so too is the set $S = \{(x', y') \in X' \times Y' \mid \mu(R^{(x',y')}) = 1\}$. Fubini's theorem ensures that $\{x' \in X' \mid \mu(C^{x'}) = 1\}$ is itself μ'-conull, and if x' is in this set, then the (E, F')-ergodicity of μ and the fact that $(\mathrm{proj}_{Y'} \circ \pi)(\cdot, x')$ is a homomorphism from $E \restriction C^{x'}$ to F' ensure that $x' \in \mathrm{proj}_{X'}(S)$, thus $\mathrm{proj}_{X'}(S)$ is a μ'-conull Borel set. As S has countable vertical sections, the Lusin-Novikov uniformization theorem yields a Borel uniformization $\phi\colon \mathrm{proj}_{X'}(S) \to Y'$ of S. Set $B = \{(x, x') \in C \cap (X \times \mathrm{proj}_{X'}(S)) \mid (\mathrm{proj}_{Y'} \circ \pi)(x, x') \ F' \ \phi(x')\}$, and note that if $w', x' \in \mathrm{proj}_{X'}(S)$, then there exists $x \in B^{w'} \cap B^{x'}$, and if $w' \ E' \ x'$, then $\phi(w') \ F' \ (\mathrm{proj}_{Y'} \circ \pi)(x, w') \ F' \ (\mathrm{proj}_{Y'} \circ \pi)(x, x') \ F' \ \phi(x')$, thus ϕ is a homomorphism from $E' \restriction \mathrm{proj}_{X'}(S)$ to F'. Suppose, towards a contradiction, that there are E'-inequivalent points $w', x' \in \mathrm{proj}_{X'}(S)$ such that $\phi(w') \ F' \ \phi(x')$, and for both $v' \in \{w', x'\}$, fix an F-quasi-invariant Borel probability measure $\nu_{v'}$ on Y such that $(\mathrm{proj}_Y \circ \pi)(\cdot, v')_* \mu \ll \nu_{v'}$ and the two measures agree on all F-invariant Borel sets. As the functions of the form $(\mathrm{proj}_Y \circ \pi)(\cdot, v') \restriction B^{v'}$ are μ-reductions of E to F and $[(\mathrm{proj}_Y \circ \pi)(B^{w'} \times \{w'\})]_F \cap [(\mathrm{proj}_Y \circ \pi)(B^{x'} \times \{x'\})]_F = \emptyset$, it follows that $\nu_{w'}$ and $\nu_{x'}$ are orthogonal measures in the μ^{th} vertical section of $\ll^{\mathcal{E}}_{E, F \restriction D}$, a contradiction. ⊠

Remark 14.5 (Conley-Miller). Proposition 9.5 ensures that if $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, and E is non-measure-hyper-$\mathcal{E}$, then there is an E-ergodic E-quasi-invariant non-E-hyper-$\mathcal{E}$ Borel probability measure on X, so if E is projectively measure-hyper-$\mathcal{E}$-separable, then Proposition 14.4 yields an E-non-measure-hyper-$\mathcal{E}$ Borel set $D \subseteq X$ with the property that for no $n \in \mathbb{Z}^+$ is $(E \restriction D) \times \Delta(n+1)$ measure reducible to $(E \restriction D) \times \Delta(n)$.

Remark 14.6 (Conley-Miller). Even if the existence of a $(\mu \times \mu')$-reduction of $E \times E'$ to $(F \restriction D) \times F'$ is weakened to the existence of a $(\mu \times \mu')$-reduction of $E \times E'$ to $F \times F'$, the above argument still yields a countable-to-one μ-homomorphism from E' to F'. In particular, it follows that if E is non-measure-hyper-$\mathcal{E}$ but projectively measure-hyper-$\mathcal{E}$-separable, E' is non-measure-$\mathcal{E}$, and F' is measure $\mathcal{E}$, then $E \times E'$ is not measure reducible to $E \times F'$.

Remark 14.7 (Conley-Miller). Under the additional assumption that $\mathcal{E}$ is dichotomous, the above argument shows that if there is an $(E, E^+_{\mathcal{E}})$-ergodic Borel probability measure, E is projectively measure-hyper-$\mathcal{E}$-separable, E' is non-measure-hyper-$\mathcal{E}$, and F' is measure-hyper-$\mathcal{E}$, then $E \times E'$ is not measure reducible to $E \times F'$.

15. Reducibility without embeddability

We say that E is *invariant-measure-$\mathcal{E}$* if $E \restriction B$ is $(\mu \restriction B)$-$\mathcal{E}$ for all Borel sets $B \subseteq X$ and $(E \restriction B)$-invariant Borel probability measures μ on B.

Question 15.1. Are measure hyperfiniteness and invariant-measure hyperfiniteness equivalent?

Question 15.2. Is invariant-measure hyperfiniteness closed downward under passage to Borel subequivalence relations?

Proposition 15.3 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, $\mathcal{E}$ is dichotomous, X and Y are standard Borel spaces, E is an invariant-measure-hyper-$\mathcal{E}$ countable Borel equivalence relation on X, and F is a non-$\mathcal{E}$ countable Borel equivalence relation on Y. Then E is measure reducible to F if and only if E is measure embeddable into F.*

Proof. It is sufficient to show that if μ is a Borel probability measure on X for which there is a μ-reduction of E to F, then there is a μ-embedding of E into F. Towards this end, suppose that $C \subseteq X$ is a μ-conull Borel set and $\phi \colon C \to Y$ is a Borel reduction of $E \restriction C$ to F. As E is countable, the Lusin-Novikov uniformization theorem yields a Borel function from $[C]_E$ to C whose graph is contained in E. Replacing C by $[C]_E$, ϕ by its composition with such a function, and μ with an E-quasi-invariant Borel probability measure ν on X for which $\mu \ll \nu$ and the two measures agree on all E-invariant Borel sets, we can assume that C is E-invariant and μ is E-quasi-invariant.

As ϕ is countable-to-one, the Lusin-Novikov uniformization theorem yields an $(E \restriction C)$-complete Borel set $B \subseteq C$ on which ϕ is injective. Fix a μ-maximal Borel set $A \subseteq B$ for which $E \restriction A$ is compressible. Replacing A by $[A]_E \cap B$, we can assume that A is $(E \restriction B)$-invariant. Proposition 2.1 then yields a Borel injection $\psi \colon [A]_E \to A$ whose graph is contained in E.

If $[A]_E$ is μ-conull, then set $A' = \emptyset$. Otherwise, Theorem 2.2 ensures that $\mu \restriction (B \setminus A)$ is equivalent to an $E \restriction (B \setminus A)$-invariant Borel probability measure ν on $B \setminus A$. As E is invariant-measure hyper-$\mathcal{E}$, there is an E-hyper-$\mathcal{E}$ ν-conull Borel set $B' \subseteq B \setminus A$. As $((E \restriction B') \times I(\mathbb{N})) \times \Delta(\mathbb{N})$ is hyper-$\mathcal{E}$, the fact that $\mathcal{E}$ is dichotomous ensures that there is a ν-conull Borel set $A' \subseteq B'$ and a Borel embedding $\phi' \colon (A' \times \mathbb{N}) \times \mathbb{N} \to Y$ of $((E \restriction A') \times I(\mathbb{N})) \times \Delta(\mathbb{N})$ into F. By the Lusin-Novikov uniformization theorem, there is a Borel injection $\psi' \colon [A']_E \to (A' \times \mathbb{N}) \times \{0\}$ for which the graph of $\mathrm{proj}_X \circ \mathrm{proj}_{X \times \mathbb{N}} \circ \psi'$ is contained in E. Let

$\pi\colon Y \to Y$ be the function supported on $\phi'((A' \times \mathbb{N}) \times \mathbb{N})$ given by $(\pi \circ \phi')((x,m),n) = \phi'((x,m),n+1)$, and note that $(\pi \circ \phi \circ \psi) \cup (\phi' \circ \psi')$ is a μ-embedding of $E \upharpoonright [A \cup A']_E$ into F. ⊠

Remark 15.4 (Conley-Miller). As proj_X is a Borel reduction of $E \times I(\mathbb{N})$ to E, Proposition 15.3 ensures that if E is invariant-measure-hyper-$\mathcal{E}$ and non-E-$\mathcal{E}$, then $E \times I(\mathbb{N})$ is measure embeddable into E.

We say that E is *invariant-measure embeddable* into F if there is a μ-embedding of $E \upharpoonright B$ into F for all Borel sets $B \subseteq X$ and $(E \upharpoonright B)$-invariant Borel probability measures μ on B.

Proposition 15.5 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, X is a standard Borel space, and E is a non-invariant-measure-hyper-$\mathcal{E}$ projectively-measure-hyper-$\mathcal{E}$-separable treeable countable Borel equivalence relation on X. Then there is a non-invariant-measure-hyper-$\mathcal{E}$ Borel equivalence relation $F \subseteq E$ with the property that for no $n \in \mathbb{Z}^+$ is $F \times I(n+1)$ invariant-measure embeddable into $F \times I(n)$.*

Proof. By passing to a Borel subset of X, we can assume that there is an E-invariant non-E-hyper-$\mathcal{E}$ Borel probability measure μ on X. As the Lusin-Novikov uniformization theorem ensures that E is the union of countably-many graphs of Borel functions, Proposition 3.3 yields a non-μ-hyper-$\mathcal{E}$ Borel subequivalence relation E' of E that is generated by finitely-many graphs of Borel functions, so that $C_\nu(E') < \infty$ for all E'-invariant Borel probability measures ν on X. By Proposition 9.5, there is an E'-ergodic E'-invariant non-E'-hyper-$\mathcal{E}$ Borel probability measure ν on X. As Proposition 13.3 ensures that E' is projectively measure-hyper-$\mathcal{E}$-separable, there is an E'-invariant ν-conull Borel set $C \subseteq X$ that is null with respect to every measure in the ν^{th} vertical section of $\ll^{\text{hyper-}\mathcal{E}}_{E',E'}$ orthogonal to ν. Set $F = E' \upharpoonright C$, and let m_n denote the uniform probability measure on n for all $n \in \mathbb{Z}^+$.

Suppose, towards a contradiction, that there exists $n \in \mathbb{N}$ for which there is a $(\nu \times m_{n+1})$-conull Borel set $B \subseteq C \times (n+1)$ and a Borel embedding $\pi\colon B \to C \times n$ of $(F \times I(n+1)) \upharpoonright B$ into $F \times I(n)$. For all $i < n+1$ and $j < n$, let $\pi_{i,j}$ be the restriction of the function $(\mathrm{proj}_X \circ \pi)(\cdot, i)$ to $\mathrm{proj}_X((C \times \{i\}) \cap \pi^{-1}(C \times \{j\}))$, and if this set is ν-positive, then fix an F-quasi-invariant Borel probability measure $\nu_{i,j}$ on C such that $(\pi_{i,j})_* \nu \ll \nu_{i,j}$ and the two measures agree on all F-invariant Borel sets. Our choice of C ensures that $\nu_{i,j} \ll \nu$. Observe that if a set $D \subseteq C \times n$ is $\pi_*(\nu \times m_{n+1})$-positive, then there exist $i < n+1$ and $j < n$ for which $\mathrm{proj}_Y(D \cap (C \times \{j\}))$ is $(\pi_{i,j})_* \nu$-positive, and therefore ν-positive, so

D is $(\nu \times m_n)$-positive, thus $\pi_*(\nu \times m_{n+1}) \ll \nu \times m_n$. As the uniform ergodic decomposition theorem ensures that any two ergodic invariant Borel probability measures are either the same or orthogonal, it follows that $\pi_*(\nu \times m_{n+1}) \restriction \pi(B)$ and $(\nu \times m_n) \restriction \pi(B)$ have the same normalizations. As F is non-ν-hyperfinite and therefore ν-aperiodic, Proposition 7.7 yields that $C_\nu(F) > 1$, in which case Remark 7.9 ensures that $C_{(\nu \times m_{n+1})/(n+1)}(F \times I(n+1)) < C_{(\nu \times m_n)/n}(F \times I(n))$ and $C_{(\nu \times m_n)/n}(F \times I(n)) \leq C_{(\nu \times m_n)/(\nu \times m_n)(\pi(B))}((F \times I(n)) \restriction \pi(B))$, contradicting the fact that the first and last quantities are the same. ⊠

16. Minimality

A *minimal* element of a set X under a quasi-order $\leq$ is a point $x \in X$ such that $\forall y \in X \ (y \leq x \Longrightarrow x \leq y)$. We say that E is *measure-minimal non-$\mathcal{E}$* if it is a minimal non-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space under measure reducibility.

Proposition 16.1 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, $\mathcal{E}$ is dichotomous, X is a standard Borel space, and E is a countable Borel equivalence relation on X. If the set of E-ergodic E-quasi-invariant non-measure-hyper-$\mathcal{E}$ Borel probability measures on X is a measure-equivalence class, then E is measure-minimal non-measure-hyper-$\mathcal{E}$.*

Proof. Suppose that Y is a standard Borel space and F is a non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relation on Y that is measure reducible to E. As in the proof of Proposition 15.3, the fact that $\mathcal{E}$ is dichotomous ensures that there is a Borel embedding $\phi \colon Y \to Y$ of F into F for which $\sim[\phi(Y)]_F$ is non-F-$\mathcal{E}$ but F-hyper-$\mathcal{E}$. By Proposition 9.5, there is an F-ergodic F-quasi-invariant non-hyper-$\mathcal{E}$ Borel probability measure ν on Y. Fix a ν-conull Borel set $D \subseteq [\phi(Y)]_F$ and a Borel reduction $\psi \colon D \to X$ of $F \restriction D$ to E, as well as an E-quasi-invariant Borel probability measure μ on X such that $\psi_*\nu \ll \mu$ but the two measures agree on all E-invariant Borel sets. Then μ is E-ergodic and non-E-measure-hyper-$\mathcal{E}$, and the Lusin-Novikov uniformization ensures that there is a Borel reduction $\pi \colon [\psi(D)]_E \to D$ of $E \restriction [\psi(D)]_E$ to $F \restriction D$.

Suppose now that μ' is a Borel probability measure on X. As usual, we can assume that μ' is E-quasi-invariant. Fix a μ'-maximal E-invariant E-hyper-$\mathcal{E}$ Borel set $B \subseteq \sim[\psi(D)]_E$. As $\mathcal{E}$ is dichotomous, there exist a $(\mu' \restriction B)$-conull Borel set $C \subseteq B$ and a Borel embedding $\pi' \colon C \to \sim[\phi(Y)]_E$ of $E \restriction C$ to $F \restriction \sim[\phi(Y)]_E$. As Proposition 9.5

ensures that $\mu' \upharpoonright \sim B \ll \mu$, it follows that $\pi \cup \pi'$ is a μ'-reduction of E to F. ⊠

Proposition 16.2 (Conley-Miller). *Suppose that $\mathcal{E}$ is a class of countable Borel equivalence relations on standard Borel spaces, X is a standard Borel space, and E is a measure-minimal non-measure-$\mathcal{E}$ projectively-$\mathcal{E}$-separable countable Borel equivalence relation on X. Then the set of E-ergodic E-quasi-invariant non-E-$\mathcal{E}$ Borel probability measures on X is a measure-equivalence class.*

Proof. Suppose, towards a contradiction, that there are orthogonal E-ergodic E-quasi-invariant non-E-$\mathcal{E}$ Borel probability measures μ and ν on X. As E is projectively $\mathcal{E}$-separable, Proposition 14.1 yields an E-invariant μ-conull Borel set $C \subseteq X$ that is null with respect to every measure in the union of the μ^{th} and ν^{th} vertical sections of $\ll^{\mathcal{E}}_{E,E}$ orthogonal to μ. By measure minimality, there exist a $(\mu+\nu)$-conull Borel set $B \subseteq X$ and a Borel reduction $\pi \colon B \to C$ of $E \upharpoonright B$ to $E \upharpoonright C$. Then $\pi_* \mu, \pi_* \nu \ll \mu$, so the E-ergodicity of μ ensures that $[\pi(B \cap C)]_E \cap [\pi(B \setminus C)]_E$ is μ-conull, thus there exist $x \in B \cap C$ and $y \in B \setminus C$ for which $\pi(x) \mathrel{E} \pi(y)$. As x and y are E-inequivalent, this contradicts the fact that π is a reduction of $E \upharpoonright B$ to $E \upharpoonright C$. ⊠

Question 16.3. Is there a measure-minimal non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relation on a standard Borel space?

Question 16.4. Is there a non-$E^{\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$-hyperfinite Borel probability measure orthogonal to $\mathbb{m}^2$?

Proposition 16.5. *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, X is a standard Borel space, and E is a countable Borel equivalence relation on X for which the set of E-ergodic E-quasi-invariant non-E-hyper-$\mathcal{E}$ Borel probability measures on X is a measure-equivalence class. Then every E-ergodic non-E-hyper-$\mathcal{E}$ Borel probability measure on X is $(E, \mathbb{E}_0)$-ergodic.*

Proof. Suppose that μ is an E-ergodic non-$(E, \mathbb{E}_0)$-ergodic Borel probability measure on X, and fix a μ-null-to-one Borel homomorphism $\phi \colon X \to 2^{\mathbb{N}}$ from E to $\mathbb{E}_0$. Then there exists $c \in 2^{\mathbb{N}}$ with the property that for all $d \in \sim[c]_{\mathbb{E}_0}$, every E-ergodic E-quasi-invariant Borel probability measure on $\phi^{-1}([d]_{\mathbb{E}_0})$ is E-hyper-$\mathcal{E}$, in which case Proposition 9.5 ensures that $\phi^{-1}([d]_{\mathbb{E}_0})$ is E-measure-hyper-$\mathcal{E}$. It then follows from Proposition 9.7 that $\sim\phi^{-1}([c]_{\mathbb{E}_0})$ is E-measure-hyper-$\mathcal{E}$, so the fact that $\phi^{-1}([c]_{\mathbb{E}_0})$ is μ-null yields that E is μ-hyper-$\mathcal{E}$. ⊠

Remark 16.6. Remark 14.6 and Propositions 16.2 and 16.5 ensure that if $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, and E is measure-minimal non-measure-hyper-$\mathcal{E}$ and projectively-measure-hyper-$\mathcal{E}$-separable, then there is no non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relation F on a standard Borel space for which $E \times F$ is measure reducible to $E \times \mathbb{E}_0$.

17. Bases

An *external basis* for a set $Y \subseteq X$ under a quasi-order $\leq$ on X is a set $B \subseteq X$ such that $\forall y \in Y \exists b \in B\ b \leq y$.

Question 17.1. Suppose that E is non-measure-hyper-$\mathcal{E}$ but projectively measure-hyper-$\mathcal{E}$-separable, and $\mathcal{F}$ is the set of restrictions of E to E-invariant non-E-measure-hyper-$\mathcal{E}$ Borel sets. Is there an external basis for $\mathcal{F}$ under measure-hyper-$\mathcal{E}$-to-one measure homomorphism whose elements are measure-minimal non-measure-hyper-$\mathcal{E}$?

Remark 17.2. Proposition 16.5 ensures that a positive answer to the special case of Question 17.1 in which $\mathcal{E}$ is the family of smooth countable Borel equivalence relations would yield a positive answer to the corresponding special case of Question 14.3. It would also allow one to drop the assumption that E is measure-minimal in Remark 16.6.

Theorem 17.3 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, $\mathcal{E}$ is dichotomous, X is a standard Borel space, E is a non-measure-hyper-$\mathcal{E}$ projectively-measure-hyper-$\mathcal{E}$-separable countable Borel equivalence relation on X, the set M of non-E-hyper-$\mathcal{E}$ Borel probability measures on X is analytic, $\mathcal{F}$ is the set of restrictions of E to E-invariant non-E-measure-hyper-$\mathcal{E}$ Borel sets, $\mathcal{B}$ is an external basis for $\mathcal{F}$ under measure-hyper-$\mathcal{E}$-to-one measure homomorphism consisting of non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relations on standard Borel spaces, and $2^{\mathbb{N}}$ is not a union of $\mathcal{B}$-many countable sets. Then E is a disjoint union of countably-many measure-minimal non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relations on standard Borel spaces.*

Proof. By Proposition 16.1, it is sufficient to show that M is a union of countably-many measure-equivalence classes. Suppose, towards a contradiction, that this is not the case. The perfect set theorem for co-analytic equivalence relations on Hausdorff spaces then yields a non-empty perfect set $P \subseteq M$ of pairwise-orthogonal measures. By Theorem 1.1, there exist a continuous injection $\pi \colon 2^{\mathbb{N}} \to P$

and a K_σ sequence $(K_c)_{c \in 2^\mathbb{N}}$ of pairwise disjoint subsets of X such that $\pi(c)(K_c) = 1$ for all $c \in 2^\mathbb{N}$. As E is projectively measure-hyper-$\mathcal{E}$-separable, it follows that for each $F \in \mathcal{B}$, the set of $c \in 2^\mathbb{N}$ for which there is an F-measure-hyper-$\mathcal{E}$-to-one $\pi(c)$-homomorphism from F to $E \upharpoonright K_c$ is countable, thus $2^\mathbb{N}$ is the union of $\mathcal{B}$-many countable sets, the desired contradiction. ⊠

Remark 17.4 (Conley-Miller). Under the stronger assumption that $\mathcal{B}$ is a countable external basis for $\mathcal{F}$ under smooth-to-one measure homomorphism, it is not difficult to see that the hypothesis that M is analytic is superfluous, as Proposition 4.2 easily implies that the family of smooth-to-one Borel homomorphisms is closed under composition.

Remark 17.5 (Conley-Miller). Even without the assumption that M is analytic, if the union of $\aleph_1$-many meager sets is always meager, then we can still conclude that there is a basis for $\mathcal{F}$ under measure embeddability consisting of $(\leq \aleph_1)$-many minimal non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relations on standard Borel spaces under measure reducibility. To see this, appeal to Proposition 9.3 to see that M is co-analytic, and use the perfect set theorem for analytic equivalence relations in place of that for co-analytic equivalence relations.

18. Antichains

We have essentially already seen one way of building antichains.

Theorem 18.1 (Conley-Miller). *Suppose that $\mathcal{E}$ is the downward closure of a countable Borel equivalence relation on a standard Borel space under smooth-to-one Borel homomorphisms, $\mathcal{E}$ is dichotomous, X is a standard Borel space, E is a non-measure-hyper-$\mathcal{E}$ projectively-measure-hyper-$\mathcal{E}$-separable countable Borel equivalence relation on X that is not a disjoint union of countably-many measure-minimal non-measure-hyper-$\mathcal{E}$ countable Borel equivalence relations on standard Borel spaces, and the set M of non-E-hyper-$\mathcal{E}$ Borel probability measures on X is analytic. Then there exist a continuous injection $\pi \colon 2^\mathbb{N} \to M$ and a K_σ sequence $(K_c)_{c \in 2^\mathbb{N}}$ of pairwise disjoint subsets of X such that $\pi(c)(K_c) = 1$ for all $c \in 2^\mathbb{N}$ and for no two distinct sequences $c, d \in 2^\mathbb{N}$ is there a measure-hyper-$\mathcal{E}$-to-one $\pi(c)$-homomorphism from E to $E \upharpoonright K_d$.*

Proof. By the proof of Theorem 17.3, we can assume that there exist a continuous injection $\phi \colon 2^\mathbb{N} \to M$ and a K_σ sequence $(K_c)_{c \in 2^\mathbb{N}}$ of pairwise disjoint subsets of X such that $\phi(c)(K_c) = 1$ for all $c \in 2^\mathbb{N}$. As E is projectively measure-hyper-$\mathcal{E}$, the vertical sections of the set $(\phi \times \phi)^{-1}(\ll_{E,F}^{\text{hyper-}\mathcal{E}})$ are countable. As Proposition 9.9 ensures that this set is analytic, and therefore meager, Mycielski's theorem yields a continuous injection $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for no two distinct sequences $c, d \in 2^{\mathbb{N}}$ is there a measure-hyper-$\mathcal{E}$-to-one $(\phi \circ \psi)(c)$-homomorphism from E to $E \upharpoonright K_{\psi(d)}$, thus $\phi \circ \psi$ and $(K_{\psi(c)})_{c \in 2^{\mathbb{N}}}$ are as desired. ⊠

Remark 18.2. This reduces the problem of building antichains to the case that E is measure-minimal non-measure-hyper-$\mathcal{E}$. When E is treeable, it is known that there is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of measure-minimal non-measure-hyper-$\mathcal{E}$ subequivalence relations of E that are pairwise incomparable under measure reducibility. However, the existence of antichains (within the treeable countable Borel equivalence relations) under countable-to-one measure homomorphism remains open.

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