An introduction to classical descriptive set theory

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Introduction

These are the notes accompanying an introductory course to descriptive set theory at the Kurt Gödel Research Center for Mathematical Logic at the University of Vienna in Fall 2015. I am grateful to the head of the KGRC, Sy Friedman, for his encouragement and many useful suggestions, as well as to all of the participants.

The goal of the course was to provide a succinct introduction to the structures underlying the main results of classical descriptive set theory. In the first half, we discuss trees, the corresponding representations of closed sets, Borel sets, analytic spaces, injectively analytic spaces, and Polish spaces, as well as Baire category. In the second half, we establish various relatives of the $\mathbb{G}_0$ dichotomy, which we then use to establish many of the primary dichotomy theorems of descriptive set theory.

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CHAPTER 1

The basics

1. Trees

Given a set X, we use $X^{<\mathbb{N}}$ to denote the set $\bigcup_{n\in\mathbb{N}} X^n$ of functions from natural numbers to X. The *length* of $t \in X^{<\mathbb{N}}$, or $|t|$, is simply its domain. We say that $s \in X^{<\mathbb{N}}$ is an *initial segment* of $t \in X^{<\mathbb{N}}$, t is an *extension* of s, or $s \sqsubseteq t$ if $|s| \leq |t|$ and $s = t \upharpoonright |s|$. A *tree* on X is a set $T \subseteq X^{<\mathbb{N}}$ which is *closed under initial segments*, in the sense that $t \upharpoonright n \in T$ whenever $t \in T$ and $n \leq |t|$.

Many problems in descriptive set theory can be reduced to analogous problems concerning the structure of trees. In this first section, we establish several basic facts about trees which we will later utilize through such reductions.

Pruning. The *pruning derivative* is the function associating with each tree T the subtree $\mathrm{Prune}(T)$ consisting of all elements of T which have proper extensions in T.

PROPOSITION 1.1.1. *Suppose that S and T are trees. If $S \subseteq T$, then* $\mathrm{Prune}(S) \subseteq \mathrm{Prune}(T)$.

PROOF. By the definition of the pruning derivative. □

A tree T is *pruned* if every $t \in T$ has a proper extension in T.

PROPOSITION 1.1.2. *Suppose that T is a tree. Then T is pruned if and only if* $T = \mathrm{Prune}(T)$.

PROOF. This follows from the definitions of the pruning derivative and pruned trees. □

Set $\mathrm{Prune}^0(T) = T$, $\mathrm{Prune}^{\alpha+1}(T) = \mathrm{Prune}(\mathrm{Prune}^\alpha(T))$ for all ordinals α, and $\mathrm{Prune}^\lambda(T) = \bigcap_{\alpha<\lambda} \mathrm{Prune}^\alpha(T)$ for limit ordinals λ.

PROPOSITION 1.1.3. *Suppose that α is an ordinal and S and T are trees. If $S \subseteq T$, then* $\mathrm{Prune}^\alpha(S) \subseteq \mathrm{Prune}^\alpha(T)$.

PROOF. By Proposition 1.1.1 and the obvious induction. □

The *pruning rank* of T is the least α with $\mathrm{Prune}^{\alpha}(T) = \mathrm{Prune}^{\alpha+1}(T)$. A *branch* through a tree T on X is a sequence $x \in X^{\mathbb{N}}$ such that $x \upharpoonright n \in T$, for all $n \in \mathbb{N}$. A tree is *well-founded* if it has no branches.

PROPOSITION 1.1.4. *Suppose that T is a tree with pruning rank α. Then T is well-founded if and only if* $\mathrm{Prune}^{\alpha}(T) = \emptyset$.

PROOF. Proposition 1.1.2 ensures that $\mathrm{Prune}^{\alpha}(T)$ is pruned. It is clear that if T is well-founded, then so too is $\mathrm{Prune}^{\alpha}(T)$. Conversely, if T is not well-founded, then there is a branch x through T, in which case the subtree $S = \{x \upharpoonright n \mid n \in \mathbb{N}\}$ is pruned, so Propositions 1.1.2 and 1.1.3 imply that $S \subseteq \mathrm{Prune}^{\alpha}(T)$, thus x is a branch through $\mathrm{Prune}^{\alpha}(T)$, hence $\mathrm{Prune}^{\alpha}(T)$ is not well-founded. As a pruned tree is well-founded if and only if it is empty, the proposition follows. □

The *pruning rank* of $t \in T$ within T is the maximal ordinal α with the property that $t \in \mathrm{Prune}^{\alpha}(T)$, or ∞ if no such ordinal exists.

PROPOSITION 1.1.5. *Suppose that T is a tree and $t \in T$. Then the pruning rank of t within T is the least ordinal strictly greater than the pruning ranks of the proper extensions of t within T.*

PROOF. If α is at most the pruning rank of some proper extension of t within T, then this proper extension is in $\mathrm{Prune}^{\alpha}(T)$, thus the pruning rank of t within T is strictly greater than α. And if α is strictly greater than the pruning ranks of the proper extensions of t within T, then no proper extension of t is in $\mathrm{Prune}^{\alpha}(T)$, thus the pruning rank of t within T is at most α. □

PROPOSITION 1.1.6. *Suppose that T is a tree. Then T has a unique branch if and only if for all $n \in \mathbb{N}$ there is a unique $t \in T$ of length n whose pruning rank within T is maximal.*

PROOF. Suppose first that T has exactly one branch x. Then for each $n \in \mathbb{N}$, the pruning rank of $x \upharpoonright n$ within T is ∞, but the pruning rank of every $t \in T \setminus \{x \upharpoonright n\}$ of length n within T is an ordinal.

Conversely, suppose that for all $n \in \mathbb{N}$, there is a unique $t_n \in T$ of length n whose pruning rank within T is maximal. Then for every $n \in \mathbb{N}$ and $t \in T \setminus \{t_n\}$ of length n, Proposition 1.1.5 yields i for which the pruning rank of $t_n \frown (i) \in T$ is at least that of t, and therefore at least that of every extension of t in T. In particular, it follows that $(t_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of elements of T, so $\bigcup_{n \in \mathbb{N}} t_n$ is a branch through T. And the fact that the pruning rank of every $t \in T \setminus \{t_n \mid n \in \mathbb{N}\}$ within T is an ordinal ensures that there are no other branches of T. □

A *homomorphism* from a binary relation R on X to a binary relation S on Y is a function $\phi\colon X \to Y$ such that $w \mathrel{R} x \implies \phi(w) \mathrel{S} \phi(x)$, for all $w, x \in X$.

Proposition 1.1.7. *Suppose that S and T are non-empty trees. Then there is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$ if and only if the pruning rank of $\emptyset$ within S is at most the pruning rank of $\emptyset$ within T.*

Proof. Suppose that $\phi\colon S \to T$ is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$. Then ϕ induces a homomorphism from $\sqsubset \restriction \mathrm{Prune}(S)$ to $\sqsubset \restriction \mathrm{Prune}(T)$. The obvious transfinite induction therefore ensures that ϕ induces a homomorphism from $\sqsubset \restriction \mathrm{Prune}^\alpha(S)$ to $\sqsubset \restriction \mathrm{Prune}^\alpha(T)$, for all ordinals α. In particular, it follows that if $\mathrm{Prune}^\alpha(S)$ is non-empty, then so too is $\mathrm{Prune}^\alpha(T)$, thus the pruning rank of $\emptyset$ within S is at most the pruning rank of $\emptyset$ within T.

Conversely, suppose that the pruning rank of $\emptyset$ within S is at most the pruning rank of $\emptyset$ within T. Recursively construct functions ϕ_n from the set of $s \in S$ of length n to the set of $t \in T$ of length n by setting $\phi_0(\emptyset) = \emptyset$ and letting $\phi_{n+1}(s \frown (i))$ be any element of T of the form $\phi_n(s) \frown (j)$ for which the pruning rank of the former within S is at most that of the latter within T (which exists by Proposition 1.1.5). Then $\bigcup_{n \in \mathbb{N}} \phi_n$ is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$. □

Perfection. We say that $s, t \in X^{<\mathbb{N}}$ are *incompatible*, or $s \perp t$, if neither is an extension of the other. The *perfection derivative* is the function associating with each tree T the subtree $\mathrm{Perfect}(T)$ consisting of all elements of T which have a pair of incompatible extensions in T.

Proposition 1.1.8. *Suppose that S and T are trees. If $S \subseteq T$, then* $\mathrm{Perfect}(S) \subseteq \mathrm{Perfect}(T)$.

Proof. By the definition of the perfection derivative. □

A tree T is *perfect* if every $t \in T$ has a pair of incompatible extensions in T.

Proposition 1.1.9. *Suppose that T is a tree. Then T is perfect if and only if* $T = \mathrm{Perfect}(T)$.

Proof. This follows from the definitions of the perfection derivative and perfect trees. □

Set $\mathrm{Perfect}^0(T) = T$, $\mathrm{Perfect}^{\alpha+1}(T) = \mathrm{Perfect}(\mathrm{Perfect}^\alpha(T))$ for all ordinals α, and $\mathrm{Perfect}^\lambda(T) = \bigcap_{\alpha<\lambda} \mathrm{Perfect}^\alpha(T)$ for limit ordinals λ.

Proposition 1.1.10. *Suppose that α is an ordinal and S and T are trees. If $S \subseteq T$, then* $\mathrm{Perfect}^\alpha(S) \subseteq \mathrm{Perfect}^\alpha(T)$.

Proof. By Proposition 1.1.8 and the obvious induction. $\square$

The *perfection rank* of T is the least α with the property that $\mathrm{Perfect}^{\alpha}(T) = \mathrm{Perfect}^{\alpha+1}(T)$. An *embedding* of a binary relation R on X into a binary relation S on Y is a function $\phi\colon X \to Y$ such that $w \mathrel{R} x \iff \phi(w) \mathrel{S} \phi(x)$, for all $w, x \in X$. A tree T is *scattered* if there is no embedding of $\sqsubseteq \restriction 2^{<\mathbb{N}}$ into $\sqsubseteq \restriction T$.

Proposition 1.1.11. *Suppose that T is a tree with perfection rank α. Then T is scattered if and only if* $\mathrm{Perfect}^{\alpha}(T) = \emptyset$.

Proof. Proposition 1.1.9 ensures that $\mathrm{Perfect}^{\alpha}(T)$ is perfect. It is clear that if T is scattered, then so too is $\mathrm{Perfect}^{\alpha}(T)$. Conversely, if T is not scattered, then there is an embedding $\phi\colon 2^{<\mathbb{N}} \to T$ of $\sqsubseteq \restriction 2^{<\mathbb{N}}$ into $\sqsubseteq \restriction T$, in which case the subtree S generated by $\phi(2^{<\mathbb{N}})$ is perfect, so Propositions 1.1.9 and 1.1.10 imply that $S \subseteq \mathrm{Perfect}^{\alpha}(T)$, thus ϕ is an embedding of $\sqsubseteq \restriction 2^{<\mathbb{N}}$ into $\sqsubseteq \restriction \mathrm{Perfect}^{\alpha}(T)$, hence $\mathrm{Perfect}^{\alpha}(T)$ is not scattered. As a perfect tree is scattered if and only if it is empty, the proposition follows. $\square$

The *perfection rank* of $t \in T$ within T is the maximal ordinal α with the property that $t \in \mathrm{Perfect}^{\alpha}(T)$, or ∞ if no such ordinal exists.

Proposition 1.1.12. *Suppose that T is a scattered tree. Then there are at most $|T|$-many branches through T.*

Proof. Proposition 1.1.11 ensures that every $t \in T$ has ordinal perfection rank within T. But any function sending x to a proper initial segment of minimal pruning rank within T is injective, so there are at most $|T|$-many branches through T. $\square$

Generalizations. We will occasionally refer to objects as *trees* even when they are not literally trees on a set as previously discussed, but can nevertheless trivially be coded as such.

The simplest example consists of sets $T \subseteq \bigcup_{n \in \mathbb{N}} X^n \times X^n$ with the property that

$$\forall n \in \mathbb{N} \forall m \leq n \forall (s, t) \in T \cap (X^n \times X^n)\ (s \restriction m, t \restriction m) \in T,$$

equipped with the order in which $(s, t) \in T \cap (X^n \times X^n)$ is extended by $(s', t') \in T \cap (X^{n'} \times X^{n'})$ if and only if $n \leq n'$ and both $s \sqsubseteq s'$ and $t \sqsubseteq t'$. By identifying $\bigcup_{n \in \mathbb{N}} X^n \times X^n$ with $(X \times X)^{<\mathbb{N}}$ in the natural fashion, one can view such sets T as trees on $\mathbb{N} \times \mathbb{N}$.

Of course, a similar comment applies to sets $T \subseteq \bigcup_{n \in \mathbb{N}} (X^n)^k$, for all natural numbers $k \geq 3$. A slightly more subtle example concerns sets $T \subseteq \bigcup_{n \in \mathbb{N}} (X^n)^n$ with the property that

$$\forall n \in \mathbb{N} \forall m \leq n \forall (t_i)_{i<n} \in T\ (t_i \restriction m)_{i<m} \in T,$$

equipped with the order in which $(t_i)_{i<n} \in T$ is extended by $(t'_i)_{i<n'} \in T$ if and only if $n \leq n'$ and $t_i \sqsubseteq t'_i$ for all $i < n$. However, since the natural way of viewing such ordered sets as trees is not significantly simpler than the most general result of this form, we now turn our attention to the latter.

A *quasi-order* on a set X is a reflexive transitive binary relation R on X. The *incomparability relation* associated with R is the binary relation $\perp_R$ on X for which $x \perp_R y$ if and only if neither $x \; R \; y$ nor $y \; R \; x$. The *strict quasi-order* associated with R is the binary relation $<_R$ on X for which $x <_R y$ if and only if $x \; R \; y$ but $\neg y \; R \; x$. The *equivalence relation* associated with R is the binary relation $\equiv_R$ on X for which $x \equiv_R y$ if and only if $x \; R \; y$ and $y \; R \; x$. We say that x and y are *R-comparable* if it is not the case that $x \perp_R y$. We say that a set $Y \subseteq X$ is an *R-antichain* if no two distinct points of Y are R-comparable, and Y is an *R-chain* if all pairs of points of Y are R-comparable. We use the usual *interval notation*, such as $(-\infty, x)_R$ to denote the set $\{y \in X \mid y <_R x\}$.

A *partial order* is a quasi-order whose associated equivalence relation is equality. A *linear order* is a partial order with respect to which every two elements are comparable. A *tree order* on a set X is a partial order R on X whose restriction to each of the sets $(-\infty, x)_R$ is a finite linear order. A tree order is *rooted* if there is an *R-minimum* element of X, that is, if there is an element of X which is R-related to every element of X. An *isomorphism* of a binary relation R on X with a binary relation S on Y is a bijection $\pi \colon X \to Y$ such that $w \; R \; x \iff \pi(w) \; R \; \pi(x)$, for all $w, x \in X$.

PROPOSITION 1.1.13. *Suppose that R is a rooted tree order on X. Then there is a tree T on X for which R is isomorphic to $\sqsubseteq \restriction T$.*

PROOF. Simply observe that the function $\pi \colon X \to X^{<\mathbb{N}}$ sending x to the strictly R-increasing enumeration of $(-\infty, x)_R$ is an embedding of R into $\sqsubseteq$ for which $\pi(X)$ is a tree on X. $\square$

2. Closed sets

Closed subsets of sequence spaces have natural tree representations. Many problems concerning the former can be reduced to analogous problems concerning the latter. In this section, we establish several basic facts about this correspondence, give an application, and consider examples of sets arising from trees.

Tree representations. Suppose that X is a discrete topological space. For each $s \in X^{<\mathbb{N}}$, let $\mathcal{N}_s$ denote the set of elements of $X^{\mathbb{N}}$ extending s. These sets form a basis for the product topology on $X^{\mathbb{N}}$.

For each tree T, let $[T]$ denote the set of branches through T.

Proposition 1.2.1. *Suppose that X is a discrete space. Then a set $Y \subseteq X^{\mathbb{N}}$ is closed if and only if there is a tree T on X with $Y = [T]$.*

Proof. Suppose first that Y is closed, and let T denote the set of proper initial segments of elements of Y. Clearly T is a tree on X and $Y \subseteq [T]$. To see that $[T] \subseteq Y$, suppose that $x \in [T]$, and note that if $n \in \mathbb{N}$, then $x \upharpoonright n \in T$, so $\mathcal{N}_{x \upharpoonright n} \cap Y \neq \emptyset$, thus $x \in Y$.

It remains to show that if T is a tree on X, then $[T]$ is closed. Towards this end, suppose that x is in the closure of $[T]$, and note that if $n \in \mathbb{N}$, then $\mathcal{N}_{x \upharpoonright n} \cap [T] \neq \emptyset$, so $x \upharpoonright n \in T$, thus $x \in [T]$. □

The *power set* of X is the set $\mathcal{P}(X)$ of all subsets of X. We endow $\mathcal{P}(X)$ with the topology it inherits from 2^X via the identification of sets with their characteristic functions.

Proposition 1.2.2. *The set of trees on X is closed in $\mathcal{P}(X^{<\mathbb{N}})$.*

Proof. The set of trees on X is the intersection of the clopen sets $\{Y \in \mathcal{P}(X^{<\mathbb{N}}) \mid t \in Y \implies s \in Y\}$, where $s \sqsubseteq t$ vary over $X^{<\mathbb{N}}$. □

Proposition 1.2.3. *Suppose that X is a discrete space. Then the set of pairs (T, x), where T is a tree on X and $x \in [T]$, is closed in $\mathcal{P}(X^{<\mathbb{N}}) \times X^{\mathbb{N}}$.*

Proof. By Proposition 1.2.2, the set of trees on X is closed, thus so too is its product with $X^{\mathbb{N}}$. But the desired set is the intersection of this product with the sets $\{(T, x) \in \mathcal{P}(X^{<\mathbb{N}}) \times X^{\mathbb{N}} \mid x \upharpoonright n \in T\}$, where $n \in \mathbb{N}$, and these latter sets are clopen, as they can be written as both $\bigcup_{t \in X^n} \{T \in \mathcal{P}(X^{<\mathbb{N}}) \mid t \in T\} \times \mathcal{N}_t$ and as the complement of $\bigcup_{t \in X^n} \{T \in \mathcal{P}(X^{<\mathbb{N}}) \mid t \notin T\} \times \mathcal{N}_t$. □

The x^{th} *vertical section* of a set $R \subseteq X \times Y$ is the set R_x of all $y \in Y$ for which $(x, y) \in R$. Similarly, the y^{th} *horizontal section* of R is the set R^y of all $x \in X$ for which $(x, y) \in R$.

Proposition 1.2.4 (Lebesgue). *Suppose that X is a discrete space. Then there is a closed set $C \subseteq \mathcal{P}(X^{<\mathbb{N}}) \times X^{\mathbb{N}}$ whose vertical sections are exactly the closed subsets of $X^{\mathbb{N}}$.*

Proof. Let C denote the set of pairs (T, x), where T is a tree on X and $x \in [T]$. Proposition 1.2.1 ensures that the vertical sections of C are exactly the closed subsets of $X^{\mathbb{N}}$, and Proposition 1.2.3 implies that C is closed. □

A *retraction* from a set X onto a subset Y is a function $\phi\colon X \to Y$ whose restriction to Y is the identity.

PROPOSITION 1.2.5. *Suppose that X is a discrete space. Then there is a continuous retraction from $X^{\mathbb{N}}$ onto every non-empty closed set $C \subseteq X^{\mathbb{N}}$.*

PROOF. Let T denote the set of proper initial segments of elements of C, associate with each $t \in T$ an extension $\beta(t) \in [T]$, let $\mathrm{proj}_T(x)$ denote the maximal initial segment of $x \in X^{\mathbb{N}} \setminus C$ in T, and let ϕ denote the retraction from X onto C which agrees with $\beta \circ \mathrm{proj}_T$ off of C. To see that ϕ is continuous, it is sufficient to show that $\phi(\mathcal{N}_{x \restriction n}) \subseteq \mathcal{N}_{\phi(x) \restriction n}$, for all $n \in \mathbb{N}$ and $x \in X^{\mathbb{N}}$. Towards this end, note that if $x \restriction n \in T$ then $\phi(\mathcal{N}_{x \restriction n}) \subseteq \mathcal{N}_{x \restriction n}$, and if $x \restriction n \notin T$ then $\phi(\mathcal{N}_{x \restriction n}) = \{\phi(x)\}$. $\square$

The perfect set theorem. The following result implies that closed subsets of $\mathbb{N}^{\mathbb{N}}$ satisfy the continuum hypothesis.

THEOREM 1.2.6 (Cantor). *Suppose that X is a discrete space and $C \subseteq X^{\mathbb{N}}$ is closed. Then at least one of the following holds:*

(1) *The cardinality of C is at most that of $X^{<\mathbb{N}}$.*
(2) *There is a continuous injection $\pi\colon 2^{\mathbb{N}} \to C$.*

PROOF. By Proposition 1.2.1, there is a tree T on X with $C = [T]$. If T is scattered, then Proposition 1.1.12 ensures that there are at most $|T|$-many branches through T, so $|C| \leq |X^{<\mathbb{N}}|$. Otherwise, there is an embedding ϕ of extension on $2^{<\mathbb{N}}$ into extension on T. Define $\pi\colon 2^{\mathbb{N}} \to C$ by $\pi(c) = \bigcup_{n \in \mathbb{N}} \phi(c \restriction n)$. To see that π is continuous, note that if $n \in \mathbb{N}$ and $c \in 2^{\mathbb{N}}$, then $\pi(\mathcal{N}_{c \restriction n}) \subseteq \mathcal{N}_{\phi(c \restriction n)} \subseteq \mathcal{N}_{\pi(c) \restriction n}$. To see that π is injective, note that if $c, d \in 2^{\mathbb{N}}$ are distinct, then there exists $n \in \mathbb{N}$ sufficiently large for which $c \restriction n$ and $d \restriction n$ are distinct, so $\phi(c \restriction n)$ and $\phi(d \restriction n)$ are incompatible, thus $\pi(c)$ and $\pi(d)$ are distinct. $\square$

Examples. Here we consider a pair of sets arising from trees.

PROPOSITION 1.2.7. *The set of (ϕ, S, T), where S and T are trees on $\mathbb{N}$ and $\phi \restriction S$ is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$, is a closed subset of $(\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}^{<\mathbb{N}}} \times \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$.*

PROOF. Proposition 1.2.2 ensures that the set of trees on $\mathbb{N}$ is closed, and $\phi \restriction S$ is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$ if and only if $s \in S \implies \phi(s) \in T$ and $(r \sqsubset s$ and $r, s \in S) \implies \phi(r) \sqsubset \phi(s)$, for all $r, s \in \mathbb{N}^{<\mathbb{N}}$. $\square$

PROPOSITION 1.2.8. *The set of (ϕ, S, T), where S and T are trees on $\mathbb{N}$ and $\phi \restriction S$ is an embedding of $\sqsubseteq \restriction S$ into $\sqsubseteq \restriction T$, is a closed subset of $(\mathbb{N}^{<\mathbb{N}})^{\mathbb{N}^{<\mathbb{N}}} \times \mathcal{P}(\mathbb{N}^{<\mathbb{N}}) \times \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$.*

PROOF. By Proposition 1.2.7, it is sufficient to observe that $\phi \upharpoonright S$ is an embedding of $\sqsubseteq \upharpoonright S$ into $\sqsubseteq \upharpoonright T$ if and only if it is a homomorphism from strict extension on S to strict extension on T and $(\phi(r) \sqsubset \phi(s)$ and $r, s \in S) \Longrightarrow r \sqsubset s$, for all $r, s \in \mathbb{N}^{<\mathbb{N}}$. $\square$

3. Borel sets

A *σ-algebra* is a family of sets which is closed under complements, countable intersections, and countable unions. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets. While the structure of Borel sets is a central focus of classical descriptive set theory, this structure is typically derived from tree representations of other types of sets. In this section, we give an alternate characterization of Borel sets, as well as several examples of Borel functions arising from trees.

An alternate characterization. Clearly one can also characterize σ-algebras as families of sets which are closed under complements and either countable intersections or countable unions. It only takes a little more work to establish yet another characterization.

PROPOSITION 1.3.1. *Suppose that X is set and $\mathcal{X} \subseteq \mathcal{P}(X)$ is closed under complements. Then the closure $\mathcal{Z}$ of $\mathcal{X}$ under countable disjoint unions and countable intersections is a σ-algebra.*

PROOF. Let $\mathcal{Y}$ denote the family of sets $Y \subseteq X$ for which both Y and $X \setminus Y$ are in $\mathcal{Z}$. Clearly $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{Z}$, so it is sufficient to show that $\mathcal{Y}$ is closed under countable unions. Towards this end, suppose that $(Y_n)_{n \in \mathbb{N}}$ is a sequence of elements of $\mathcal{Y}$. Then both Y_n and $X \setminus Y_n$ are in $\mathcal{Z}$ for all $n \in \mathbb{N}$, so $Y_n \setminus \bigcup_{m<n} Y_m = Y_n \cap \bigcap_{m<n} X \setminus Y_m$ is in $\mathcal{Z}$ for all $n \in \mathbb{N}$, thus both $\bigcup_{n \in \mathbb{N}} Y_n = \bigcup_{n \in \mathbb{N}} Y_n \setminus \bigcup_{m<n} Y_m$ and $X \setminus \bigcup_{n \in \mathbb{N}} Y_n = \bigcap_{n \in \mathbb{N}} X \setminus Y_n$ are in $\mathcal{Z}$, hence $\bigcup_{n \in \mathbb{N}} Y_n \in \mathcal{Y}$. $\square$

Examples. A function $\phi \colon X \to Y$ between topological spaces is *Borel* if preimages of open sets are Borel. Note that if $Y = \mathcal{P}(\mathbb{N})$, then ϕ is Borel if and only if the sets $\{x \in X \mid n \in \phi(x)\}$ are Borel.

PROPOSITION 1.3.2. *The restriction of the pruning derivative to trees on $\mathbb{N}$ is Borel.*

PROOF. As $t \in \mathrm{Prune}(T) \iff \exists n \in \mathbb{N}\ t \frown (n) \in T$, it follows that the set of trees T on $\mathbb{N}$ for which $t \in \mathrm{Prune}(T)$ is open. $\square$

PROPOSITION 1.3.3. *The function $\phi \colon \mathcal{P}(\mathbb{N})^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ given by $\phi((X_k)_{k \in \mathbb{N}}) = \bigcap_{k \in \mathbb{N}} X_k$ is Borel.*

PROOF. As $n \in \bigcap_{k \in \mathbb{N}} X_k \iff \forall k \in \mathbb{N}\ n \in X_k$, it follows that the set of sequences $(X_k)_{k \in \mathbb{N}}$ for which $n \in \bigcap_{k \in \mathbb{N}} X_k$ is closed. □

PROPOSITION 1.3.4. *Suppose that α is a countable ordinal. Then the restriction of* Prune^α *to trees on $\mathbb{N}$ is Borel.*

PROOF. By Propositions 1.3.2 and 1.3.3 and induction. □

PROPOSITION 1.3.5. *Suppose that $C \subseteq \mathbb{N}^\mathbb{N}$, X is a topological space, and $\phi \colon C \to X$. Then the map $T \colon X \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ given by $t \in T(x) \iff x \in \overline{\phi(C \cap \mathcal{N}_t)}$ is Borel.*

PROOF. The set of $x \in X$ for which $t \in T(x)$ is closed. □

4. Analytic spaces

A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$. Tree representations of closed sets give rise to tree representations of analytic sets, and many of the most important properties of Borel sets come from the latter. In this section, we establish the basic properties of analytic sets.

Closure properties. Here we establish the basic closure properties of analytic sets, beginning with the simplest operations.

PROPOSITION 1.4.1. *Suppose that X is a topological space. Then $\bigcup_{n \in \mathbb{N}} A_n$ is analytic for all sequences $(A_n)_{n \in \mathbb{N}}$ of analytic subsets of X.*

PROOF. Fix closed sets $C_n \subseteq \mathbb{N}^\mathbb{N}$ for which there are continuous surjections $\phi_n \colon C_n \to A_n$. Then the map $\phi(n, x) = \phi_n(x)$ is a continuous surjection from $\{(n, x) \in \mathbb{N} \times \mathbb{N}^\mathbb{N} \mid x \in C_n\}$ onto $\bigcup_{n \in \mathbb{N}} A_n$. □

PROPOSITION 1.4.2. *Suppose that $(X_n)_{n \in \mathbb{N}}$ is a sequence of analytic spaces. Then $\prod_{n \in \mathbb{N}} X_n$ is analytic.*

PROOF. Fix closed sets $C_n \subseteq \mathbb{N}^\mathbb{N}$ for which there are continuous surjections $\phi_n \colon C_n \to X_n$. Then $\prod_{n \in \mathbb{N}} \phi_n$ is a continuous surjection from $\prod_{n \in \mathbb{N}} C_n$ onto $\prod_{n \in \mathbb{N}} X_n$. □

PROPOSITION 1.4.3. *Suppose that X is a Hausdorff space. Then $\bigcap_{n \in \mathbb{N}} A_n$ is analytic for all sequences $(A_n)_{n \in \mathbb{N}}$ of analytic subsets of X.*

PROOF. Proposition 1.4.2 ensures that there is a closed set $C \subseteq \mathbb{N}^\mathbb{N}$ for which there is a continuous surjection $\phi \colon C \to \prod_{n \in \mathbb{N}} A_n$. As X is Hausdorff, the set $D = \{x \in X^\mathbb{N} \mid \forall m, n \in \mathbb{N}\ x(m) = x(n)\}$ is closed, thus so too is the set $D' = \phi^{-1}(D)$. But the composition of the projection onto any coordinate with ϕ is a continuous surjection of D' onto $\bigcap_{n \in \mathbb{N}} A_n$. □

We will later see that the family of analytic subsets of a Hausdorff space is usually not closed under complements. Next, we note that the analytic sets are closed under continuous images and preimages.

Proposition 1.4.4. *Suppose that X is an analytic space, Y is a topological space, and $\phi\colon X \to Y$ is continuous. Then $\phi(X)$ is analytic.*

Proof. Fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous surjection $\psi\colon C \to X$. Then $\phi \circ \psi$ is a continuous surjection from C to $\phi(X)$. $\square$

Proposition 1.4.5. *Suppose that X is an analytic space, Y is a Hausdorff space, $\phi\colon X \to Y$ is continuous, and $A \subseteq Y$ is analytic. Then $\phi^{-1}(A)$ is analytic.*

Proof. Fix closed sets $C_A, C_X \subseteq \mathbb{N}^{\mathbb{N}}$ for which there are continuous surjections $\phi_A\colon C_A \to A$ and $\phi_X\colon C_X \to X$. As Y is Hausdorff, the set $\{(y,z) \in Y \times Y \mid y = z\}$ is closed, thus so too is the set $C = \{(c_A, c_X) \in C_A \times C_X \mid \phi_A(c_A) = (\phi \circ \phi_X)(c_X)\}$. Then $\phi^{-1}(A) = (\phi_X \circ \mathrm{proj}_{C_X})(C)$, so $\phi^{-1}(A)$ is analytic. $\square$

Finally, we note that the simplest subsets of analytic spaces are themselves analytic.

Proposition 1.4.6. *Suppose that X is an analytic space. Then every closed set $C \subseteq X$ is analytic.*

Proof. Fix a closed set $D \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous surjection $\phi\colon D \to X$. Then $\phi^{-1}(C)$ is closed and its image under ϕ is C. $\square$

Proposition 1.4.7. *Suppose that X is an analytic space. Then every open set $U \subseteq X$ is analytic.*

Proof. Fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous surjection $\phi\colon C \to X$. Then $\phi^{-1}(U)$ is an open subset of C, and is therefore a union of countably-many clopen subsets C_n of C. As $U = \bigcup_{n \in \mathbb{N}} \phi(C_n)$, Proposition 1.4.1 ensures that it is analytic. $\square$

Alternative representations. Here we mention several other representations of analytic sets.

Proposition 1.4.8. *Every non-empty analytic space is a continuous image of $\mathbb{N}^{\mathbb{N}}$.*

Proof. Suppose that X is a non-empty analytic space, and fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous surjection $\phi\colon C \to X$. By Proposition 1.2.5, there is a continuous retraction ψ from $\mathbb{N}^{\mathbb{N}}$ onto C. Then $\phi \circ \psi$ is a continuous surjection of $\mathbb{N}^{\mathbb{N}}$ onto X. $\square$

PROPOSITION 1.4.9. *Suppose that X is an analytic Hausdorff space and $Y \subseteq X$. Then Y is analytic if and only if there is a closed set $C \subseteq \mathbb{N}^{\mathbb{N}} \times X$ whose projection onto X is Y.*

PROOF. If Y is analytic, then there is a closed set $D \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous surjection $\phi \colon D \to Y$. Our assumption that X is Hausdorff ensures that the set $C = \mathrm{graph}(\phi)$ is closed, and clearly its projection onto X is Y.

Conversely, if $C \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is a closed set whose projection onto X is Y, then Proposition 1.4.2 ensures that $\mathbb{N}^{\mathbb{N}} \times X$ is analytic, so Proposition 1.4.6 implies that C is analytic, thus Proposition 1.4.4 yields that Y is analytic. □

In order to provide one more useful representation of analytic sets, we first need the following simple observation.

PROPOSITION 1.4.10. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ and $D \subseteq \mathbb{N}^{\mathbb{N}}$, X is a Hausdorff space, and $\phi \colon C \to X$ and $\psi \colon D \to X$ are continuous. Then for all $c \in C$ and $d \in D$ with the property that $\phi(c) \neq \psi(d)$, there exists $n \in \mathbb{N}$ such that $\overline{\phi(\mathcal{N}_{c \upharpoonright n})} \cap \psi(\mathcal{N}_{d \upharpoonright n}) = \emptyset$.*

PROOF. As X is Hausdorff, there are disjoint open neighborhoods U of $\phi(c)$ and V of $\psi(d)$. As ϕ and ψ are continuous, there exists $n \in \mathbb{N}$ sufficiently large such that $\phi(\mathcal{N}_{c \upharpoonright n}) \subseteq U$ and $\psi(\mathcal{N}_{d \upharpoonright n}) \subseteq V$. But then $\overline{\phi(\mathcal{N}_{c \upharpoonright n})}$ is disjoint from V, and therefore disjoint from $\psi(\mathcal{N}_{d \upharpoonright n})$. □

A *reduction* of a set $A \subseteq X$ to a set $B \subseteq Y$ is a function $\pi \colon X \to Y$ with the property that $x \in A \iff \phi(x) \in B$, for all $x \in X$.

PROPOSITION 1.4.11. *Suppose that X is a Hausdorff space. Then there is a Borel reduction of every analytic set $A \subseteq X$ to the set of ill-founded trees on $\mathbb{N}$ within the set of all trees on $\mathbb{N}$.*

PROOF. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to A$. Proposition 1.3.5 ensures that the function $T \colon X \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ given by $t \in T(x) \iff x \in \overline{\phi(\mathcal{N}_t)}$ is Borel. It only remains to note that by Proposition 1.4.10, a point of x is in A if and only if $T(x)$ is ill-founded. □

Analytic vs. Borel. We already have the machinery in hand to establish one half of the relationship between analytic and Borel sets.

PROPOSITION 1.4.12. *Suppose that X is an analytic Hausdorff space. Then every Borel subset of X is analytic.*

PROOF. By Proposition 1.3.1, it is sufficient to show that the family of analytic subsets of X is closed under countable unions and countable intersections, and contains every closed subset and every open subset of X. This follows from Propositions 1.4.1, 1.4.3, 1.4.6, and 1.4.7. $\square$

We will later see that the converse of this statement is usually false. Nevertheless, there are several useful weak converses.

PROPOSITION 1.4.13 (Lusin-Sierpiński). *Suppose that X is a Hausdorff space and $A \subseteq X$ is analytic. Then A is both an intersection of $\aleph_1$-many Borel sets and a union of $\aleph_1$-many Borel sets.*

PROOF. By Proposition 1.4.11, we need only establish that the set of ill-founded trees on $\mathbb{N}$, viewed as a subset of the set of all trees on $\mathbb{N}$, has the desired property. For each $\alpha < \omega_1$, let B_α denote the set of all trees T on $\mathbb{N}$ for which the pruning rank of $\emptyset$ within T is at least α, and let C_α denote the set of all $T \in B_\alpha$ for which the pruning rank of T is at most α. Proposition 1.3.4 ensures that these sets are Borel, and clearly the set of ill-founded trees on $\mathbb{N}$ is both $\bigcap_{\alpha<\omega_1} B_\alpha$ and $\bigcup_{\alpha<\omega_1} C_\alpha$. $\square$

We say that sets $A, B \subseteq X$ are *separated* by a set $C \subseteq X$ if $A \subseteq C$ and $B \cap C = \emptyset$.

THEOREM 1.4.14 (Lusin). *Suppose that X is a Hausdorff space and $A, B \subseteq X$ are disjoint analytic sets. Then there is a Borel set $C \subseteq X$ separating A from B.*

PROOF. By Proposition 1.4.8, we can assume that there are continuous surjections $\phi\colon \mathbb{N}^\mathbb{N} \to A$ and $\psi\colon \mathbb{N}^\mathbb{N} \to B$. Define $A_t = \phi(\mathcal{N}_t)$ and $B_t = \psi(\mathcal{N}_t)$, for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let T denote the tree on $\mathbb{N} \times \mathbb{N}$ consisting of all $(s,t) \in \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \times \mathbb{N}^n$ such that $\overline{A_s} \cap B_t \neq \emptyset$. Proposition 1.4.10 ensures that T is well-founded, so the pruning rank of every $(s,t) \in T$ within T is a countable ordinal, thus we can recursively construct Borel sets $C_{(s,t)} \subseteq X$ separating A_s from B_t with each $(s,t) \in \bigcup_{n\in\mathbb{N}} \mathbb{N}^n \times \mathbb{N}^n$ by setting $C_{(s,t)} = \overline{A_s}$ for all $(s,t) \notin T$, and $C_{(s,t)} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} C_{(s\frown(m),t\frown(n))}$ for all $(s,t) \in T$. Set $C = C_{(\emptyset,\emptyset)}$. $\square$

We say that a subset of a topological space is *bi-analytic* if both it and its complement are analytic.

THEOREM 1.4.15 (Souslin). *Suppose that X is a Hausdorff space. Then every bi-analytic subset of X is Borel.*

PROOF. This is just a rephrasing of the special case of Theorem 1.4.14 in which $A = X \setminus B$. $\square$

THEOREM 1.4.16 (Souslin). *Suppose that X is an analytic Hausdorff space. Then a subset of X is bi-analytic if and only if it is Borel.*

PROOF. By Proposition 1.4.12 and Theorem 1.4.15. $\square$

We say that a function $\phi\colon X \to Y$ is *analytic* if preimages of analytic sets are analytic.

THEOREM 1.4.17. *Suppose that X and Y are analytic Hausdorff spaces and $\phi\colon X \to Y$. Then the following are equivalent:*

(1) *The function ϕ is Borel.*
(2) *The graph of ϕ is Borel.*
(3) *The graph of ϕ is analytic.*
(4) *The function ϕ is analytic.*

PROOF. Proposition 1.4.10 ensures that there is a sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of Y which separates points. To see (1) $\Longrightarrow$ (2), it is sufficient to check that

$$\mathrm{graph}(\phi) = \bigcap_{n\in\mathbb{N}} (\phi^{-1}(B_n) \times B_n) \cup (\phi^{-1}(Y \setminus B_n) \times (Y \setminus B_n)).$$

Towards this end, note that if $n \in \mathbb{N}$ and $x \in X$, then $\phi(x)$ is in B_n or $Y \setminus B_n$, so $(x, \phi(x))$ is in $\phi^{-1}(B_n) \times B_n$ or $\phi^{-1}(Y \setminus B_n) \times (Y \setminus B_n)$. Conversely, if $(x, y) \notin \mathrm{graph}(\phi)$, then there exists $n \in \mathbb{N}$ for which $\phi(x) \in B_n$ and $y \notin B_n$, so $x \notin \phi^{-1}(Y \setminus B_n)$, thus (x, y) is in neither $\phi^{-1}(B_n) \times B_n$ nor $\phi^{-1}(Y \setminus B_n) \times (Y \setminus B_n)$.

To see (2) $\Longrightarrow$ (3), note that Proposition 1.4.2 ensures that $X \times Y$ is analytic, so Proposition 1.4.12 implies that every Borel subset of $X \times Y$ is analytic.

To see (3) $\Longrightarrow$ (4), note that if $A \subseteq Y$ is analytic, then $\phi^{-1}(A)$ is the projection of $\mathrm{graph}(\phi) \cap (X \times A)$ onto X, and since Propositions 1.4.2, 1.4.3, and 1.4.4 ensure that the family of analytic sets is closed under products, intersections, and images under continuous functions, it follows that the latter set is analytic.

To see (4) $\Longrightarrow$ (1), note that if $B \subseteq Y$ is Borel, then Proposition 1.4.12 ensures that it is bi-analytic, thus so too is $\phi^{-1}(B)$, in which case Theorem 1.4.15 implies that $\phi^{-1}(B)$ is Borel. □

As corollaries, we can now extend our earlier results on the closure of the family of analytic sets under continuous images and preimages.

PROPOSITION 1.4.18. *Suppose that X is an analytic space, Y is an analytic Hausdorff space, and $\phi\colon X \to Y$ is Borel. Then the set $\phi(X)$ is analytic.*

PROOF. By Proposition 1.4.8, we can assume that there is a continuous surjection $\psi\colon \mathbb{N}^{\mathbb{N}} \to X$. Then $\phi \circ \psi$ is Borel, so Theorem 1.4.17 ensures that its graph is analytic. Proposition 1.4.4 therefore ensures that $\mathrm{proj}_Y(\mathrm{graph}(\phi \circ \psi))$ is analytic. But this latter set is $\phi(X)$. □

PROPOSITION 1.4.19. *Suppose that X and Y are analytic Hausdorff spaces, $\phi\colon X \to Y$ is Borel, and $A \subseteq Y$ is analytic. Then so is $\phi^{-1}(A)$.*

Proof. This is just a rephrasing of (1) $\Longrightarrow$ (4) in the statement of Theorem 1.4.17. □

A Borel bijection $\phi \colon X \to Y$ is a *Borel isomorphism* if it sends Borel sets to Borel sets.

Proposition 1.4.20. *Every Borel bijection $\phi \colon X \to Y$ between analytic Hausdorff spaces is a Borel isomorphism.*

Proof. Suppose that $B \subseteq X$ is Borel. Proposition 1.4.12 then ensures that B is bi-analytic, so Proposition 1.4.18 implies that $\phi(B)$ is bi-analytic, thus Proposition 1.4.15 yields that $\phi(B)$ is Borel. □

We next turn our attention to generalizations of Theorem 1.4.14.

Theorem 1.4.21 (Lusin). *Suppose that X is a Hausdorff space, n is a natural number, and $(A_i)_{i \leq n}$ is a sequence of analytic subsets of X with empty intersection. Then there is a sequence $(B_i)_{i \leq n}$ of Borel subsets of X with empty intersection such that $A_i \subseteq B_i$, for all $i \leq n$.*

Proof. While the proof of Theorem 1.4.14 easily generalizes, we will instead obtain the desired result as a consequence of Theorem 1.4.14 itself. Towards this end, note that Proposition 1.4.12 and the latter allow us to recursively construct Borel sets $B_i \subseteq X_i$ separating A_i from $\bigcap_{j<i} B_j \cap \bigcap_{j>i} A_j$, for all $i < n$, and define $B_n = X \setminus \bigcap_{i<n} B_i$. □

The generalization to countably-infinite sequences requires a little more thought.

Theorem 1.4.22 (Novikov). *Suppose that X is a Hausdorff space and $(A_n)_{n \in \mathbb{N}}$ is a sequence of analytic subsets of X with empty intersection. Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel subsets of X with empty intersection such that $A_n \subseteq B_n$, for all $n \in \mathbb{N}$.*

Proof. The most obvious attempt at generalizing the proof of Theorem 1.4.14 is to fix continuous surjections $\phi_k \colon \mathbb{N}^{\mathbb{N}} \to A_k$ and to apply the same argument to the tree of all sequences $(t_k)_{k \in \mathbb{N}} \in \bigcup_{n \in \mathbb{N}} (\mathbb{N}^n)^{\mathbb{N}}$ for which $\phi_0(\mathcal{N}_{t_0}) \cap \bigcap_{k>0} \overline{\phi_k(\mathcal{N}_{t_k})} \neq \emptyset$. Unfortunately, this tree is uncountably branching, so there is little reason to believe that the sets defined in the course of the construction are Borel. Fortunately, this problem is easily remedied by instead focusing upon the tree of all $(t_k)_{k<n} \in (\mathbb{N}^n)^n$ with $n \neq 0 \Longrightarrow \phi_0(\mathcal{N}_{t_0}) \cap \bigcap_{0<k<n} \overline{\phi_k(\mathcal{N}_{t_k})} \neq \emptyset$, where n varies over $\mathbb{N}$. □

We next note another kind of generalization of Theorem 1.4.14. A set $\mathcal{X}$ of subsets of an analytic Hausdorff space X is $\mathbf{\Pi}^1_1$-*on*-$\mathbf{\Sigma}^1_1$ if for every analytic Hausdorff space W and analytic set $R \subseteq W \times X$, the corresponding set $\{w \in W \mid R_w \in \mathcal{X}\}$ is co-analytic.

THEOREM 1.4.23. *Suppose that X is an analytic Hausdorff space, $\mathcal{X} \subseteq \mathcal{P}(X)$ is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$, and $A \in \mathcal{X}$ is analytic. Then there is a Borel set $B \in \mathcal{X}$ containing A.*

PROOF. By Propositions 1.4.11, 1.4.18, and 1.4.19, we can assume that X is a set of trees on $\mathbb{N}$, and A consists of the ill-founded trees in X. Proposition 1.3.4 ensures that the set B_α of all trees T in X for which the pruning rank of $\emptyset$ within T is at least α is a Borel subset of X, for all $\alpha < \omega_1$. Note A is contained in each of these sets. We will show that if none of them is in $\mathcal{X}$, then A is Borel.

Let R denote the set of all pairs $(S, T) \in X \times X$ for which there is a homomorphism $\phi \colon S \to T$ from $\sqsubset \upharpoonright S$ to $\sqsubset \upharpoonright T$. Propositions 1.2.7 and 1.4.4 ensure that R is analytic. Proposition 1.1.7 implies that if $T \in A$ then $R_T = A$, and if $T \in X \setminus A$ then $R_T = B_\alpha$, where α is the pruning rank of $\emptyset$ within T. It therefore follows that $A = \{T \in X \mid R_T \in \mathcal{X}\}$. The fact that $\mathcal{X}$ is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$ then ensures that A is co-analytic, so Theorem 1.4.15 implies that A is Borel. $\square$

The perfect set theorem. Here we show that analytic Hausdorff spaces satisfy the continuum hypothesis.

THEOREM 1.4.24 (Souslin). *Suppose that X is an analytic Hausdorff space. Then exactly one of the following holds:*

(1) *The set X is countable.*
(2) *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \to X$.*

PROOF. We will simply repeat the proof of Theorem 1.2.6, albeit utilizing a slightly modified version of the perfection derivative. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$. Recursively define a decreasing sequence of trees T^α on $\mathbb{N}$, along with a decreasing sequence of sets $X^\alpha = X \setminus \bigcup_{t \in \mathbb{N}^{<\mathbb{N}} \setminus T^\alpha} \phi(\mathcal{N}_t)$, by setting $T^0 = \mathbb{N}^{<\mathbb{N}}$, $T^{\alpha+1} = \{t \in T^\alpha \mid |\phi(\mathcal{N}_t) \cap X^\alpha| \geq 2\}$ for all ordinals $\alpha < \omega_1$, and $T^\lambda = \bigcap_{\alpha < \lambda} T^\alpha$ for all limit ordinals $\lambda < \omega_1$. Let α denote the least ordinal for which $T^\alpha = T^{\alpha+1}$.

If $T^\alpha = \emptyset$, then $X^\alpha = \emptyset$, so for all $x \in X$, there exists $\beta < \alpha$ with $x \in X^\beta \setminus X^{\beta+1}$. As $X^\beta \setminus X^{\beta+1} = \bigcup_{t \in T^\beta \setminus T^{\beta+1}} \phi(\mathcal{N}_t) \cap X^\beta$, it follows that X is countable.

To handle the case that $T^\alpha \neq \emptyset$, we will recursively construct a function $f \colon 2^{<\mathbb{N}} \to T^\alpha$ such that:

(a) $\forall i < 2 \forall t \in 2^{<\mathbb{N}}\ f(t) \sqsubset f(t \frown (i))$.
(b) $\forall t \in 2^{<\mathbb{N}}\ \phi(\mathcal{N}_{f(t \frown (0))}) \cap \phi(\mathcal{N}_{f(t \frown (1))}) = \emptyset$.

We begin by setting $f(\emptyset) = \emptyset$. Given $t \in 2^{<\mathbb{N}}$ for which we have already found $f(t)$, fix distinct points $x_{0,t}, x_{1,t} \in \phi(\mathcal{N}_{f(t)}) \cap X^\alpha$. Then there exist

$b_{0,t}, b_{1,t} \in \mathcal{N}_{f(t)}$ such that $x_{i,t} = \phi(b_{i,t})$. As ϕ is continuous and X is Hausdorff, there exists $n_t \in \mathbb{N}$ for which $\phi(\mathcal{N}_{b_{0,t} \restriction n_t}) \cap \phi(\mathcal{N}_{b_{1,t} \restriction n_t}) = \emptyset$. Setting $f(t \frown (i)) = b_{i,t} \restriction n_t$, this completes the recursive construction.

Condition (a) ensures that we obtain a function $\psi \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by setting $\psi(c) = \bigcup_{n \in \mathbb{N}} f(c \restriction n)$, and moreover, that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $\psi(\mathcal{N}_{c \restriction n}) \subseteq \mathcal{N}_{f(c \restriction n)} \subseteq \mathcal{N}_{\psi(c) \restriction n}$, so ψ is continuous, thus the function $\pi = \phi \circ \psi$ is also continuous. To see that π is injective, note that if $c, d \in 2^{\mathbb{N}}$ are distinct and $n \in \mathbb{N}$ is minimal for which $c \restriction n \neq d \restriction n$, then $\pi(c) \in \phi(\mathcal{N}_{f(c \restriction n)})$ and $\pi(d) \in \phi(\mathcal{N}_{f(d \restriction n)})$, so condition (b) ensures that $\pi(c) \neq \pi(d)$. □

As a corollary, we obtain the following.

Proposition 1.4.25. *Suppose that X and Y are analytic Hausdorff spaces. Then there is an injection of X into Y if and only if there is a Borel injection of X into Y.*

Proof. We can clearly assume that X is uncountable and there is an injection of X into Y, in which case Y is uncountable as well. Theorem 1.4.24 then yields a continuous injection of $2^{\mathbb{N}}$ into Y, so we need only show that there is a Borel injection of X into $2^{\mathbb{N}}$. Towards this end, note that by Proposition 1.4.8, there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$, and observe that the function $T \colon X \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ given by $t \in T(x) \iff x \in \overline{\phi(\mathcal{N}_t)}$ yields such an injection, by Propositions 1.3.5 and 1.4.10. □

Complete and universal sets. Here we establish the analog of Proposition 1.2.4 for analytic sets.

Proposition 1.4.26. *Suppose that X is an analytic Hausdorff space. Then there is an analytic set $A \subseteq 2^{\mathbb{N}} \times X$ whose vertical sections are exactly the analytic subsets of X.*

Proof. We first handle the case that $X = \mathbb{N}^{\mathbb{N}}$. By Proposition 1.2.4, there is a closed set $C \subseteq 2^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}})$ whose vertical sections are exactly the closed subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$. Proposition 1.4.9 then ensures that the set $A = \{(c, b) \in 2^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \mid \exists a \in \mathbb{N}^{\mathbb{N}}\ (c, (b, a)) \in C\}$ is as desired.

We now take care of the general case. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$. As Proposition 1.4.5 ensures that every analytic subset of X is the image of an analytic subset of $\mathbb{N}^{\mathbb{N}}$ under ϕ, Proposition 1.4.4 implies that the set $A_X = \{(c, \phi(b)) \mid (c, b) \in A\}$ is as desired. □

As a corollary, we obtain the following.

PROPOSITION 1.4.27. *Suppose that X is an analytic Hausdorff space. Then exactly one of the following holds:*

(1) *The set X is countable.*
(2) *There is an analytic subset of X which is not Borel/co-analytic.*

PROOF. It is sufficient to show that if X is uncountable, then there is an analytic subset of X which is not co-analytic.

We first consider the case that $X = 2^{\mathbb{N}}$. By Proposition 1.4.26, there is an analytic set $A \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ whose vertical sections are exactly the analytic subsets of $2^{\mathbb{N}}$. Proposition 1.4.5 then yields that the set $B = \{c \in 2^{\mathbb{N}} \mid (c, c) \in A\}$ is analytic. Suppose, towards a contradiction, that B is also co-analytic. Fix $c \in 2^{\mathbb{N}}$ with $A_c = 2^{\mathbb{N}} \setminus B$, and note that $c \in B \iff (c, c) \in A \iff c \notin B$.

For the general case, appeal to Theorem 1.4.24 to obtain a continuous injection $\phi \colon 2^{\mathbb{N}} \to X$. Proposition 1.4.4 then ensures that $\phi(B)$ is analytic. But if $\phi(B)$ were also co-analytic, then Proposition 1.4.5 would imply that so too is B. $\square$

We next turn our attention to another sort of universality. We say that an analytic subset of $\mathbb{N}^{\mathbb{N}}$ is *complete analytic* if it continuously reduces every analytic subset of $\mathbb{N}^{\mathbb{N}}$.

PROPOSITION 1.4.28. *The set of ill-founded trees on $\mathbb{N}$ is complete analytic.*

PROOF. To see that the set of ill-founded trees on $\mathbb{N}$ is analytic, note that a tree on $\mathbb{N}$ is ill-founded if and only if it is in the projection onto $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ of the set of all pairs (T, x), where T is a tree on $\mathbb{N}$ and $x \in [T]$, and appeal to Propositions 1.2.3 and 1.4.4.

To see that every analytic set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is continuously reducible to the set of ill-founded trees on $\mathbb{N}$, appeal to Proposition 1.4.9 to obtain a closed set $C \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ whose projection onto the leftmost coordinate is A, appeal to Proposition 1.2.1 to obtain a tree T on $\mathbb{N} \times \mathbb{N}$ with the property that $C = [T]$, and observe that the function $\phi \colon \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ given by $t \in \phi(b) \iff (b \upharpoonright |t|, t) \in T$ is as desired, since $t \in \phi(b) \iff b \in \bigcup_{s \in T^t} \mathcal{N}_s \iff b \in \mathbb{N}^{\mathbb{N}} \setminus \bigcup_{s \in \mathbb{N}^{|t|} \setminus T^t} \mathcal{N}_s$. $\square$

PROPOSITION 1.4.29. *The set of trees on $\mathbb{N}$ with uncountably-many branches is complete analytic.*

PROOF. To see that the set of trees on $\mathbb{N}$ with uncountably-many branches is analytic, note that by Proposition 1.1.12, a tree on $\mathbb{N}$ has uncountably-many branches if and only if it is in the projection onto $\mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ of the set of all pairs $(\phi, T) \in (\mathbb{N}^{<\mathbb{N}})^{2^{<\mathbb{N}}} \times \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ for which T is a tree on $\mathbb{N}$ and ϕ is an embedding of $\sqsubseteq \upharpoonright 2^{<\mathbb{N}}$ into $\sqsubseteq \upharpoonright T$. As

Proposition 1.2.8 ensures that the latter set is closed, Proposition 1.4.4 yields that the desired set is analytic.

To see that every analytic set $A \subseteq \mathbb{N}^\mathbb{N}$ is continuously reducible to the set of trees on $\mathbb{N}$ with uncountably-many branches, it is sufficient to handle the special case that A is the set of ill-founded trees on $\mathbb{N}$, by Proposition 1.4.28. Towards this end, consider the map sending a tree S on $\mathbb{N}$ to the tree T on $\mathbb{N} \times 2$ consisting of all pairs $(s, t) \in S \times 2^{<\mathbb{N}}$ for which $|s| = |t|$. □

Proposition 1.4.30. *The set of trees on* $\mathbb{N}$ *which do not have exactly one branch is complete analytic.*

Proof. To see that the set of trees on $\mathbb{N}$ which do not have exactly one branch is analytic, appeal first to Proposition 1.1.6 to see that a tree T on $\mathbb{N}$ does not have exactly one branch if and only if there exists $n \in \mathbb{N}$ such that for all $s \in \mathbb{N}^n \cap T$, there exists a distinct $t \in \mathbb{N}^n \cap T$ of pruning rank at least that of s within T. By Proposition 1.1.7, this is equivalent to the statement that for all $s \in \mathbb{N}^n \cap T$, there exists a distinct $t \in \mathbb{N}^n \cap T$ for which there is a homomorphism from $\sqsubset \restriction T_s$ to $\sqsubset \restriction T_t$, where T_r denotes the tree of all sequences in T extending r. As Propositions 1.2.7 and 1.4.4 ensure that the existence of such a homomorphism is an analytic condition, Propositions 1.4.1 and 1.4.3 yield the desired result.

To see that every analytic set $A \subseteq \mathbb{N}^\mathbb{N}$ is continuously reducible to the set of trees on $\mathbb{N}$ which do not have exactly one branch, note that any map sending a tree on $\mathbb{N}$ to one with exactly one more branch is a reduction of the set of ill-founded trees on $\mathbb{N}$ to this set, and apply Proposition 1.4.28. □

Well-founded relations. Given sets $R \subseteq X \times X$ and $Y \subseteq X$, we say that a point $y \in Y$ is an R-*minimal* element of Y if no element of Y is R-related to y. We say that R is *well-founded* if every non-empty subset of X has an R-minimal element. The *rank* of a point x with respect to such a relation is the least ordinal α strictly greater than the rank of every $y \mathrel{R} x$, and the *rank* of R is the least ordinal strictly greater than all of these ranks.

Theorem 1.4.31 (Kunen-Martin). *Suppose that* X *is a Hausdorff space and* $R \subseteq X \times X$ *is analytic and well-founded. Then the rank of* R *is countable.*

Proof. By Proposition 1.4.8, we can assume that there is a continuous surjection $(\phi, \psi) \colon \mathbb{N}^\mathbb{N} \to R$. Let S denote the well-founded tree on $\mathbb{N}^\mathbb{N}$ consisting of all $s \in (\mathbb{N}^\mathbb{N})^{<\mathbb{N}}$ such that $\phi(s(i)) = \psi(s(i + 1))$, for all $i < |s| - 1$. A straightforward induction using Proposition 1.1.5 shows that the pruning rank of each non-empty sequence $s \in S$ within S is the rank of $\phi(s(|s| - 1))$ with respect to R. It follows that the rank of each $x \in X$ with respect to R is at most the pruning rank of $\emptyset$ within S, thus the rank of R is at most the pruning rank of S.

Let T denote the well-founded tree consisting of all $t \in \bigcup_{n \in \mathbb{N}} (\mathbb{N}^n)^n$ with the property that $\phi(\mathcal{N}_{t(i)}) \cap \psi(\mathcal{N}_{t(i+1)}) \neq \emptyset$, for all $i < |t| - 1$. As there is a homomorphism from $\sqsubset \restriction S$ to $\sqsubset \restriction T$, Proposition 1.1.7 ensures that the pruning rank of S is at most that of T. But T is countable, thus so too is its pruning rank. $\square$

5. Injectively analytic spaces

A topological space is *injectively analytic* if it is a continuous injective image of a closed subset of $\mathbb{N}^\mathbb{N}$. Some properties of Borel sets come from those of injectively analytic sets. In this section, we establish the basic properties of the latter.

Closure properties. Here we establish the basic closure properties of injectively analytic sets, beginning with the simplest operations.

Proposition 1.5.1. *Suppose that X is a topological space and $(A_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint injectively analytic subsets of X. Then $\bigcup_{n \in \mathbb{N}} A_n$ is injectively analytic.*

Proof. Fix closed sets $C_n \subseteq \mathbb{N}^\mathbb{N}$ for which there are continuous bijections $\phi_n \colon C_n \to A_n$. Then the function $\phi(n, x) = \phi_n(x)$ is a continuous bijection from $\{(n, x) \in \mathbb{N} \times \mathbb{N}^\mathbb{N} \mid x \in C_n\}$ to $\bigcup_{n \in \mathbb{N}} A_n$. $\square$

Proposition 1.5.2. *Suppose that $(X_n)_{n \in \mathbb{N}}$ is a sequence of injectively analytic spaces. Then $\prod_{n \in \mathbb{N}} X_n$ is injectively analytic.*

Proof. Fix closed sets $C_n \subseteq \mathbb{N}^\mathbb{N}$ for which there are continuous bijections $\phi_n \colon C_n \to X_n$. Then $\prod_{n \in \mathbb{N}} \phi_n$ is a continuous bijection from $\prod_{n \in \mathbb{N}} C_n$ to $\prod_{n \in \mathbb{N}} X_n$. $\square$

Proposition 1.5.3. *Suppose that X is a Hausdorff space. Then $\bigcap_{n \in \mathbb{N}} A_n$ is injectively analytic for every sequence $(A_n)_{n \in \mathbb{N}}$ of injectively analytic subsets of X.*

Proof. Proposition 1.5.2 ensures that there is a closed set $C \subseteq \mathbb{N}^\mathbb{N}$ for which there is a continuous bijection $\phi \colon C \to \prod_{n \in \mathbb{N}} A_n$. As X is Hausdorff, the set $D = \{x \in X^\mathbb{N} \mid \forall m, n \in \mathbb{N}\ x(m) = x(n)\}$ is closed, thus so too is the set $D' = \phi^{-1}(D)$. But the composition of the projection onto any coordinate with ϕ is a continuous bijection from D' to $\bigcap_{n \in \mathbb{N}} A_n$. $\square$

We next note that the simplest subsets of injectively analytic spaces are themselves injectively analytic.

Proposition 1.5.4. *Suppose that X is an injectively analytic space. Then every closed set $C \subseteq X$ is injectively analytic.*

Proof. Fix a closed set $D \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous bijection $\phi \colon D \to X$. Then $\phi^{-1}(C)$ is closed and its ϕ-image is C. □

Proposition 1.5.5. *Suppose that X is an injectively analytic space. Then every open set $U \subseteq X$ is injectively analytic.*

Proof. Fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous bijection $\phi \colon C \to X$. Then $\phi^{-1}(U)$ is an open subset of C, and is therefore a union of countably-many pairwise disjoint clopen subsets C_n of C. As $U = \bigcup_{n \in \mathbb{N}} \phi(C_n)$, Proposition 1.5.1 ensures that it is injectively analytic. □

Borel vs. injectively analytic. We already have the machinery in hand to establish one half of the relationship between Borel and injectively analytic sets.

Proposition 1.5.6. *Suppose that X is an injectively analytic Hausdorff space. Then every Borel subset of X is injectively analytic.*

Proof. By Proposition 1.3.1, it is sufficient to show that the family of injectively analytic subsets of X is closed under countable disjoint unions and countable intersections, and contains every closed subset and every open subset of X. This follows from Propositions 1.5.1, 1.5.3, 1.5.4, and 1.5.5. □

In order to establish the converse, we will use the following consequence of separation.

Theorem 1.5.7. *Suppose that X is an injectively analytic space, Y is an analytic Hausdorff space, and $\phi \colon X \to Y$ is a Borel injection. Then $\phi(X)$ is Borel.*

Proof. Fix a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous bijection $\psi \colon C \to X$. Proposition 1.4.18 then ensures that for each $t \in \mathbb{N}^{<\mathbb{N}}$, the set $A_t = (\phi \circ \psi)(C \cap \mathcal{N}_t)$ is analytic. We will recursively construct sequences $(B_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of Borel subsets of Y such that:

(1) $\forall t \in \mathbb{N}^{<\mathbb{N}}\ A_t \subseteq B_t$.
(2) $\forall t \in \mathbb{N}^{<\mathbb{N}}\ (B_{t \frown (n)})_{n \in \mathbb{N}}$ partitions a subset of B_t.

We begin by setting $B_\emptyset = Y$. Given $t \in \mathbb{N}^{<\mathbb{N}}$ for which we have found B_t, appeal to Theorem 1.4.14 to find Borel sets $B_{m,n,t} \subseteq B_t$ separating

$A_{t\frown(m)}$ from $A_{t\frown(n)}$, for all $m < n$. To complete the recursive definition, set $B_{t\frown(m)} = \bigcap_{n>m} B_{m,n,t} \setminus \bigcup_{n<m} B_{n,m,t}$, for all $m \in \mathbb{N}$.

For each $t \in \mathbb{N}^{<\mathbb{N}}$, set $C_t = \overline{A_t} \cap B_t$. As Y is Hausdorff, it follows that $\phi(X) = \bigcup_{b \in \mathbb{N}^\mathbb{N}} \bigcap_{n \in \mathbb{N}} C_{b \upharpoonright n} = \bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} C_t$, thus $\phi(X)$ is Borel. $\square$

In particular, we have the following.

Theorem 1.5.8. *Suppose that X is an analytic Hausdorff space. Then every injectively analytic set $B \subseteq X$ is Borel.*

Proof. Fix a closed set $C \subseteq \mathbb{N}^\mathbb{N}$ for which there is a continuous bijection $\phi\colon C \to B$, and appeal to Theorem 1.5.7. $\square$

Remark 1.5.9. By appealing to the proof of Theorem 1.5.7 rather than its statement, one can eliminate the assumption that X is analytic in the statement of Theorem 1.5.8.

Theorem 1.5.10. *Suppose that X is an injectively analytic Hausdorff space. Then a set $Y \subseteq X$ is Borel if and only if it is injectively analytic.*

Proof. By Proposition 1.5.6 and Theorem 1.5.8. $\square$

Theorem 1.5.11. *Suppose that X and Y are injectively analytic Hausdorff spaces and $R \subseteq X \times Y$ is Borel. Then $\{x \in X \mid |R_x| = 1\}$ is co-analytic.*

Proof. As Proposition 1.5.2 ensures that $X \times Y$ is injectively analytic, Proposition 1.5.6 yields a closed set $C \subseteq \mathbb{N}^\mathbb{N}$ for which there is a continuous bijection $(\phi, \psi)\colon C \to R$. Proposition 1.3.5 ensures that the function $T\colon X \to \mathcal{P}(\mathbb{N}^{<\mathbb{N}})$ given by $t \in T(x) \iff x \in \overline{\phi(C \cap \mathcal{N}_t)}$ is Borel, and Proposition 1.4.10 implies $|R_x| = 1$ if and only if there is a unique branch through $T(x)$, so the desired result follows from Propositions 1.4.19 and 1.4.30. $\square$

Remark 1.5.12. The above argument goes through under the somewhat more general assumption that X is an analytic Hausdorff space, Y is a topological space, and $R \subseteq X \times Y$ is injectively analytic.

The isomorphism theorem. Here we establish a version of Proposition 1.4.25 for injectively analytic spaces.

Theorem 1.5.13 (Schröder-Bernstein). *Suppose that X and Y are injectively analytic Hausdorff spaces for which there are Borel injections $\phi\colon X \to Y$ and $\psi\colon Y \to X$. Then there is a Borel bijection $\pi\colon X \to Y$.*

Proof. We will show the stronger fact that there is a Borel set $X' \subseteq X$ for which the function $\pi = (\phi \restriction X') \cup (\psi^{-1} \restriction (X \setminus X'))$ is as desired. Towards this end, note that the existence of such a set necessitates that the set $X'_0 = X \setminus \psi(Y)$ is contained in X'. Recursively defining $Y'_n = \phi(X'_n)$ and $X'_{n+1} = \psi(Y'_n)$, a straightforward induction then shows that $X'_n \subseteq X'$, for all $n \in \mathbb{N}$. We will show that the set $X' = \bigcup_{n \in \mathbb{N}} X'_n$ is as desired. Towards this end, note first that the corresponding function π has full domain, since $X \setminus X' \subseteq \mathrm{dom}(\psi^{-1})$. As $\phi(X') = \bigcup_{n \in \mathbb{N}} Y'_n = \psi^{-1}(X')$, the sets $\phi(X')$ and $\psi^{-1}(X \setminus X')$ partition Y, thus π is bijective. □

Theorem 1.5.14. *Suppose that X and Y are injectively analytic Hausdorff spaces. Then there is a bijection between X and Y if and only if there is a Borel bijection between X and Y.*

Proof. By Proposition 1.4.25 and Theorem 1.5.13. □

6. Polish spaces

A *Polish space* is a completely metrizable second countable topological space. The structure of such spaces is the primary focus of classical descriptive set theory. Here we note several of their properties.

Closure properties. Here we establish a pair of basic facts concerning complete metric spaces and completely metrizable spaces.

Proposition 1.6.1. *Suppose that X is a complete metric space. Then a set $Y \subseteq X$ is complete if and only if it is closed.*

Proof. If Y is closed, then every Cauchy sequence of points of Y converges to a point of Y, thus Y is complete. Conversely, if Y is not closed, then there is a sequence $(y_n)_{n \in \mathbb{N}}$ of points of Y converging to a point of $X \setminus Y$, and since every such sequence is necessarily Cauchy, it follows that Y is not complete. □

A set is G_δ if it is an intersection of countably-many open sets.

Proposition 1.6.2. *Suppose that X is a metric space. Then every closed set $C \subseteq X$ is G_δ.*

Proof. Simply note that $C = \bigcap_{n \in \mathbb{N}} \mathcal{B}(C, 1/n)$. □

Proposition 1.6.3. *Suppose that X is a completely metrizable space. Then a set $Y \subseteq X$ is completely metrizable if and only if it is G_δ.*

PROOF. Suppose first that there is a compatible complete metric d_Y on Y. For each $\epsilon > 0$, let U_ϵ denote the union of all open sets $U \subseteq X$ for which $U \cap Y$ has d_Y-diameter at most ϵ. As every point of $\bigcap_{\epsilon>0} U_\epsilon \cap \overline{Y}$ is a limit of a d_Y-Cauchy sequence, it follows that $Y = \bigcap_{\epsilon>0} U_\epsilon \cap \overline{Y}$, so Proposition 1.6.2 yields that Y is G_δ.

Conversely, to show that every G_δ set $Y \subseteq X$ is completely metrizable, it is sufficient to show that there is a closed continuous injection $\pi \colon Y \to X \times \mathbb{R}^\mathbb{N}$, as Proposition 1.6.1 then ensures that we will obtain a compatible complete metric on Y by pulling back the product of any compatible complete metric d_X on X with the usual complete metric on $\mathbb{R}^\mathbb{N}$ through π. Towards this end, fix open sets $U_k \subseteq X$ with the property that $Y = \bigcap_{k\in\mathbb{N}} U_k$, and define $\pi \colon Y \to X \times \mathbb{R}^\mathbb{N}$ by $\pi(y) = (y, (1/d_X(y, X \setminus U_k))_{k\in\mathbb{N}})$. It is clear that π is injective. To see that π is continuous, note that $\pi(y_n) \to \pi(y)$ whenever $y_n \to y$. To see that π is closed, suppose that $C \subseteq Y$ is closed, and observe that if $(c_n)_{n\in\mathbb{N}}$ is a sequence of points of C for which $(\pi(c_n))_{n\in\mathbb{N}}$ converges to some point (x, r), then $c_n \to x$. It also follows that for each $k \in \mathbb{N}$, the sequence $(d_X(c_n, X \setminus U_k))_{n\in\mathbb{N}}$ is bounded away from zero, so $x \in U_k$, thus $x \in Y$, hence $x \in C$. The continuity of π therefore ensures that $\pi(c_n) \to \pi(x)$, so $\pi(x) = (x, r)$, thus the latter is in $\pi(C)$. □

Injectively analytic spaces vs. Polish spaces. We begin this section with a useful lemma concerning complete metric spaces.

PROPOSITION 1.6.4. *Suppose that X is a complete metric space and $(X_n)_{n\in\mathbb{N}}$ is a sequence of non-empty subsets of X with the property that* $\operatorname{diam}(X_n) \to 0$ *and* $\forall n \in \mathbb{N}\ \overline{X_{n+1}} \subseteq X_n$. *Then* $|\bigcap_{n\in\mathbb{N}} X_n| = 1$.

PROOF. The fact that $\operatorname{diam}(X_n) \to 0$ ensures that $|\bigcap_{n\in\mathbb{N}} X_n| \leq 1$. To see that $|\bigcap_{n\in\mathbb{N}} X_n| \geq 1$, fix $x_n \in X_n$ for all $n \in \mathbb{N}$. The fact that $\operatorname{diam}(X_n) \to 0$ ensures that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, so completeness yields $x \in X$ for which $x_n \to x$, in which case $x \in \bigcap_{n\in\mathbb{N}} \overline{X_n} \subseteq \bigcap_{n\in\mathbb{N}} X_n$. □

We next characterize a strengthening of injective analyticity.

PROPOSITION 1.6.5. *Suppose that X is a topological space. Then X is homeomorphic to a closed subspace of $\mathbb{N}^\mathbb{N}$ if and only if X is a zero-dimensional Polish space.*

PROOF. By Proposition 1.6.1, it is sufficient to show that if X is a zero-dimensional Polish space, then X is homeomorphic to a closed subspace of $\mathbb{N}^\mathbb{N}$. Towards this end, fix a compatible complete metric d on X, and recursively construct a sequence $(U_t)_{t\in\mathbb{N}^{<\mathbb{N}}}$ of clopen subsets of X such that:

(1) $U_\emptyset = X$.

(2) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $(U_{t \frown (n)})_{n \in \mathbb{N}}$ is a partition of U_t.
(3) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $\operatorname{diam}(U_t) \leq 1/|t|$.

Define $T = \{t \in \mathbb{N}^{<\mathbb{N}} \mid U_t \neq \emptyset\}$ and $C = [T]$. Conditions (2) and (3) and Proposition 1.6.4 allow us to define a function $\pi \colon C \to X$ by letting $\pi(c)$ be the unique point of $\bigcap_{n \in \mathbb{N}} U_{c \upharpoonright n}$. To see that π is bijective, note that if $x \in X$, then condition (1) and repeated application of condition (2) yield a unique $c \in C$ for which $\pi(c) = x$. To see that π is a homeomorphism, observe that $\{U_t \mid t \in T\}$ is a clopen basis for X and $\pi^{-1}(U_t) = C \cap \mathcal{N}_t$, for all $t \in T$. □

Along similar lines, we have the following.

Proposition 1.6.6. *Every Polish space is injectively analytic.*

Proof. Suppose that X is a Polish space, and fix a compatible complete metric d on X. We say that a subset of a topological space is F_σ if it is a union of countably-many closed sets. We will recursively construct a sequence $(F_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of F_σ subsets of X such that:

(1) $F_\emptyset = X$.
(2) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $(F_{t \frown (n)})_{n \in \mathbb{N}}$ is a partition of F_t.
(3) $\forall n \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$ $\overline{F_{t \frown (n)}} \subseteq F_t$.
(4) $\forall n \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$ $\operatorname{diam}(F_t) \leq 1/|t|$.

Given $t \in \mathbb{N}^{<\mathbb{N}}$ for which we have found F_t, fix closed sets $C_{n,t} \subseteq X$, of diameter at most $1/|t|$, whose union is F_t. By Proposition 1.6.2, the sets $F_{t \frown (n)} = C_{n,t} \setminus \bigcup_{m<n} C_{m,t}$ are F_σ.

Define $T = \{t \in \mathbb{N}^{<\mathbb{N}} \mid F_t \neq \emptyset\}$ and $C = [T]$. Conditions (3) and (4) and Proposition 1.6.4 allow us to define a function $\pi \colon C \to X$ by letting $\pi(c)$ be the unique point of $\bigcap_{n \in \mathbb{N}} F_{c \upharpoonright n}$. To see that π is bijective, note that if $x \in X$, then condition (1) and repeated application of condition (2) yield a unique $c \in C$ for which $\pi(c) = x$. To see that π is continuous, observe that every open subset of X is a union of sets of the form F_t, and note that $\pi^{-1}(F_t) = C \cap \mathcal{N}_t$, for all $t \in T$. □

As a corollary, we have the following.

Proposition 1.6.7. *Suppose that X is a topological space. Then X is an injectively analytic Hausdorff space if and only if there is a finer zero-dimensional Polish topology on X consisting of sets which are Borel with respect to the original topology.*

Proof. By Proposition 1.6.6, we need only show that if X is an injectively analytic Hausdorff space, then there is a finer zero-dimensional Polish topology on X consisting of sets which are Borel with respect to the original topology. But if $C \subseteq \mathbb{N}^{\mathbb{N}}$ is a closed set for which there is a continuous bijection $\phi\colon C \to X$, then Theorem 1.5.7 and Proposition 1.6.1 ensure that the pushforward of the topology on C through ϕ is as desired. □

Change of topology. Here we establish a number of strengthenings of Proposition 1.6.6 which allow one to treat countable families of Borel sets and Borel functions on injectively analytic Hausdorff spaces as if they consist solely of clopen sets and continuous functions on zero-dimensional Polish spaces.

PROPOSITION 1.6.8. *Suppose that X is an injectively analytic Hausdorff space and $\mathcal{B}$ is a partition of X into countably-many Borel sets. Then there is a finer zero-dimensional Polish topology τ on X, consisting of sets which are Borel with respect to the original topology, with the property that every set in $\mathcal{B}$ is τ-clopen.*

PROOF. Fix $N \subseteq \mathbb{N}$ for which there is an injective enumeration $(B_n)_{n \in N}$ of $\mathcal{B}$. By Proposition 1.5.10, there are closed sets $C_n \subseteq \mathbb{N}^{\mathbb{N}}$ for which there exist continuous bijections $\phi_n\colon C_n \to B_n$, for all $n \in N$. Set $C = \{(n, x) \mid n \in N \text{ and } x \in C_n\}$, define a function $\phi\colon C \to X$ by $\phi(n, x) = \phi_n(x)$, and observe that Theorem 1.5.7 ensures that the pushforward of the topology on C through ϕ is as desired. □

The following fact will allow us to amalgamate topologies obtained in this fashion.

PROPOSITION 1.6.9. *Suppose that $(\tau_n)_{n \in \mathbb{N}}$ is an increasing sequence of Polish topologies on a set X. Then the topology τ on X generated by $\bigcup_{n \in \mathbb{N}} \tau_n$ is also Polish.*

PROOF. As the families of complete metric spaces and second countable topological spaces are closed under countable products, it follows that $\prod_{n \in \mathbb{N}} (X, \tau_n)$ is Polish. Our assumption that $(\tau_n)_{n \in \mathbb{N}}$ is increasing ensures that the subspace $D = \{x \in X^{\mathbb{N}} \mid \forall m, n \in \mathbb{N}\ x(m) = x(n)\}$ of this product is closed, so Proposition 1.6.1 therefore implies that the subspace topology that D inherits from this product is Polish. To see that (X, τ) is Polish, it only remains to check that the projection function from D to the latter is a homeomorphism. But this follows from the observation that if $n \in \mathbb{N}$ and $U \subseteq X$ is τ_n-open, then the pullback of U under the projection is the intersection of D with the open set $X^n \times U \times X^{\mathbb{N}}$. □

As corollaries, we obtain the following.

PROPOSITION 1.6.10. *Suppose that X is an injectively analytic Hausdorff space and $(B_n)_{n \in \mathbb{N}}$ is a sequence of Borel subsets of X. Then* *there is a finer zero-dimensional Polish topology τ on X, consisting of sets which are Borel with respect to the original topology, such that each B_n is τ-clopen.*

PROOF. By recursively appealing to Proposition 1.6.8, we obtain an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of finer zero-dimensional Polish topologies on X, consisting of sets which are Borel with respect to the original topology, such that each B_n is τ_n-clopen. Proposition 1.6.9 then ensures that the topology τ generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is as desired. $\square$

PROPOSITION 1.6.11. *Suppose that X is an injectively analytic Hausdorff space, Y is a second countable space, and $\phi\colon X \to Y$ is Borel. Then there is a finer zero-dimensional Polish topology τ on X, consisting of sets which are Borel with respect to the original topology, such that ϕ is τ-continuous.*

PROOF. Fix a countable open basis $\mathcal{V}$ for Y, let $\mathcal{U}$ denote the set of ϕ-preimages of elements of $\mathcal{V}$, and appeal to Proposition 1.6.10 to obtain a finer zero-dimensional Polish topology τ on X, consisting of sets which are Borel with respect to the original topology, such that each of the sets in $\mathcal{U}$ is τ-open. $\square$

PROPOSITION 1.6.12. *Suppose that X is an injectively analytic Hausdorff space and $\phi\colon X \to X$ is Borel. Then there is a finer zero-dimensional Polish topology τ on X, consisting of sets which are Borel with respect to the original topology, such that ϕ is τ-continuous.*

PROOF. By Proposition 1.6.7, we can assume that X is second countable. By recursively applying Proposition 1.6.11, we obtain an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of finer zero-dimensional Polish topologies on X, consisting of sets which are Borel with respect to the original topology, such that ϕ is continuous when viewed as a function from (X, τ_{n+1}) to (X, τ_n). Proposition 1.6.9 then ensures that the topology τ generated by $\bigcup_{n\in\mathbb{N}} \tau_n$ is as desired. $\square$

Putting these results together, we obtain the following.

PROPOSITION 1.6.13. *Suppose that $\mathcal{X}$ is a countable family of injectively analytic Hausdorff spaces, $\mathcal{B}$ is a countable family of Borel subsets of these spaces, and $\mathcal{F}$ is a countably family of functions between these spaces. Then there are finer zero-dimensional Polish topologies on the spaces in $\mathcal{X}$, consisting of sets which are Borel with respect to the original topologies, such that the sets in $\mathcal{B}$ are clopen and the functions in $\mathcal{F}$ are continuous with respect to the corresponding topologies.*

PROOF. By Propositions 1.6.7 and 1.6.10, there are finer zero-dimensional Polish topologies $\tau_0(X)$ on each $X \in \mathcal{X}$, consisting of sets which are Borel with respect to the original topologies, such that each $B \in \mathcal{B}$ is $\tau_0(X)$-clopen, where $B \subseteq X$. Fix an enumeration $(f_n\colon X_n \to Y_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$ in which every element of $\mathcal{F}$ appears infinitely often, and recursively appeal to Propositions 1.6.11 and 1.6.12 to obtain finer zero-dimensional Polish topologies $\tau_{n+1}(X)$ on each $X \in \mathcal{X}$, consisting of sets which are Borel with respect to $\tau_n(X)$, such that f_n is continuous when viewed as a function from $\tau_{n+1}(X_n)$ to $\tau_{n+1}(Y_n)$. Proposition 1.6.9 then ensures that the topologies $\tau(X)$ generated by $\bigcup_{n\in\mathbb{N}} \tau_n(X)$ are as desired. □

7. Baire category

Here we discuss a fundamental tool in the study of Polish spaces.

The Baire property. A subset of a topological space is *meager* if it is a union of countably-many nowhere dense sets. A subset of a topological space is *comeager* if its complement is meager, or equivalently, if it contains an intersection of countably-many dense open sets. A *Baire space* is a topological space whose comeager subsets are dense.

THEOREM 1.7.1 (Baire). *Every complete metric space is Baire.*

PROOF. Suppose that X is a complete metric space and $C \subseteq X$ is comeager. To see that C intersects every non-empty open set $U \subseteq X$, fix positive real numbers $\epsilon_n \to 0$, as well as dense open sets $U_n \subseteq X$ for which $\bigcap_{n\in\mathbb{N}} U_n \subseteq C$. Set $V_0 = U$, and recursively choose non-empty open sets $V_{n+1} \subseteq U_n$, of diameter at most ϵ_n, for which $\overline{V_{n+1}} \subseteq V_n$. Proposition 1.6.4 ensures that there is a unique point in $\bigcap_{n\in\mathbb{N}} V_n$, and this point is clearly in $C \cap U$. □

PROPOSITION 1.7.2. *Suppose that X is a Baire space. Then every non-empty open set $U \subseteq X$ is a Baire space.*

PROOF. Suppose, towards a contradiction, that there is a sequence $(U_n)_{n\in\mathbb{N}}$ of dense open subsets of U whose intersection is not dense in U. Let V denote the interior of the complement of U, and define $V_n = U_n \cup V$ for all $n \in \mathbb{N}$. Then $(V_n)_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X whose intersection is not dense, a contradiction. □

PROPOSITION 1.7.3. *Suppose that X is a topological space, $U \subseteq X$ is a non-empty open set, and $Y \subseteq U$. Then Y is meager in U if and only if Y is meager in X.*

PROOF. If Y is meager in U, then there is a sequence $(U_n)_{n\in\mathbb{N}}$ of dense open subsets of U whose intersection is disjoint from Y. Let V denote the interior of the complement of U. Then the sets $V_n = U_n \cup V$

are dense and open in X, and since their intersection is disjoint from Y, it follows that Y is meager in X.

Conversely, if Y is meager in X, then there is a sequence $(V_n)_{n\in\mathbb{N}}$ of dense open subsets of X whose intersection is disjoint from Y. Then the sets $U_n = U \cap V_n$ are dense and open in U, and their intersection is disjoint from Y, thus Y is meager in U. □

A subset of a topological space has the *Baire property* if its symmetric difference with some open set is meager.

PROPOSITION 1.7.4. *Suppose that X is a topological space and $B \subseteq X$ has the Baire property. Then at least one of the following holds:*

1. *The set B is meager.*
2. *There is a non-empty open set $U \subseteq X$ with the property that $B \cap U$ is comeager in U.*

Moreover, if X is a Baire space, then exactly one of these holds.

PROOF. Fix an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager. If U is empty, then B is meager. Otherwise, since $U \setminus B$ is meager in X, Proposition 1.7.3 ensures that it is meager in U, and it follows that $B \cap U$ is comeager in U.

To see that conditions (1) and (2) are mutually exclusive, suppose that there is a non-empty open set $U \subseteq X$ with the property that $B \cap U$ is comeager in U. Let V denote the interior of $X \setminus U$, and note that $(B \cap U) \cup V$ is comeager. It follows that if B is also meager, then V is a comeager set disjoint from U, contradicting Theorem 1.7.1. □

We say that a function $\phi \colon X \to Y$ is *Baire measurable* if preimages of open sets have the Baire property.

PROPOSITION 1.7.5. *Suppose that X is a topological space, Y is a second countable topological space, and $\phi \colon X \to Y$ is Baire measurable. Then there is a dense G_δ set $C \subseteq X$ such that $\phi \restriction C$ is continuous.*

PROOF. Fix an enumeration $(V_n)_{n\in\mathbb{N}}$ of an open basis for Y, fix open sets $U_n \subseteq X$ such that $U_n \bigtriangleup f^{-1}(V_n)$ is meager for all $n \in \mathbb{N}$, fix dense G_δ sets $C_n \subseteq X$ disjoint from $U_n \bigtriangleup \phi^{-1}(V_n)$ for all $n \in \mathbb{N}$, and observe that the set $C = \bigcap_{n\in\mathbb{N}} C_n$ is as desired. □

PROPOSITION 1.7.6. *Suppose that X and Y are topological spaces and $\phi \colon X \to Y$ is a continuous open surjection. Then a set $D \subseteq Y$ is comeager if and only if the corresponding set $C = \phi^{-1}(D)$ is comeager.*

PROOF. Suppose first that D is comeager. Then there are dense open sets $V_n \subseteq Y$ such that $\bigcap_{n\in\mathbb{N}} V_n \subseteq D$. The fact that ϕ is continuous ensures that the sets $U_n = \phi^{-1}(V_n)$ are open, and the fact that ϕ is open implies that they are dense, thus C is comeager.

Conversely, suppose that C is comeager. Then there are dense open sets $U_n \subseteq X$ such that $\bigcap_{n\in\mathbb{N}} U_n \subseteq C$. The fact that ϕ is open ensures that the sets $V_n = \phi(U_n)$ are open, and the fact that ϕ is a continuous surjection implies that they are dense, thus D is comeager. $\square$

Analytic and Borel vs. the Baire property. Here we examine the relationship between analytic and Borel sets and those with the Baire property.

Proposition 1.7.7. *Suppose that X is a topological space. Then the family of subsets of X with the Baire property is a σ-algebra containing the open subsets of X.*

Proof. As the empty set is meager, it follows that every open set has the Baire property.

To see that the family of subsets of X with the Baire property is closed under complements, suppose that $B \subseteq X$ has the Baire property, fix an open set $U \subseteq X$ such that $B \bigtriangleup U$ is meager, let C denote the complement of B, let V denote the interior of $X \setminus U$, and note that $C \bigtriangleup V \subseteq (C \bigtriangleup (X \setminus U)) \cup ((X \setminus U) \setminus V) = (B \bigtriangleup U) \cup ((X \setminus U) \setminus V)$, thus C has the Baire property.

To see that the family of subsets of X with the Baire property is closed under countable unions, suppose that $B_n \subseteq X$ has the Baire property for all $n \in \mathbb{N}$, fix open sets $U_n \subseteq X$ such that $B_n \bigtriangleup U_n$ is meager for all $n \in \mathbb{N}$, set $B = \bigcup_{n\in\mathbb{N}} B_n$ and $U = \bigcup_{n\in\mathbb{N}} U_n$, and observe that $B \bigtriangleup U \subseteq \bigcup_{n\in\mathbb{N}} B_n \bigtriangleup U_n$, thus B has the Baire property. $\square$

A set $B \subseteq X$ is a *Baire envelope* for a set $Y \subseteq X$ if $Y \subseteq B$ and every subset of $B \setminus Y$ with the Baire property is meager.

Proposition 1.7.8. *Suppose that X is a Baire space. Then every set $Y \subseteq X$ has an F_σ Baire envelope.*

Proof. Fix a maximal sequence $(U_i)_{i\in I}$ of pairwise disjoint non-empty open subsets of X within which Y is meager, fix dense G_δ sets $C_i \subseteq U_i$ disjoint from Y for all $i \in I$, and define $B = X \setminus \bigcup_{i\in I} C_i$.

To see that B is F_σ, it is enough to show that $\bigcup_{i\in I} C_i$ is G_δ. Towards this end, fix dense open sets $V_{i,n} \subseteq U_i$ such that $\bigcap_{n\in\mathbb{N}} V_{i,n} \subseteq C_i$ for all $i \in I$, set $V_n = \bigcup_{i\in I} V_{i,n}$ for all $n \in \mathbb{N}$, and note that $\bigcup_{i\in I} C_i = \bigcap_{n\in\mathbb{N}} V_n$.

To see that B is a Baire envelope for Y, suppose, towards a contradiction, that $A \subseteq B \setminus Y$ is a non-meager set with the Baire property. Proposition 1.7.4 then yields a non-empty open set $U \subseteq X$ in which $A \cap U$ is comeager, and the maximality of $(U_i)_{i\in I}$ ensures the existence of $i \in I$ for which $U \cap U_i \neq \emptyset$. As both $A \cap U \cap U_i$ and $C_i \cap U$ are comeager in $U \cap U_i$, so too is $A \cap C_i \cap U$. As Proposition 1.7.2 implies that the latter set is non-empty, this contradicts the fact that $B \cap C_i = \emptyset$. $\square$

PROPOSITION 1.7.9. *Suppose that X is a Baire Hausdorff space. Then every analytic set $A \subseteq X$ has the Baire property.*

PROOF. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to A$. For each $t \in \mathbb{N}^{<\mathbb{N}}$, set $A_t = \phi(\mathcal{N}_t)$ and appeal to Proposition 1.7.8 to obtain a Baire envelope $B_t \subseteq \overline{A_t}$ for A_t with the Baire property. As $A = \bigcup_{b \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} A_{b \upharpoonright n} = \bigcup_{b \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} B_{b \upharpoonright n}$ and the latter set is contained in $\bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} B_t$, to see that A has the Baire property, it is certainly sufficient to show that the difference $\bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} B_t \setminus \bigcup_{b \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} B_{b \upharpoonright n}$ is meager. And for this, it is sufficient to show that for each $t \in \mathbb{N}^{<\mathbb{N}}$, the set $B_t \setminus \bigcup_{n \in \mathbb{N}} B_{t \frown (n)}$ is meager. But $B_t \setminus \bigcup_{n \in \mathbb{N}} B_{t \frown (n)} \subseteq B_t \setminus \bigcup_{n \in \mathbb{N}} A_{t \frown (n)} = B_t \setminus A_t$, and is therefore meager by the definition of Baire envelope. $\square$

PROPOSITION 1.7.10. *Suppose that X is a topological space, Y is a second countable Baire space, and $R \subseteq X \times Y$ is Borel. Then the sets $\{x \in X \mid R_x$ is non-meager$\}$ and $\{x \in X \mid R_x$ is comeager$\}$ are Borel.*

PROOF. It is sufficient to show that if $R \subseteq X \times Y$ is Borel, then so too are the sets of the form $\{x \in X \mid R_x \cap V$ is non-meager in $V\}$, for all non-empty open sets $V \subseteq Y$. Clearly the family of Borel sets with this property contains every Borel rectangle and is closed under countable unions, and therefore contains every open set. To see that it is closed under complements, suppose that $R \subseteq X \times Y$ is a Borel set with this property, let S denote the complement of R, and fix $x \in X$. Then S_x is non-meager in V if and only if there exists $n \in \mathbb{N}$ such that $V_n \subseteq V$ and $S_x \cap V_n$ is comeager in V_n by Proposition 1.7.4, and the latter holds if and only if $R_x \cap V_n$ is meager in V_n. $\square$

In order to establish the analogous fact about analytic sets, we first note the following characterization of the circumstances under which an analytic set is comeager.

PROPOSITION 1.7.11. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ is closed, X is a Baire Hausdorff space, $\phi \colon C \to X$ is continuous, and $U \subseteq X$ is a non-empty open set. Then $\phi(C)$ is comeager in U if and only if there is a sequence $(U_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of open subsets of X satisfying the following conditions:*

(1) *$U_\emptyset$ is dense in U.*
(2) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ *$U_t \subseteq \overline{\phi(C \cap \mathcal{N}_t)}$.*
(3) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ *$\{U_{t \frown (n)} \mid n \in \mathbb{N}\}$ partitions a dense subset of U_t.*

PROOF. We first note that if $\phi(C)$ is comeager in U, then there is a sequence $(U_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ satisfying conditions (1) and (3), as well as the following strengthening of condition (2):

(2′) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $\phi(C \cap \mathcal{N}_t)$ is comeager in U_t.

Towards this end, we begin by setting $U_\emptyset = U$. Given $t \in \mathbb{N}^{<\mathbb{N}}$ for which we have already found U_t, recursively define $U_{t \frown (n)} = \bigcup \mathcal{U}_{t \frown (n)}$, where $\mathcal{U}_{t \frown (n)}$ is a maximal family of pairwise disjoint non-empty open subsets of $U_t \setminus \bigcup_{m<n} \overline{U_{t \frown (m)}}$ in which $\phi(C \cap \mathcal{N}_{t \frown (n)})$ is comeager.

Conversely, note that if $(U_t)_{t \in \mathbb{N}^{\mathbb{N}}}$ satisfies conditions (1) – (3), then $\bigcap_{n \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^n} U_t = \bigcup_{b \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} U_{b \restriction n}$, the former set is comeager in U, and the latter set is contained in $\phi(C)$, thus $\phi(C)$ is comeager in U. □

PROPOSITION 1.7.12. *Suppose that X is an analytic Hausdorff space, Y is a Polish space, and $R \subseteq X \times Y$ is analytic. Then $\{x \in X \mid R_x$ is comeager$\}$ and $\{x \in X \mid R_x$ is non-meager$\}$ are analytic as well.*

PROOF. By Proposition 1.4.8, we can assume that there is a continuous surjection $(\phi, \psi) \colon \mathbb{N}^{\mathbb{N}} \to R$. Our assumption that X is Hausdorff ensures that the set $C = \mathrm{graph}(\phi)$ is closed. Note that $R_x = \psi(C^x)$, for all $x \in X$. Fix an enumeration $(V_n)_{n \in \mathbb{N}}$ of an open basis for Y consisting of non-empty sets, and appeal to Proposition 1.7.11 to obtain that, for all $x \in X$, the set R_x is comeager in a non-empty open set $V \subseteq Y$ if and only if there is a function $\nu \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ such that, setting $U_t = \bigcup_{n \in \nu(t)} V_n$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, the following conditions hold:

(1) $U_\emptyset$ is dense in V.
(2) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $U_t \subseteq \overline{\psi(C^x \cap \mathcal{N}_t)}$.
(3) $\forall t \in \mathbb{N}^{<\mathbb{N}}$ $\{U_{t \frown (n)} \mid n \in \mathbb{N}\}$ partitions a dense subset of U_t.

Propositions 1.4.3 and 1.4.12 ensure that the set of pairs (x, ν) satisfying these conditions is analytic, thus so too is $\{x \in X \mid R_x$ is comeager$\}$. As Proposition 1.7.4 ensures that $\{x \in X \mid R_x$ is non-meager$\}$ is the union of the sets of the form $\{x \in X \mid R_x \cap V_n$ is comeager in $V_n\}$ for $n \in \mathbb{N}$, it follows that the former is analytic as well. □

Quantifier exchange. Here we establish the fundamental fact relating Baire category in a product to Baire category on its components.

THEOREM 1.7.13 (Kuratowski-Ulam). *Suppose that X is a Baire space, Y is second countable Baire space, and $R \subseteq X \times Y$ has the Baire property.*

(1) *The set $\{x \in X \mid R_x$ has the Baire property$\}$ is comeager.*
(2) *The set R is comeager if and only if $\{x \in X \mid R_x$ is comeager$\}$ is comeager.*

PROOF. We first establish the special case of ($\Longrightarrow$) in (2) for which R is dense and open. Associate with each non-empty open set $V \subseteq Y$ the open set $V' = \mathrm{proj}_X(R \cap (X \times V))$. Note that if $U \subseteq X$ is a non-empty open set, then $R \cap (U \times V) \neq \emptyset$, so $U \cap V' \neq \emptyset$, thus V' is dense. Fix an enumeration $(V_n)_{n\in\mathbb{N}}$ of an open basis for Y consisting of non-empty sets, and observe that the set $C = \bigcap_{n\in\mathbb{N}} V_n'$ is comeager, and R_x is dense and open for all $x \in C$.

We next establish ($\Longrightarrow$) in (2). Fix dense open sets $U_n \subseteq X \times Y$ with $\bigcap_{n\in\mathbb{N}} U_n \subseteq R$. Then the sets $C_n = \{x \in X \mid (U_n)_x \text{ is dense and open}\}$ are comeager, thus so too is the set $C = \bigcap_{n\in\mathbb{N}} C_n$, and R_x is comeager for all $x \in C$.

To see (1), fix an open set $U \subseteq X \times Y$ such that $R \mathbin{\triangle} U$ is meager, note that the set $C = \{x \in X \mid R_x \mathbin{\triangle} U_x \text{ is meager}\}$ is comeager, and observe that R_x has the Baire property for all $x \in C$.

It only remains to establish ($\Longleftarrow$) in (2). Towards this end, note that $U \setminus (R \mathbin{\triangle} U) \subseteq R$, so if U is dense, then R is comeager. But if U is not dense, then there are non-empty open sets $V \subseteq X$ and $W \subseteq Y$ such that $U \cap (V \times W) = \emptyset$. Note that if $x \in V$, then $R_x \cap W \subseteq R_x \setminus U_x \subseteq R_x \mathbin{\triangle} U_x$, so $\{x \in V \mid R_x \cap W \text{ is meager}\}$ is comeager in V, thus there are comeagerly many $x \in V$ with the property that $R_x \cap W$ is comeager and meager in W, contradicting Propositions 1.7.2 and 1.7.4. $\square$

Perfect set theorems. Here we establish several perfect set theorems related to Baire category.

The *diagonal* on a set X is the set $\Delta(X)$ consisting of all pairs of the form (x, x), where $x \in X$.

THEOREM 1.7.14 (Mycielski). *Suppose that X is a complete metric space and $R \subseteq X \times X$ is meager. Then there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \Delta(2^{\mathbb{N}})$ to $(X \times X) \setminus R$.*

PROOF. Fix a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of dense open subsets of $X \times X$ such that $R \cap \bigcap_{n\in\mathbb{N}} V_n = \emptyset$. We will recursively find non-empty open sets $U_t \subseteq X$, for all $t \in 2^{<\mathbb{N}}$, satisfying the following conditions:

(1) $\forall n \in \mathbb{N} \forall s, t \in 2^{n+1}\ (s \neq t \Longrightarrow U_s \times U_t \subseteq V_n)$.
(2) $\forall i < 2 \forall t \in 2^{<\mathbb{N}}\ \overline{U_{t \frown (i)}} \subseteq U_t$.
(3) $\forall t \in 2^{<\mathbb{N}}\ \mathrm{diam}(U_t) \leq 1/|t|$.

We begin by setting $U_\emptyset = X$. Given $n \in \mathbb{N}$ for which we have found $(U_t)_{t\in 2^n}$, fix non-empty open sets $U_t' \subseteq U_t$ such that $\overline{U_t'} \subseteq U_t$ and $\mathrm{diam}(U_t') \leq 1/(n+1)$ for all $t \in 2^n$. For all distinct $s, t \in 2^{n+1}$, the corresponding set $V_{s,t} = \{x \in X^{2^{n+1}} \mid (x(s), x(t)) \in V_n\}$ is dense and open, thus so too is the intersection of all such sets. In particular, it follows that there are non-empty open sets $U_{t \frown (i)} \subseteq U'_t$ such that $\prod_{t \in 2^{n+1}} U_t \subseteq \bigcap \{V_{s,t} \mid (s,t) \in (2^{n+1} \times 2^{n+1}) \setminus \Delta(2^{n+1})\}$. This completes the recursive construction.

Conditions (2) and (3) and Proposition 1.6.4 ensure that we obtain a map $\pi \colon 2^{\mathbb{N}} \to X$ by letting $\pi(c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} U_{c \upharpoonright n}$. Condition (3) implies that π is continuous, and it follows from condition (1) that ϕ is a homomorphism from $(2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \Delta(2^{\mathbb{N}})$ to $(X \times X) \setminus R$. $\square$

For each $n \in \mathbb{N} \cup \{\mathbb{N}\}$, the *lexicographic order* on 2^n is given by

$$c <_{\text{lex}} d \iff \exists k \in \mathbb{N}\ (c \upharpoonright k = d \upharpoonright k \text{ and } c(k) < d(k)).$$

Theorem 1.7.15 (Galvin). *Suppose that X is a complete metric space and $R \subseteq X \times X$ is a set with the Baire property which is not comeager in any set of the form $U \times U$, where $U \subseteq X$ is non-empty and open. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $<_{\text{lex}}$ to $(X \times X) \setminus R$.*

Proof. Given distinct $s, t \in 2^{<\mathbb{N}}$, let $s \wedge t$ denote the maximal common initial segment of s and t. We will recursively find non-empty open sets $U_t \subseteq X$ and decreasing sequences $(V_{k,t})_{k \in \mathbb{N}}$ of dense open subsets of $U_{t \frown (0)} \times U_{t \frown (1)}$ with the property that $R \cap \bigcap_{k \in \mathbb{N}} V_{k,t} = \emptyset$, for all $t \in 2^{<\mathbb{N}}$, such that:

(1) $\forall n \in \mathbb{N} \forall s, t \in 2^{n+1}\ (s <_{\text{lex}} t \Longrightarrow U_s \times U_t \subseteq V_{n, s \wedge t})$.
(2) $\forall i < 2 \forall t \in 2^{<\mathbb{N}}\ \overline{U_{t \frown (i)}} \subseteq U_t$.
(3) $\forall t \in 2^{<\mathbb{N}}\ \operatorname{diam}(U_t) \leq 1/|t|$.

We begin by setting $U_\emptyset = X$. Given $n \in \mathbb{N}$ for which we have found $(U_t)_{t \in 2^{\leq n}}$ and $(V_{k,t})_{(k,t) \in \mathbb{N} \times 2^{<n}}$, fix non-empty open sets $U'_{t \frown (i)} \subseteq U_t$ with the property that $\overline{U'_{t \frown (i)}} \subseteq U_t$, $\operatorname{diam}(U'_{t \frown (i)}) \leq 1/(n+1)$, and R is meager in $U'_{t \frown (0)} \times U'_{t \frown (1)}$ for all $i < 2$ and $t \in 2^n$. In addition, fix decreasing sequences $(V_{k,t})_{k \in \mathbb{N}}$ of dense open subsets of $U_{t \frown (0)} \times U_{t \frown (1)}$ such that $R \cap \bigcap_{k \in \mathbb{N}} V_{k,t} = \emptyset$ for all $t \in 2^n$. For all $s <_{\text{lex}} t$ in 2^{n+1}, the set $V_{s,t} = \{x \in \prod_{t \in 2^{n+1}} U'_t \mid (x(s), x(t)) \in V_{n, s \wedge t}\}$ is dense and open in $\prod_{t \in 2^{n+1}} U'_t$, thus so too is the intersection of all such sets. In particular, it follows that there are non-empty open sets $U_t \subseteq U'_t$ with the property that $\prod_{t \in 2^{n+1}} U_t \subseteq \bigcap \{V_{s,t} \mid (s,t) \in <_{\text{lex}} \upharpoonright 2^{n+1}\}$. This completes the recursive construction.

Conditions (2) and (3) and Proposition 1.6.4 ensure that we obtain a map $\pi \colon 2^{\mathbb{N}} \to X$ by letting $\pi(c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} U_{c \upharpoonright n}$. Condition (3) implies that π is continuous, and it follows from condition (1) that ϕ is a homomorphism from $<_{\text{lex}}$ to $(X \times X) \setminus R$. $\square$

The equivalence relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ is given by

$$x \mathbin{\mathbb{E}_0} y \iff \exists n \in \mathbb{N} \forall m \geq n \; x(m) = y(m).$$

A *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on X to a sequence $(S_i)_{i \in I}$ of binary relations on Y is a function $\phi \colon X \to Y$ which is a homomorphism from R_i to S_i, for all $i \in I$. We say that a relation $R \subseteq X \times X$ is *generated by homeomorphisms* if it is a union of graphs of homeomorphisms of X.

Theorem 1.7.16 (Becker-Kechris). *Suppose that X is a complete metric space, $D \subseteq X \times X$ is nowhere dense, E is a dense equivalence relation on X generated by homeomorphisms, and $R \subseteq X \times X$ is meager. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}), (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $(E \setminus D, (X \times X) \setminus R)$.*

Proof. By replacing D with $D \cup D^{-1}$, we can assume that D is symmetric. Fix a decreasing sequence $(V_n)_{n \in \mathbb{N}}$ of dense open subsets of $X \times X$, disjoint from D, whose intersection is disjoint from R. By replacing each V_n with $V_n \cap V_n^{-1}$, we can assume that they are all symmetric as well. We will recursively find non-empty open sets $U_n \subseteq X$ in addition to homeomorphisms $\gamma_n \colon X \to X$ whose graphs are contained in E, with which we associate the homeomorphisms $\gamma_t \colon X \to X$ given by $\gamma_t = \prod_{n < |t|} \gamma_n^{t(n)}$ for all $t \in 2^{<\mathbb{N}}$, satisfying the following conditions:

1. $\forall n \in \mathbb{N} \forall s, t \in 2^n \; \gamma_{s \frown (0)}(U_{n+1}) \times \gamma_{t \frown (1)}(U_{n+1}) \subseteq V_n$.
2. $\forall i < 2 \forall n \in \mathbb{N} \; \overline{U_{n+1}} \cup \overline{\gamma_n(U_{n+1})} \subseteq U_n$.
3. $\forall n \in \mathbb{N} \forall t \in 2^n \; \operatorname{diam}(\gamma_t(U_n)) \leq 1/|t|$.

We begin by setting $U_0 = X$. Given $n \in \mathbb{N}$ for which we have found U_n and $(\gamma_m)_{m<n}$, fix a non-empty open set $U_n' \subseteq U_n$ with $\overline{U_n'} \subseteq U_n$ and $\operatorname{diam}(\gamma_t(U_n')) \leq 1/(n+1)$ for all $t \in 2^n$. For all $s, t \in 2^n$, the corresponding set $V_{s,t} = (\gamma_s \times \gamma_t)^{-1}(V_n)$ is dense and open, thus so too is the intersection of all such sets. It follows that there are non-empty open sets $U_{i,n} \subseteq U_n'$ such that $U_{0,n} \times U_{1,n} \subseteq \bigcap_{s,t \in 2^n} V_{s,t}$. As E is dense, there exist $x_{i,n} \in U_{i,n}$ such that $x_{0,n} \mathrel{E} x_{1,n}$. As E is generated by homeomorphisms, there is a homeomorphism $\gamma_n \colon X \to X$ such that $\operatorname{graph}(\gamma_n) \subseteq E$ and $\gamma_n \cdot x_{0,n} = x_{1,n}$. We complete the recursive construction by setting $U_{n+1} = U_{0,n} \cap \gamma_n^{-1}(U_{1,n})$.

Conditions (2) and (3) and Proposition 1.6.4 ensure that we obtain a map $\phi \colon 2^{\mathbb{N}} \to X$ by letting $\phi(c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} U_{c \restriction n}$. Condition (3) implies that ϕ is continuous, and it follows from condition (1) that ϕ is a homomorphism from $(\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}), (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $(E \setminus D, (X \times X) \setminus R)$. $\square$

We use $\mathbb{F}_0$ to denote the equivalence relation of index two below $\mathbb{E}_0$ given by

$$x \mathbin{\mathbb{F}_0} y \iff \exists n \in \mathbb{N} \forall m > n \ \sum_{k<m} x(k) \equiv \sum_{k<m} y(k) \pmod 2.$$

Theorem 1.7.17. *Suppose that X is a complete metric space, $D \subseteq X \times X$ is nowhere dense, E is an equivalence relation on X, F is an index two subequivalence relation of E for which $E \setminus F$ is dense and generated by homeomorphisms, and $R \subseteq X \times X$ is meager. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $(F \setminus D, E \setminus (D \cup F), (X \times X) \setminus R)$.*

Proof. The proof is essentially the same as that of Theorem 1.7.16, except that we ensure that $x_{0,n} \ (E \setminus F) \ x_{1,n}$ and $\mathrm{graph}(\gamma_n) \subseteq E \setminus F$. $\square$

The isomorphism theorem. Here we note that, modulo trivialities, there is only one notion of Baire category on a Polish space.

Proposition 1.7.18. *Suppose that X is a non-empty perfect Polish space. Then there is a dense G_δ set $C \subseteq X$ homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

Proof. We will recursively construct non-empty open sets $U_t \subseteq X$, for all $t \in \mathbb{N}^{<\mathbb{N}}$, which satisfy the following conditions:

(1) $\forall t \in \mathbb{N}^{<\mathbb{N}} \ \{U_{t\frown(n)} \mid n \in \mathbb{N}\}$ partitions a dense open subset of U_t.
(2) $\forall n \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \ \overline{U_{t\frown(n)}} \subseteq U_t$.
(3) $\forall t \in \mathbb{N}^{<\mathbb{N}} \ \mathrm{diam}(U_t) \leq 1/|t|$.

We begin by setting $U_\emptyset = X$. Given $t \in \mathbb{N}^{<\mathbb{N}}$ for which we have already found U_t, note that for every non-empty open set $U \subseteq U_t$, there is a non-empty open set $V \subseteq U_t$ such that $\overline{V} \subset U$, $U \setminus \overline{V}$ is non-empty, and $\mathrm{diam}(V) \leq 1/(|t|+1)$. By recursively applying this fact, we obtain a maximal infinite family $\mathcal{U}_t$ of pairwise disjoint open sets $U \subseteq U_t$ such that $\overline{U} \subseteq U_t$ and $\mathrm{diam}(U) \leq 1/(|t|+1)$. As any such family is countable and has dense union in U_t, we can take $(U_{t\frown(n)})_{n \in \mathbb{N}}$ to be any enumeration of $\mathcal{U}_t$. This completes the recursive construction.

Conditions (2) and (3) ensure that we obtain a map $\pi \colon \mathbb{N}^{\mathbb{N}} \to X$ by letting $\pi(c)$ be the unique element of $\bigcap_{n \in \mathbb{N}} U_{c \restriction n}$. Condition (3) implies that π is continuous, and condition (1) yields that π is a bijection onto a dense G_δ set. As $\pi(\mathcal{N}_t) = \pi(\mathbb{N}^{\mathbb{N}}) \cap U_t$ for all $t \in \mathbb{N}^{\mathbb{N}}$, it follows that π is a homeomorphism of $\mathbb{N}^{\mathbb{N}}$ with $\pi(\mathbb{N}^{\mathbb{N}})$. $\square$

CHAPTER 2

Dichotomies

1. The baby $\mathbb{G}_0$ dichotomy

Digraphs. Like the perfect set theorem, descriptive set-theoretic dichotomy theorems are typically proven by first applying a derivative to transform the problem at hand into a relatively simple special case, and then appealing to a topological construction. As it turns out, these derivatives are often quite similar, and can be compartmentalized into a small family of graph-theoretic dichotomy theorems, which simultaneously allow one to gauge the difficulty of the derivative portion of the corresponding arguments. Here we present the simplest such graph-theoretic dichotomy, and give a pair of applications.

A *digraph* on a set X is an irreflexive set $G \subseteq X \times X$, a set $Y \subseteq X$ is *G-independent* if $G \upharpoonright Y = \emptyset$, and an *$\mathbb{N}$-coloring* of G is a function $c \colon X \to \mathbb{N}$ for which preimages of singletons are G-independent. A *homomorphism* from a digraph G on X to a digraph H on Y is a function $\phi \colon X \to Y$ such that $(w, x) \in G \Longrightarrow (\phi(w), \phi(x)) \in H$, for all $w, x \in X$.

For each set $S \subseteq 2^{<\mathbb{N}}$ and function $\beta \colon S \to 2^{\mathbb{N}}$, let $G_{S,\beta}(2^{\mathbb{N}})$ denote the digraph on $2^{\mathbb{N}}$ given by

$$G_{S,\beta}(2^{\mathbb{N}}) = \{(s \frown (0) \frown \beta(s), s \frown (1) \frown \beta(s)) \mid s \in S\},$$

and let $S * 2 * \beta$ denote the set of sequences $s \frown (i) \frown (\beta(s) \upharpoonright k)$, where $i < 2$, $k \in \mathbb{N}$, and $s \in S$. We say that β is *sparse* if $S \cap (S * 2 * \beta) = \emptyset$.

Theorem 2.1.1 (Lecomte). *Suppose that X is an analytic Hausdorff space and G is a digraph on X. Then for all $S \subseteq 2^{<\mathbb{N}}$ and all sparse functions $\beta \colon S \to 2^{\mathbb{N}}$, at least one of the following holds:*

1. *There is an $\mathbb{N}$-coloring of G.*
2. *There is a continuous homomorphism from $G_{S,\beta}(2^{\mathbb{N}})$ to G.*

Proof. We will essentially repeat the proof of Theorem 1.4.24, albeit using a slightly modified version of the derivative utilized there. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$. Recursively define a decreasing sequence of trees T^α on $\mathbb{N}$, along with a decreasing sequence of sets $X^\alpha = X \setminus \bigcup_{t \in \mathbb{N}^{<\mathbb{N}} \setminus T^\alpha} \phi(\mathcal{N}_t)$, by setting $T^0 = \mathbb{N}^{<\mathbb{N}}$, $T^{\alpha+1} = \{t \in T^\alpha \mid \phi(\mathcal{N}_t) \cap X^\alpha \text{ is } G\text{-dependent}\}$ for all ordinals $\alpha < \omega_1$, and $T^\lambda = \bigcap_{\alpha<\lambda} T^\alpha$ for all limit ordinals $\lambda < \omega_1$. Let α denote the least ordinal for which $T^\alpha = T^{\alpha+1}$.

If $T^\alpha = \emptyset$, then $X^\alpha = \emptyset$, so for all $x \in X$, there exists $\beta < \alpha$ with $x \in X^\beta \setminus X^{\beta+1}$. As $X^\beta \setminus X^{\beta+1} = \bigcup_{t \in T^\beta \setminus T^{\beta+1}} \phi(\mathcal{N}_t) \cap X^\beta$, it follows that X is a countable union of G-independent sets.

To handle the case that $T^\alpha \neq \emptyset$, we will recursively construct three functions $f \colon 2^{<\mathbb{N}} \to T^\alpha$ and $g_0, g_1 \colon S \to [T^\alpha]$ such that:

(a) $\forall i < 2 \forall t \in 2^{<\mathbb{N}}\ f(t) \sqsubset f(t \frown (i))$.
(b) $\forall i < 2 \forall k \in \mathbb{N} \forall s \in S\ f(s \frown (i) \frown (\beta(s) \upharpoonright k)) \sqsubset g_i(s)$.
(c) $\forall s \in S\ ((\phi \circ g_0)(s), (\phi \circ g_1)(s)) \in G$.

We begin by setting $f(\emptyset) = \emptyset$. Suppose now that $t \in 2^{<\mathbb{N}}$, and we have already defined $f(t)$ and $g_i(s)$, for all $i < 2$ and $s \in S$ with the property that $s \frown (i) \sqsubseteq t$. If $t \in S$, then fix $(x_{0,t}, x_{1,t}) \in G \upharpoonright (\phi(\mathcal{N}_{f(t)}) \cap X^\alpha)$, as well as $g_i(t) \in \mathcal{N}_{f(t)}$ such that $x_{i,t} = (\phi \circ g_i)(t)$, for all $i < 2$. Regardless of whether $t \in S$, suppose that u is a minimal proper extension of t. If $u \in S * 2 * \beta$, then we take $f(u)$ to be any proper extension of $f(t)$ which is an initial segment of $g_i(s)$, where $u = s \frown (i) \frown (\beta(s) \upharpoonright k)$ for some $k \in \mathbb{N}$. Otherwise, we take $f(u)$ to be any proper extension of $f(t)$. This completes the recursive construction.

Condition (a) ensures that we obtain a function $\psi \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by setting $\psi(c) = \bigcup_{n\in\mathbb{N}} f(c \upharpoonright n)$, and moreover, that if $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, then $\psi(\mathcal{N}_{c\upharpoonright n}) \subseteq \mathcal{N}_{f(c\upharpoonright n)} \subseteq \mathcal{N}_{\psi(c)\upharpoonright n}$, so ψ is continuous, thus the function $\pi = \phi \circ \psi$ is also continuous. To see that π is a homomorphism from $G_{S,\beta}(2^\mathbb{N})$ to G, suppose that $s \in S$, appeal to condition (b) to see that $\psi(s \frown (i) \frown \beta(s)) = g_i(s)$ for all $i < 2$, and observe that condition (c) then ensures that $(\pi(s \frown (0) \frown \beta(s)), \pi(s \frown (1) \frown \beta(s))) \in G$. □

Remark 2.1.2. A set $T \subseteq 2^{<\mathbb{N}}$ is *dense* if $\forall s \in 2^{<\mathbb{N}} \exists t \in T\ s \sqsubseteq t$. A straightforward construction shows that if $S \subseteq 2^{<\mathbb{N}}$, $\alpha \colon S \to 2^\mathbb{N}$ is sparse, $T \subseteq 2^{<\mathbb{N}}$ is dense, $\beta \colon T \to 2^\mathbb{N}$, and $\phi \colon 2^\mathbb{N} \to X$ is a continuous homomorphism from $G_{T,\beta}(2^\mathbb{N})$ to G, then there is a continuous homomorphism $\psi \colon 2^\mathbb{N} \to 2^\mathbb{N}$ from $G_{S,\alpha}(2^\mathbb{N})$ to $G_{T,\beta}(2^\mathbb{N})$ with the property that $\phi \circ \psi$ is injective. As one can trivially find dense sets $T \subseteq 2^{<\mathbb{N}}$ for which there are sparse functions $\beta \colon T \to 2^\mathbb{N}$, this yields the strengthening of Theorem 2.1.1 for which the homomorphism in condition (2) is required to be injective.

As an application of Theorem 2.1.1, we obtain the following.

Theorem 2.1.3 (Feng, Todorcevic). *Suppose that X is an analytic Hausdorff space and G is an open digraph on X. Then exactly one of the following holds:*

(1) *There is an* $\mathbb{N}$*-coloring of* G*.*
(2) *There is a continuous homomorphism from* $<_{\mathrm{lex}}$ *to* G*.*

Proof. It is clear that the two conditions are mutually exclusive. To see that at least one holds, suppose that there is no $\mathbb{N}$-coloring of G, fix a dense set $S \subseteq 2^{<\mathbb{N}}$ for which there is a sparse function $\beta \colon S \to 2^{\mathbb{N}}$, and appeal to Theorem 2.1.1 to obtain a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $G_{S,\beta}(2^{\mathbb{N}})$ to G.

We will recursively construct a function $f \colon 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ such that:

(a) $\forall i < 2 \forall t \in 2^{<\mathbb{N}}\ f(t) \sqsubset f(t \frown (i))$.
(b) $\forall t \in 2^{<\mathbb{N}}\ \phi(\mathcal{N}_{f(t \frown (0))}) \times \phi(\mathcal{N}_{f(t \frown (1))}) \subseteq G$.

We begin by setting $f(\emptyset) = \emptyset$. Given $t \in 2^{<\mathbb{N}}$ for which we have found $f(t)$, fix $s_t \in S$ with $f(t) \sqsubseteq s_t$. The fact that ϕ is a homomorphism then ensures that $(\phi(s_t \frown (0) \frown \beta(s_t)), \phi(s_t \frown (1) \frown \beta(s_t))) \in G$, so the fact that ϕ is continuous and G is open yields $k_t \in \mathbb{N}$ with the property that $\phi(\mathcal{N}_{s_t \frown (0) \frown (\beta(s_t) \upharpoonright k_t)}) \times \phi(\mathcal{N}_{s_t \frown (1) \frown (\beta(s_t) \upharpoonright k_t)}) \subseteq G$. We complete the recursive construction by setting $f(t \frown (i)) = s_t \frown (i) \frown (\beta(s_t) \upharpoonright k_t)$.

Condition (a) ensures that we obtain a function $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting $\psi(c) = \bigcup_{n \in \mathbb{N}} f(c \upharpoonright n)$, and moreover, that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $\psi(\mathcal{N}_{c \upharpoonright n}) \subseteq \mathcal{N}_{f(c \upharpoonright n)} \subseteq \mathcal{N}_{\psi(c) \upharpoonright n}$, so ψ is continuous, thus the function $\pi = \phi \circ \psi$ is also continuous. To see that the latter is a homomorphism from $<_{\mathrm{lex}}$ to G, note that if $c <_{\mathrm{lex}} d$ and $n \in \mathbb{N}$ is minimal for which $c \upharpoonright n \neq d \upharpoonright n$, then condition (b) yields that $\phi(\mathcal{N}_{f(c \upharpoonright n)}) \times \phi(\mathcal{N}_{f(d \upharpoonright n)}) \subseteq G$, so $(\pi(c), \pi(d)) \in G$. □

As an application of Theorem 2.1.3, we obtain another proof of the perfect set theorem for analytic Hausdorff spaces.

Theorem 2.1.4 (Souslin). *Suppose that* X *is an analytic Hausdorff space. Then exactly one of the following holds:*

(1) *The set* X *is countable.*
(2) *There is a continuous injection* $\pi \colon 2^{\mathbb{N}} \to X$*.*

Proof. This is just the special case of Theorem 2.1.3 for which G is the complement of the diagonal. □

Dihypergraphs. An $\mathbb{N}$-*dimensional dihypergraph* on a set X is a set $G \subseteq X^{\mathbb{N}}$ of non-constant sequences, a set $Y \subseteq X$ is G*-independent* if $G \upharpoonright Y = \emptyset$, and an $\mathbb{N}$-*coloring* of G is a function $c \colon X \to \mathbb{N}$ for which preimages of singletons are G-independent. A *homomorphism* from an $\mathbb{N}$-dimensional dihypergraph G on X to an $\mathbb{N}$-dimensional dihypergraph H on Y is a function $\phi \colon X \to Y$ with $(x_i)_{i \in \mathbb{N}} \in G \implies (\phi(x_i))_{i \in \mathbb{N}} \in H$, for all $(x_i)_{i \in \mathbb{N}} \in X^{\mathbb{N}}$.

For each set $S \subseteq \mathbb{N}^{<\mathbb{N}}$ and function $\beta \colon S \to \mathbb{N}^{\mathbb{N}}$, let $G_{S,\beta}(\mathbb{N}^{\mathbb{N}})$ denote the digraph on $\mathbb{N}^{\mathbb{N}}$ given by

$$G_{S,\beta}(\mathbb{N}^{\mathbb{N}}) = \{(s \frown (i) \frown \beta(s))_{i \in \mathbb{N}} \mid s \in S\},$$

and let $S * \mathbb{N} * \beta$ denote the set of sequences $s \frown (i) \frown (\beta(s) \upharpoonright k)$, where $i, k \in \mathbb{N}$ and $s \in S$. We say that β is *sparse* if $S \cap (S * \mathbb{N} * \beta) = \emptyset$.

Theorem 2.1.5. *Suppose that X is an analytic Hausdorff space and G is an $\mathbb{N}$-dimensional dihypergraph on X. Then for all $S \subseteq \mathbb{N}^{<\mathbb{N}}$ and all sparse functions $\beta \colon S \to \mathbb{N}^{\mathbb{N}}$, at least one of the following holds:*

(1) *There is an $\mathbb{N}$-coloring of G.*
(2) *There is a continuous homomorphism from $G_{S,\beta}(\mathbb{N}^{\mathbb{N}})$ to G.*

Proof. The proof of Theorem 2.1.1 works just as well here. □

Remark 2.1.6. A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is *dense* if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T\ s \sqsubseteq t$. A straightforward construction shows that if $S \subseteq \mathbb{N}^{<\mathbb{N}}$, $\alpha \colon S \to \mathbb{N}^{\mathbb{N}}$ is sparse, $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is dense, $\beta \colon T \to \mathbb{N}^{\mathbb{N}}$, and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is a continuous homomorphism from $G_{T,\beta}(\mathbb{N}^{\mathbb{N}})$ to G, then there is a continuous homomorphism $\psi \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ from $G_{S,\alpha}(\mathbb{N}^{\mathbb{N}})$ to $G_{T,\beta}(\mathbb{N}^{\mathbb{N}})$ with the property that $\phi \circ \psi$ is injective. As one can trivially find dense sets $T \subseteq \mathbb{N}^{<\mathbb{N}}$ for which there are sparse functions $\beta \colon T \to \mathbb{N}^{\mathbb{N}}$, this yields the strengthening of Theorem 2.1.5 for which the homomorphism in condition (2) is required to be injective.

The *evenly-splitting $\mathbb{N}$-dimensional dihypergraph* is the $\mathbb{N}$-dimensional dihypergraph on $\mathbb{N}^{\mathbb{N}}$ consisting of all sequences $(s \frown (i) \frown b(i))_{i \in \mathbb{N}}$, where s varies over $\mathbb{N}^{<\mathbb{N}}$ and b varies over $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$.

Theorem 2.1.7. *Suppose that X is an analytic Hausdorff space and G is a box-open $\mathbb{N}$-dimensional dihypergraph on X. Then exactly one of the following holds:*

(1) *There is an $\mathbb{N}$-coloring of G.*
(2) *There is a continuous homomorphism from the evenly-splitting $\mathbb{N}$-dimensional dihypergraph to G.*

Proof. The proof of Theorem 2.1.3 works just as well here. □

A set is K_σ if it is a union of countably-many compact sets.

Theorem 2.1.8 (Hurewicz, Kechris, Saint Raymond). *Suppose that X is a metric space and $A \subseteq X$ is analytic. Then exactly one of the following holds:*

(1) *The set A is contained in a K_σ subset of X.*
(2) *There is a closed continuous injection of $\mathbb{N}^{\mathbb{N}}$ into X whose image is contained in A.*

PROOF. As every compact subset of $\mathbb{N}^{\mathbb{N}}$ is contained in a set of the form $\prod_{n \in \mathbb{N}} d(n)$ for some $d \in \mathbb{N}^{\mathbb{N}}$, it follows that $\mathbb{N}^{\mathbb{N}}$ is not K_σ. As the family of compact sets is closed under preimages by closed injections, it follows that the two conditions are mutually exclusive.

To see that at least one of the two conditions holds, let G denote the $\mathbb{N}$-dimensional dihypergraph on A consisting of all injective sequences $(x_n)_{n \in \mathbb{N}}$ with no subsequences converging to a point of X.

LEMMA 2.1.9. *The $\mathbb{N}$-dimensional dihypergraph G is box open.*

PROOF. Suppose that $(x_n)_{n \in \mathbb{N}} \in G$, and observe that if $(\epsilon_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers with the property that $\epsilon_n \to 0$ and $\epsilon_n \leq d(x_m, x_n)/2$ for all distinct $m, n \in \mathbb{N}$, then $A \cap \prod_{n \in \mathbb{N}} \mathcal{B}(x_n, \epsilon_n)$ is a box-open subset of G. ❐

As every G-independent set has compact closure within X, it follows that if there is an $\mathbb{N}$-coloring of G, then A is contained in a K_σ subset of X. Otherwise, an application of Theorem 2.1.5 yields a continuous homomorphism $\phi \colon \mathbb{N}^{\mathbb{N}} \to A$ from the evenly-splitting $\mathbb{N}$-dimensional dihypergraph to G.

LEMMA 2.1.10. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ is closed. Then so is $\phi(C)$.*

PROOF. Suppose that $(c_n)_{n \in \mathbb{N}}$ is a sequence of points of C for which $(\phi(c_n))_{n \in \mathbb{N}}$ converges to some point $x \in X$. Note that if $k \in \mathbb{N}$ and $(b_n)_{n \in \mathbb{N}}$ is a subsequence of $(c_n)_{n \in \mathbb{N}}$ consisting of points which differ from one another for the first time on their k^{th} coordinates, then there is a further subsequence $(a_n)_{n \in \mathbb{N}}$ of $(b_n)_{n \in \mathbb{N}}$ which is also a subsequence of an element of the evenly-splitting $\mathbb{N}$-dimensional dihypergraph, so $(\phi(a_n))_{n \in \mathbb{N}}$ is a subsequence of an element of G, contradicting the fact that $\phi(a_n) \to x$. A straightforward recursive construction therefore yields $d \in \mathbb{N}^{\mathbb{N}}$ for which every c_n is in $\prod_{k \in \mathbb{N}} d(k)$. As the latter space is compact, by passing to a subsequence we can assume that $(c_n)_{n \in \mathbb{N}}$ converges to some point c. As C is closed, it follows that $c \in C$, so the continuity of ϕ implies that $x = \phi(c)$, thus $x \in \phi(C)$. ❐

It only remains to note that the fact that G consists solely of injective sequences ensures that ϕ is injective. □

We equip $\mathbb{N}^{\leq \mathbb{N}}$ with the smallest topology making clopen the sets $\mathcal{N}_s^*$ consisting of all extensions of s in $\mathbb{N}^{\leq \mathbb{N}}$. One obtains a homeomorphism $\phi \colon \mathbb{N}^{\leq \mathbb{N}} \to 2^{\mathbb{N}}$ by setting

$$\phi(t) = \begin{cases} \bigoplus_{n \in \mathbb{N}} (1)^{t(n)} \frown (0) & \text{if } t \in \mathbb{N}^{\mathbb{N}} \text{ and} \\ (\bigoplus_{n < |t|} (1)^{t(n)} \frown (0)) \frown (1)^\infty & \text{otherwise.} \end{cases}$$

THEOREM 2.1.11 (Hurewicz, Kechris-Louveau-Woodin). *Suppose that X is a metric space and $A \subseteq X$ is analytic. Then exactly one of the following holds:*

(1) *The set A is F_σ.*
(2) *There is a continuous reduction of $\mathbb{N}^\mathbb{N} \subseteq \mathbb{N}^{\leq\mathbb{N}}$ to $A \subseteq X$.*

PROOF. As every closed subset of $\mathbb{N}^{\leq\mathbb{N}}$ disjoint from $\mathbb{N}^{<\mathbb{N}}$ is contained in a set of the form $\prod_{n\in\mathbb{N}} d(n)$ for some $d \in \mathbb{N}^\mathbb{N}$, and a straightforward diagonalization shows that $\mathbb{N}^\mathbb{N}$ is not a union of countably-many such sets, it follows that the two conditions are mutually exclusive.

To see that at least one of the two conditions holds, let G denote the $\mathbb{N}$-dimensional dihypergraph on A consisting of all sequences $(x_n)_{n\in\mathbb{N}}$ of points in A converging to a point in $X \setminus A$. As the closure within X of every G-independent set is contained in A, it follows that if there is an $\mathbb{N}$-coloring of G, then A is F_σ.

LEMMA 2.1.12. *The $\mathbb{N}$-dimensional dihypergraph G is box open.*

PROOF. Suppose that $(x_n)_{n\in\mathbb{N}} \in G$, and note that if $(\epsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to zero, then $A \cap \prod_{n\in\mathbb{N}} \mathcal{B}(x_n, \epsilon_n)$ is a box-open subset of G. ◻

By Theorem 2.1.5 and Lemma 2.1.12, we can assume that there is a continuous homomorphism $\phi\colon \mathbb{N}^\mathbb{N} \to A$ from the evenly-splitting $\mathbb{N}$-dimensional dihypergraph to G.

LEMMA 2.1.13. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $x_t \in X \setminus A$ for which $\phi(\mathcal{N}_{t\frown(n)}) \to x_t$.*

PROOF. As ϕ is a homomorphism from the evenly-splitting $\mathbb{N}$-dimensional dihypergraph to G, if $(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \phi(\mathcal{N}_{t\frown(n)})$ then there exists $x_t \in X \setminus A$ for which $x_n \to x_t$. If it is not the case that $\phi(\mathcal{N}_{t\frown(n)}) \to x_t$, then there is an open neighborhood U of x_t for which there is an infinite set $N \subseteq \mathbb{N}$ such that for all $n \in N$, there exists $y_n \in \phi(\mathcal{N}_{t\frown(n)}) \setminus U$. By shrinking N if necessary, we can assume that it is also co-infinite, in which case $(x_n)_{n\in\mathbb{N}\setminus N} \cup (y_n)_{n\in N}$ does not converge, a contradiction. ◻

Extend ϕ to a function on $\mathbb{N}^{\leq\mathbb{N}}$ by setting $\phi(t) = x_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Clearly ϕ is a reduction of $\mathbb{N}^\mathbb{N}$ to A.

LEMMA 2.1.14. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$. Then $\phi(\mathcal{N}_t^*) \subseteq \overline{\phi(\mathcal{N}_t)}$.*

PROOF. Note that $\phi(\mathcal{N}_t^* \setminus \mathcal{N}_t) \subseteq \bigcup_{s\in\mathcal{N}_t^*\setminus\mathcal{N}_t} \overline{\phi(\mathcal{N}_s)} \subseteq \overline{\phi(\mathcal{N}_t)}$. ◻

To see that ϕ is continuous, we will show that if $c \in \mathbb{N}^{\leq\mathbb{N}}$ and $U \subseteq X$ is an open neighborhood of $\phi(c)$, then there is an open neighborhood of c whose image under ϕ is contained in $\overline{U}$. If $c \in \mathbb{N}^{<\mathbb{N}}$, then Lemma 2.1.13 yields $n \in \mathbb{N}$ for which $\bigcup_{m \geq n} \phi(\mathcal{N}_{c \frown (m)}) \subseteq U$, so Lemma 2.1.14 ensures that

$$\phi(\mathcal{N}_c^* \setminus \bigcup_{m<n} \mathcal{N}_{c \frown (m)}^*) = \{x_c\} \cup \bigcup_{m \geq n} \phi(\mathcal{N}_{c \frown (m)}^*) \subseteq \{x_c\} \cup \bigcup_{m \geq n} \overline{\phi(\mathcal{N}_{c \frown (m)})} \subseteq \overline{U}.$$

If $c \in \mathbb{N}^{\mathbb{N}}$, then the continuity of $\phi \restriction \mathbb{N}^{\mathbb{N}}$ yields $n \in \mathbb{N}$ for which $\phi(\mathcal{N}_{c \restriction n}) \subseteq U$, so Lemma 2.1.14 ensures that $\phi(\mathcal{N}_{c \restriction n}^*) \subseteq \overline{\phi(\mathcal{N}_{c \restriction n})} \subseteq \overline{U}$. □

Remark 2.1.15. A simple construction can be used to establish that if $\phi \colon \mathbb{N}^{\leq \mathbb{N}} \to X$ is a continuous reduction of $\mathbb{N}^{\mathbb{N}}$ to A, then there is a continuous reduction $\psi \colon \mathbb{N}^{\leq \mathbb{N}} \to \mathbb{N}^{\leq \mathbb{N}}$ of $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$ for which $\phi \circ \psi$ is injective. In particular, this yields the strengthening of Theorem 2.1.11 for which the reduction in condition (2) is required to be injective.

Measurability. A set is $\mathbf{\Delta}_2^0$ if it is both F_σ and G_δ, and a function is $\mathbf{\Delta}_2^0$-*measurable* if preimages of open sets are $\mathbf{\Delta}_2^0$.

Theorem 2.1.16 (Lecomte). *Suppose that X is an analytic metric space, G is a digraph on X, $S \subseteq 2^{<\mathbb{N}}$ is dense, and $\beta \colon S \to 2^{\mathbb{N}}$ is sparse. Then exactly one of the following holds:*

(1) *There is a $\mathbf{\Delta}_2^0$-measurable $\mathbb{N}$-coloring of G.*
(2) *There is a continuous homomorphism from $G_{S,\beta}(2^{\mathbb{N}})$ to G.*

Proof. As the density of S ensures that every $G_{S,\beta}(2^{\mathbb{N}})$-independent closed set is nowhere dense, the two conditions are mutually exclusive. To see that at least one of them holds, we will essentially repeat the proof of Theorem 2.1.1, albeit using a slightly modified version of the derivative utilized there. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$. Recursively define decreasing sequences of trees T^α on $\mathbb{N}$ and sets $X^\alpha \subseteq X$ by setting $T^0 = \mathbb{N}^{<\mathbb{N}}$ and $X^0 = X$, $T^{\alpha+1} = \{t \in T^\alpha \mid \overline{\phi(\mathcal{N}_t) \cap X^\alpha} \text{ is } G\text{-dependent}\}$ and $X^{\alpha+1} = X^\alpha \setminus \bigcup_{t \in T^\alpha \setminus T^{\alpha+1}} \overline{\phi(\mathcal{N}_t) \cap X^\alpha}$ for all ordinals $\alpha < \omega_1$, as well as $T^\lambda = \bigcap_{\alpha < \lambda} T^\alpha$ and $X^\lambda = \bigcap_{\alpha < \lambda} X^\alpha$ for all limit ordinals $\lambda < \omega_1$. Let α denote the least ordinal for which $T^\alpha = T^{\alpha+1}$.

If $T^\alpha = \emptyset$, then $X^\alpha = \emptyset$, so for all $x \in X$, there is an ordinal $\beta < \alpha$ with $x \in X^\beta \setminus X^{\beta+1}$. It follows that X is a countable union of G-independent closed sets, thus G has a $\mathbf{\Delta}_2^0$-measurable coloring.

To handle the case that $T^\alpha \neq \emptyset$, fix a decreasing sequence $(\epsilon_k)_{k \in \mathbb{N}}$ of positive real numbers converging to zero. We will recursively construct functions $f \colon 2^{<\mathbb{N}} \to T^\alpha$, $g_i \colon S \to \overline{X^\alpha}$, and $g_{i,k} \colon S \to T^\alpha$ for all $i < 2$ and $k \in \mathbb{N}$ such that:

(a) $\forall i < 2 \forall t \in 2^{<\mathbb{N}} \ (t \frown (i) \notin S * 2 * \beta \Longrightarrow f(t) \sqsubset f(t \frown (i)))$.

(b) $\forall i < 2 \forall k \in \mathbb{N} \forall s \in S\ f(s) \sqsubseteq g_{i,k}(s)$.
(c) $\forall i < 2 \forall k \in \mathbb{N} \forall s \in S\ f(s \frown (i) \frown (\beta(s) \upharpoonright k)) = g_{i,k}(s)$.
(d) $\forall i < 2 \forall k \in \mathbb{N} \forall s \in S\ \phi(\mathcal{N}_{g_{i,k}(s)}) \subseteq \mathcal{B}(g_i(s), \epsilon_k)$.
(e) $\forall s \in S\ (g_0(s), g_1(s)) \in G$.

We begin by setting $f(\emptyset) = \emptyset$. Suppose now that $t \in 2^{<\mathbb{N}}$, and we have already defined $f(t)$, as well as $g_i(s)$ and $g_{i,j+1}(s)$ in case t is of the form $s \frown (i) \frown (\beta(s) \upharpoonright j)$ for some $i < 2$, $j \in \mathbb{N}$, and $s \in S$. If $t \in S$, then fix $(g_0(t), g_1(t)) \in G \upharpoonright \overline{\phi(\mathcal{N}_{f(t)}) \cap X^\alpha}$, in addition to points $x_{i,k,t} \in \mathcal{B}(g_i(t), \epsilon_k) \cap \phi(\mathcal{N}_{f(t)}) \cap X^\alpha$ and $b_{i,k,t} \in \mathcal{N}_{f(t)}$ such that $x_{i,k,t} = \phi(b_{i,k,t})$ for all $i < 2$ and $k \in \mathbb{N}$, and take each $g_{i,k}(t)$ to be any proper initial segment of $b_{i,k,t}$ with the property that $f(t) \sqsubseteq g_{i,k}(t)$ and $\phi(\mathcal{N}_{g_{i,k}(t)}) \subseteq \mathcal{B}(g_i(t), \epsilon_k)$. Regardless of whether $t \in S$, suppose that u is a minimal proper extension of t. If $u \in S * 2 * \beta$, set $f(u) = g_{i,k}(s)$, where $u = s \frown (i) \frown (\beta(s) \upharpoonright k)$. Otherwise, take $f(u)$ to be any proper extension of $f(t)$. This completes the recursive construction.

Let $[S * 2 * \beta]$ denote the set of points of the form $s \frown (i) \frown \beta(s)$, where $i < 2$ and $s \in S$. Note that if $c \in 2^{\mathbb{N}} \setminus [S*2*\beta]$, then the sparsity of β ensures that the set $N_c = \{n \in \mathbb{N} \mid c \upharpoonright (n+1) \notin S*2*\beta\}$ is infinite. As conditions (a), (b), and (c) ensure that $f(c \upharpoonright m) \sqsubset f(c \upharpoonright n)$ whenever $m \in N_c$ and $n > m$, we obtain a function $\psi \colon 2^{\mathbb{N}} \setminus [S * 2 * \beta] \to \mathbb{N}^{\mathbb{N}}$ by setting $\psi(c) = \bigcup_{n \in N_c} f(c \upharpoonright n)$. Let $\pi \colon 2^{\mathbb{N}} \to X$ be the extension of $\phi \circ \psi$ given by $\pi(s \frown (i) \frown \beta(s)) = g_i(s)$, for all $i < 2$ and $s \in S$. To see that π is continuous, note that if $c \in 2^{\mathbb{N}} \setminus [S*2*\beta]$ then $\pi(\mathcal{N}_{c \upharpoonright n}) \subseteq \overline{\phi(\mathcal{N}_{f(c \upharpoonright n)})}$ whenever $n \in N_c$, and if $i < 2$, $k \in \mathbb{N}$, and $s \in S$ then $\pi(\mathcal{N}_{s \frown (i) \frown (\beta(s) \upharpoonright k)})$ is contained in $\overline{\mathcal{B}(g_i(s), \epsilon_k)}$ by conditions (a), (b), (c), and (d). To see that π is a homomorphism from $G_{S,\beta}(2^{\mathbb{N}})$ to G, note that if $s \in S$, then $\pi(s \frown (i) \frown \beta(s)) = g_i(s)$ for all $i < 2$ by conditions (c) and (d), so $(\pi(s \frown (0) \frown \beta(s)), \pi(s \frown (1) \frown \beta(s))) \in G$ by condition (e). $\square$

2. The $\mathbb{G}_0$ dichotomy

Digraphs. Here we note that, under the additional assumption that the digraph in question is analytic, the inexistence of $\mathbb{N}$-colorings leads to continuous homomorphisms from much larger graphs, and as a result, has far stronger consequences.

For each set $S \subseteq 2^{<\mathbb{N}}$, let $G_S(2^{\mathbb{N}})$ denote $\bigcup_{\beta \colon S \to 2^{\mathbb{N}}} G_{S,\beta}(2^{\mathbb{N}})$. We say that a set $S \subseteq 2^{\mathbb{N}}$ is *sparse* if $|S \cap 2^n| \leq 1$ for all $n \in \mathbb{N}$.

THEOREM 2.2.1 (Kechris-Solecki-Todorcevic). *Suppose that X is a Hausdorff space and G is an analytic digraph on X. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

(1) *There is an $\mathbb{N}$-coloring of G.*

(2) *There is a continuous homomorphism* $\pi \colon 2^{\mathbb{N}} \to X$ *from* $G_S(2^{\mathbb{N}})$ *to* G.

Proof. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi_G \colon \mathbb{N}^{\mathbb{N}} \to G$. By Propositions 1.4.1, 1.4.4, and 1.4.8, we can assume that there is a continuous function $\phi_X \colon \mathbb{N}^{\mathbb{N}} \to X$ for which $\phi_X(\mathbb{N}^{\mathbb{N}})$ is the set of points in at least one of the projections of G. Fix sequences $s_n \in 2^n$ such that $S \subseteq \{s_n \mid n \in \mathbb{N}\}$.

We will recursively define a decreasing sequence $(X^\alpha)_{\alpha<\omega_1}$ of subsets of X, off of which there are $\mathbb{N}$-colorings of G. We begin with $X^0 = X$, and we set $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ for all limit ordinals $\lambda < \omega_1$. To describe the construction of $X^{\alpha+1}$ from X^α, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{n^a}$, and $\psi^a_n \colon 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for $n < n^a$. We say that an approximation b is a *one-step extension* of an approximation a if the following conditions hold:

(a) $n^a = n^b - 1$.
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}$ $(s \sqsubseteq t \Longrightarrow \phi^a(s) \sqsubseteq \phi^b(t))$.
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}$ $(s \sqsubseteq t \Longrightarrow \psi^a_n(s) \sqsubseteq \psi^b_n(t))$.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^{\mathbb{N}}$, and $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \mathbb{N}^{\mathbb{N}}$ for $n < n^\gamma$, such that

$$(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown t)),$$

for all $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. We say that a configuration γ is *compatible* with an approximation a if the following conditions hold:

(i) $n^a = n^\gamma$.
(ii) $\forall t \in 2^{n^a}$ $\phi^a(t) \sqsubseteq \phi^\gamma(t)$.
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}$ $\psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

We say that a configuration γ is *compatible* with a set $Y \subseteq X$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma})$ is contained in Y. We say that an approximation a is Y*-terminal* if no configuration is compatible both with a one-step extension of a and with Y. Let $A(a, Y)$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(s_{n^a})$, where γ varies over configurations compatible with both a and Y.

Lemma 2.2.2. *Suppose that* $Y \subseteq X$ *and* a *is a* Y*-terminal approximation. Then* $A(a, Y)$ *is* G*-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and Y, with the property that $((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a})) \in G$. Then there exists $d \in \mathbb{N}^{\mathbb{N}}$ such that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_0})(s_{n^a}), (\phi_X \circ \phi^{\gamma_1})(s_{n^a}))$. Let γ denote the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and

$t \in 2^{n^a}$, $\psi_n^\gamma(t \frown (i)) = \psi_n^{\gamma_i}(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi_{n^a}^\gamma(\emptyset) = d$. Then the unique approximation b with which γ is compatible is a one-step extension of a, contradicting the fact that a is Y-terminal. ❒

We finally define $X^{\alpha+1}$ to be the difference of X^α and the union of the sets $A(a, X^\alpha)$, where a varies over all X^α-terminal approximations.

LEMMA 2.2.3. *Suppose that $\alpha < \omega_1$ and a is an approximation which is not $X^{\alpha+1}$-terminal. Then there is a one-step extension of a which is not X^α-terminal.*

PROOF. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $X^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in X^{\alpha+1}$, so $A(b, X^\alpha) \cap X^{\alpha+1} \neq \emptyset$, thus b is not X^α-terminal. ❒

Fix $\alpha < \omega_1$ such that the families of X^α-terminal approximations and $X^{\alpha+1}$-terminal approximations are the same, let a_0 denote the unique approximation a with the property that $n^a = 0$, and observe that $A(a_0, Y) = Y$ for all $Y \subseteq X$. In particular, it follows that if a_0 is X^α-terminal, then $X^{\alpha+1} = \emptyset$, so there is an $\mathbb{N}$-coloring of G.

Otherwise, by recursively applying Lemma 2.2.3, we obtain one-step extensions a_{n+1} of a_n, for all $n \in \mathbb{N}$, which are not X^α-terminal. Define $\phi\colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \restriction n)$, as well as $\psi_n\colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_n(c) = \bigcup_{m>n} \psi_n^{a_m}(c \restriction (m-(n+1)))$, for all $n \in \mathbb{N}$. Clearly ϕ is continuous. We will complete the proof by showing that the function $\pi = \phi_X \circ \phi$ is a homomorphism from $G_S(2^\mathbb{N})$ to G. For this, it is sufficient to show the stronger fact that if $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, then

$$(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c)).$$

And for this, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(s_n \frown (0) \frown c), (\phi_X \circ \phi)(s_n \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (0) \frown s)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(s_n \frown (1) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \restriction (m-(n+1))$. The fact that a_m is not X^α-terminal yields a configuration γ compatible with a_m. Then $((\phi_X \circ \phi^\gamma)(s_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(s_n \frown (1) \frown s)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$. □

While conditions (1) and (2) of Theorem 2.2.1 are not mutually exclusive, we at least have the following.

PROPOSITION 2.2.4. *Suppose that $S \subseteq 2^{<\mathbb{N}}$ is dense. Then no nonmeager set $B \subseteq 2^\mathbb{N}$ with the Baire property is $G_S(2^\mathbb{N})$-independent.*

PROOF. Fix $r \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_r$, as well as $s \in S$ with $r \sqsubseteq s$. As B is comeager in $\mathcal{N}_s$ and the function $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ flipping the $|s|^{\text{th}}$ coordinate of its input is a homeomorphism, the set $B \cap \phi^{-1}(B) \cap \mathcal{N}_s$ is comeager in $\mathcal{N}_s$. But $(x, \phi(x)) \in G_S(2^{\mathbb{N}}) \restriction B$ for all $x \in B \cap \phi^{-1}(B) \cap \mathcal{N}_{s \frown (0)}$, thus B is $G_S(2^{\mathbb{N}})$-dependent. $\square$

PROPOSITION 2.2.5. *Suppose that $S \subseteq 2^{<\mathbb{N}}$ is dense. Then there is no Baire measurable coloring $c \colon X \to \mathbb{N}$ of $G_S(2^{\mathbb{N}})$.*

PROOF. If $c \colon X \to \mathbb{N}$ is Baire measurable, then there exists $n \in \mathbb{N}$ for which $c^{-1}(\{n\})$ is non-meager, so Proposition 2.2.4 implies that $c^{-1}(\{n\})$ is $G_S(2^{\mathbb{N}})$-dependent, thus c is not a coloring of $G_S(2^{\mathbb{N}})$. $\square$

A *partial transversal* of an equivalence relation is a set intersecting every equivalence class in at most one point.

THEOREM 2.2.6 (Silver). *Suppose that X is a Hausdorff space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:*

1. *The equivalence relation E has only countably-many classes.*
2. *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \to X$ of $2^{\mathbb{N}}$ into a partial transversal of E.*

PROOF. It is clear that the two conditions are mutually exclusive. To see that at least one of them holds, let G denote the complement of E. As every G-independent set is contained in a single E-class, it follows that if there is an $\mathbb{N}$-coloring of G, then E has only countably-many classes. By Theorem 2.2.1, we can therefore assume that there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G, for some dense set $S \subseteq 2^{<\mathbb{N}}$.

Let E' denote the pullback of E through ϕ. Note that each E'-class is $G_S(2^{\mathbb{N}})$-independent, and therefore meager. But then Theorem 1.7.13 ensures that E' itself must be meager, in which case Theorem 1.7.14 yields a continuous injection $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $2^{\mathbb{N}}$ into a partial transversal of E'. Set $\pi = \phi \circ \psi$. $\square$

More generally, we have the following.

THEOREM 2.2.7 (Louveau). *Suppose that X is a Hausdorff space and E is a co-analytic quasi-order on X. Then exactly one of the following holds:*

1. *The equivalence relation $\equiv_R$ has only countably-many classes.*
2. *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \to X$ of $2^{\mathbb{N}}$ into an R-antichain or a continuous homomorphism $\pi \colon 2^{\mathbb{N}} \to X$ from $<_{\text{lex}}$ to $<_R$.*

Proof. It is clear that the two conditions are mutually exclusive. To see that at least one of them holds, let G denote the complement of R. As every G-independent set is contained in a single $\equiv_R$-class, it follows that if there is an $\mathbb{N}$-coloring of G, then $\equiv_R$ has only countably-many classes. By Theorem 2.2.1, we can therefore assume that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G, for some dense set $S \subseteq 2^{<\mathbb{N}}$.

Let R' denote the pullback of R through ϕ. Note that each $\equiv_{R'}$-class is $G_S(2^{\mathbb{N}})$-independent, and therefore meager. But then Theorem 1.7.13 ensures that $\equiv_{R'}$ itself must be meager. If there exists $t \in 2^{<\mathbb{N}}$ such that R' is meager in $\mathcal{N}_t \times \mathcal{N}_t$ as well, then Theorem 1.7.14 yields a continuous injection $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $2^{\mathbb{N}}$ into an R'-antichain. Otherwise, Theorem 1.7.15 yields a continuous homomorphism $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $<_{\text{lex}}$ to $<_{R'}$. Set $\pi = \phi \circ \psi$. □

A similar result concerns metric spaces. A *pseudo-metric* is a function $d\colon X \times X \to \{r \in \mathbb{R} \mid r \geq 0\}$ such that $d(x,x) = 0$, $d(x,y) = d(y,x)$, and $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$. The *uniformity* associated with d is the sequence $(U_\epsilon)_{\epsilon>0}$ given by $U_\epsilon = \{(x,y) \in X \times X \mid d(x,y) < \epsilon\}$. We say that $Y \subseteq X$ is *ϵ-discrete* if $d(y,z) \geq \epsilon$, for all $y, z \in Y$. We say that $Y \subseteq X$ is *dense* if for all $\epsilon > 0$ and $x \in X$, there exists $y \in Y$ for which $d(x,y) < \epsilon$. We say that (X,d) is *separable* if it admits a countable dense set.

Theorem 2.2.8 (Friedman, Harrington, Kechris). *Suppose that X is a Hausdorff space and d is a pseudo-metric on X such that the sets of its uniformity are co-analytic. Then exactly one of the following holds:*

(1) *The space (X,d) is separable.*
(2) *There is a continuous injection $\pi\colon 2^{\mathbb{N}} \to X$ of $2^{\mathbb{N}}$ into an ϵ-discrete subspace of (X,d), for some $\epsilon > 0$.*

Proof. It is clear that the two conditions are mutually exclusive. To see that at least one of them holds, let G_ϵ denote the complement of U_ϵ, for all $\epsilon > 0$. If c is an $\mathbb{N}$-coloring of G_ϵ and $Y \subseteq X$ has the property that for all $x \in X$ there exists $y \in Y$ with $c(x) = c(y)$, then for all $x \in X$ there exists $y \in Y$ with $d(x,y) < \epsilon$. In particular, it follows that if there are $\mathbb{N}$-colorings of every G_ϵ, then (X,d) is separable. By Theorem 2.2.1, we can therefore assume that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to $G_{2\epsilon}$, for some $\epsilon > 0$ and dense set $S \subseteq 2^{<\mathbb{N}}$.

Let U'_ϵ denote the pullback of U_ϵ through ϕ. Note that every vertical section of U'_ϵ is $G_S(2^{\mathbb{N}})$-independent, and therefore meager. But then Theorem 1.7.13 ensures that U'_ϵ itself must be meager, in which case Theorem 1.7.14 yields a continuous function $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ sending distinct points to U'_ϵ-unrelated points. Set $\pi = \phi \circ \psi$. □

Another related theorem concerns linear orders. We say that a set $Y \subseteq X$ is *dense* with respect to a linear quasi-order R on X if it intersects every non-empty open interval, and we say that a family of sets is *intersecting* if no two sets in the family have empty intersection.

PROPOSITION 2.2.9. *A quasi-order R on a set X has a dense set of cardinality κ if and only if the family of closed R-intervals with non-empty interiors is a union of κ-many intersecting families.*

PROOF. Note first that for each $x \in X$, the family $\mathcal{F}_x$ of closed R-intervals containing x is an intersecting family. And if $C \subseteq X$ is dense, then every closed R-interval with non-empty intersection is in a set of the form $\mathcal{F}_x$, for some $x \in C$.

Conversely, suppose that $(\mathcal{F}_\alpha)_{\alpha<\kappa}$ is a sequence of intersecting families whose union is the family of all closed R-intervals with non-empty interiors. For each $\alpha < \kappa$, fix a maximal strictly decreasing sequence $([x_{\alpha,\beta}, y_{\alpha,\beta}]_R)_{\beta<\gamma_\alpha}$ consisting of closed intervals in F_α.

LEMMA 2.2.10. *Suppose that $\alpha < \kappa$. Then $\gamma_\alpha < \kappa^+$.*

PROOF. Suppose, towards a contradiction, that $\gamma_\alpha \geq \kappa^+$. Then there is a subsequence $([x'_{\alpha,\beta}, y'_{\alpha,\beta}]_R)_{\beta<\kappa^+}$ of $([x_{\alpha,\beta}, y_{\alpha,\beta}]_R)_{\beta<\gamma_\alpha}$ with the property that $(x'_{\alpha,\beta})_{\beta<\kappa^+}$ is strictly R-increasing or $(y'_{\alpha,\beta})_{\beta<\kappa^+}$ is strictly R-decreasing. In the former case, the closed intervals of the form $[x'_{\alpha,\lambda+3n}, x'_{\alpha,\lambda+3n+2}]_R$ have non-empty interiors and are pairwise disjoint. In the latter case, the closed intervals of the form $[y'_{\alpha,\lambda+3n+2}, y'_{\alpha,\lambda+3n}]_R$ have non-empty interiors and are pairwise disjoint. And in neither case are all of these intervals in $\bigcup_{\alpha<\kappa} \mathcal{F}_\alpha$, a contradiction. ❒

For each $\alpha < \kappa$ with the property that γ_α is a successor ordinal, fix $z_\alpha \in (x_{\alpha,\beta_\alpha}, y_{\alpha,\beta_\alpha})_R$, where β_α is the predecessor of γ_α. Then the set of points of the form $x_{\alpha,\beta}$, $y_{\alpha,\beta}$, and z_α is an R-dense set. □

We say that R is *separable* if there is a countable R-dense set.

THEOREM 2.2.11 (Friedman, Shelah). *Suppose that X is a Hausdorff space and R is a linear co-analytic quasi-order on X. Then exactly one of the following holds:*

(1) *The quasi-order R is separable.*
(2) *There is a continuous injection $\pi\colon 2^{\mathbb{N}} \to X \times X$ of $2^{\mathbb{N}}$ into pairs whose corresponding closed R-intervals have non-empty interior and are pairwise disjoint.*

PROOF. Clearly the two conditions are mutually exclusive. To see that at least one holds, note first that the set $A \subseteq X \times X$ given by

$$A = \{(x, y) \in X \times X \mid (x, y)_R \neq \emptyset\}$$

is analytic, as is the graph G on A given by

$$G = \{((x_1, y_1), (x_2, y_2)) \in A \times A \mid [x_1, y_1]_R \cap [x_2, y_2]_R = \emptyset\}.$$

We say that a family of sets is *intersecting* if no two sets in the family are disjoint. If there is an $\mathbb{N}$-coloring of G, then the set of closed intervals with non-empty interiors is a union of countably-many intersecting families, so Proposition 2.2.9 ensures that R is separable. By Theorem 2.2.1, we can therefore assume that there is a continuous homomorphism $(\phi, \psi)\colon 2^{\mathbb{N}} \to A \times A$ from $G_S(2^{\mathbb{N}})$ to G, for some dense set $S \subseteq 2^{<\mathbb{N}}$.

Note that for each $x \in X$, the set $\{c \in 2^{\mathbb{N}} \mid x \in [\phi(c), \psi(c)]_R\}$ is meager, since otherwise we could find $(c, d) \in G_S(2^{\mathbb{N}})$ for which $x \in [\phi(c), \psi(c)]_R \cap [\phi(d), \psi(d)]_R$, contradicting the fact that ϕ is a homomorphism from $G_S(2^{\mathbb{N}})$ to G. Then the vertical sections of the sets $\{(c, d) \in 2^{\mathbb{N}} \mid \phi(c) \in [\phi(d), \psi(d)]_R\}$, $\{(c, d) \in 2^{\mathbb{N}} \mid \psi(c) \in [\phi(d), \psi(d)]_R\}$, $\{(c, d) \in 2^{\mathbb{N}} \mid \phi(d) \in [\phi(c), \psi(c)]_R\}$, and $\{(c, d) \in 2^{\mathbb{N}} \mid \psi(d) \in [\phi(c), \psi(c)]_R\}$ are all meager. But then Theorem 1.7.13 ensures that all of these sets themselves must be meager, thus so too is their union. As the complement of this union is the pullback G' of G through ϕ, Theorem 1.7.14 yields a continuous injection $\pi'\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $2^{\mathbb{N}}$ into a G'-clique. Set $\pi = (\phi \circ \pi', \psi \circ \pi')$. □

Finite-dimensional dihypergraphs. Suppose that $d \geq 2$ is a natural number. A *d-dimensional dihypergraph* on X is a set $G \subseteq X^d$ of non-constant sequences, a set $Y \subseteq X$ is *G-independent* if $G \upharpoonright Y = \emptyset$, and an *$\mathbb{N}$-coloring* of G is a function $c\colon X \to \mathbb{N}$ for which preimages of singletons are G-independent.

For each set $S \subseteq d^{<\mathbb{N}}$ and function $\beta\colon S \to d^{\mathbb{N}}$, let $G_{S,\beta}(d^{\mathbb{N}})$ denote the digraph on $d^{\mathbb{N}}$ given by

$$G_{S,\beta}(d^{\mathbb{N}}) = \{(s \frown (i) \frown \beta(s))_{i<d} \mid s \in S\},$$

and let $G_S(d^{\mathbb{N}})$ denote $\bigcup_{\beta\colon S \to d^{\mathbb{N}}} G_{S,\beta}(d^{\mathbb{N}})$.

We say that a set $S \subseteq d^{\mathbb{N}}$ is *sparse* if $|S \cap d^n| \leq 1$ for all $n \in \mathbb{N}$.

THEOREM 2.2.12. *Suppose that $d \geq 2$, X is a Hausdorff space, and G is an analytic d-dimensional dihypergraph on X. Then for each sparse set $S \subseteq d^{<\mathbb{N}}$, at least one of the following holds:*

(1) *There is an $\mathbb{N}$-coloring of G.*

(2) *There is a continuous homomorphism* $\pi \colon d^{\mathbb{N}} \to X$ *from* $G_S(d^{\mathbb{N}})$ *to* G.

Proof. This follows from the proof of Theorem 2.2.1. □

A *matroid* on X is a function $M \colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following conditions:

(1) $\forall Y \subseteq X \ Y \subseteq M(Y)$.
(2) $\forall Y \subseteq X \ M(Y) = M(M(Y))$.
(3) $\forall Z \subseteq Y \subseteq X \ M(Z) \subseteq M(Y)$.
(4) $\forall Y \subseteq X \forall x \in X \forall y \in M(\{x\} \cup Y) \setminus M(Y) \ x \in M(\{y\} \cup Y)$.

We say that a sequence $(x_i)_{i \in I}$ of points of X is M*-independent* if $x_i \notin M(\{x_j \mid j \in I \setminus \{i\}\})$, for all $i \in I$. We say that a set $Y \subseteq X$ has M*-dimension* at most κ if there is a set $Z \subseteq X$ of cardinality at most κ for which $Y \subseteq M(Z)$. A standard argument reveals that for all $d \in \mathbb{N}$, every M-independent sequence of points in a set of M-dimension at most d has length at most d.

Theorem 2.2.13. *Suppose that* X *is a Hausdorff space,* $M \colon \mathcal{P}(X) \to \mathcal{P}(X)$ *is a matroid on* X*, and* d *is a positive natural number such that the graph of* $M \restriction [X]^d$ *is co-analytic when viewed as a subset of* X^{d+1}*. Then for every analytic set* $A \subseteq X$*, exactly one of the following holds:*

(1) *The set* A *is a union of countably-many sets of* M*-dimension at most* d*.*
(2) *There is a continuous injection* $\pi \colon 2^{\mathbb{N}} \to A$ *sending sets of cardinality* $d+1$ *to* M*-independent sets.*

Proof. It is clear that the two conditions are mutually exclusive. To see that at least one holds, let G be the $(d+1)$-dimensional dihypergraph on A consisting of all M-independent sequences of length $d+1$. Note that if a set $Y \subseteq X$ is G-independent, then there is a maximal M-independent set $Z \subseteq Y$ of cardinality at most d, in which case $Y = M(Z)$. It follows that if there is an $\mathbb{N}$-coloring of G, then A is a union of countably-many sets of M-dimension at most d. By Theorem 2.2.12, we can therefore assume that there is a continuous homomorphism $\phi \colon (d+1)^{\mathbb{N}} \to X$ from $G_S((d+1)^{\mathbb{N}})$ to G, for some set $S \subseteq (d+1)^{<\mathbb{N}}$ which is *dense* in the natural sense.

Let G' denote the pullback of G through ϕ. Note that for all sequences $(c_i)_{i<d}$ of elements of $(d+1)^{\mathbb{N}}$, the set of $c \in (d+1)^{\mathbb{N}}$ for which $\phi(c) \in M(\{\phi(c_i) \mid i < d\})$ is necessarily meager, since otherwise the straightforward analog of Proposition 2.2.4 for $G_S((d+1)^{\mathbb{N}})$ would yield a sequence of $G_S((d+1)^{\mathbb{N}})$-related such points, and the fact that ϕ is a homomorphism from $G_S((d+1)^{\mathbb{N}})$ to G would ensure that the image of this sequence under ϕ would be an M-independent sequence of points of $M(\{\phi(c_i) \mid i < d\})$ of length $d+1$, which is impossible. It then follows that for all $j \leq d$, the set of sequences $(c_i)_{i \leq d}$ of points in $(d+1)^{\mathbb{N}}$ with $\phi(c_j) \in M(\{\phi(c_i) \mid i \leq d \text{ and } i \neq j\})$ is meager, from which it follows that G' is comeager. The straightforward generalization of Theorem 1.7.14 to $(d+1)$-fold powers therefore yields a continuous injection $\psi \colon 2^{\mathbb{N}} \to (d+1)^{\mathbb{N}}$ sending injective sequences of length $d+1$ to elements of G'. Define $\pi = \phi \circ \psi$. □

The special case of Theorem 2.2.13 for $d = 1$ is a rephrasing of the perfect set theorem for co-analytic equivalence relations. But it yields a wealth of additional results, such as the van Engelen-Kunen-Miller characterization of analytic subsets of $\mathbb{R}^2$ which can be covered by countably-many lines.

Measurability. While Theorem 2.2.1 is already quite powerful, it is often useful to know whether there is an $\mathbb{N}$-coloring of the graph in question which is Borel. In order to obtain an analogous result characterizing the existence of such colorings, we first need to establish a corollary of yet another generalization of Theorem 1.4.14.

Proposition 2.2.14. *Suppose that X and Y are Hausdorff spaces, $A \subseteq X$ is analytic, $B \subseteq Y$ is analytic, $R \subseteq X \times Y$ is analytic, and $(A \times B) \cap R = \emptyset$. Then there are Borel sets $A' \subseteq X$ and $B' \subseteq Y$ such that $A \subseteq A'$, $B \subseteq B'$, and $(A' \times B') \cap R = \emptyset$.*

Proof. As A is disjoint from $\mathrm{proj}_X(R \cap (X \times B))$, there is a Borel set $A' \subseteq X$ separating the former from the latter. As B is disjoint from $\mathrm{proj}_Y(R \cap (A' \times Y))$, there is a Borel set $B' \subseteq X$ separating the former from the latter. Clearly the sets A' and B' are as desired. □

Proposition 2.2.15. *Suppose that X is a Hausdorff space, G is an analytic digraph on X, and $A \subseteq X$ is a G-independent analytic set. Then there is a G-independent Borel set $B \subseteq X$ such that $A \subseteq B$.*

Proof. The fact that A is G-independent ensures that $A \times A$ and G are disjoint, so Proposition 2.2.14 yields Borel sets $C, D \subseteq X$ such that $A \subseteq C$, $A \subseteq D$, and $(C \times D) \cap G = \emptyset$. But then the Borel set $B = C \cap D$ is as desired. □

We can now establish the promised strengthening of Theorem 2.2.1.

Theorem 2.2.16 (Kechris-Solecki-Todorcevic). *Suppose that X is a Hausdorff space and G is an analytic digraph on X. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

(1) *There is a Borel $\mathbb{N}$-coloring of G.*

(2) *There is a continuous homomorphism from $G_S(2^{\mathbb{N}})$ to G.*

Moreover, for each dense set $S \subseteq 2^{<\mathbb{N}}$, at most one of these holds.

Proof. Proposition 2.2.5 ensures that when S is dense, the two conditions are mutually exclusive. To see that one of them holds, note that, in the proof of Theorem 2.2.1, if the set X^α is Borel, then the sets of the form $A(a, X^\alpha)$, where a is an approximation, are analytic. When a is X^α-terminal, Proposition 2.2.15 therefore yields a G-independent Borel set $B(a, X^\alpha) \subseteq X$ such that $A(a, X^\alpha) \subseteq B(b, X^\alpha)$. We can therefore simply repeat the proof of Theorem 2.2.1, with the caveat that $X^{\alpha+1}$ is the difference of X^α and the union of the sets of the form $B(a, X^\alpha)$, where a is an X^α-terminal approximation, for all $\alpha < \omega_1$. □

As $G_S(2^{\mathbb{N}}) \subseteq \mathbb{E}_0$ for every set $S \subseteq 2^{<\mathbb{N}}$, it follows from Proposition 2.2.4 that every partial transversal of $\mathbb{E}_0$ with the Baire property is meager. We say that an equivalence relation is *countable* if all of its equivalence classes are countable.

Theorem 2.2.17 (Glimm, Effros, Jackson-Kechris-Louveau, Shelah-Weiss). *Suppose that X is a Hausdorff space and E is a countable analytic equivalence relation on X. Then exactly one of the following holds:*

(1) *The set X is the union of countably-many Borel partial transversals of E.*
(2) *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E.*

Proof. The fact that every Borel partial transversal of $\mathbb{E}_0$ is meager ensures that the two conditions are mutually exclusive. To see that at least one holds, let G denote the difference of E and the diagonal on X. Clearly any Borel $\mathbb{N}$-coloring of G gives rise to countably many Borel partial transversals of E whose union is X. We say that a set $S \subseteq 2^{<\mathbb{N}}$ is *full* if $|S \cap 2^n| \geq 1$ for all $n \in \mathbb{N}$. A straightforward induction shows that if S is such a set, then the connected components of $G_S(2^{\mathbb{N}})$ are exactly the equivalence classes of $\mathbb{E}_0$. By Theorem 2.2.16, we can assume that there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G, for some dense full set $S \subseteq 2^{<\mathbb{N}}$.

Let D' denote the pullback of the diagonal on X through ϕ, and let E' denote the pullback of E through ϕ. Then every equivalence class of D' is meager, since otherwise we could find $G_S(2^{\mathbb{N}})$-related points $c, d \in 2^{\mathbb{N}}$ such that $\phi(c) = \phi(d)$, contradicting the fact that ϕ is a homomorphism from $G_S(2^{\mathbb{N}})$ to G. As every E'-class is the union of countably many D'-classes, it follows that every E'-class is also meager, thus so too is E' itself. As $G_S(2^{\mathbb{N}}) \subseteq E'$, it follows that $\mathbb{E}_0 \subseteq E'$. Theorem 1.7.16 therefore yields a continuous homomorphism

$\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{E}_0 \setminus \Delta(2^{\mathbb{N}}), (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $(E' \setminus D', (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus E')$. Set $\pi = \phi \circ \psi$. □

A subset Y of a set X is a *partial transversal* of an equivalence relation E on X over a subequivalence relation F if $E \upharpoonright Y = F \upharpoonright Y$. As $G_S(2^{\mathbb{N}}) \subseteq \mathbb{E}_0 \setminus \mathbb{F}_0$ for every $S \subseteq 2^{<\mathbb{N}}$, every partial transversal of $\mathbb{E}_0$ over $\mathbb{F}_0$ with the Baire property is meager.

THEOREM 2.2.18 (Louveau). *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a co-analytic equivalence relation on X, and $E \cap F$ has index two below E. Then exactly one of the following holds:*

(1) *The set X is the union of countably many Borel partial transversals of E over $E \cap F$.*
(2) *There is a continuous homomorphism from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $((E \cap F) \setminus \Delta(X), E \setminus F, (X \times X) \setminus (E \cup F))$.*

PROOF. The fact that every Borel partial transversal of $\mathbb{E}_0$ over $\mathbb{F}_0$ is meager ensures that the two conditions are mutually exclusive. To see that at least one holds, consider the digraph $G = E \setminus F$ on X. Clearly every Borel $\mathbb{N}$-coloring of G gives rise to a countable family of Borel partial transversals of E over F whose union is X. By Theorem 2.2.16, we can therefore assume that there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G, for some dense full set $S \subseteq 2^{<\mathbb{N}}$.

Let D' denote the pullback of the diagonal on X through ϕ, let E' denote the pullback of E through ϕ, and let F' denote the pullback of F through ϕ. Then every equivalence class of F' is meager, since otherwise we could find $G_S(2^{\mathbb{N}})$-related points $c, d \in 2^{\mathbb{N}}$ whose images under ϕ are F-related, contradicting the fact that ϕ is a homomorphism from $G_S(2^{\mathbb{N}})$ to the complement of F. As every E'-class is the union of two $(E' \cap F')$-classes, it follows that every E'-class is also meager, thus so too is $E' \cup F'$. As $G_S(2^{\mathbb{N}}) \subseteq E' \setminus F'$, it follows that $\mathbb{F}_0 \subseteq E' \cap F'$ and $\mathbb{E}_0 \setminus \mathbb{F}_0 \subseteq E' \setminus F'$. Theorem 1.7.17 therefore yields a continuous homomorphism $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{F}_0 \setminus \Delta(2^{\mathbb{N}}), \mathbb{E}_0 \setminus \mathbb{F}_0, (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus \mathbb{E}_0)$ to $((E' \cap F') \setminus D', E' \setminus F', (2^{\mathbb{N}} \times 2^{\mathbb{N}}) \setminus (E' \cup F'))$. Set $\pi = \phi \circ \psi$. □

The special case of Theorem 2.2.18 in which $F \subseteq E$ can be viewed as stating that exactly one of the following holds:

(1) The set X/F is the union of countably-many partial transversals of E with Borel liftings.
(2) There is an embedding $\pi\colon 2^{\mathbb{N}}/\mathbb{F}_0 \to X/F$ of $\mathbb{E}_0/\mathbb{F}_0$ into E/F with a continuous lifting.

Parametrizations. A *uniformization* of a set $R \subseteq X \times Y$ is a function $\phi\colon \mathrm{proj}_X(R) \to Y$ whose graph is contained in R. A set is $\sigma(\mathbf{\Sigma}^1_1)$ if it is in the smallest σ-algebra containing the analytic sets.

THEOREM 2.2.19 (Jankov-von Neumann). *Suppose that X is a Hausdorff space, Y is a topological space, and $R \subseteq X \times Y$ is analytic. Then there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization of R.*

PROOF. By Proposition 1.4.8, we can assume that there is a continuous surjection $\phi\colon \mathbb{N}^\mathbb{N} \to R$. Define $\psi\colon \mathrm{proj}_X(R) \to Y$ by letting $\psi(x)$ be the lexicographically minimal $b \in \mathbb{N}^\mathbb{N}$ for which $(\mathrm{proj}_X \circ \phi)(b) = x$. Then ψ is $\sigma(\mathbf{\Sigma}^1_1)$-measurable, so $\mathrm{proj}_Y \circ \phi \circ \psi$ is as desired. $\square$

The following observation rules out the strengthening of Theorem 2.2.19 in which the uniformization is required to be Borel.

PROPOSITION 2.2.20. *Suppose that X and Y are injectively analytic Hausdorff spaces, $R \subseteq X \times Y$ is Borel, and $\mathrm{proj}_X(R)$ is the union of a sequence $(A_n)_{n\in\mathbb{N}}$ of analytic sets for which each $R \cap (A_n \times Y)$ has a Borel uniformization $\phi_n\colon A_n \to Y$. Then $\mathrm{proj}_X(R)$ is Borel.*

PROOF. Let G denote the digraph on R given by

$$G = \{((x, y), (x, z)) \in R \times R \mid x \in X \text{ and } y \neq z\}.$$

Then a set is G-independent if and only if it is the graph of a partial function. As Theorem 1.4.17 ensures that the graph of each ϕ_n is analytic, Theorem 2.2.15 yields G-independent Borel sets $R_n \subseteq R$ containing the graph of ϕ_n, for all $n \in \mathbb{N}$. As Theorem 1.5.7 ensures that the projections of these sets onto X are Borel, it follows that so too is $\mathrm{proj}_X(R)$. $\square$

We say that a digraph G on a set $R \subseteq X \times Y$ is *vertically invariant* if $x_1 = x_2$ for all $((x_1, y_1), (x_2, y_2)) \in G$. We use $G(x)$ to denote the graph on R_x consisting of all pairs (y, z) for which $((x, y), (x, z)) \in G$.

THEOREM 2.2.21. *Suppose that X and Y are Hausdorff spaces, $R \subseteq X \times Y$, and G is a vertically-invariant analytic digraph on R. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

(1) *There is a Borel $\mathbb{N}$-coloring of G.*
(2) *There is a continuous homomorphism from $G_S(2^\mathbb{N})$ to $G(x)$ for some $x \in X$.*

Moreover, for each dense set $S \subseteq 2^{<\mathbb{N}}$, at most one of these holds.

PROOF. By Theorem 2.2.16, it is enough to observe that if $S \subseteq 2^\mathbb{N}$ is full and $\phi\colon 2^\mathbb{N} \to X \times Y$ is a continuous homomorphism from $G_S(2^\mathbb{N})$ to G, then $\mathrm{proj}_X \circ \phi$ is constant. This is a direct consequence of the inexistence of non-trivial $\mathbb{E}_0$-invariant open sets. $\square$

As a corollary, we obtain the parametrized analog of the perfect set theorem for analytic sets.

THEOREM 2.2.22 (Lusin-Novikov). *Suppose that X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an analytic set whose vertical sections are countable. Then there are partial functions ϕ_n, whose graphs are Borel subsets of R, such that $R = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$.*

PROOF. Let G denote the vertically-invariant digraph on R given by

$$G = \{((x, y), (x, z)) \in R \times R \mid x \in X \text{ and } y \neq z\}.$$

If there is a Borel $\mathbb{N}$-coloring of G, then there are partial functions ϕ_n, whose graphs are Borel subsets of R, such that $R = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$.

Suppose, towards a contradiction, that there is no Borel $\mathbb{N}$-coloring of G. Theorem 2.2.21 then ensures that there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to Y$ from $G_S(2^{\mathbb{N}})$ to $G(x)$, for some dense set $S \subseteq 2^{<\mathbb{N}}$ and $x \in X$. One can now repeat the second half of the proof of Theorem 2.1.3 to obtain a continuous injection $\psi \colon 2^{\mathbb{N}} \to R_x$, a contradiction. □

Proposition 2.2.20 and Theorem 2.2.22 easily imply that countable-to-one images of Borel subsets of injectively analytic Hausdorff spaces are Borel. This can also be seen without the former using the following.

PROPOSITION 2.2.23 (Lusin-Novikov). *Suppose that X and Y are injectively analytic Hausdorff spaces and $\phi \colon X \to Y$ is a countable-to-one Borel function. Then X is a union of countably-many Borel sets on which ϕ is injective.*

PROOF. As Theorem 1.4.17 ensures that $\operatorname{graph}(\phi)$ is Borel, Theorem 2.2.22 yields partial functions ϕ_n, whose graphs are Borel, such that $\operatorname{graph}(\phi^{-1}) = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$. Theorem 1.5.7 ensures that the projections of the graphs of these functions onto X are Borel, and are therefore as desired. □

The *orbit equivalence relation* induced by a group Γ of permutations of a set X is given by $x \mathrel{E_\Gamma^X} y \iff \exists \gamma \in \Gamma \; \gamma \cdot x = y$.

THEOREM 2.2.24 (Feldman-Moore). *Suppose that X is an injectively analytic Hausdorff space and E is a countable Borel equivalence relation on X. Then there is a countable group Γ of Borel automorphisms of X for which $E = E_\Gamma^X$.*

PROOF. By Proposition 2.2.23, there are Borel sets $B_n \subseteq X$ and Borel injections $\phi_n \colon B_n \to X$ such that $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$. Fix Borel sets $C_n, D_n \subseteq X$ such that $\bigcup_{n \in \mathbb{N}} C_n \times D_n$ is the complement of the diagonal on X. For all $m, n \in \mathbb{N}$, set $A_{m,n} = B_m \cap C_n \cap \phi_m^{-1}(D_n)$, and let $\iota_{m,n}$ be the involution of X, supported by $A_{m,n} \cup \phi_m(A_{m,n})$, which agrees with ϕ_m on $A_{m,n}$. Then the group Γ generated by these involutions is as desired. □

We close with the following generalization of Theorem 2.2.22.

Theorem 2.2.25. *Suppose that X and Y are Hausdorff spaces, F is a co-analytic equivalence relation on Y, and $R \subseteq X \times Y$ is an analytic set whose vertical sections intersect only countably-many F-classes. Then there are Borel sets $R_n \subseteq R$, whose vertical sections intersect at most one F-class, such that $R = \bigcup_{n \in \mathbb{N}} R_n$.*

Proof. The proof is nearly identical to that of Theorem 2.2.22, except that we set

$$G = \{((x, y), (x, z)) \in R \times R \mid x \in X \text{ and } \neg y \ F \ z\},$$

and then appeal to the second half of the proof of Theorem 2.2.6. □

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