COMPOSITIONS OF PERIODIC AUTOMORPHISMS

BENJAMIN D. MILLER

Abstract. We introduce a notion of separability that holds of all Borel automorphisms of standard Borel spaces and automorphisms of complete Boolean algebras. We then prove that separable automorphisms of σ-complete Boolean algebras are products of various types of periodic automorphisms in their full groups. As applications, we show that a wide variety of groups of automorphisms consist solely of commutators and satisfy the Bergman property, that natural strengthenings of the Bergman property characterize the inexistence of invariant Borel probability measures in standard Borel spaces and standard measure spaces, and that the length four normal closure of any Borel automorphism of $\mathbb{R}$ with uncountable support is the group of all Borel automorphisms of $\mathbb{R}$.

Introduction

A *Borel space* is a set X equipped with a σ-algebra of *Borel* subsets. A function between Borel spaces is *Borel* if preimages of Borel sets are Borel. A *Borel automorphism* of a Borel space is a Borel bijection of the space with itself whose inverse is also Borel.

Given a binary relation R on a set X, we say that a family of subsets of X *separates R-related points* if, for all distinct R-related points x and y, there is a set in the family that contains x but not y. When X is a Borel space, we say that R is *separable* if there is a countable family of Borel subsets of X that separates R-related points. We say that a Borel automorphism $T\colon X \to X$ is *separable* if the graphs of its powers are separable. If there is a countable separating family of Borel subsets of X, then every Borel automorphism of X is separable.

We say that a subrelation R of the orbit equivalence relation E_Γ^X generated by a countable group Γ of Borel automorphisms of X is Γ-*decomposable* if there is a sequence $(B_\gamma)_{\gamma\in\Gamma}$ of Borel subsets of X with the property that $R = \bigcup_{\gamma\in\Gamma} \operatorname{graph}(\gamma \upharpoonright B_\gamma)$. We say that a partial function $T\colon X \rightharpoonup X$ is Γ-*decomposable* if its graph is Γ-decomposable. It is easy to see that the set $[\Gamma]$ of all Γ-decomposable Borel automorphisms

2010 *Mathematics Subject Classification.* Primary 03E15, 28A05, 37B05.
Key words and phrases. Bergman property, Borel automorphisms, full groups.
The author was partially supported by FWF grant P29999.

of X forms a group under composition; we refer to it as the *full group* of Γ. In the special case that there is a single Borel automorphism $T\colon X \to X$ for which Γ is the group $\langle T \rangle$ generated by T, we say that a partial function $S\colon X \rightharpoonup X$ is *T-decomposable* if it is Γ-decomposable, we define the *full group* of T to be the full group of Γ, and we use $[T]$ to denote $[\Gamma]$.

In what follows, we establish various algebraic properties of full groups of separable Borel automorphisms. While these results essentially appeared some time ago in [Mil04, §1], here we follow a more systematic approach that yields substantially more readable proofs of somewhat stronger theorems. This is only part of the justification for publishing these results now, however, as questions have recently arisen concerning analogs of properties of the group of Borel automorphisms of $\mathbb{R}$ for the group of permutations of $\mathbb{R}/\mathbb{Q}$ with Borel graphs, and the answers to such questions—which will appear in a subsequent paper—rely heavily upon the arguments presented here.

In §1, we consider *involutions* (i.e., functions $I\colon X \to X$ for which $I^2 = \mathrm{id}_X$). Although only the simplest Borel automorphisms are compositions of two Borel involutions in their full groups, we establish:

Theorem 1. *Suppose that X is a Borel space and $T\colon X \to X$ is a separable Borel automorphism. Then there are Borel involutions $I_1, I_2, I_3 \in [T]$ for which $T = I_3 \circ I_2 \circ I_1$.*

In §2, we consider more general products. We say that a permutation is *(a)periodic* if all of its orbits are (in)finite. We show that separability is not only sufficient for the above result, but necessary, even to obtain an ostensibly weaker conclusion:

Theorem 2. *Suppose that $k \geq 2$, X is a Borel space, $T\colon X \to X$ is an aperiodic Borel automorphism, and there are periodic Borel automorphisms $S_1, \ldots, S_k \in [T]$ with the property that $T = S_k \circ \cdots \circ S_1$. Then T is separable.*

We then determine the circumstances under which an aperiodic separable Borel automorphism is a composition of two Borel automorphisms with prescribed periods in its full group:

Theorem 3. *Suppose that $k_1 \geq 2$, $k_2 \geq 3$, X is a Borel space, and $T\colon X \to X$ is an aperiodic separable Borel automorphism. Then there exist $S_1, S_2 \in [T]$ such that every orbit of S_1 has cardinality k_1, every orbit of S_2 has cardinality 1 or k_2, and $T = S_2 \circ S_1$.*

In §3, we show that every aperiodic separable Borel automorphism is a special kind of commutator:

Theorem 4. *Suppose that $k \geq 3$, X is a Borel space, and $T \colon X \to X$ is an aperiodic separable Borel automorphism. Then there exist $S_1, S_2 \in [T]$ such that S_1^{-1} and S_2 are conjugate in $[T]$, every orbit of S_1 and S_2 has cardinality 1 or k, and $T = S_2 \circ S_1$.*

We say that a countable group of permutations is *aperiodic* if all of its orbits are infinite. More generally, we show that every Borel automorphism in the full group of an aperiodic countable group of separable Borel automorphisms is a special kind of commutator:

Theorem 5. *Suppose that $k \geq 3$, X is a Borel space, Γ is an aperiodic countable group of separable Borel automorphisms of X, and $T \in [\Gamma]$. Then there exist $S_1, S_2 \in [T]$ such that S_1^{-1} and S_2 are conjugate in $[\Gamma]$, every orbit of S_1 and S_2 has cardinality 1, 2, or k, and $T = S_2 \circ S_1$.*

A *Borel measure* on X is a measure μ on the Borel subsets of X. We say that such a measure is Γ-*invariant* if $\mu(B) = \mu(\gamma B)$ for all Borel sets $B \subseteq X$ and $\gamma \in \Gamma$. As this easily implies that $\mu(B) = \mu(T(B))$ for all Borel sets $B \subseteq X$ and $T \in [\Gamma]$, the special cases of Theorems 1, 3, and 5 for Lebesgue-measure-preserving Borel automorphisms of $[0, 1]$ easily yield the results of [Ryz85].

In §4, we focus on Borel spaces X that are *standard*, in the sense that the Borel structure on X is generated by a second-countable complete metric on X. The *support* of a function $T \colon X \to X$ is given by $\mathrm{supp}(T) = \{x \in X \mid x \neq T(x)\}$. Our main result is the following strengthening of Shortt's theorem that the quotient of the group of Borel automorphisms of a standard Borel space by the subgroup of permutations with countable supports is simple (see [Sho90]):

Theorem 6. *Suppose that X is a standard Borel space and $T \colon X \to X$ is a Borel automorphism with uncountable support. Then every Borel automorphism $S \colon X \to X$ is a composition of four conjugates of $T^{\pm 1}$ by Borel automorphisms of X.*

In §5, we consider algebraic properties originating in [Ber06]. We say that a group Γ of Borel automorphisms of X is *closed under countable decomposition* if $[\Delta] \subseteq \Gamma$ for all countable subgroups Δ of Γ. Given a group Γ, we say that an increasing sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of subsets of Γ is *exhaustive* if $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$. We say that Γ has the *Bergman property* if, for every exhaustive increasing sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of subsets of Γ, there exists $k \in \mathbb{N}$ such that $\Gamma = (\Gamma_k)^k$. The following fact implies Bergman's theorem that the symmetric group of permutations of $\mathbb{N}$ has the latter property (see [Ber06, Theorem 6]), as well as the analogous fact for the group of Borel automorphisms of $\mathbb{R}$ (see [DG05, Theorem 3.4]):

Theorem 7. *Suppose that X is a Borel space and Γ is a group of separable Borel automorphisms of X that is closed under countable decomposition and has an aperiodic countable subgroup. Then Γ has the Bergman property.*

For each $k \in \mathbb{N}$, we say that Γ has the *k-Bergman property* if, for every exhaustive increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of Γ, there exists $n \in \mathbb{N}$ such that $\Gamma = (\Gamma_n)^k$. When Γ is a countable group of permutations of X, we say that a set $Y \subseteq X$ is *Γ-complete* if $X = \Gamma Y$. We say that a countable group Γ of Borel automorphisms of X is *compressible* if there is a Γ-decomposable injection $T\colon X \to X$ for which $\sim T(X)$ is Γ-complete. The proofs of [Ber06, Theorem 6] and [DG05, Theorem 3.4] show that the corresponding groups have the 17-Bergman property, which is also an easy consequence of:

Theorem 8. *Suppose that X is a Borel space and Γ is a group of separable Borel automorphisms of X that is closed under countable decomposition and has a compressible countable subgroup. Then Γ has the 14-Bergman property.*

As a corollary, we obtain the following characterization of the existence of invariant Borel probability measures:

Theorem 9. *Suppose that $k \geq 14$, X is a standard Borel space, and Γ is an aperiodic countable group of Borel automorphisms of X. Then exactly one of the following holds:*

1. *There is a Γ-invariant Borel probability measure on X.*
2. *The group $[\Gamma]$ has the k-Bergman property.*

We say that a Borel measure μ on X is *Γ-quasi-invariant* if $\mu(B) = 0 \iff \mu(\gamma B) = 0$ for all Borel sets $B \subseteq X$ and $\gamma \in \Gamma$. Given such a μ, let $[\Gamma]/\mu$ denote the quotient of $[\Gamma]$ by the subgroup of Borel automorphisms in $[\Gamma]$ with μ-null supports. We also obtain:

Theorem 10. *Suppose that $k \geq 14$, X is a standard Borel space, Γ is an aperiodic countable group of Borel automorphisms of X, and μ is a Γ-quasi-invariant σ-finite Borel measure on X. Then exactly one of the following holds:*

1. *There is a Γ-invariant Borel probability measure $\nu \ll \mu$.*
2. *The group $[\Gamma]/\mu$ has the k-Bergman property.*

In §6, we generalize Theorems 1–5 and 7–8 to σ-complete Boolean algebras. To achieve this, we use essentially the same arguments as those utilized in [Fre04, §382M] to obtain the main result of [Ryz93] as a consequence of Theorem 1. However, by avoiding the use of Stone spaces, we eliminate the need for choice (beyond `DC`).

1. Products of involutions

The *saturation* of a set $Y \subseteq X$ under a partial injection $T\colon X \rightharpoonup X$ is given by $[Y]_T = \bigcup_{n\in\mathbb{Z}} T^n(\mathrm{dom}(T^n) \cap Y)$ (where T^0 is the identity function on X), the T-*orbit* of a point $x \in X$ is given by $[x]_T = [\{x\}]_T$, and a *transversal* of T is a set $Y \subseteq X$ whose intersection with each T-orbit is a singleton. When X is a Borel space, we say that a Borel automorphism $T\colon X \to X$ is *smooth* if it admits a Borel transversal. We begin this section by noting that the existence of involutions $I, J \in [T]$ for which $T = I \circ J$ is equivalent to a slight weakening of smoothness:

Proposition 1.1. *Suppose that X is a Borel space and $T\colon X \to X$ is a Borel automorphism. Then the following are equivalent:*

1. *There are involutions $I_0, I_1 \in [T]$ for which $T = I_1 \circ I_0$.*
2. *There is a Borel set $B \subseteq X$ whose intersection with each T-orbit is a singleton or doubleton.*

Proof. To see (1) $\implies$ (2), define a function $D\colon X \to \mathcal{P}(\mathbb{Z})$ by setting $D(x) = \{n \in \mathbb{Z} \mid I_0(x) = T^n(x)\}$ for all $x \in X$.

Lemma 1.2. *Suppose that $x \in X$. Then there is at most one point $y \in [x]_T \setminus \{x\}$ for which $D(x) \cap D(y) \neq \emptyset$.*

Proof. For all $M \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$, define $M - n = \{m - n \mid m \in M\}$.

Sublemma 1.3. *If $n \in \mathbb{Z}$, then $D(T^n(x)) = D(x) - 2n$.*

Proof. It is sufficient to show that $D(x) - 2n \subseteq D(T^n(x))$ for all $n \in \mathbb{Z}$, as this implies that $D(T^n(x)) - 2(-n) \subseteq D((T^{-n} \circ T^n)(x)) = D(x)$, so $D(T^n(x)) \subseteq D(x) - 2n$, thus $D(T^n(x)) = D(x) - 2n$ for all $n \in \mathbb{Z}$. But if $m \in D(x)$ and $n \in \mathbb{Z}$, then

$$
\begin{aligned}
(I_0 \circ T^n)(x) &= (I_0 \circ (I_1 \circ I_0)^n)(x) \\
&= ((I_0 \circ I_1)^n \circ I_0)(x) \\
&= (T^{-n} \circ I_0)(x) \\
&= T^{m-n}(x) \\
&= (T^{m-2n} \circ T^n)(x),
\end{aligned}
$$

so $m - 2n \in D(T^n(x))$, thus $D(x) - 2n \subseteq D(T^n(x))$. ⊠

If $k = |[x]_T|$ is infinite, then $D(y)$ is a singleton for all $y \in [x]_T$, so Sublemma 1.3 ensures that $\forall y \in [x]_T \setminus \{x\}\ D(x) \cap D(y) = \emptyset$. Otherwise, $D(y)$ is a translate of $k\mathbb{Z}$ for all $y \in [x]_T$, in which case Sublemma 1.3 implies that $\forall y \in [x]_T \setminus \{x\}\ D(x) \cap D(y) = \emptyset$ if k is odd, whereas $\forall y \in [x]_T \setminus \{x, T^{k/2}(x)\}\ D(x) \cap D(y) = \emptyset$ if k is even. ⊠

Fix Borel sets $B_n \subseteq X$ for which $I_0 = \bigcup_{n\in\mathbb{Z}} T^n \restriction B_n$. Note that if $n \in \mathbb{N}$ and $x \in B_n$, then $n \in D(x)$, so Lemma 1.2 ensures that B_n intersects each T-orbit in at most two points. Fix an enumeration $(k_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}$ and define $B'_n = B_{k_n} \setminus \bigcup_{m<n} [B_{k_m}]_T$ for all $n \in \mathbb{N}$. Then the set $B = \bigcup_{n\in\mathbb{N}} B'_n$ intersects each T-orbit in a singleton or doubleton.

To see (2) $\Longrightarrow$ (1), first recall that the *hitting time function* associated with B and T is the map $h^T_B \colon X \to \mathbb{Z}^+ \cup \{\infty\}$ given by

$$h^T_B(x) = \begin{cases} n & \text{if } n \in \mathbb{Z}^+ \text{ and } x \in T^{-n}(B) \setminus \bigcup_{0<m<n} T^{-m}(B) \text{ and} \\ \infty & \text{if } x \notin \bigcup_{n>0} T^{-n}(B), \end{cases}$$

and the *return time function* is the restriction r^T_B of h^T_B to B. Define $\overline{T}_B \colon (h^T_B)^{-1}(\mathbb{Z}^+) \to B$ by $\overline{T}_B(x) = T^{h^T_B(x)}(x)$. The corresponding *induced transformation* is given by $T_B = \overline{T}_B \restriction (r^T_B)^{-1}(\mathbb{Z}^+)$. These functions are clearly Borel (when $\mathbb{Z}^+ \cup \{\infty\}$ is endowed with the power set Borel structure).

Note that the set $B_\infty = (r^T_B)^{-1}(\{\infty\})$ intersects each infinite T-orbit in a singleton and misses each finite T-orbit, whereas the set $B_{<\infty} = \{x \in (r^T_B)^{-1}(\mathbb{Z}^+) \mid r^T_B(x) \geq (r^T_B \circ T_B)(x)\}$ misses each infinite T-orbit and intersects each finite T-orbit in a singleton or a set of the form $\{x, T^{k/2}(x)\}$, where $x \in X$ and $k = |[x]_T|$ is even. Set $A = B_{<\infty} \cup B_\infty$.

Lemma 1.4. *Suppose that $m, n \in \mathbb{Z}$, $x, y \in A$, and $T^{-m}(x) = T^{-n}(y)$. Then $T^m(x) = T^n(y)$.*

Proof. If $x = y$, then $T^{-m}(x) = T^{-n}(y) = T^{-n}(x)$, so $x = T^{m-n}(x)$, thus $T^n(y) = T^n(x) = T^m(x)$. Otherwise, set $k = |[x]_T| = |[y]_T|$. Then $y = T^{-k/2}(x) = T^{k/2}(x)$, so $T^{-m}(x) = T^{-n}(y) = T^{-k/2-n}(x)$, thus $x = T^{-k/2+m-n}(x)$, hence $T^{k/2-m+n}(x) = x$, and it follows that $T^n(y) = T^{k/2+n}(x) = T^m(x)$. ⊠

Define an involution $I_1 \in [T]$ by $I_1 = \bigcup_{n\in\mathbb{Z}} T^{2n} \restriction T^{-n}(A)$, as well as

$$\begin{aligned} I_0 &= I_1 \circ T \\ &= \textstyle\bigcup_{n\in\mathbb{Z}} (T^{2n} \restriction T^{-n}(A)) \circ T \\ &= \textstyle\bigcup_{n\in\mathbb{Z}} T^{2n} \circ (T \restriction T^{-(n+1)}(A)) \\ &= \textstyle\bigcup_{n\in\mathbb{Z}} T^{2n+1} \restriction T^{-(n+1)}(A). \end{aligned}$$

Then $I_1 \circ I_0 = I_1 \circ I_1 \circ T = T$ and $I_0^2 = I_0 \circ I_1 \circ T = T^{-1} \circ T = \mathrm{id}$. ⊠

Remark 1.5. In the special case that no T-orbit has even finite cardinality, condition (2) of Proposition 1.1 is equivalent to the smoothness of T. This can be established by noting that, in this case, the set A from the proof of (2) $\Longrightarrow$ (1) is a transversal of E^X_T.

Remark 1.6. For future arguments, it will be important to note the number of fixed points of the involutions I_0 and I_1 arising in the proof of (2) $\implies$ (1) along the T-orbit of each $x \in A$. Towards this end, first observe that if $n \in \mathbb{Z}$ and $y = T^{-n}(x)$, then

$$\begin{aligned} I_1(y) = y &\iff I_1(T^{-n}(x)) = T^{-n}(x) \\ &\iff T^n(x) = T^{-n}(x) \\ &\iff T^{2n}(x) = x \end{aligned}$$

and

$$\begin{aligned} I_0(y) = y &\iff (I_1 \circ T)(T^{-n}(x)) = T^{-n}(x) \\ &\iff I_1(T^{-(n-1)}(x)) = T^{-n}(x) \\ &\iff T^{n-1}(x) = T^{-n}(x) \\ &\iff T^{2n-1}(x) = x. \end{aligned}$$

If $k = |[x]_T|$ is infinite, then $n = 0$ is the unique solution to the first equation and there are no solutions to the second, so x is the unique fixed point of $I_1 \restriction [x]_T$ and $I_0 \restriction [x]_T$ has no fixed points. Otherwise, the solutions to the first equation are given by $2n \equiv 0 \pmod{k}$ and the solutions to the second are given by $2n - 1 \equiv 0 \pmod{k}$. If k is odd, then $n = 0$ is the unique solution in k to the first congruence and $n = (k+1)/2$ is the unique solution in k to the second, so x is the unique fixed point of $I_1 \restriction [x]_T$ and $T^{-(k+1)/2}(x)$ is the unique fixed point of $I_0 \restriction [x]_T$. If k is even, then $n = 0$ and $n = k/2$ are the unique solutions in k to the first congruence and there are no solutions to the second, so x and $T^{-k/2}(x)$ are the unique fixed points of $I_1 \restriction [x]_T$ and $I_0 \restriction [x]_T$ has no fixed points.

Remark 1.7. By applying (2) $\implies$ (1) to T^{-1}, one obtains involutions $I_0, I_1 \in [T]$ for which $T = I_1 \circ I_0$ and the number of fixed points of I_1 and I_0 along each T-orbit is as in Remark 1.6, but with the roles of I_1 and I_0 reversed.

We next turn our attention to writing automorphisms as compositions of three involutions.

Proposition 1.8. *Suppose that X is a Borel space and $T \colon X \to X$ is a Borel automorphism whose graph is separable. Then the support of T is Borel.*

Proof. Fix a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel sets separating graph(T)-related points, and observe that $\mathrm{supp}(T) = \bigcup_{n \in \mathbb{N}} B_n \setminus T^{-1}(B_n)$. ⊠

For all cardinals κ and sets X, let $[X]^\kappa$ denote the set of all subsets of X of cardinality κ.

Proposition 1.9. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X with Borel supports, $B \subseteq X$ is Borel, and $k \leq \aleph_0$. Then $\{x \in X \mid |B \cap \Gamma x| = k\}$ is Borel.*

Proof. It is enough to show that if $k < \aleph_0$, then the corresponding set $A = \{x \in X \mid |B \cap \Gamma x| \geq k\}$ is Borel. Towards this end, define $C_\Delta = \bigcap_{\delta \in \Delta} \delta^{-1} B$ and $D_\Delta = \bigcap_{\gamma \in \Delta} \bigcap_{\delta \in \Delta \setminus \{\gamma\}} \mathrm{supp}(\gamma^{-1}\delta)$ for all $\Delta \in [\Gamma]^k$, and observe that $A = \bigcup_{\Delta \in [\Gamma]^k} C_\Delta \cap D_\Delta$. ⊠

Given a Borel automorphism $T \colon X \to X$, let $\leq_T$ denote the quasi-order on X given by $x \leq_T y \iff \exists n \in \mathbb{N}\, T^n(x) = y$ and $<_T$ denote the strict quasi-order on X given by $x <_T y \iff (x \leq_T y \text{ and } \neg y \leq_T x)$. Given $x, y \in X$, set $(x, y)_T = \{z \in X \mid x <_T z <_T y\}$ (and define $[x, y)_T$, $(x, y]_T$, and $[x, y]_T$ analogously). Given $S \in [T]$, we say that a point $x \in X$ is *covered* by a point $y \in [x]_T$ if $y \leq_T x <_T S(y)$. Observe that if such a y exists, then $[x]_T$ is infinite (since otherwise $<_T \restriction [x]_T = \emptyset$), so there is a $\leq_T$-maximal such y. We use $c_S(x)$ to denote this point. We say that S is *T-covering* if every point in X is covered by a point in its T-orbit. Observe that the existence of such an element of $[T]$ ensures that T is aperiodic. We say that S is *T-non-crossing* if $(x, S(x))_T$ is S-invariant for all $x \in X$. We say that S is *T-oriented* if $S(x) \leq_T x \implies \forall y \in [x]_S\, y \leq_T x$ for all $x \in X$.

A *reduction* of an equivalence relation E on a set X to an equivalence relation F on a set Y is a function $\pi \colon X \to Y$ with the property that $w \mathrel{E} x \iff \pi(w) \mathrel{F} \pi(x)$ for all $w, x \in X$.

Proposition 1.10. *Suppose that X is a Borel space, $T \colon X \to X$ is a Borel automorphism, $S \in [T]$ is T-covering, T-non-crossing, and T-oriented, and $R = S^{-1} \circ T$. Then c_S is a reduction of E_R^X to equality.*

Proof. Fix $x, y \in X$. To see that $c_S(x) \neq c_S(y) \implies \neg x \mathrel{E_R^X} y$, it is enough to handle the case that $x \mathrel{E_T^X} y$. By reversing the roles of x and y if necessary, we can assume that $x <_T y$. Then $y <_T (S \circ c_S)(x) \implies c_S(x) \leq_T c_S(y)$ and $c_S(y) \leq_T x \implies c_S(y) \leq_T c_S(x)$, so the fact that $c_S(x) \neq c_S(y)$ ensures that $(S \circ c_S)(x) \leq_T y$ or $x <_T c_S(y)$, in which case $y \notin [c_S(x), (S \circ c_S)(x))_T$ or $x \notin [c_S(y), (S \circ c_S)(y))_T$. To see that $\neg x \mathrel{E_R^X} y$, it is therefore sufficient to show the following:

Lemma 1.11. *For all $z \in X$, the set $[c_S(z), (S \circ c_S)(z))_T$ is R-invariant.*

Proof. If $z' \in [c_S(z), (S \circ c_S)(z))_T$, then $T(z') \in (c_S(z), (S \circ c_S)(z)]_T$, so $T(z') \in (c_S(z), (S \circ c_S)(z))_T$ or $T(z') = (S \circ c_S)(z)$. The S-invariance of $(c_S(z), (S \circ c_S)(z))_T$ ensures that $R(z') \in (c_S(z), (S \circ c_S)(z))_T$ in the former case, and clearly $R(z') = c_S(z)$ in the latter. ⊠

To see that $c_S(x) = c_S(y) \implies x \mathrel{E_R^X} y$, note that c_S is finite-to-one, so—by the obvious induction—it is sufficient to establish the special case where y is the $\leq_T$-minimal element of $[x]_T$ for which $x <_T y$ and $c_S(x) = c_S(y)$. Observe that $c_S(z) \in (x, y)_T$ for all $z \in (x, y)_T$.

Lemma 1.12. *There exists $n \in \mathbb{N}$ with the property that $(S^n \circ T)(x) = y$ and $(S^m \circ T)(x) <_T (S^{m+1} \circ T)(x)$ for all $m < n$.*

Proof. If $n \in \mathbb{N}$, $(S^m \circ T)(x) <_T (S^{m+1} \circ T)(x)$ for all $m < n$, and $(S^n \circ T)(x) <_T y$, then $(S^n \circ T)(x) \in (x, y)_T$, so $(c_S \circ S^n \circ T)(x) \in (x, y)_T$. As $((S^m \circ T)(x), (S^{m+1} \circ T)(x))_T$ is S-invariant for all $m < n$, it follows that $(c_S \circ S^n \circ T)(x) = (S^n \circ T)(x)$, so $(S^n \circ T)(x) <_T (S^{n+1} \circ T)(x)$, and the fact that $c_S(y) <_T (S^n \circ T)(x) <_T y$ ensures that $(S^{n+1} \circ T)(x) \leq_T y$. The obvious induction therefore yields the desired result. ⊠

As $c_S(y) <_T y$, it follows that $S(y) \leq_T y$, in which case the fact that $((S^m \circ T)(x), (S^{m+1} \circ T)(x))_T$ is S-invariant for all $m < n$ ensures that $S(y) \leq_T T(x)$. If $S(y) <_T T(x)$, then $y \neq (S^{-1} \circ T)(x)$, so the fact that S is T-oriented ensures that $(S^{-1} \circ T)(x) <_T T(x)$, thus the S-invariance of $((S^{-1} \circ T)(x), T(x))_T$ implies that $c_S(x) = (S^{-1} \circ T)(x)$; but $T(x) \leq_T y$, so $c_S(y) \neq (S^{-1} \circ T)(x)$, contradicting the fact that $c_S(x) = c_S(y)$. It follows that $S(y) = T(x)$, hence $y = R(x)$. ⊠

Remark 1.13. As c_S is finite-to-one, the conclusion of Proposition 1.10 immediately implies that R is periodic.

The *vertical sections* of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y \mid x \mathrel{R} y\}$, where $x \in X$. The *restriction* of a binary relation R on X to a set $Y \subseteq X$ is the binary relation $R \restriction Y$ on Y given by $R \restriction Y = R \cap (Y \times Y)$. A *digraph* on X is an irreflexive binary relation G on X, and a *graph* is a symmetric digraph. A set $Y \subseteq X$ is *G-independent* if $G \restriction Y = \emptyset$, and an *$I$-coloring* of G is a function $c \colon X \to I$ such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$. Equip $\mathbb{N}$ with the power set Borel structure, and note that the existence of a Borel $\mathbb{N}$-coloring of a digraph G on a Borel space X is equivalent to the existence of a cover $(B_n)_{n \in \mathbb{N}}$ of X by G-independent Borel sets. The following facts are simple generalizations of results from [KST99, §4]:

Proposition 1.14. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, and G is a separable Γ-decomposable digraph on X whose vertical sections are finite. Then there is a Borel $\mathbb{N}$-coloring of G.*

Proof. Fix Borel sets $A_\gamma \subseteq X$ for which $G = \bigcup_{\gamma \in \Gamma} \operatorname{graph}(\gamma \restriction A_\gamma)$, as well as an enumeration $(B_n)_{n \in \mathbb{N}}$ of a family of Borel sets that separates

G-related points and is closed under finite intersections. Then the set

$$C_n = \{x \in B_n \mid B_n \cap G_x = \emptyset\} = B_n \setminus (\bigcup_{\gamma \in \Gamma} A_\gamma \cap \gamma^{-1} B_n)$$

is Borel and G-independent for all $n \in \mathbb{N}$. But $X = \bigcup_{n \in \mathbb{N}} C_n$. ⊠

Proposition 1.15. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, and G is a Γ-decomposable graph on X that admits a Borel $\mathbb{N}$-coloring. Then every G-independent Borel set $B \subseteq X$ is contained in a Borel maximal G-independent set.*

Proof. Fix Borel sets $A_\gamma \subseteq X$ for which $G = \bigcup_{\gamma \in \Gamma} \operatorname{graph}(\gamma \restriction A_\gamma)$, as well as a cover $(B_n)_{n \in \mathbb{N}}$ of X by G-independent Borel sets. Set $C_0 = B$ and recursively define

$$\begin{aligned} C_{n+1} &= C_n \cup \{x \in B_n \mid \neg \exists y \in C_n \ x \ G \ y\} \\ &= C_n \cup (B_n \setminus \bigcup_{\gamma \in \Gamma} A_\gamma \cap \gamma^{-1} C_n) \end{aligned}$$

for all $n \in \mathbb{N}$. As each of these sets is Borel and G-independent, so too is the set $C_\infty = \bigcup_{n \in \mathbb{N}} C_n$. To see that C_∞ is a maximal G-independent set, suppose that $x \in X$ and $C_\infty \cup \{x\}$ is G-independent, fix $n \in \mathbb{N}$ for which $x \in B_n$, and observe that $x \in C_{n+1} \subseteq C_\infty$. ⊠

The *diagonal* on X is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$. A *transversal* of an equivalence relation is a set that intersects every equivalence class in a single point.

Proposition 1.16. *Suppose that X is a Borel space, Γ is a countable group of separable Borel automorphisms of X, and $B \subseteq X$ is a Borel set whose intersection with every Γ-orbit is finite. Then there is a Borel transversal of $E_\Gamma^X \restriction B$.*

Proof. Proposition 1.8 ensures that the graph

$$\begin{aligned} G &= (E_\Gamma^X \setminus \Delta(X)) \cap (B \times B) \\ &= \bigcup_{\gamma \in \Gamma} \operatorname{graph}(\gamma \restriction (\operatorname{supp}(\gamma) \cap B \cap \gamma^{-1} B)) \end{aligned}$$

is Γ-decomposable. As it also has finite vertical sections and is separable, Propositions 1.14 and 1.15 yield a Borel maximal G-independent set $A \subseteq X$. But the intersection of any such set with B is a transversal of $E_\Gamma^X \restriction B$. ⊠

Given $n > 0$ and an aperiodic Borel automorphism $T : X \to X$, we say that a set $Y \subseteq X$ is $T^{\leq n}$*-independent* (or $T^{<(n+1)}$*-independent*) if it is independent with respect to the digraph $G = \bigcup_{1 \leq m \leq n} \operatorname{graph}(T^m)$. We also use R_T^X to denote $\leq_T$. Given a quasi-order R on X, we say that a set $Y \subseteq X$ is R*-complete* if it intersects every vertical section of

R. A *balanced marker sequence* for T is a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of $R^X_{T^{-1}}$- and R^X_T-complete Borel subsets of X such that

$$\forall x \in X \exists n \in \mathbb{N}\ (x \notin B_n \text{ and } \forall i \in \{\pm 1\}\ \overline{(T^i)}_{B_n}(x) \notin B_{n+1}).$$

Proposition 1.17. *Suppose that X is a Borel space and $T \colon X \to X$ is an aperiodic separable Borel automorphism. Then T has a balanced marker sequence.*

Proof. We first consider the special case of the proposition where there is an E^X_T-complete Borel set $B \subseteq X$ with the property that the set $B_n = \{x \in X \mid d_T(x, B) \geq n\}$ is $R^X_{T^{-1}}$- and R^X_T-complete for all $n \in \mathbb{N}$, where $d_T(x, B) = \min\{|i| \mid i \in \mathbb{Z} \text{ and } T^i(x) \in B\}$. Clearly $(B_n)_{n \in \mathbb{N}}$ is decreasing. As $|d_T(x, B) - d_T(T(x), B)| \leq 1$ for all $x \in X$, it follows that, if $x \in X$ and $n > d_T(x, B)$, then $x \notin B_n$ and $d_T(\overline{(T^i)}_{B_n}(x), B) = n$ for all $i \in \{\pm 1\}$, so $\overline{(T^i)}_{B_n}(x) \notin B_{n+1}$ for all $i \in \{\pm 1\}$, thus $(B_n)_{n \in \mathbb{N}}$ is a balanced marker sequence for T.

For the general case, define $B_0 = X$. Given $n \in \mathbb{N}$ and an $R^X_{T^{-1}}$- and R^X_T-complete Borel set $B_n \subseteq X$, appeal to Propositions 1.14 and 1.15 to obtain a Borel maximal $(T_{B_n})^{\leq 2}$-independent set $B_{n+1} \subseteq B_n$. A straightforward induction shows that $\overline{r^T_{B_n}}$ is bounded below by 3^n, so the set $A_n = \{x \in \sim B_n \mid \forall i \in \{\pm 1\}\ \overline{(T^i)}_{B_n}(x) \notin B_{n+1}\}$ contains $\bigcup_{0<i<3^n} (T^i \circ T_{B_n})(B_{n+1})$ for all $n \in \mathbb{N}$. Set $B = \sim \bigcup_{n \in \mathbb{N}} A_n$, and note that the special case yields a balanced marker sequence for $T \restriction [B]_T$, whereas $(B_n \setminus [B]_T)_{n \in \mathbb{N}}$ is a balanced marker sequence for $T \restriction \sim [B]_T$. ⊠

Proposition 1.18. *Suppose that X is a Borel space and $T \colon X \to X$ is an aperiodic separable Borel automorphism. Then there is a T-covering T-non-crossing involution $I \in [T]$.*

Proof. By Proposition 1.17, there is a balanced marker sequence $(B_n)_{n \in \mathbb{N}}$ for T. Let I be the involution agreeing with $(T^{-i})_{B_n} \circ (T^i)_{B_{n+1}} \circ (T^{-i})_{B_n}$ on $(T^i)_{B_n}(B_{n+1}) \setminus B_{n+1}$ for all $i \in \{\pm 1\}$ and $n \in \mathbb{N}$, and fixed elsewhere. To see that I is T-covering, note that if $n \in \mathbb{N}$, $x \in \sim B_n$, $\overline{(T^i)}_{B_n}(x) \notin B_{n+1}$ for all $i \in \{\pm 1\}$, and $x_i = ((T^{-i})_{B_n} \circ \overline{(T^i)}_{B_{n+1}})(x)$ for all $i \in \{\pm 1\}$, then $x_{-1} <_T x <_T x_1$ and $I(x_{-1}) = x_1$. To see that I is T-non-crossing, suppose that $y \in (x, I(x))_T$ and let n be the maximal natural number for which $x \in B_n$. Then $B_{n+1} \cap [x, I(x)]_T = \emptyset$. Moreover, if $I(y) \neq y$ and m is the maximal natural number for which $y \in B_m$, then $m < n$ and $B_{m+1} \cap ([y, I(y)]_T \cup [I(y), y]_T) = \emptyset$. As $x, I(x) \in B_n \subseteq B_{m+1}$, it follows that $x, I(x) \notin [y, I(y)]_T \cup [I(y), y]_T$, so $I(y) \in (x, I(x))_T$. ⊠

We can now give the following:

Proof of Theorem 1. By Propositions 1.8 and 1.9, the *periodic part* of T, given by $\mathrm{Per}(T) = \{x \in X \mid |[x]_T| < \aleph_0\}$, is Borel. As Proposition 1.16 ensures that $T \restriction \mathrm{Per}(T)$ is smooth, Proposition 1.1 implies that it is the composition of two involutions in its full group. We can therefore assume that T is aperiodic, so Proposition 1.18 yields a T-covering T-non-crossing involution $I_2 \in [T]$. As every involution in $[T]$ is T-oriented, Remark 1.13 ensures that $I_2 \circ T$ is periodic, so Propositions 1.1 and 1.16 yields involutions $I_0, I_1 \in [I_2 \circ T] \subseteq [T]$ with the property that $I_2 \circ T = I_1 \circ I_0$, thus $T = I_2 \circ I_1 \circ I_0$. ⊠

2. Products of periodic automorphisms

We begin this section with the following:

Proof of Theorem 2. Define $\phi \colon X \to \mathbb{N}$ by

$$\phi(x) = \min\{n \in \mathbb{N} \mid \textstyle\bigcup_{1 \leq i \leq k} [x]_{S_i} \subseteq [T^{-n}(x), T^n(x)]_T\}.$$

As $S_1, \ldots, S_k \in [T]$, it follows that ϕ is Borel, thus so too is the set

$$B = \{x \in X \mid \forall m \in \mathbb{Z} \exists n \in \mathbb{Z}\ (\phi \circ T^n)(x) \neq (\phi \circ T^{m+n})(x)\}.$$

Clearly B is T-invariant and $(B \cap (\phi \circ T^n)^{-1}(\{i\}))_{(i,n) \in \mathbb{N} \times \mathbb{Z}}$ separates $E^B_{T \restriction B}$-related points, so $T \restriction B$ is separable. Suppose, towards a contradiction, that $B \neq X$. Then there exists $x \in {\sim}B$, in which case the function $n \mapsto (\phi \circ T^n)(x)$ is periodic, and therefore bounded. Define $m = \max \phi([x]_T)$.

For all $S \in [T]$ and $y \in [x]_T$, let $n_S(y)$ be the unique integer with the property that $S(y) = T^{n_S(y)}(y)$. Define $\overline{n}_S = \lim_{n \to \infty} \frac{1}{n} \sum_{i<n} n_S(T^i(x))$. As $\overline{n}_T = 1$, we need only show that $\overline{n}_{S_k \circ \cdots \circ S_1} \neq 1$, since this contradicts our assumption that $T = S_k \circ \cdots \circ S_1$.

Towards this end, note that if $1 \leq j \leq k$, $n \geq 2(2k-1)m$, and $I_\ell = [T^{\ell m}(x), T^{n - \ell m}(x))_T$ for all $0 \leq \ell \leq 2k-1$, then

$$\begin{aligned}
&(S_{j-1} \circ \cdots \circ S_1)^{-1}([(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j}) \\
&\quad \subseteq (S_{j-1} \circ \cdots \circ S_1)^{-1}([I_k]_{S_j}) \\
&\quad \subseteq (S_{j-1} \circ \cdots \circ S_1)^{-1}(I_{k-1}) \\
&\quad \subseteq I_0,
\end{aligned}$$

so $[(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j} \subseteq (S_{j-1} \circ \cdots \circ S_1)(I_0)$. Define

$$D_j = (S_{j-1} \circ \cdots \circ S_1)(I_0) \setminus [(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j}$$

and observe that

$$\begin{aligned}|D_j| &\leq |(S_{j-1} \circ \cdots \circ S_1)(I_0) \setminus (S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})| \\ &= |(S_{j-1} \circ \cdots \circ S_1)(I_0 \setminus I_{2k-1})| \\ &= |I_0 \setminus I_{2k-1}| \\ &= 2(2k-1)m.\end{aligned}$$

Moreover, the S_j-invariance of $[(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j}$ ensures that $\sum_{y \in [(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j}} n_{S_j}(y) = 0$, for if $y \in X$ and $\ell = |[y]_{S_j}|$, then

$$\textstyle\sum_{z \in [y]_{S_j}} n_{S_j}(z) = \sum_{i<\ell} (n_{S_j} \circ S_j^i)(y) = n_{S_j^\ell}(y) = 0.$$

It now follows that if $n \geq 2(2k-1)m$, then

$$\begin{aligned}&\tfrac{1}{n} \textstyle\sum_{i<n} n_{S_k \circ \cdots \circ S_1}(T^i(x)) \\ &\quad = \tfrac{1}{n} \textstyle\sum_{y \in I_0} n_{S_k \circ \cdots \circ S_1}(y) \\ &\quad = \tfrac{1}{n} \textstyle\sum_{1 \leq j \leq k} \sum_{y \in I_0} (n_{S_j} \circ S_{j-1} \circ \cdots \circ S_1)(y) \\ &\quad = \tfrac{1}{n} \textstyle\sum_{1 \leq j \leq k} \sum_{y \in (S_{j-1} \circ \cdots \circ S_1)(I_0)} n_{S_j}(y) \\ &\quad = \tfrac{1}{n} \textstyle\sum_{1 \leq j \leq k} \sum_{y \in D_j} n_{S_j}(y) + \tfrac{1}{n} \sum_{1 \leq j \leq k} \sum_{y \in [(S_{j-1} \circ \cdots \circ S_1)(I_{2k-1})]_{S_j}} n_{S_j}(y) \\ &\quad \leq \tfrac{1}{n} \textstyle\sum_{1 \leq j \leq k} |D_j| m \\ &\quad \leq \tfrac{2k(2k-1)m^2}{n},\end{aligned}$$

so $\overline{n}_{S_k \circ \cdots \circ S_1} \leq \lim_{n \to \infty} \frac{2k(2k-1)m^2}{n} = 0$. ⊠

In order to construct T-covering T-non-crossing T-oriented elements of $[T]$ with prescribed finite periods, we will need analogous finitary notions. Given integers a and b, we will use the notation (a, b), $(a, b]$, $[a, b)$, and $[a, b]$ to denote the corresponding intervals of integers. Given a permutation σ of $[a, b]$, we say that a point $i \in [a, b)$ is *covered* by a point $j \in [a, b)$ if $j \leq i < \sigma(j)$. If such a j exists, then we use $c_\sigma(i)$ to denote the maximal such j. We say that σ is *covering* if every point in $[a, b)$ is covered by a point in $[a, b)$.

Proposition 2.1. *Suppose that $a \leq b$ are integers and σ is a permutation of $[a, b]$ for which $\sigma(b) = a$. Then σ is covering.*

Proof. Given $i \in [a, b)$, fix the least positive integer n with the property that $i < \sigma^n(b)$. As $\sigma(b) = a$, it follows that $n > 1$, so $n - 1$ is a positive integer, thus $\sigma^{n-1}(b) \leq i$, hence $\sigma^{n-1}(b)$ covers i. ⊠

We say that σ is *non-crossing* if $\forall i \in [a, b]$ $(i, \sigma(i))$ is σ-invariant.

Proposition 2.2. *Suppose that $a \leq b$ are integers and σ is a covering non-crossing permutation of $[a, b]$. Then $\sigma(b) = a$.*

Proof. Suppose, towards a contradiction, that $a < \sigma(b)$. As b does not cover any point of $[a, b)$, it follows that $b \neq c_\sigma(\sigma(b) - 1)$, so $\sigma(b) \neq (\sigma \circ c_\sigma)(\sigma(b) - 1)$, thus $\sigma(b) \in (c_\sigma(\sigma(b) - 1), (\sigma \circ c_\sigma)(\sigma(b) - 1))$. As $b \notin (c_\sigma(\sigma(b) - 1), (\sigma \circ c_\sigma)(\sigma(b) - 1))$, this contradicts the σ-invariance of $(c_\sigma(\sigma(b) - 1), (\sigma \circ c_\sigma)(\sigma(b) - 1))$. ⊠

The following observation eliminates the need to verify T-orientation, thereby explaining why we do not consider its finitary analog:

Proposition 2.3. *Suppose that $T \colon X \to X$ is an aperiodic Borel automorphism of a Borel space and $S \in [T]$ is periodic and T-non-crossing. Then S is T-oriented.*

Proof. Suppose, towards a contradiction, that there exists $x \in X$ for which $S(x) \leq_T x$ but x is not the $\leq_T$-maximal element of $[x]_S$. As S is periodic, there is a least $n \in \mathbb{N}$ for which $x <_T S^n(x)$, in which case $S^n(x) \not\leq_T S^{n-1}(x)$, so $x \neq S^{n-1}(x)$, thus $x \in (S^{n-1}(x), S^n(x))_T$. As $S^{n-1}(x), S^n(x) \notin (S^{n-1}(x), S^n(x))_T$, this contradicts the S-invariance of $(S^{n-1}(x), S^n(x))_T$. ⊠

Given integers $a \leq i \leq b$ and $c \leq d$, define $\phi_{[a,b],[c,d],i} \colon [a, i) \cup (i, b] \to [a, i) \cup (i + d - c, b + d - c]$ by

$$\phi_{[a,b],[c,d],i}(j) = \begin{cases} j & \text{if } j \in [a, i) \text{ and} \\ j + d - c & \text{if } j \in (i, b], \end{cases}$$

and $\phi_{[c,d],i} \colon [c, d] \to [i, i + d - c]$ by $\phi_{[c,d],i}(j) = i + j - c$. Given a permutation σ of $[a, b]$ fixing i and a permutation τ of $[c, d]$, the *amalgamation* of σ and τ at i is the permutation of $[a, b + d - c]$ given by $\sigma *_i \tau = (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i}) \cup (\phi_{[c,d],i} \circ \tau \circ \phi^{-1}_{[c,d],i})$.

Proposition 2.4. *Suppose that $a \leq i \leq b$ and $c \leq d$ are integers, σ is a covering non-crossing permutation of $[a, b]$ fixing i, and τ is a covering non-crossing permutation of $[c, d]$. Then $\sigma *_i \tau$ is covering and non-crossing. Moreover, the corresponding function $c_{\sigma *_i \tau}$ is the unique extension of $(\phi_{[a,b],[c,d],i} \circ c_\sigma \circ \phi^{-1}_{[a,b],[c,d],i}) \cup (\phi_{[c,d],i} \circ c_\tau \circ \phi^{-1}_{[c,d],i})$ with the property that $c_{\sigma *_i \tau}(i + d - c) = c_\sigma(i)$.*

Proof. If $i \in \{a, b\}$, then $a = \sigma(a)$ or $b = \sigma(b)$. As Proposition 2.2 ensures that $a = \sigma(b)$, it follows that $a = b$, so $\sigma *_i \tau = \phi_{[c,d],i} \circ \tau \circ \phi^{-1}_{[c,d],i}$, which easily yields the desired conclusion. We can therefore assume that $i \in (a, b)$.

To see that $\sigma *_i \tau$ is covering, appeal to Proposition 2.2 to see that

$$\begin{aligned}(\sigma *_i \tau)(b+d-c) &= (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i})(b+d-c)\\ &= (\phi_{[a,b],[c,d],i} \circ \sigma)(b)\\ &= \phi_{[a,b],[c,d],i}(a)\\ &= a,\end{aligned}$$

and apply Proposition 2.1.

To see that $\sigma *_i \tau$ is non-crossing, note first that if $j \in [a, i)$ and $(\sigma *_i \tau)(j) \in (i+d-c, b+d-c]$, then the fact that σ is non-crossing ensures that $(j, \sigma(j))$ is σ-invariant, in which case the fact that σ fixes i implies that $(j, i) \cup (i, \sigma(j))$ is σ-invariant, so

$$\begin{aligned}&(\sigma *_i \tau)((j, (\sigma *_i \tau)(j)))\\ &\quad = (\sigma *_i \tau)((j, i) \cup [i, i+d-c] \cup (i+d-c, (\sigma *_i \tau)(j)))\\ &\quad = (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i})((j, i)) \cup [i, i+d-c] \cup\\ &\qquad\quad (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i})((i+d-c, (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i})(j)))\\ &\quad = (\phi_{[a,b],[c,d],i} \circ \sigma)((j, i)) \cup [i, i+d-c] \cup (\phi_{[a,b],[c,d],i} \circ \sigma)((i, \sigma(j)))\\ &\quad = (\phi_{[a,b],[c,d],i} \circ \sigma)((j, i) \cup (i, \sigma(j))) \cup [i, i+d-c]\\ &\quad = \phi_{[a,b],[c,d],i}((j, i) \cup (i, \sigma(j))) \cup [i, i+d-c]\\ &\quad = \phi_{[a,b],[c,d],i}((j, i)) \cup [i, i+d-c] \cup \phi_{[a,b],[c,d],i}((i, \sigma(j)))\\ &\quad = (j, i) \cup [i, i+d-c] \cup (i+d-c, (\phi_{[a,b],[c,d],i} \circ \sigma \circ \phi^{-1}_{[a,b],[c,d],i})(j))\\ &\quad = (j, (\sigma *_i \tau)(j)).\end{aligned}$$

But if $j \in [a, b+d-c]$, $j < i \iff (\sigma *_i \tau)(j) < i$, and

$$(\phi_j, \rho_j) = \begin{cases} (\phi_{[a,b],[c,d],i}, \sigma) & \text{if } j \in [a, i) \cup (i+d-c, b+d-c] \text{ and}\\ (\phi_{[c,d],i}, \tau) & \text{if } j \in [i, i+d-c],\end{cases}$$

then the fact that ρ_j is non-crossing ensures that $(\phi_j^{-1}(j), (\rho_j \circ \phi_j^{-1})(j))$ is ρ_j-invariant, so

$$\begin{aligned}&(\sigma *_i \tau)((j, (\sigma *_i \tau)(j)))\\ &\quad = (\phi_j \circ \rho_j \circ \phi_j^{-1})((j, (\phi_j \circ \rho_j \circ \phi_j^{-1})(j)))\\ &\quad = (\phi_j \circ \rho_j)((\phi_j^{-1}(j), (\rho_j \circ \phi_j^{-1})(j)))\\ &\quad = \phi_j((\phi_j^{-1}(j), (\rho_j \circ \phi_j^{-1})(j)))\\ &\quad = (j, (\sigma *_i \tau)(j)).\end{aligned}$$

To see that $c_{\sigma *_i \tau}(i+d-c) = c_\sigma(i)$, observe that no element of $(c_\sigma(i), i)$ covers $i+d-c$ (with respect to $\sigma *_i \tau$), no element of $[i, i+d-c]$ covers

$i+d-c$ (again with respect to $\sigma *_i \tau$), and $c_\sigma(i) < i < (\sigma \circ c_\sigma)(i)$, so $(\phi_{[a,b],[c,d],i} \circ c_\sigma)(i) < i+d-c < (\phi_{[a,b],[c,d],i} \circ \sigma \circ c_\sigma)(i)$, thus $c_\sigma(i) < i+d-c < ((\sigma *_i \tau) \circ c_\sigma)(i)$.

Finally, suppose that $j \in [a, b+d-c) \setminus \{i+d-c\}$. Then $(c_{\rho_j} \circ \phi_j^{-1})(j)$ is the maximal element k of $\mathrm{dom}(\rho_j)$ for which $k \leq \phi_j^{-1}(j) < \rho_j(k)$, so $(\phi_j \circ c_{\rho_j} \circ \phi_j^{-1})(j)$ is the maximal element ℓ of $\phi_j(\mathrm{dom}(\rho_j))$ for which $\ell \leq j < (\phi_j \circ \rho_j \circ \phi_j^{-1})(\ell)$. If $j \in [a,i) \cup (i+d-c, b+d-c)$, then the fact that no element of $[i, i+d-c]$ covers j therefore ensures that $(\phi_j \circ c_{\rho_j} \circ \phi_j^{-1})(j)$ is the maximal element ℓ of $[a, b+d-c]$ for which $\ell \leq j < (\sigma *_i \tau)(\ell)$. If $j \in [i, i+d-c)$, then the fact that no element of $(i+d-c, b+d-c]$ covers j therefore ensures that $(\phi_j \circ c_{\rho_j} \circ \phi_j^{-1})(j)$ is the maximal element ℓ of $[a, b+d-c]$ for which $\ell \leq j < (\sigma *_i \tau)(\ell)$. In both cases, it follows that $c_{\sigma *_i \tau}(j) = (\phi_j \circ c_{\rho_j} \circ \phi_j^{-1})(j)$. ⊠

Given natural numbers $k_1, k_2 \geq 2$, we say that a permutation σ of a finite subinterval of $\mathbb{Z}$ is (k_1,k_2)-*dromedary* if it is covering and non-crossing, the c_σ-preimage of every singleton has cardinality 0 or k_1, and every σ-orbit has cardinality 1 or k_2. Let Succ denote the successor function on $\mathbb{Z}$.

Proposition 2.5. *Suppose that $k_1 \geq 2$ and $k_2 \geq 3$. Then there is a function $f \colon \mathbb{N} \setminus 2 \to \{1,2,3,4\}$ such that the following hold for all integers $a < b$ and (k_1,k_2)-dromedary permutations σ of $(a,b]$:*

1. *For all $n \in \{1,2,3,4\}$, there is an extension of σ to a (k_1,k_2)-dromedary permutation of $[a, b+nk_1(k_2-1))$.*
2. *For all $i > b+1$, there is an extension of σ to a (k_1,k_2)-dromedary permutation τ of $[a, b+f(i-b)k_1(k_2-1))$ for which $i \notin \mathrm{supp}(\tau) \cup \{b+f(i-b)k_1(k_2-1), b+f(i-b)k_1(k_2-1)+1\}$.*

Proof. We first show that it is sufficient to establish the special case of the proposition where $a = 0$. To see (1), note that if $n \in \{1,2,3,4\}$, then the special case yields an extension τ' of $\mathrm{Succ}^{-a} \circ \sigma \circ \mathrm{Succ}^a$ to a (k_1,k_2)-dromedary permutation of $[0, b-a+nk_1(k_2-1))$, in which case $\mathrm{Succ}^a \circ \tau' \circ \mathrm{Succ}^{-a}$ is an extension of σ to a (k_1,k_2)-dromedary permutation of $[a, b+nk_1(k_2-1))$. To see (2), note that if $i > b+1$, then $i-a > b-a+1$ and $f((i-a)-(b-a)) = f(i-b)$, so the special case yields an extension τ' of $\mathrm{Succ}^{-a} \circ \sigma \circ \mathrm{Succ}^a$ to a (k_1,k_2)-dromedary permutation of $[0, b-a+f(i-b)k_1(k_2-1))$ for which $i-a \notin \mathrm{supp}(\tau') \cup \{b-a+f(i-b)k_1(k_2-1), b-a+f(i-b)k_1(k_2-1)+1\}$, in which case $\mathrm{Succ}^a \circ \tau' \circ \mathrm{Succ}^{-a}$ is an extension of σ to a (k_1,k_2)-dromedary permutation τ of $[a, b+f(i-b)k_1(k_2-1))$ with the property that $i \notin \mathrm{supp}(\tau) \cup \{b+f(i-b)k_1(k_2-1), b+f(i-b)k_1(k_2-1)+1\}$.

We next show that it is sufficient to establish the further special case of the proposition where $b = 1$. To see (1), note that if $n \in \{1, 2, 3, 4\}$, then the further special case yields a (k_1, k_2)-dromedary permutation τ' of $[0, 1+nk_1(k_2-1))$ for which $\tau'(1) = 1$, so Proposition 2.4 ensures that $\tau' *_1 \sigma$ is an extension of σ to a (k_1, k_2)-dromedary permutation of $[0, b+nk_1(k_2-1))$. To see (2), note that if $i > b+1$, then $i-(b-1) > 2$ and $f((i-(b-1))-1) = f(i-b)$, so the further special case yields a (k_1, k_2)-dromedary permutation τ' of $[0, 1+f(i-b)k_1(k_2-1))$ with $\tau'(1) = 1$ and $i-(b-1) \notin \mathrm{supp}(\tau') \cup \{1+f(i-b)k_1(k_2-1), 1+f(i-b)k_1(k_2-1)+1\}$, in which case Proposition 2.4 ensures that $\tau' *_1 \sigma$ is an extension of σ to a (k_1, k_2)-dromedary permutation τ of $[0, b+f(i-b)k_1(k_2-1))$ with the property that $i \notin \mathrm{supp}(\tau) \cup \{b+f(i-b)k_1(k_2-1), b+f(i-b)k_1(k_2-1)+1\}$.

To establish the further special case, define $f \colon \mathbb{N} \setminus 2 \to \{1, 2, 3, 4\}$ by

$$f(i) = \begin{cases} 1 & \text{if } i \notin \{jk_1 - 1 \mid 0 < j < k_2\} \cup \{k_1(k_2-1), k_1(k_2-1)+1\}, \\ 2 & \text{if } i \in \{jk_1 - 1 \mid 0 < j < k_2\}, \\ 3 & \text{if } i = k_1(k_2-1), \text{ and} \\ 4 & \text{if } i = k_1(k_2-1)+1. \end{cases}$$

Define σ_1 on $[0, k_1(k_2-1)]$ by $\sigma_1 = (0\ k_1\ \cdots\ k_1(k_2-1))$. Proposition 2.1 ensures that σ_1 is covering, the fact that $\sigma_1 \restriction (jk_1, (j+1)k_1)$ is the identity for all $j < k_2 - 1$ implies that σ_1 is non-crossing and the c_{σ_1}-preimage of jk_1 is $[jk_1, (j+1)k_1)$ for all $j < k_2-1$, and the unique non-trivial σ_1-orbit is $\{jk_1 \mid j < k_2\}$, thus σ_1 is (k_1, k_2)-dromedary. If $0 < j < n$, then jk_1-1 is fixed by σ_1, so Proposition 2.4 ensures that the permutation $\sigma_{2,j} = \sigma_1 *_{jk_1-1} \sigma_1$ is (k_1, k_2)-dromedary. As 1 is fixed by σ_1, it follows that jk_1 is fixed by $\sigma_{2,j}$ for all $0 < j < k_2$, so Proposition 2.4 ensures that the permutation $\sigma_3 = \sigma_{2,k_2-1} *_{k_1(k_2-1)} \sigma_1$ is (k_1, k_2)-dromedary. As 1 is fixed by σ_1, it follows that $k_1(k_2-1)+1$ is fixed by σ_3, so Proposition 2.4 ensures that the permutation $\sigma_4 = \sigma_3 *_{k_1(k_2-1)+1} \sigma_1$ is (k_1, k_2)-dromedary. As 1 is fixed by σ_1, it follows that $k_1(k_2-1)+2$ is fixed by σ_4.

To see (1), observe that σ_1, $\sigma_{2,j}$ (for any $0 < j < k_2$), σ_3, and σ_4 are as desired. To see (2), suppose that $i > 2$. If $f(i-1) = 1$, then $i \notin \{jk_1 \mid 0 < j < k_2\} \cup \{k_1(k_2-1)+1, k_1(k_2-1)+2\}$, so the permutation $\tau = \sigma_1$ is as desired. If $f(i-1) = 2$, then $i = jk_1$ for some $0 < j < k_2$, in which case the permutation $\tau = \sigma_{2,j}$ is as desired. If $f(i-1) = 3$, then $i = k_1(k_2-1)+1$, so the permutation $\tau = \sigma_3$ is as desired. And if $f(i-1) = 4$, then $i = k_1(k_2-1)+2$, so the permutation $\tau = \sigma_4$ is as desired. ⊠

Given integers $a \leq b$ and a permutation σ of $[a, b]$, let $\overline{\sigma}$ denote the permutation of $[-b, -a]$ given by $\overline{\sigma}(i) = -\sigma^{-1}(-i)$.

Proposition 2.6. *Suppose that $a \leq b$ are integers and σ is a covering non-crossing permutation of $[a, b]$. Then $\overline{\sigma}$ is a covering non-crossing permutation of $[-b, -a]$ with the property that $c_{\overline{\sigma}}(i) = -(\sigma \circ c_\sigma)(-i-1)$ for all $i \in [-b, -a)$.*

Proof. As Proposition 2.2 ensures that $\overline{\sigma}(-a) = -\sigma^{-1}(a) = -b$, Proposition 2.1 implies that $\overline{\sigma}$ is covering. To see that $\overline{\sigma}$ is non-crossing, note that if $i \in [-b, -a]$, then the σ-invariance of $(\sigma^{-1}(-i), -i)$ ensures that

$$\begin{aligned} \overline{\sigma}((i, \overline{\sigma}(i))) &= \overline{\sigma}((i, -\sigma^{-1}(-i))) \\ &= -\sigma^{-1}((\sigma^{-1}(-i), -i)) \\ &= -(\sigma^{-1}(-i), -i) \\ &= (i, -\sigma^{-1}(-i)) \\ &= (i, \overline{\sigma}(i)). \end{aligned}$$

Finally, note that if $i \in [-b, -a)$ and j' is the least element of $(a, b]$ for which $\sigma^{-1}(j') \leq -i-1 < j'$, then $j' = (\sigma \circ c_\sigma)(-i-1)$, since otherwise $\sigma^{-1}(j') < c_\sigma(-i-1) \leq -i-1 < j' < (\sigma \circ c_\sigma)(-i-1)$, contradicting the σ-invariance of $(\sigma^{-1}(j'), j')$. It therefore follows that

$$\begin{aligned} c_{\overline{\sigma}}(i) &= \max\{j \in [-b, -a) \mid j \leq i < \overline{\sigma}(j)\} \\ &= \max\{j \in [-b, -a) \mid j \leq i < -\sigma^{-1}(-j)\} \\ &= -\min\{j' \in (a, b] \mid -j' \leq i < -\sigma^{-1}(j')\} \\ &= -\min\{j' \in (a, b] \mid \sigma^{-1}(j') < -i \leq j'\} \\ &= -\min\{j' \in (a, b] \mid \sigma^{-1}(j') \leq -i-1 < j'\} \\ &= -(\sigma \circ c_\sigma)(-i-1), \end{aligned}$$

which completes the proof. ⊠

In what follows, we will implicitly use the straightforward observation that the map $\sigma \mapsto \overline{\sigma}$ is an involution.

Proposition 2.7. *Suppose that $k_1 \geq 2$ and $k_2 \geq 3$. Then there is a function $f \colon \mathbb{N} \setminus 2 \to \{1, 2, 3, 4\}$ such that the following hold for all integers $a < b$ and (k_1, k_2)-dromedary permutations σ of $[a, b)$:*

1. *For all $n \in \{1, 2, 3, 4\}$, there is an extension of σ to a (k_1, k_2)-dromedary permutation of $(a - nk_1(k_2 - 1), b]$.*
2. *For all $i < a - 1$, there is an extension of σ to a (k_1, k_2)-dromedary permutation τ of $(a - f(a-i)k_1(k_2-1), b]$ for which $i \notin \operatorname{supp}(\tau) \cup \{a - f(a-i)k_1(k_2-1) - 1, a - f(a-i)k_1(k_2-1)\}$.*

Proof. To see (1), suppose that $n \in \{1, 2, 3, 4\}$, appeal to Proposition 2.6 to see that $\overline{\sigma}$ is a (k_1, k_2)-dromedary permutation of $(-b, -a]$, appeal to Proposition 2.5 to obtain an extension of $\overline{\sigma}$ to a (k_1, k_2)-dromedary permutation τ' of $[-b, -a + nk_1(k_2 - 1))$, and appeal once more to Proposition 2.6 to see that $\overline{\tau'}$ is an extension of σ to a (k_1, k_2)-dromedary permutation of $(a - nk_1(k_2 - 1), b]$.

To see (2), fix $f\colon \mathbb{N} \setminus 2 \to \{1, 2, 3, 4\}$ as in Proposition 2.5, appeal to Proposition 2.6 to see that $\overline{\sigma}$ is a (k_1, k_2)-dromedary permutation of $(-b, -a]$, and note that if $i < a - 1$, then $-i > -a + 1$ and $f(a - i) = f((-i) - (-a))$, so there is an extension of $\overline{\sigma}$ to a (k_1, k_2)-dromedary permutation τ' of $[-b, -a + f(a - i)k_1(k_2 - 1))$ with the property that $-i \notin \operatorname{supp}(\tau') \cup \{-a+f(a-i)k_1(k_2-1), -a+f(a-i)k_1(k_2-1)+1\}$, thus one more application of Proposition 2.6 ensures that $\overline{\tau'}$ is an extension of σ to a (k_1, k_2)-dromedary permutation τ of $(a - f(a-i)k_1(k_2-1), b]$ with $i \notin \operatorname{supp}(\tau) \cup \{a - f(a-i)k_1(k_2-1) - 1, a - f(a-i)k_1(k_2-1)\}$. ⊠

Proposition 2.8. *Suppose that $a < b$, $i < a - 1$, $j > b + 1$, $k_1 \geq 2$, and $k_2 \geq 3$ are integers and σ is a (k_1, k_2)-dromedary permutation of $(a, b]$. Then there exist $n \in \{1, 2, \ldots, 10\}$ and an extension of σ to a (k_1, k_2)-dromedary permutation τ of $(a - nk_1(k_2 - 1), b + nk_1(k_2 - 1)]$ with the property that $i \notin \operatorname{supp}(\tau) \cup \{a - nk_1(k_2-1) - 1, a - nk_1(k_2-1)\}$ and $j \notin \operatorname{supp}(\tau) \cup \{b + nk_1(k_2 - 1) + 1\}$.*

Proof. If $i < a - 4k_1(k_2 - 1) - 1$ and $f\colon \mathbb{N} \setminus 2 \to \{1, 2, 3, 4\}$ is the function given by Proposition 2.5, then there is an extension of σ to a (k_1, k_2)-dromedary permutation τ' of $[a, b+f(j-b)k_1(k_2-1))$ for which $j \notin \operatorname{supp}(\tau') \cup \{b+f(j-b)k_1(k_2-1), b+f(j-b)k_1(k_2-1)+1\}$, so part (1) of Proposition 2.7 yields the desired extension of τ' to a (k_1, k_2)-dromedary permutation of $(a-f(j-b)k_1(k_2-1), b+f(j-b)k_1(k_2-1)]$.

Similarly, if $j > b + 4k_1(k_2 - 1) + 1$ and $f\colon \mathbb{N} \setminus 2 \to \{1, 2, 3, 4\}$ is the function given by Proposition 2.7, then part (1) of Proposition 2.5 yields an extension of σ to a (k_1, k_2)-dromedary permutation τ' of $[a, b + f(a - i)k_1(k_2 - 1))$, in which case the defining property of f yields the desired extension of τ' to a (k_1, k_2)-dromedary permutation τ of $(a - f(a - i)k_1(k_2 - 1), b + f(a - i)k_1(k_2 - 1)]$ with the property that $i \notin \operatorname{supp}(\tau) \cup \{a - f(a - i)k_1(k_2 - 1) - 1, a - f(a - i)k_1(k_2 - 1)\}$.

It only remains to handle the case that $a - 4k_1(k_2 - 1) - 1 \leq i$ and $j \leq b+4k_1(k_2-1)+1$. We will recursively construct integers $a_e \leq a$ and $b_e \geq b$, as well as extensions of σ to (k_1, k_2)-dromedary permutations σ_e of $(a_e, b_e]$ with the property that $i \notin \operatorname{supp}(\sigma_e) \cup \{a_e - 1, a_e\}$ and $j \notin \operatorname{supp}(\sigma_e) \cup \{b_e + 1\}$ for all natural numbers $e \leq 4$. We begin by setting $a_0 = a$, $b_0 = b$, and $\sigma_0 = \sigma$. Suppose now that $e < 4$ and we have already found a_e, b_e, and σ_e. If there is an extension of σ_e to a (k_1, k_2)-dromedary permutation τ_e of $[a_e, b_e + k_1(k_2 - 1))$ for which $j \notin \operatorname{supp}(\tau_e) \cup \{b_e + k_1(k_2 - 1), b_e + k_1(k_2 - 1) + 1\}$, then set $m_{e+1} = 1$. Otherwise, part (2) of Proposition 2.5 yields $m_{e+1} \in \{2, 3, 4\}$ with the property that there is an extension of σ_e to a (k_1, k_2)-dromedary permutation τ_e of $[a_e, b_e + m_{e+1} k_1(k_2 - 1))$ for which $j \notin \operatorname{supp}(\tau_e) \cup \{b_e + m_{e+1} k_1(k_2 - 1), b_e + m_{e+1} k_1(k_2 - 1) + 1\}$. If there is an extension of τ_e to a (k_1, k_2)-dromedary permutation σ_{e+1} of $(a_e - k_1(k_2 - 1), b_e + m_{e+1} k_1(k_2 - 1)]$ for which $i \notin \operatorname{supp}(\sigma_{e+1}) \cup \{a_e - k_1(k_2 - 1) - 1, a_e - k_1(k_2 - 1)\}$, then set $\ell_{e+1} = 1$. Otherwise, part (2) of Proposition 2.7 yields $\ell_{e+1} \in \{2, 3, 4\}$ with the property that there is an extension of τ_e to a (k_1, k_2)-dromedary permutation σ_{e+1} of $(a_e - \ell_{e+1} k_1(k_2 - 1), b_e + m_{e+1} k_1(k_2 - 1)]$ for which $i \notin \operatorname{supp}(\sigma_{e+1}) \cup \{a_e - \ell_{e+1} k_1(k_2 - 1) - 1, a_e - \ell_{e+1} k_1(k_2 - 1)\}$. Setting $a_{e+1} = a_e - \ell_{e+1} k_1(k_2 - 1)$ and $b_{e+1} = b_e + m_{e+1} k_1(k_2 - 1)$, this completes the recursive construction.

Lemma 2.9. *There is at most one $e \in \{1, 2, 3, 4\}$ for which $\ell_e \neq 1$, as well as at most one $e \in \{1, 2, 3, 4\}$ for which $m_e \neq 1$.*

Proof. If $e \in \{1, 2, 3, 4\}$ has the property that $m_e \neq 1$, then part (1) of Proposition 2.5 ensures that $j \in (b_{e-1} + 1, b_{e-1} + k_1(k_2 - 1) + 1]$, and since the intervals of this form are pairwise disjoint, there is at most one such e. Similarly, if $e \in \{1, 2, 3, 4\}$ has the property that $\ell_e \neq 1$, then part (1) of Proposition 2.7 ensures that $i \in [a_{e-1} - k_1(k_2 - 1) - 1, a_{e-1} - 1)$, and since the intervals of this form are also pairwise disjoint, there is again at most one such e. ⊠

Define $\ell_{\max} = \max\{\ell_1, \ell_2, \ell_3, \ell_4\}$ and $m_{\max} = \max\{m_1, m_2, m_3, m_4\}$. If $\ell_{\max} = m_{\max}$, then set $\ell = 3 + \ell_{\max} = 3 + m_{\max}$ and observe that $a_4 = a - \ell k_1(k_2 - 1)$ and $b_4 = b + m k_1(k_2 - 1)$ by Lemma 2.9, so the permutation $\tau = \sigma_4$ is as desired. Otherwise, set $\ell = 3 + \ell_{\max} + m_{\max}$ and observe that $a_4 - m_{\max} k_1(k_2 - 1) = a - \ell k_1(k_2 - 1)$ and $b_4 + \ell_{\max} k_1(k_2 - 1) = b + \ell k_1(k_2 - 1)$ by Lemma 2.9, so an application of part (1) of Propositions 2.5 and 2.7 yields the desired extension of σ_4 to a (k_1, k_2)-dromedary permutation τ of $(a_4 - m_{\max} k_1(k_2 - 1), b_4 + \ell_{\max} k_1(k_2 - 1)]$. ⊠

Given an aperiodic Borel automorphism $T \colon X \to X$, a *T-gap* in a set $Y \subseteq X$ is an interval of the form $(y, T_Y(y))_T$, where $y \in Y$ and $1 < r_Y^T(y) < \infty$.

Proposition 2.10. *Suppose that a, $b > a + 1$, $k_1 \geq 2$, and $k_2 \geq 3$ are integers, $J \subseteq (a, b)$ is a set whose* Succ*-gaps have cardinality at least $10k_1(k_2 - 1) - 1$, and σ is a permutation of J whose restriction to each* Succ*-gap in $\sim J$ is (k_1, k_2)-dromedary. Then there exist $c \in$*

$[a - 10k_1(k_2 - 1) - 4, a]$, $d \in [b, b + 10k_1(k_2 - 1) + 4]$, and an extension of σ to a (k_1, k_2)-dromedary permutation of $(c, d]$.

Proof. We first show that it is sufficient to establish the special case of the proposition where $a = 10k_1(k_2 - 1) - 4$. Let c', d', and τ' be the result of applying this special case to $a' = 10k_1(k_2 - 1) - 4$, $b' = b - (a - a')$, $J' = \mathrm{Succ}^{a'-a}(J)$, and $\sigma' = \mathrm{Succ}^{a'-a} \circ \sigma \circ \mathrm{Succ}^{a-a'}$, and observe that the integers $c = c' + (a - a')$ and $d = d' + (a - a')$ and the permutation $\tau = \mathrm{Succ}^{a-a'} \circ \tau' \circ \mathrm{Succ}^{a'-a}$ are as desired.

We next show that it is sufficient to establish the further special case of the proposition where J is $\mathrm{Succ}^{<10k_1(k_2-1)}$-independent. Define $\phi \colon \mathbb{N} \setminus (J \cap \mathrm{Succ}(J)) \to \mathbb{N}$ by $\phi(j) = |j \setminus (J \cap \mathrm{Succ}(J))|$. Then the set $J' = \phi(J \setminus \mathrm{Succ}(J))$ is $\mathrm{Succ}^{<10k_1(k_2-1)}$-independent, so the further special case yields $c' \in [0, 10k_1(k_2 - 1) - 4]$, $d' \in [\phi(b), \phi(b) + 10k_1(k_2 - 1) + 4]$, and an (k_1, k_2)-dromedary permutation τ' of $(c', d']$ whose support is disjoint from J'. Let $(j_i)_{i<\ell}$ be the strictly increasing enumeration of $J \setminus \mathrm{Succ}(J)$ and define $c_0 = c'$, $d_0 = d'$, and $\tau_0 = \tau'$. For all $i < \ell$, set $C_i = \phi^{-1}(\{\phi(j_i)\})$, $c_{i+1} = c_i$, and $d_{i+1} = d_i + |C_i \setminus \{j_i\}|$, and appeal to Proposition 2.4 to see that the function $\tau_{i+1} = \tau_i *_{j_i} (\sigma \upharpoonright C_i)$ is a (k_1, k_2)-dromedary permutation of $(c_{i+1}, d_{i+1}]$. Then the permutation $\tau = \tau_\ell$ is the desired extension of σ.

Finally, we establish the special case of the proposition where J is $\mathrm{Succ}^{<10k_1(k_2-1)}$-independent (but a need not be $10k_1(k_2 - 1) - 4$). Set $m = a + \lfloor (b - a)/2 \rfloor$. As J is $T^{<8}$-independent, by setting $c_0 = m - 2$ if J intersects $[m+1, m+4]$ and $c_0 = m+2$ otherwise, we can ensure that $[c_0 - 1, c_0 + 2]$ is disjoint from J. As $m - a$ and $b - (m+1)$ are within 1 of one another, it follows that $c_0 - a$ and $b - (c_0 + 1)$ are within 5 of one another. Note that $a < c_0$ or $c_0 + 1 < b$, since otherwise $b - 1 \leq c_0 \leq a$, contradicting the fact that $b > a + 1$. Set $d_0 = c_0 + 1$, let τ_0 be the unique permutation of $(c_0, d_0]$, and recursively apply Proposition 2.8 to c_i, d_i, the maximum element of J below c_i (or any integer strictly below $c_i - 1$ if there is no such element of J), the minimum element of J above d_i (or any integer strictly above $d_i + 1$ if there is no such element of J), k_1, k_2, and τ_i to obtain $n_i \in \{1, \ldots, 10\}$ and an extension of τ_i to a (k_1, k_2)-dromedary permutation τ_{i+1} of $(c_{i+1}, d_{i+1}]$ for which $c_{i+1} = c_i - n_i k_1(k_2 - 1)$, $d_{i+1} = d_i + n_i k_1(k_2 - 1)$, and $\mathrm{supp}(\tau_{i+1}) \cup \{c_{i+1} - 1, c_{i+1}, d_{i+1} + 1\}$ is disjoint from J (since J is $T^{<10k_1(k_2-1)}$-independent), noting that $c_{i+1} - a$ and $b - d_{i+1}$ are within 5 of one another. Let i be the maximal natural number for which $a < c_i$ or $d_i < b$. If $a < c_i$, then

$c_{i+1} \geq c_i - 10k_1(k_2-1) > a - 10k_1(k_2-1)$ and

$$\begin{aligned} d_{i+1} &\leq d_i + 10k_1(k_2-1) \\ &= b + 10k_1(k_2-1) + (d_i - b) \\ &\leq b + 10k_1(k_2-1) + (a - c_i) + 5 \\ &\leq b + 10k_1(k_2-1) + 4. \end{aligned}$$

If $d_i < b$, then

$$\begin{aligned} c_{i+1} &\geq c_i - 10k_1(k_2-1) \\ &= a - 10k_1(k_2-1) + (c_i - a) \\ &\geq a - 10k_1(k_2-1) + (b - d_i) - 5 \\ &\geq a - 10k_1(k_2-1) - 4 \end{aligned}$$

and $d_{i+1} \leq d_i + 10k_1(k_2-1) < b + 10k_1(k_2-1)$. In both cases, it follows that the integers $c = c_{i+1}$ and $d = d_{i+1}$ and the permutation $\tau = \tau_{i+1}$ are as desired. ⊠

Given $n \geq 1$, $k_1 \geq 2$, $k_2 \geq 3$, an aperiodic bijection $T \colon X \to X$, and $x \in X$, we say that a permutation σ of $[x, T^n(x))_T$ is T-(k_1, k_2)-*dromedary* if the corresponding permutation $\theta^{-1} \circ \sigma \circ \theta$ of n is (k_1, k_2)-dromedary, where $\theta \colon n \to [x, T^n(x))_T$ is given by $\theta(i) = T^i(x)$ for all $i < n$. We can now give the following:

Proof of Theorem 3. We will find a T-covering T-non-crossing Borel automorphism $S \in [T]$ such that the preimage of every singleton under c_S has cardinality 0 or k_1 and the orbit of every point under S has cardinality 1 or k_2. To see that this is sufficient, set $S_2 = S$, appeal to Proposition 2.3 to see that any such automorphism is T-oriented, so Proposition 1.10 ensures that every orbit of the automorphism $S_1 = S_2^{-1} \circ T$ has cardinality k_1, and the fact that $S_2 \in [T]$ easily implies that $S_1 \in [T]$.

We will construct an exhaustive increasing sequence $(X_i)_{i \in \mathbb{N}}$ of Borel subsets of X, whose complements are $R^X_{T^{-1}}$- and R^X_T-complete, as well as T-decomposable bijections $S_i \colon X_i \to X_i$ such that:

(1) $\forall i \in \mathbb{N}\ S_i = S_{i+1} \restriction X_i$.
(2) $\forall i \in \mathbb{N} \forall T$-gaps I in $\sim X_i$ $S_i \restriction I$ is T-(k_1, k_2)-dromedary.

To see that this is sufficient, appeal to condition (1) to see that we obtain a function $S \colon X \to X$ by setting $S = \bigcup_{i \in \mathbb{N}} S_i$. To see that S is injective, note that if $x, y \in X$ have the property that $S(x) = S(y)$ and $i \in \mathbb{N}$ is sufficiently large that $x, y \in X_i$, then $S_i(x) = S_i(y)$, so $x = y$. To see that S is surjective, note that if $y \in X$ and $i \in \mathbb{N}$ is sufficiently large that $y \in X_i$, then there exists $x \in X_i$ for which

$S_i(x) = y$, so $S(x) = y$. To see that S is T-decomposable, fix Borel sets $B_{i,n} \subseteq X_i$ with the property that $S_i = \bigcup_{n\in\mathbb{Z}} T^n \restriction B_{i,n}$ for all $i \in \mathbb{N}$, set $B_n = \bigcup_{i\in\mathbb{N}} B_{i,n}$ for all $n \in \mathbb{Z}$, and observe that $S = \bigcup_{n\in\mathbb{Z}} T^n \restriction B_n$. To see that the preimage of every singleton under c_S has cardinality 0 or k_1, note that if $x \in X$ and $i \in \mathbb{N}$ is sufficiently large that $[x, S(x))_T \subseteq X_i$, then $c_S^{-1}(\{x\}) = c_{S_i}^{-1}(\{x\})$, and condition (2) ensures that the latter set has cardinality 0 or k_1. To see that the orbit of every point under S has cardinality 1 or k_2, note that if $x \in X$ and $i \in \mathbb{N}$ is sufficiently large that $x \in X_i$, then condition (2) implies that the orbit of x under S_i has cardinality 1 or k_2 and coincides with the orbit of x under S.

Appeal to Propositions 1.14 and 1.15 to obtain an $R^X_{T^{-1}}$- and R^X_T-complete $T^{<30k_1(k_2-1)+7}$-independent Borel set $D \subseteq X$. By Proposition 1.17, there is a balanced marker sequence $(D_i)_{i\in\mathbb{N}}$ for T_D. For all $a > 0$, $b > a + 1$, and $i \in \mathbb{N}$, let $D_{a,b,i+1}$ be the set of $x \in D_{i+1}$ for which $T_{D_i}(x), (T_{D_i})^2(x) \notin D_{i+1}$, $a = r^T_{D_i}(x)$, and $b = h^T_{T^{-1}_{D_i}(D_{i+1})}(x)$.

To guarantee that $(X_i)_{i\in\mathbb{N}}$ is exhaustive, it is enough to ensure that $(T^a(x), T^b(x))_T \subseteq X_{i+1}$ for all $a > 0$, $b > a+1$, $i \in \mathbb{N}$, and $x \in D_{a,b,i+1}$, since the fact that $(D_i)_{i\in\mathbb{N}}$ is a balanced marker sequence ensures that every point of X appears in an interval of this form. To guarantee that the sets X_i are Borel and the recursive construction goes through, we will construct Borel functions $a_{i+1}, b_{i+1} \colon \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1} \to \mathbb{N}$ such that:

(3) $\forall a > 0 \forall b > a+1 \forall x \in D_{a,b,i+1}\ a_{i+1}(x) \in [a - 10k_1(k_2-1) - 4, a]$.
(4) $\forall a > 0 \forall b > a+1 \forall x \in D_{a,b,i+1}\ b_{i+1}(x) \in [b, b + 10k_1(k_2-1) + 4]$.
(5) $X_{i+1} \setminus X_i = \bigcup_{a>0} \bigcup_{b>a+1} \bigcup_{x\in D_{a,b,i+1}} (T^{a_{i+1}(x)}(x), T^{b_{i+1}(x)}(x)]_T \setminus X_i$.

We begin by setting $S_0 = X_0 = \emptyset$. Suppose now that $i \in \mathbb{N}$ and we have already found $(a_j)_{1\leq j\leq i}$, $(b_j)_{1\leq j\leq i}$, and S_i.

Lemma 2.11. *The size of each T-gap in X_i is at least $10k_1(k_2-1)-1$.*

Proof. Suppose that $n > 1$, $x \in X_i$, and $r^T_{X_i}(x) = n$, so that $(x, T^n(x))_T$ is a T-gap in X_i. Then x and $T^n(x)$ are right and left endpoints of the sorts of intervals appearing in condition (5), so conditions (3)–(5) yield integers $b \in [0, 10k_1(k_2-1)+4]$ and $a \in [-1, 10k_1(k_2-1)+3]$ for which $T^{-b}(x), T^{a+n}(x) \in D$. As the fact that $n > 1$ ensures that $-b \leq 0 < a+n$, the fact that D is $T^{<30k_1(k_2-1)+7}$-independent implies that $30k_1(k_2-1) + 7 \leq b + a + n \leq n + 20k_1(k_2-1) + 7$, thus $n \geq 10k_1(k_2-1)$. ⊠

Let Q_{i+1} be the set of quadruples $q = (a^q, b^q, J^q, \sigma^q)$ with the property that $a^q > 0$, $b^q > a^q + 1$, $J^q \subseteq (a^q, b^q)$ is a set whose Succ-gaps have cardinality at least $10k_1(k_2-1)-1$, and σ^q is a permutation of J^q whose restriction to each Succ-gap in $\sim J^q$ is (k_1, k_2)-dromedary. For all $q \in Q_{i+1}$, Proposition 2.10 yields $c^q \in [a^q - 10k_1(k_2-1) - 4, a^q]$, $d^q \in [b^q, b^q + 10k_1(k_2-1)+4]$, and an extension of σ^q to a (k_1,k_2)-dromedary permutation τ^q of $(c^q, d^q]$. Let $D_{i+1,q}$ be the set of all $x \in D_{a^q,b^q,i+1}$ satisfying the following conditions:

(a) $\forall j \in (a^q, b^q)\ (j \in J^q \iff T^j(x) \in X_i)$.
(b) $\forall j \in J^q\ (S_i \circ T^j)(x) = T^{\sigma^q(j)}(x)$.

Clearly $\bigcup_{q \in Q_{i+1}} D_{i+1,q} \subseteq \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1}$ and Lemma 2.11 ensures that the reverse inclusion holds. Define $a_{i+1}(x) = c^q$ and $b_{i+1}(x) = d^q$ for all $q \in Q_{i+1}$ and $x \in D_{i+1,q}$.

Lemma 2.12. *Suppose that $a > 0$, $b > a+1$, and $x \in D_{a,b,i+1}$. Then $(T^{a_{i+1}(x)-1}(x), T^{b_{i+1}(x)+1}(x)]_T \cap X_i = (T^a(x), T^b(x)]_T \cap X_i$.*

Proof. As condition (3) ensures that $a_{i+1}(x) \leq a$ and condition (4) implies that $b \leq b_{i+1}(x)$, it is sufficient to show that

$$(T^{a_{i+1}(x)-1}(x), T^a(x)]_T \cap X_i = (T^b(x), T^{b_{i+1}(x)+1}(x)]_T \cap X_i = \emptyset.$$

Suppose, towards a contradiction, that this is false. Then condition (5) yields $a' > 0$, $b' > a'+1$, $i' < i$, and $x' \in D_{a',b',i'+1}$ for which

$$(T^{a_{i+1}(x)-1}(x), T^a(x)]_T \cap (T^{a_{i'+1}(x')}(x'), T^{b_{i'+1}(x')}(x')]_T \neq \emptyset \qquad (\dagger)$$

or

$$(T^b(x), T^{b_{i+1}(x)+1}(x)]_T \cap (T^{a_{i'+1}(x')}(x'), T^{b_{i'+1}(x')}(x')]_T \neq \emptyset. \qquad (\ddagger)$$

To handle the case that ($\dagger$) holds, note first that $T^{a_{i'+1}(x')}(x') <_T T^a(x)$ and $T^{a_{i+1}(x)-1}(x) <_T T^{b_{i'+1}(x')}(x')$. As $x', T^{a'}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 10k_1(k_2-1)+4}$-independent, condition (3) implies that $x' <_T T^{a_{i'+1}(x')}(x')$, so $x' <_T T^a(x)$. As $x', T^a(x) \in D_{i'+1}$, it follows that $T_{D_{i'+1}}(x') \leq_T T^a(x)$. As $T^{b'}(x'), T_{D_{i'+1}}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 10k_1(k_2-1)+4}$-independent, condition (4) ensures that $T^{b_{i'+1}(x')}(x') <_T T_{D_{i'+1}}(x')$, in which case $T^{b_{i'+1}(x')}(x') <_T T^a(x)$, so $T^{b_{i'+1}(x')}(x') \in [T^{a_{i+1}(x)}(x), T^a(x))_T$. But $T^a(x), T^{b'}(x') \in D_{i'}$, condition (3) yields $c \in (0, 10k_1(k_2-1)+4]$ for which $T^a(x) = T^{b_{i'+1}(x')+c}(x')$, and condition (4) ensures that $b_{i'+1}(x') - b' \in [0, 10k_1(k_2-1)+4]$, contradicting the $T^{\leq 20k_1(k_2-1)+8}$-independence of $D_{i'}$.

To handle the case that ($\ddagger$) holds, note first that $T^b(x) <_T T^{b_{i'+1}(x')}(x')$ and $T^{a_{i'+1}(x')}(x') <_T T^{b_{i+1}(x)+1}(x)$. As $T^{b'}(x'), T_{D_{i'+1}}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 10k_1(k_2-1)+4}$-independent, condition (4) implies that $T^{b_{i'+1}(x')}(x') <_T T_{D_{i'+1}}(x')$, so $T^b(x) <_T T_{D_{i'+1}}(x')$. As $T^b(x), T_{D_{i'+1}}(x') \in D_{i'+1}$, it follows that $T^b(x) \leq_T x'$. As $x', T^{a'}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 10k_1(k_2-1)+4}$-independent, condition (3) ensures that $x' <_T T^{a_{i'+1}(x')}(x')$, in which case $T^b(x) <_T T^{a_{i'+1}(x')}(x')$, so $T^{a_{i'+1}(x')}(x') \in (T^b(x), T^{b_{i+1}(x)}(x)]_T$.

But $T^b(x), T^{a'}(x') \in D_{i'}$, condition (3) ensures that $a' - a_{i'+1}(x') \in [0, 10k_1(k_2-1)+4]$, and condition (4) yields $c \in (0, 10k_1(k_2-1)+4]$ for which $T^{b+c}(x) = T^{a_{i'+1}(x')}(x')$, contradicting the $T^{\leq 20k_1(k_2-1)+8}$-independence of $D_{i'}$. ⊠

As D is $T^{\leq 10k_1(k_2-1)+4}$-independent, conditions (3) and (4) imply that if $x \in \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1}$, then $(T^{a_{i+1}}(x), T^{b_{i+1}}(x)]_T \subseteq (x, T_{D_{i+1}}(x))_T$. As the intervals of the latter form are pairwise disjoint, those of the former form are not only pairwise disjoint, but are not adjacent to one another. Lemma 2.12 therefore ensures that the function $S_{i+1} = S_i \cup \bigcup_{q \in Q_{i+1}} \bigcup_{j \in (c^q, d^q]} T^{\tau^q(j)-j} \upharpoonright T^j(D_{i+1,q})$ is well-defined and as desired. ⊠

3. Commutators

An *isomorphism* of a permutation σ of a set X with a permutation τ of a set Y is a bijection $\pi \colon X \to Y$ such that $\pi \circ \sigma = \tau \circ \pi$. Given integers $a \leq b$ and $k \geq 2$, a *pointed k-dromedary dyad* on $[a,b]$ is a triple of the form $u = (f^u, \sigma^u, \pi^u)$, where $f^u \in [a,b]$, σ^u is a covering non-crossing permutation of $[a,b]$ whose orbits all have cardinality 1 or k and which fixes f^u, and $\pi^u \colon [a,b] \setminus \{f^u\} \to [a,b)$ is an isomorphism of $\sigma^u \upharpoonright ([a,b] \setminus \{f^u\})$ with $(\mathrm{Succ}^{-1} \circ \sigma^u) \upharpoonright [a,b)$.

Proposition 3.1. *Suppose that $a \leq i \leq b$, $c \leq d$, and $n \geq 2$ are integers, u is a pointed k-dromedary dyad on $[a,b]$ for which $f^u \neq i$ but σ^u fixes i, and v is a pointed k-dromedary dyad on $[c,d]$. Then:*

1. *The integer i is not in $\{b, \pi^u(i), (\pi^u)^{-1}(i)\}$.*
2. *The domain of $\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i}$ is*
$$([a,i) \cup (i+d-c, b+d-c]) \setminus \{\phi_{[a,b],[c,d],i}(f^u), (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)\}.$$
3. *The domain of $\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i}$ is $[i, i+d-c] \setminus \{\phi_{[c,d],i}(f^v)\}$.*
4. *The triple $w = (f^w, \sigma^w, \pi^w)$, where $f^w = \phi_{[a,b],[c,d],i}(f^u)$, $\sigma^w = \sigma^u *_i \sigma^v$, and π^w is the extension of*
$$(\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i}) \cup (\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})$$
given by $(\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \mapsto i+d-c$ and $\phi_{[c,d],i}(f^v) \mapsto (\phi_{[a,b],[c,d],i} \circ \pi^u)(i)$, is a pointed k-dromedary dyad on $[a, b+d-c]$.

Proof. We begin with (1). To see that $i \neq b$, note that Proposition 2.2 would otherwise imply that $a = \sigma^u(b) = b$, contradicting the fact that $f^u \neq i$. To see that $i \neq \pi^u(i)$, note that i is a fixed point of σ^u, so $\pi^u(i)$ is a fixed point of $\mathrm{Succ}^{-1} \circ \sigma^u$, thus it cannot be a fixed point of σ^u. To see that $i \neq (\pi^u)^{-1}(i)$, note that i is a fixed point of σ^u, so it cannot be a fixed point of $\mathrm{Succ}^{-1} \circ \sigma^u$, thus $(\pi^u)^{-1}(i)$ is not a fixed point of σ^u.

Lemma 3.2. *Suppose that f and g are functions. Then* $\mathrm{dom}(f \circ g) = g^{-1}(\mathrm{dom}(f))$.

Proof. Simply observe that

$$\begin{aligned} x \in \mathrm{dom}(f \circ g) &\iff (x \in \mathrm{dom}(g) \text{ and } g(x) \in \mathrm{dom}(f)) \\ &\iff x \in \mathrm{dom}(g) \cap g^{-1}(\mathrm{dom}(f)) \\ &\iff x \in g^{-1}(\mathrm{dom}(f)), \end{aligned}$$

since $g^{-1}(\mathrm{dom}(f)) \subseteq \mathrm{dom}(g)$. ⊠

To see (2), appeal to Lemma 3.2 to obtain that

$$\begin{aligned} &\mathrm{dom}(\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i}) \\ &\quad = (\pi^u \circ \phi^{-1}_{[a,b],[c,d],i})^{-1}(\mathrm{dom}(\phi_{[a,b],[c,d],i})) \\ &\quad = (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})([a,b] \setminus \{i\}) \\ &\quad = \phi_{[a,b],[c,d],i}([a,b] \setminus \{f^u, (\pi^u)^{-1}(i)\}) \\ &\quad = ([a,i) \cup (i+d-c, b+d-c]) \setminus \\ &\qquad\quad \{\phi_{[a,b],[c,d],i}(f^u), (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)\}. \end{aligned}$$

To see (3), appeal to Lemma 3.2 to obtain that

$$\begin{aligned} \mathrm{dom}(\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i}) &= (\pi^v \circ \phi^{-1}_{[c,d],i})^{-1}(\mathrm{dom}(\phi_{[c,d],i})) \\ &= (\phi_{[c,d],i} \circ (\pi^v)^{-1})([c,d]) \\ &= \phi_{[c,d],i}([c,d] \setminus \{f^v\}) \\ &= [i, i+d-c] \setminus \{\phi_{[c,d],i}(f^v)\}. \end{aligned}$$

To see (4), first note that

$$\begin{aligned} \sigma^w(f^w) &= (\sigma^w \circ \phi_{[a,b],[c,d],i})(f^u) \\ &= (\phi_{[a,b],[c,d],i} \circ \sigma^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \phi_{[a,b],[c,d],i})(f^u) \\ &= (\phi_{[a,b],[c,d],i} \circ \sigma^u)(f^u) \\ &= \phi_{[a,b],[c,d],i}(f^u) \\ &= f^w. \end{aligned}$$

By Proposition 2.4, it only remains to show that π^w is an isomorphism of $\sigma^w \restriction ([a, b+d-c] \setminus \{f^w\})$ with $(\mathrm{Succ}^{-1} \circ \sigma^w) \restriction [a, b+d-c)$.

As the sets $[a, b+d-c] \setminus \{f^w\}$ and $[a, b+d-c)$ are finite and have the same cardinality, to see that π^w is a bijection between them, it is sufficient to establish that $\mathrm{dom}(\pi^w) = [a, b+d-c] \setminus \{f^w\}$ and $\pi^w([a, b+d-c] \setminus \{f^w\}) = [a, b+d-c)$. But the domain of π^w is the union of $([a,i) \cup (i+d-c, b+d-c]) \setminus \{f^w, (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)\}$,

$[i, i+d-c] \setminus \{\phi_{[c,d],i}(f^v)\}$, and $\{(\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i), \phi_{[c,d],i}(f^v)\}$ by (2) and (3), whereas

$$\begin{aligned}
&\pi^w(([a,i) \cup (i+d-c, b+d-c]) \setminus \{f^w, (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)\}) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(([a,i) \cup (i+d-c, b+d-c]) \setminus \\
&\qquad\qquad\qquad\qquad \{f^w, (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)\}) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \pi^u)([a,b] \setminus \{f^u, i, (\pi^u)^{-1}(i)\}) \\
&\quad = \phi_{[a,b],[c,d],i}([a,b) \setminus \{i, \pi^u(i)\}) \\
&\quad = ([a,i) \cup (i+d-c, b+d-c)) \setminus \{(\phi_{[a,b],[c,d],i} \circ \pi^u)(i)\}
\end{aligned}$$

and

$$\begin{aligned}
&\pi^w([i, i+d-c] \setminus \{\phi_{[c,d],i}(f^v)\}) \\
&\quad = (\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})([i, i+d-c] \setminus \{\phi_{[c,d],i}(f^v)\}) \\
&\quad = (\phi_{[c,d],i} \circ \pi^v)([c,d] \setminus \{f^v\}) \\
&\quad = \phi_{[c,d],i}([c,d)) \\
&\quad = [i, i+d-c),
\end{aligned}$$

and the image of $[a, b+d-c] \setminus \{f^w\}$ under π^w is the union of these sets with $\{i+d-c, (\phi_{[a,b],[c,d],i} \circ \pi^u)(i)\}$. It therefore only remains to show that $(\pi^w \circ \sigma^w)(j) = (\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w)(j)$ for all $j \in [a, b+d-c] \setminus \{f^w\}$.

We begin with the case that $j \in \mathrm{dom}(\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})$, which ensures that

$$\begin{aligned}
&(\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w)(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \sigma^w \circ \phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(j).
\end{aligned}$$

If $\sigma^w(j) \in \mathrm{dom}(\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})$, then

$$\begin{aligned}
&(\pi^w \circ \sigma^w)(j) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \sigma^w)(j) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \phi^{-1}_{[a,b],[c,d],i})(j) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \sigma^u \circ \phi^{-1}_{[a,b],[c,d],i})(j) \\
&\quad = (\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1} \circ \sigma^u \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(j),
\end{aligned}$$

since (2) ensures that $\phi^{-1}_{[a,b],[c,d],i}(j) \neq f^u$, in which case the fact that $\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i}$ and $\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1}$ agree on the intersection of their domains yields the desired conclusion. Otherwise, (2) ensures that $\sigma^w(j)$ is f^w or $(\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)$. The former cannot happen, as it would imply that $j = (\sigma^w)^{-1}(f^w) = f^w$, and (2) ensures that f^w is not in $\mathrm{dom}(\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})$. But the latter implies that $(\pi^w \circ \sigma^w)(j) = (\pi^w \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) = i + d - c$, and since (1) ensures that $i \neq b$, it follows that

$$
\begin{aligned}
&(\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i})(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \\
&\qquad\quad (\sigma^w)^{-1} \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ \phi^{-1}_{[a,b],[c,d],i} \circ \\
&\qquad\quad \phi_{[a,b],[c,d],i} \circ (\sigma^u)^{-1} \circ \phi^{-1}_{[a,b],[c,d],i} \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ (\sigma^u)^{-1} \circ (\pi^u)^{-1})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ (\pi^u \circ \sigma^u)^{-1})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ (\mathrm{Succ}^{-1} \circ \sigma^u \circ \pi^u)^{-1})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u \circ (\pi^u)^{-1} \circ (\sigma^u)^{-1} \circ \mathrm{Succ})(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \mathrm{Succ})(i) \\
&\quad = i + d - c,
\end{aligned}
$$

which yields the desired conclusion.

We next consider the case that $j \in \mathrm{dom}(\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})$, which ensures that

$$
\begin{aligned}
&(\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w)(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \sigma^w \circ \phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[c,d],i} \circ \sigma^v \circ \phi^{-1}_{[c,d],i} \circ \phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[c,d],i} \circ \sigma^v \circ \pi^v \circ \phi^{-1}_{[c,d],i})(j).
\end{aligned}
$$

Note that $\sigma^w(j) \in \mathrm{dom}(\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})$, since otherwise (3) implies that $\sigma^w(j) = \phi_{[c,d],i}(f^v)$, in which case $j = ((\sigma^w)^{-1} \circ \phi_{[c,d],i})(f^v) = (\phi_{[c,d],i} \circ (\sigma^v)^{-1} \circ \phi^{-1}_{[c,d],i} \circ \phi_{[c,d],i})(f^v) = (\phi_{[c,d],i} \circ (\sigma^v)^{-1})(f^v) = \phi_{[c,d],i}(f^v)$, and (3) ensures that $\phi_{[c,d],i}(f^v)$ is not in $\mathrm{dom}(\phi_{[c,d],i} \circ \pi^v \circ \phi^{-1}_{[c,d],i})$. As

(3) also implies that $\phi_{[c,d],i}^{-1}(j) \neq f^v$, it follows that

$$\begin{aligned}
&(\pi^w \circ \sigma^w)(j) \\
&\quad = (\phi_{[c,d],i} \circ \pi^v \circ \phi_{[c,d],i}^{-1} \circ \sigma^w)(j) \\
&\quad = (\phi_{[c,d],i} \circ \pi^v \circ \phi_{[c,d],i}^{-1} \circ \phi_{[c,d],i} \circ \sigma^v \circ \phi_{[c,d],i}^{-1})(j) \\
&\quad = (\phi_{[c,d],i} \circ \pi^v \circ \sigma^v \circ \phi_{[c,d],i}^{-1})(j) \\
&\quad = (\phi_{[c,d],i} \circ \mathrm{Succ}^{-1} \circ \sigma^v \circ \pi^v \circ \phi_{[c,d],i}^{-1})(j),
\end{aligned}$$

and since $\mathrm{Succ}^{-1} \circ \phi_{[c,d],i}$ and $\phi_{[c,d],i} \circ \mathrm{Succ}^{-1}$ agree on the intersection of their domains, the desired conclusion follows.

To handle the case that $j = (\phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i)$, note first that $(\pi^u)^{-1}(i) \neq f^u$, since $f^u \notin \mathrm{dom}(\pi^u)$. It follows that $(\sigma^u \circ (\pi^u)^{-1})(i) \neq f^u$, since otherwise $(\pi^u)^{-1}(i) = (\sigma^u)^{-1}(f^u) = f^u$. As we already showed that $(\sigma^u \circ (\pi^u)^{-1})(i) \neq (\pi^u)^{-1}(i)$ at the end of the first paragraph of the proof, (2) ensures that $(\phi_{[a,b],[c,d],i} \circ \sigma^u \circ (\pi^u)^{-1})(i)$ is in the domain of $\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi_{[a,b],[c,d],i}^{-1}$. As Proposition 2.2 implies that $c = \sigma^v(d)$, it follows that

$$\begin{aligned}
(\pi^w \circ \sigma^w)(j) &= (\pi^w \circ \sigma^w \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \\
&= (\pi^w \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \phi_{[a,b],[c,d],i}^{-1} \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \\
&= (\pi^w \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ (\pi^u)^{-1})(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \phi_{[a,b],[c,d],i}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ (\pi^u)^{-1})(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \sigma^u \circ (\pi^u)^{-1})(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1} \circ \sigma^u \circ \pi^u \circ (\pi^u)^{-1})(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1} \circ \sigma^u)(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1})(i) \\
&= \mathrm{Succ}^{-1}(i) \\
&= (\mathrm{Succ}^{-1} \circ \phi_{[c,d],i})(c) \\
&= (\mathrm{Succ}^{-1} \circ \phi_{[c,d],i} \circ \sigma^v)(d) \\
&= (\mathrm{Succ}^{-1} \circ \phi_{[c,d],i} \circ \phi_{[c,d],i}^{-1} \circ \sigma^w \circ \phi_{[c,d],i})(d) \\
&= (\mathrm{Succ}^{-1} \circ \sigma^w \circ \phi_{[c,d],i})(d) \\
&= (\mathrm{Succ}^{-1} \circ \sigma^w)(i + d - c) \\
&= (\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w \circ \phi_{[a,b],[c,d],i} \circ (\pi^u)^{-1})(i) \\
&= (\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w)(j).
\end{aligned}$$

In order to handle the final case that $j = \phi_{[c,d],i}(f^v)$, note that

$$\begin{aligned}
&(\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w)(j) \\
&\quad = (\mathrm{Succ}^{-1} \circ \sigma^w \circ \pi^w \circ \phi_{[c,d],i})(f^v) \\
&\quad = (\mathrm{Succ}^{-1} \circ \sigma^w \circ \phi_{[a,b],[c,d],i} \circ \pi^u)(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \phi_{[a,b],[c,d],i}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \pi^u)(i) \\
&\quad = (\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i} \circ \sigma^u \circ \pi^u)(i)
\end{aligned}$$

and

$$\begin{aligned}
(\pi^w \circ \sigma^w)(j) &= (\pi^w \circ \sigma^w \circ \phi_{[c,d],i})(f^v) \\
&= (\pi^w \circ \phi_{[c,d],i} \circ \sigma^v \circ \phi_{[c,d],i}^{-1} \circ \phi_{[c,d],i})(f^v) \\
&= (\pi^w \circ \phi_{[c,d],i} \circ \sigma^v)(f^v) \\
&= (\pi^w \circ \phi_{[c,d],i})(f^v) \\
&= (\phi_{[a,b],[c,d],i} \circ \pi^u)(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \pi^u \circ \sigma^u)(i) \\
&= (\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1} \circ \sigma^u \circ \pi^u)(i),
\end{aligned}$$

so the fact that $\mathrm{Succ}^{-1} \circ \phi_{[a,b],[c,d],i}$ and $\phi_{[a,b],[c,d],i} \circ \mathrm{Succ}^{-1}$ agree on the intersection of their domains yields the desired result. ⊠

Given integers and pointed k-dromedary dyads satisfying the hypotheses of Proposition 3.1, the *amalgamation* of u and v at i, denoted by $u *_i v$, is the pointed k-dromedary dyad w appearing in the conclusion of Proposition 3.1. We say that a pointed k-dromedary dyad u is *extended* by a pointed k-dromedary dyad v if $f^u \neq f^v$, $\sigma^u \sqsubseteq \sigma^v$, and $\pi^u \sqsubseteq \pi^v$.

Proposition 3.3. *Suppose that $k \geq 3$. Then there is a function $g \colon \mathbb{N} \setminus 2 \to \{1, 2, 3\}$ such that the following hold for all integers $a < b$ and pointed k-dromedary dyads u on $(a, b]$:*

1. *For all $n \in \{1, 2, 3\}$, there is an extension of u to a pointed k-dromedary dyad on $[a, b + 2n(k - 1))$.*
2. *For all $i > b + 1$, there is an extension of u to a pointed k-dromedary dyad v on $[a, b + 2g(i - b)(k - 1))$ with $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{b + 2g(i - b)(k - 1), b + 2g(i - b)(k - 1) + 1\}$.*

Proof. We first show that it is sufficient to establish the special case of the proposition where $a = 0$. To see (1), note that if $n \in \{1, 2, 3\}$, then the special case yields an extension v' of $(f^u - a, \mathrm{Succ}^{-a} \circ \sigma^u \circ \mathrm{Succ}^a, \mathrm{Succ}^{-a} \circ \pi^u \circ \mathrm{Succ}^a)$ to a pointed k-dromedary dyad on $[0, b - a + 2n(k - 1))$, in which case $(f^{v'} + a, \mathrm{Succ}^a \circ \sigma^{v'} \circ \mathrm{Succ}^{-a}, \mathrm{Succ}^a \circ \pi^{v'} \circ \mathrm{Succ}^{-a})$

is an extension of u to a pointed k-dromedary dyad on $[a, b + 2n(k - 1))$. To see (2), note that if $i > b + 1$, then $i - a > b - a + 1$ and $g((i - a) - (b - a)) = g(i - b)$, so the special case yields an extension v' of $(f^u - a, \mathrm{Succ}^{-a} \circ \sigma^u \circ \mathrm{Succ}^a, \mathrm{Succ}^{-a} \circ \pi^u \circ \mathrm{Succ}^a)$ to a pointed k-dromedary dyad on $[0, b - a + 2g(i - b)(k - 1))$ for which $i - a \notin \{f^{v'}\} \cup \mathrm{supp}(\sigma^{v'}) \cup \{b - a + 2g(i - b)(k - 1), b - a + 2g(i - b)(k - 1) + 1\}$, thus $(f^{v'} + a, \mathrm{Succ}^a \circ \sigma^{v'} \circ \mathrm{Succ}^{-a}, \mathrm{Succ}^a \circ \pi^{v'} \circ \mathrm{Succ}^{-a})$ is an extension of u to a pointed k-dromedary dyad v on $[a, b + 2g(i - b)(k - 1))$ with the property that $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{b + 2g(i - b)(k - 1), b + 2g(i - b)(k - 1) + 1\}$.

We next show that it is sufficient to establish the further special case where $b = 1$. To see (1), note that if $n \in \{1, 2, 3\}$, then the further special case gives rise to a pointed k-dromedary dyad v' on $[0, 1 + 2n(k - 1))$ for which $f^{v'} \neq 1$ but $\sigma^{v'}(1) = 1$, so Proposition 3.1 ensures that $v' *_1 u$ is an extension of u to a pointed k-dromedary dyad on $[0, b + 2n(k - 1))$. To see (2), note that if $i > b + 1$, then $i - (b - 1) > 2$ and $g((i - (b - 1)) - 1) = g(i - b)$, so the further special case yields a pointed k-dromedary dyad v' on $[0, 1 + 2g(i - b)(k - 1))$ with the property that $f^{v'} \neq 1$, $\sigma^{v'}(1) = 1$, and $i - (b - 1) \notin \{f^{v'}\} \cup \mathrm{supp}(\sigma^{v'}) \cup \{1 + 2g(i - b)(k - 1), 1 + 2g(i - b)(k - 1) + 1\}$, thus Proposition 3.1 ensures that $v' *_1 u$ is an extension of u to a pointed k-dromedary dyad v on $[a, b + 2g(i - b)(k - 1))$ with the property that $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{b + 2g(i - b)(k - 1), b + 2g(i - b)(k - 1) + 1\}$.

To establish the further special case, define $g \colon \mathbb{N} \setminus 2 \to \{1, 2, 3\}$ by

$$g(i) = \begin{cases} 1 & \text{if } i \notin [k - 1, 2k), \\ 2 & \text{if } i \in [k - 1, 2k - 2), \text{ and} \\ 3 & \text{if } i \in \{2k - 2, 2k - 1\}. \end{cases}$$

Define σ_1 on $[0, 2k - 2]$ by $\sigma_1 = (0 \; k \; k+1 \; \cdots \; 2k - 2)$. Proposition 2.1 ensures that σ_1 is covering, and the fact that $\sigma_1 \restriction (0, k)$ is the identity implies that σ_1 is non-crossing. A straightforward calculation reveals that $(\mathrm{Succ}^{-1} \circ \sigma_1) \restriction [0, 2k - 2) = (k - 1 \; k - 2 \; \cdots \; 0)$.

Lemma 3.4. *There is a pointed k-dromedary dyad u_1 with the property that $f^{u_1} = 2$ and $\sigma^{u_1} = \sigma_1$.*

Proof. It is sufficient to observe that σ_1 has one orbit of cardinality k that does not include 2 and fixes all other points of its domain, and $(\mathrm{Succ}^{-1} \circ \sigma_1) \restriction [0, 2k - 2)$ has one orbit of cardinality k and fixes all other points of its domain. ⊠

Proposition 2.4 ensures that the permutations

$$\begin{aligned}\sigma_2 &= \sigma_1 *_{k-1} \sigma_1 \\ &= (0\ 3k-2\ 3k-1\ \cdots\ 4k-4)(k-1\ 2k-1\ 2k\ \cdots\ 3k-3)\end{aligned}$$

and

$$\begin{aligned}\sigma_3 &= \sigma_2 *_{2k-2} \sigma_1 \\ &= (0\ 5k-4\ 5k-3\ \cdots\ 6k-6)(k-1\ 4k-3\ 4k-2\ \cdots\ 5k-5) \\ &\quad (2k-2\ 3k-2\ 3k-1\ \cdots\ 4k-4)\end{aligned}$$

are covering and non-crossing. Another straightforward calculation reveals that

$$\begin{aligned}(\mathrm{Succ}^{-1} \circ \sigma_2) \restriction [0, 4k-4) = {} & (3k-3\ k-2\ k-3\ \cdots\ 0) \\ & (2k-2\ 2k-3\ \cdots\ k-1)\end{aligned}$$

and

$$\begin{aligned}(\mathrm{Succ}^{-1} \circ \sigma_3) \restriction [0, 6k-6) = {} & (5k-5\ k-2\ k-3\ \cdots\ 0) \\ & (4k-4\ 2k-3\ 2k-4\ \cdots\ k-1) \\ & (3k-3\ 3k-4\ \cdots\ 2k-2).\end{aligned}$$

Lemma 3.5. *Suppose that $n \in \{2, 3\}$. Then there are pointed k-dromedary dyads u_n and v_n for which 1, f^{u_n}, and f^{v_n} are pairwise distinct but $\sigma^{u_n} = \sigma^{v_n} = \sigma_n$.*

Proof. It is enough to note that σ_n and $(\mathrm{Succ}^{-1} \circ \sigma_n) \restriction [0, 2nk-2n)$ have n orbits of cardinality k and fix all other points of their domains, as this ensures that the former has at least $1 + 2nk - 2n - nk = 1 + nk - 2n = 1 + n(k-2) \geq 3$ fixed points. ⊠

To see (1), observe that if $w_n \in \{u_n, v_n\}$ for all $n \in \{2, 3\}$, then u_1, w_2, and w_3 are as desired. To see (2), suppose that $i > 2$. If $g(i-1) = 1$, then $i \notin [k, 2k+1)$, so $v = u_1$ is as desired. If $g(i-1) = 2$, then $i \in [k, 2k-1)$, so $v = u_2$ or $v = v_2$ is as desired. And if $g(i-1) = 3$, then $i \in \{2k-1, 2k\}$, so $v = u_3$ or $v = v_3$ is as desired. ⊠

Given integers $a \leq b$ and a pointed n-dromedary dyad u on $[a, b]$, let $\overline{u}$ denote the triple $(f^{\overline{u}}, \sigma^{\overline{u}}, \pi^{\overline{u}})$, where $f^{\overline{u}} = -f^u$, $\sigma^{\overline{u}} = \overline{\sigma^u}$, and $\pi^{\overline{u}} \colon [-b, -a] \setminus \{f^{\overline{u}}\} \to [-b, -a)$ is given by $\pi^{\overline{u}}(i) = -(\mathrm{Succ} \circ \pi^u)(-i)$.

Proposition 3.6. *Suppose that $a \leq b$ and $k \geq 2$ are integers and u is a pointed k-dromedary dyad on $[a, b]$. Then $\overline{u}$ is a pointed k-dromedary dyad on $[-b, -a]$.*

Proof. Proposition 2.6 ensures that $\sigma^{\overline{u}}$ is a covering non-crossing permutation of $[-b,-a]$ whose orbits all have cardinality 1 or k. To see that $f^{\overline{u}}$ is a fixed point of $\sigma^{\overline{u}}$, note that

$$\begin{aligned}\sigma^{\overline{u}}(f^{\overline{u}}) &= -(\sigma^u)^{-1}(-f^{\overline{u}})\\ &= -(\sigma^u)^{-1}(f^u)\\ &= -f^u\\ &= f^{\overline{u}}.\end{aligned}$$

Finally, note that $\pi^u = \pi^u \circ \sigma^u \circ (\sigma^u)^{-1} = \mathrm{Succ}^{-1} \circ \sigma^u \circ \pi^u \circ (\sigma^u)^{-1}$, so $(\sigma^u)^{-1} \circ \mathrm{Succ} \circ \pi^u = \pi^u \circ (\sigma^u)^{-1}$, thus

$$\begin{aligned}(\pi^{\overline{u}} \circ \sigma^{\overline{u}})(i) &= \pi^{\overline{u}}(-(\sigma^u)^{-1}(-i))\\ &= -(\mathrm{Succ} \circ \pi^u \circ (\sigma^u)^{-1})(-i)\\ &= -(\mathrm{Succ} \circ (\sigma^u)^{-1} \circ \mathrm{Succ} \circ \pi^u)(-i)\\ &= \mathrm{Succ}^{-1}(-((\sigma^u)^{-1} \circ \mathrm{Succ} \circ \pi^u)(-i))\\ &= (\mathrm{Succ}^{-1} \circ \sigma^{\overline{u}})(-(\mathrm{Succ} \circ \pi^u)(-i))\\ &= (\mathrm{Succ}^{-1} \circ \sigma^{\overline{u}} \circ \pi^{\overline{u}})(i)\end{aligned}$$

for all $i \in [-b,-a] \setminus \{f^{\overline{u}}\}$. ⊠

Proposition 3.7. *Suppose that $k \geq 3$. Then there is a function $g \colon \mathbb{N} \setminus 2 \to \{1,2,3\}$ such that the following hold for all integers $a < b$ and pointed k-dromedary dyads u on $[a,b)$:*

(1) *For all $n \in \{1,2,3\}$, there is an extension of u to a pointed k-dromedary dyad on $(a-2n(k-1),b]$.*
(2) *For all $i < a-1$, there is an extension of u to a pointed k-dromedary dyad v on $(a-2g(a-i)(k-1),b]$ with $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{a-2g(a-i)(k-1)-1, a-2g(a-i)(k-1)\}$.*

Proof. To see (1), suppose that $n \in \{1,2,3\}$, appeal to Proposition 3.6 to see that $\overline{u}$ is a pointed k-dromedary dyad on $(-b,-a]$, appeal to Proposition 3.3 to obtain an extension of $\overline{u}$ to a pointed k-dromedary dyad v' on $[-b,-a+2n(k-1))$, and appeal once more to Proposition 3.6 to see that $\overline{v'}$ is an extension of u to a pointed k-dromedary dyad on $(a-2n(k-1),b]$.

To see (2), fix $g \colon \mathbb{N} \setminus 2 \to \{1,2,3\}$ as in Proposition 3.3, appeal to Proposition 3.6 to see that $\overline{u}$ is a pointed k-dromedary dyad on $(-b,a]$, and note that if $i < a-1$, then $-i > -a+1$ and $g(a-i) = g((-i)-(-a))$, so there is an extension of $\overline{u}$ to a pointed k-dromedary dyad v' on $[-b,-a+2g(a-i)(k-1))$ for which $-i \notin \{f^{v'}\} \cup \mathrm{supp}(\sigma^{v'}) \cup \{-a+2g(a-i)(k-1), -a+2g(a-i)(k-1)+1\}$, thus one more application of Proposition 3.6 ensures that $\overline{v'}$ is an extension of u to a pointed k-dromedary dyad v on $(a-2g(a-i)(k-1), b]$ for which $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{a-2g(a-i)(k-1)-1, a-2g(a-i)(k-1)\}$. ⊠

Proposition 3.8. *Suppose that $a < b$, $i < a-1$, $j > b+1$, and $k \geq 3$ are integers and u is a pointed k-dromedary dyad on $(a, b]$. Then there exist $n \in \{1, 2, \ldots, 7\}$ and an extension of u to a pointed k-dromedary dyad v on $(a-2n(k-1), b+2n(k-1)]$ with the property that $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{a-2n(k-1)-1, a-2n(k-1)\}$ and $j \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{b+2n(k-1)+1\}$.*

Proof. If $i < a-6(k-1)-1$ and $g \colon \mathbb{N} \setminus 2 \to \{1, 2, 3\}$ is the function given by Proposition 3.3, then there is an extension of u to a pointed k-dromedary dyad v' on $[a, b+2g(j-b)(k-1))$ for which $j \notin \{f^{v'}\} \cup \mathrm{supp}(\sigma^{v'}) \cup \{b+2g(j-b)(k-1), b+2g(j-b)(k-1)+1\}$, so part (1) of Proposition 3.7 yields the desired extension of v' to a pointed k-dromedary dyad on $(a-2g(j-b)(k-1), b+2g(j-b)(k-1)]$.

Similarly, if $j > b+6(k-1)+1$ and $g \colon \mathbb{N} \setminus 2 \to \{1, 2, 3\}$ is the function given by Proposition 3.7, then part (1) of Proposition 3.3 yields an extension of u to a pointed k-dromedary permutation v' on $[a, b+2g(a-i)(k-1))$, in which case the defining property of g yields the desired extension of v' to a pointed k-dromedary dyad v on $(a-2g(a-i)(k-1), b+2g(a-i)(k-1)]$ with the property that $i \notin \{f^v\} \cup \mathrm{supp}(\sigma^v) \cup \{a-2g(a-i)(k-1)-1, a-2g(a-i)(k-1)\}$.

It only remains to handle the case that $a-6(k-1)-1 \leq i$ and $j \leq b+6(k-1)+1$. We will recursively construct integers $a_e \leq a$ and $b_e \geq b$, as well as extensions of u to pointed k-dromedary dyads u_e on $(a_e, b_e]$ with the property that $i \notin \{f^{u_e}\} \cup \mathrm{supp}(\sigma^{u_e}) \cup \{a_e - 1, a_e\}$ and $j \notin \{f^{u_e}\} \cup \mathrm{supp}(\sigma^{u_e}) \cup \{b_e + 1\}$ for all natural numbers $e \leq 3$. We begin by setting $a_0 = a$, $b_0 = b$, and $u_0 = u$. Suppose now that $e < 3$ and we have already found a_e, b_e, and u_e. If there is an extension of u_e to a pointed k-dromedary dyad v_e on $[a_e, b_e + 2(k-1))$ for which $j \notin \{f^{v_e}\} \cup \mathrm{supp}(\sigma^{v_e}) \cup \{b_e + 2(k-1), b_e + 2(k-1)+1\}$, then set $m_{e+1} = 1$. Otherwise, part (2) of Proposition 3.3 yields $m_{e+1} \in \{2, 3\}$ with the property that there is an extension of u_e to a pointed k-dromedary dyad v_e on $[a_e, b_e + 2m_{e+1}(k-1))$ for which $j \notin \{f^{v_e}\} \cup \mathrm{supp}(\sigma^{v_e}) \cup \{b_e + 2m_{e+1}(k-1), b_e + 2m_{e+1}(k-1)+1\}$. If there is an extension of v_e to a pointed k-dromedary dyad u_{e+1} on $(a_e - 2(k-1), b_e + 2m_{e+1}(k-1)]$ for which $i \notin \{f^{u_{e+1}}\} \cup \mathrm{supp}(\sigma^{u_{e+1}}) \cup \{a_e - 2(k-1)-1, a_e - 2(k-1)\}$, then set $\ell_{e+1} = 1$. Otherwise, part (2) of Proposition 3.7 yields $\ell_{e+1} \in \{2, 3\}$ with the property that there is an extension of v_e to a pointed k-dromedary dyad u_{e+1} on $(a_e - 2\ell_{e+1}(k-1), b_e + 2m_{e+1}(k-1)]$ for which $i \notin \{f^{u_{e+1}}\} \cup \mathrm{supp}(\sigma^{u_{e+1}}) \cup \{a_e - 2\ell_{e+1}(k-1)-1, a_e - 2\ell_{e+1}(k-1)\}$.

Setting $a_{e+1} = a_e - 2\ell_{e+1}(k-1)$ and $b_{e+1} = b_e + 2m_{e+1}(k-1)$, this completes the recursive construction.

Lemma 3.9. *There is at most one $e \in \{1, 2, 3\}$ for which $\ell_e \neq 1$, as well as at most one $e \in \{1, 2, 3\}$ for which $m_e \neq 1$.*

Proof. If $e \in \{1, 2, 3\}$ has the property that $m_e \neq 1$, then part (1) of Proposition 3.3 ensures that $j \in (b_{e-1}+1, b_{e-1}+2(k-1)+1]$, and since the intervals of this form are pairwise disjoint, there is at most one such e. Similarly, if $e \in \{1, 2, 3\}$ has the property that $\ell_e \neq 1$, then part (1) of Proposition 3.7 ensures that $i \in [a_{e-1} - 2(k-1) - 1, a_{e-1} - 1)$, and since the intervals of this form are also pairwise disjoint, there is again at most one such e. ⊠

Define $\ell_{\max} = \max\{\ell_1, \ell_2, \ell_3\}$ and $m_{\max} = \max\{m_1, m_2, m_3\}$. If $\ell_{\max} = m_{\max}$, then set $n = 2 + \ell_{\max} = 2 + m_{\max}$ and note that $a_3 = a - 2n(k-1)$ and $b_3 = b + 2n(k-1)$ by Lemma 3.9, so the pointed k-dromedary dyad $v = u_3$ is as desired. Otherwise, set $n = 2 + \ell_{\max} + m_{\max}$ and note that $a_3 - 2m_{\max}(k-1) = a - 2n(k-1)$ and $b_3 + 2\ell_{\max}(k-1) = b + 2n(k-1)$ by Lemma 3.9, so an application of part (1) of Propositions 3.3 and 3.7 yields the desired extension of u_3 to a pointed k-dromedary dyad v on $(a_3 - 2m_{\max}(k-1), b_3 + 2\ell_{\max}(k-1)]$. ⊠

Proposition 3.10. *Suppose that $a > 0$, $b > a+1$, and $k \geq 3$ are integers, $J \subseteq (a, b)$ is a set whose* Succ*-gaps have cardinality at least $14(k-1)-1$, $I \subseteq J$ intersects each T-gap in $\sim J$ in a singleton, σ is a permutation of J, $\pi \colon J \setminus I \to J \cap \mathrm{Succ}^{-1}(J)$, and $u^C = (f^C, \sigma^C, \pi^C)$ is a pointed k-dromedary dyad on C, where f^C is the unique element of $C \cap I$, $\sigma^C = \sigma \upharpoonright C$, and $\pi^C = \pi \upharpoonright (C \setminus I)$ for every* Succ*-gap C in $\sim J$. Then there exist $c \in [a - 14(k-1) - 4, a]$, $d \in [b, b + 14(k-1) + 4]$, and a simultaneous extension of each u^C to a pointed k-dromedary dyad v on $(c, d]$.*

Proof. We first show that it is sufficient to establish the special case of the proposition where $a = 14(k-1) - 4$. Let c', d', and v' be the result of applying this special case to $a' = 14(k-1) - 4$, $b' = b - (a - a')$, $J' = \mathrm{Succ}^{a'-a}(J)$, $I' = \mathrm{Succ}^{a'-a}(I)$, $\sigma' = \mathrm{Succ}^{a'-a} \circ \sigma \circ \mathrm{Succ}^{a-a'}$, and $\pi' = \mathrm{Succ}^{a'-a} \circ \pi \circ \mathrm{Succ}^{a-a'}$, and observe that the integers $c = c' + (a - a')$ and $d = d' + (a - a')$ and the triple $v = (f^{v'} + (a - a'), \mathrm{Succ}^{a-a'} \circ \sigma^{v'} \circ \mathrm{Succ}^{a'-a}, \mathrm{Succ}^{a-a'} \circ \pi^{v'} \circ \mathrm{Succ}^{a'-a})$ are as desired.

We next show that it is sufficient to establish the further special case of the proposition where J is $\mathrm{Succ}^{<14(k-1)}$-independent. Define $\phi \colon \mathbb{N} \setminus (J \cap \mathrm{Succ}(J)) \to \mathbb{N}$ by $\phi(j) = |j \setminus (J \cap \mathrm{Succ}(J))|$. Then the set $J' = \phi(J \setminus \mathrm{Succ}(J))$ is $\mathrm{Succ}^{<14(k-1)}$-independent, so the further special case yields $c' \in [0, 14(k-1)-4]$, $d' \in [\phi(b), \phi(b)+14(k-1)+4]$, and a pointed k-dromedary dyad v' on $(c', d']$ for which $\{f^{v'}\} \cup \mathrm{supp}(\sigma^{v'})$ is disjoint from J'. Let $(j_i)_{i<\ell}$ be the strictly increasing enumeration of $J \setminus \mathrm{Succ}(J)$ and define $c_0 = c'$, $d_0 = d'$, and $v_0 = v'$. For all $i < \ell$, set $C_i = \phi^{-1}(\{\phi(j_i)\})$, $c_{i+1} = c_i$, and $d_{i+1} = d_i + |C_i \setminus \{j_i\}|$, and appeal to Proposition 3.1 to see that the triple $v_{i+1} = v_i *_{j_i} u^{C_i}$ is a pointed k-dromedary dyad on $(c_{i+1}, d_{i+1}]$. Then the triple $v = v_\ell$ is the desired extension of u.

Finally, we establish the special case of the proposition where J is $\mathrm{Succ}^{<14(k-1)}$-independent (but a need not be $14(k-1)-4$). Set $m = a + \lfloor (b-a)/2 \rfloor$. As J is $T^{<8}$-independent, by setting $c_0 = m-2$ if J intersects $[m+1, m+4]$ and $c_0 = m+2$ otherwise, we can ensure that $[c_0 - 1, c_0 + 2]$ is disjoint from J. As $m-a$ and $b-(m+1)$ are within 1 of one another, it follows that $c_0 - a$ and $b - (c_0+1)$ are within 5 of one another. Note that $a < c_0$ or $c_0 + 1 < b$, since otherwise $b - 1 \leq c_0 \leq a$, contradicting the fact that $b > a+1$. Set $d_0 = c_0 + 1$, let v_0 be the unique pointed k-dromedary dyad on $(c_0, d_0]$, and recursively apply Proposition 3.8 to c_i, d_i, the maximum element of J below c_i (or any integer strictly below $c_i - 1$ if there is no such element of J), the minimum element of J above d_i (or any integer strictly above $d_i + 1$ if there is no such element of J), k, and v_i to obtain $n_i \in \{1, \ldots, 7\}$ and an extension of v_i to a pointed k-dromedary dyad v_{i+1} on $(c_{i+1}, d_{i+1}]$ with the property that $c_{i+1} = c_i - 2n_i(k-1)$, $d_{i+1} = d_i + 2n_i(k-1)$, and $\{f^{v_{i+1}}\} \cup \mathrm{supp}(\sigma^{v_{i+1}}) \cup \{c_{i+1} - 1, c_{i+1}, d_{i+1} + 1\}$ is disjoint from J (since J is $T^{<14(k-1)}$-independent), noting that $c_{i+1} - a$ and $b - d_{i+1}$ are within 5 of one another. Let i be the maximal natural number for which $a < c_i$ or $d_i < b$. If $a < c_i$, then $c_{i+1} \geq c_i - 14(k-1) > a - 14(k-1)$ and

$$\begin{aligned} d_{i+1} &\leq d_i + 14(k-1) \\ &= b + 14(k-1) + (d_i - b) \\ &\leq b + 14(k-1) + (a - c_i) + 5 \\ &\leq b + 14(k-1) + 4. \end{aligned}$$

If $d_i < b$, then

$$\begin{aligned} c_{i+1} &\geq c_i - 14(k-1) \\ &= a - 14(k-1) + (c_i - a) \\ &\geq a - 14(k-1) + (b - d_i) - 5 \\ &\geq a - 14(k-1) - 4 \end{aligned}$$

and $d_{i+1} \leq d_i + 14(k-1) < b + 14(k-1)$. In both cases, it follows that the integers $c = c_{i+1}$ and $d = d_{i+1}$ and the pointed k-dromedary dyad $v = v_{i+1}$ are as desired. ⊠

Given $k \geq 3$, $n \geq 1$, an aperiodic bijection $T \colon X \to X$, and $x \in X$, we say that a triple (f, σ, π) is a *pointed k-dromedary T-dyad* on $[x, T^n(x))_T$ if $f \in [x, T^n(x))_T$, σ is a permutation of $[x, T^n(x))_T$, $\pi \colon [x, T^n(x))_T \setminus \{f\} \to [x, T^{n-1}(x))_T$, and $(\theta^{-1}(f), \theta^{-1} \circ \sigma \circ \theta, \theta^{-1} \circ \pi \circ \theta)$ is a pointed k-dromedary dyad on n, where $\theta \colon n \to [x, T^n(x))_T$ is given by $\theta(i) = T^i(x)$ for all $i < n$. We can now give the following:

Proof of Theorem 4. We will find $S \in [T]$, whose orbits all have cardinality 1 or k, that is conjugate to $T^{-1} \circ S$ in $[T]$. To see that this is sufficient, set $S_2 = S$ and observe that every orbit of the automorphism $S_1 = S_2^{-1} \circ T$ has cardinality 1 or k and the fact that $S_2 \in [T]$ easily implies that $S_1 \in [T]$..

We will construct an exhaustive increasing sequence $(X_i)_{i \in \mathbb{N}}$ of Borel subsets of X whose complements are $R^X_{T^{-1}}$- and R^X_T-complete, Borel sets $F_i \subseteq X_i$ intersecting every T-gap in $\sim X_i$ in a singleton, T-decomposable injections $P_i \colon X_i \setminus F_i \to X_i \cap T^{-1}(X_i)$, and T-decomposable bijections $S_i \colon X_i \to X_i$ such that:

(1) $\forall i \in \mathbb{N}\ P_i = P_{i+1} \restriction (X_i \setminus F_i)$.
(2) $\forall i \in \mathbb{N}\ S_i = S_{i+1} \restriction X_i$.
(3) $\forall i \in \mathbb{N} \forall x \in X_{i+1} \setminus X_i\ C^x_{i+1} \cap F_{i+1} \cap X_i = \emptyset$, where C^x_i is the unique T-gap in $\sim X_i$ containing x for all $i \in \mathbb{N}$ and $x \in X_i$, and
(4) $\forall i \in \mathbb{N} \forall x \in X_i\ (f^x_i, S^x_i, P^x_i)$ is a pointed k-dromedary T-dyad on C^x_i, where $f^x_i \in C^x_i \cap F_i$, $S^x_i = S_i \restriction C^x_i$, and $P^x_i = P_i \restriction (C^x_i \setminus F_i)$.

To see that this is sufficient, note first that each $x \in X$ appears in at most finitely many F_i, for if $i \in \mathbb{N}$ is sufficiently large that $x \in X_i$ and $j > i$ is sufficiently large that $C^x_i \neq C^x_j$, then condition (3) ensures that $x \notin F_j$. Conditions (1) and (2) therefore ensure that we obtain functions $P, S \colon X \to X$ by setting $P = \bigcup_{i \in \mathbb{N}} P_i$ and $S = \bigcup_{i \in \mathbb{N}} S_i$. To see that P and S are injective, note that if $x, y \in X$ have the property that $P(x) = P(y)$ or $S(x) = S(y)$ and $i \in \mathbb{N}$ has the property that $x, y \in X_i \setminus F_i$, then $P_i(x) = P_i(y)$ or $S_i(x) = S_i(y)$, so $x = y$. To see that P and S are surjective, note that if $z \in X$ and $i \in \mathbb{N}$ is sufficiently large that $z \in X_i \cap T^{-1}(X_i)$, then there exist $x, y \in X_i$ for which $P_i(x) = z$ and $S_i(y) = z$, so $P(x) = z$ and $S(y) = z$. To see that P and S are T-decomposable, fix Borel sets $A_{i,n}, B_{i,n} \subseteq X_i$ with the property that $P_i = \bigcup_{n \in \mathbb{Z}} T^n \restriction A_{i,n}$ and $S_i = \bigcup_{n \in \mathbb{Z}} T^n \restriction B_{i,n}$ for all $i \in \mathbb{N}$, set $A_n = \bigcup_{i \in \mathbb{N}} A_{i,n}$ and $B_n = \bigcup_{i \in \mathbb{N}} B_{i,n}$ for all $n \in \mathbb{Z}$, and observe that $P = \bigcup_{n \in \mathbb{Z}} T^n \restriction A_n$ and $S = \bigcup_{n \in \mathbb{Z}} T^n \restriction B_n$. To see that P is an isomorphism of S with $T^{-1} \circ S$, note that if $x \in X$ and $i \in \mathbb{N}$ has the property that $x \in X_i \setminus F_i$, then condition (4) ensures that $(P_i \circ S_i)(x) = (T^{-1} \circ S_i \circ P_i)(x)$, so $(P \circ S)(x) = (T^{-1} \circ S \circ P)(x)$. To see that the orbit of every point under S has cardinality 1 or k, note that if $x \in X$ and $i \in \mathbb{N}$ is sufficiently large that $x \in X_i$, then condition (4) implies that the orbit of x under S_i has cardinality 1 or k and coincides with the orbit of x under S.

Appeal to Propositions 1.14 and 1.15 to obtain an $R^X_{T^{-1}}$- and R^X_T-complete $T^{<42(k-1)+7}$-independent Borel set $D \subseteq X$. By Proposition 1.17, there is a balanced marker sequence $(D_i)_{i \in \mathbb{N}}$ for T_D. For all $a > 0$, $b > a+1$, and $i \in \mathbb{N}$, let $D_{a,b,i+1}$ be the set of $x \in D_{i+1}$ for which $T_{D_i}(x), (T_{D_i})^2(x) \notin D_{i+1}$, $a = r^T_{D_i}(x)$, and $b = h^T_{T^{-1}_{D_i}(D_{i+1})}(x)$.

To guarantee that $(X_i)_{i \in \mathbb{N}}$ is exhaustive, it is enough to ensure that $(T^a(x), T^b(x))_T \subseteq X_{i+1}$ for all $a > 0$, $b > a+1$, $i \in \mathbb{N}$, and $x \in D_{a,b,i+1}$, since the fact that $(D_i)_{i \in \mathbb{N}}$ is a balanced marker sequence ensures that every point of X appears in an interval of this form. To guarantee that the sets X_i are Borel and the recursive construction goes through, we will construct Borel functions $a_{i+1}, b_{i+1} \colon \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1} \to \mathbb{N}$ such that:

(5) $\forall a > 0 \forall b > a+1 \forall x \in D_{a,b,i+1}\ a_i(x) \in [a - 14(k-1) - 4, a]$.
(6) $\forall a > 0 \forall b > a+1 \forall x \in D_{a,b,i+1}\ b_i(x) \in [b, b + 14(k-1) + 4]$.
(7) $X_{i+1} \setminus X_i = \bigcup_{a>0} \bigcup_{b>a+1} \bigcup_{x \in D_{a,b,i+1}} (T^{a_{i+1}(x)}(x), T^{b_{i+1}(x)}(x)]_T \setminus X_i$.

We begin by setting $F_0 = P_0 = S_0 = X_0 = \emptyset$. Suppose now that $i \in \mathbb{N}$ and we have already found $(a_j)_{1 \leq j \leq i}$, $(b_j)_{1 \leq j \leq i}$, F_i, P_i, and S_i.

Lemma 3.11. *The size of each T-gap in X_i is at least $14(k-1) - 1$.*

Proof. Suppose that $n > 1$, $x \in X_i$, and $r^T_{X_i}(x) = n$, so that $(x, T^n(x))_T$ is a T-gap in X_i. Then x and $T^n(x)$ are right and left endpoints of the sorts of intervals appearing in condition (7), so conditions (5)–(7) yield integers $b \in [0, 14(k-1)+4]$ and $a \in [-1, 14(k-1)+3]$ for which $T^{-b}(x), T^{a+n}(x) \in D$. As the fact that $n > 1$ ensures that $-b \leq 0 < a + n$, the fact that D is $T^{<42(k-1)+7}$-independent implies that $42(k-1)+7 \leq b+a+n \leq n+28(k-1)+7$, thus $n \geq 14(k-1)$. ⊠

Let Q_{i+1} be the set of quadruples $q = (a^q, b^q, J^q, (I^q, \sigma^q, \pi^q))$ with the property that $a^q > 0$, $b^q > a^q + 1$, $J^q \subseteq (a^q, b^q)$ is a set whose Succ-gaps have cardinality at least $14(k-1) - 1$, $I^q \subseteq J^q$ intersects each Succ-gap in $\sim J^q$ in a singleton, σ^q is a permutation of J^q, $\pi^q \colon J^q \setminus I^q \to J^q \cap \mathrm{Succ}^{-1}(J^q)$, and $u^C = (f^C, \sigma^C, \pi^C)$ is a pointed k-dromedary dyad on C, where f^C is the unique element of $C \cap I^q$, $\sigma^C = \sigma \restriction C$, and $\pi^C = \pi \restriction (C \setminus I^q)$ for every Succ-gap C in $\sim J^q$. For all $q \in Q_{i+1}$, Proposition 3.10 yields $c^q \in [a^q - 14(k-1) - 4, a^q]$, $d^q \in [b^q, b^q + 14(k-1) + 4]$, and a simultaneous extension of each u^C to a pointed k-dromedary dyad v^q on $(c^q, d^q]$. Let $D_{i+1,q}$ be the set of all $x \in D_{a^q,b^q,i+1}$ such that:

(a) $\forall j \in (a^q, b^q)$ $(j \in J^q \iff T^j(x) \in X_i)$.
(b) $\forall j \in J^q$ $(j \in I^q \iff T^j(x) \in F_i)$.
(c) $\forall j \in J^q \setminus I^q$ $(P_i \circ T^j)(x) = T^{\pi^q(j)}(x)$.
(d) $\forall j \in J^q$ $(S_i \circ T^j)(x) = T^{\sigma^q(j)}(x)$.

Clearly $\bigcup_{q \in Q_{i+1}} D_{i+1,q} \subseteq \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1}$ and Lemma 3.11 ensures that the reverse inclusion holds. Define $a_{i+1}(x) = c^q$ and $b_{i+1}(x) = d^q$ for all $q \in Q_{i+1}$ and $x \in D_{i+1,q}$.

Lemma 3.12. *Suppose that $a > 0$, $b > a + 1$, and $x \in D_{a,b,i+1}$. Then $(T^{a_{i+1}(x)-1}(x), T^{b_{i+1}(x)+1}(x)]_T \cap X_i = (T^a(x), T^b(x)]_T \cap X_i$.*

Proof. As condition (5) ensures that $a_{i+1}(x) \leq a$ and condition (6) implies that $b \leq b_{i+1}(x)$, it is sufficient to show that

$$(T^{a_{i+1}(x)-1}(x), T^a(x)]_T \cap X_i = (T^b(x), T^{b_{i+1}(x)+1}(x)]_T \cap X_i = \emptyset.$$

Suppose, towards a contradiction, that this is false. Then condition (7) yields $a' > 0$, $b' > a' + 1$, $i' < i$, and $x' \in D_{a',b',i'+1}$ for which

$$(T^{a_{i+1}(x)-1}(x), T^a(x)]_T \cap (T^{a_{i'+1}(x')}(x'), T^{b_{i'+1}(x')}(x')]_T \neq \emptyset \qquad (\dagger)$$

or

$$(T^b(x), T^{b_{i+1}(x)+1}(x)]_T \cap (T^{a_{i'+1}(x')}(x'), T^{b_{i'+1}(x')}(x')]_T \neq \emptyset. \qquad (\ddagger)$$

To handle the case that (†) holds, note first that $T^{a_{i'+1}(x')}(x') <_T T^a(x)$ and $T^{a_{i+1}(x)-1}(x) <_T T^{b_{i'+1}(x')}(x')$. As $x', T^{a'}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 14(k-1)+4}$-independent, condition (5) implies that $x' <_T T^{a_{i'+1}(x')}(x')$, so $x' <_T T^a(x)$. As $x', T^a(x) \in D_{i'+1}$, it follows that $T_{D_{i'+1}}(x') \leq_T T^a(x)$. As $T^{b'}(x'), T_{D_{i'+1}}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 14(k-1)+4}$-independent, condition (6) ensures that $T^{b_{i'+1}(x')}(x') <_T T_{D_{i'+1}}(x')$, in which case $T^{b_{i'+1}(x')}(x') <_T T^a(x)$, so $T^{b_{i'+1}(x')}(x') \in [T^{a_{i+1}(x)}(x), T^a(x))_T$. But $T^a(x), T^{b'}(x') \in D_{i'}$, condition (5) yields $c \in (0, 14(k-1)+4]$ for which $T^a(x) = T^{b_{i'+1}(x')+c}(x')$, and condition (6) ensures that $b_{i'+1}(x') - b' \in [0, 14(k-1)+4]$, contradicting the $T^{\leq 28(k-1)+8}$-independence of $D_{i'}$.

To handle the case that (‡) holds, note first that $T^b(x) <_T T^{b_{i'+1}(x')}(x')$ and $T^{a_{i'+1}(x')}(x') <_T T^{b_{i+1}(x)+1}(x)$. As $T^{b'}(x'), T_{D_{i'+1}}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 14(k-1)+4}$-independent, condition (6) implies that $T^{b_{i'+1}(x')}(x') <_T T_{D_{i'+1}}(x')$, so $T^b(x) <_T T_{D_{i'+1}}(x')$. As $T^b(x), T_{D_{i'+1}}(x') \in D_{i'+1}$, it follows that $T^b(x) \leq_T x'$. As $x', T^{a'}(x') \in D_{i'}$ and $D_{i'}$ is $T^{\leq 14(k-1)+4}$-independent, condition (5) ensures that $x' <_T T^{a_{i'+1}(x')}(x')$, in which case $T^b(x) <_T T^{a_{i'+1}(x')}(x')$, so $T^{a_{i'+1}(x')}(x') \in (T^b(x), T^{b_{i+1}(x)}(x)]_T$.

But $T^b(x), T^{a'}(x') \in D_{i'}$, condition (5) ensures that $a' - a_{i'+1}(x') \in [0, 14(k-1)+4]$, and condition (6) yields $c \in (0, 14(k-1)+4]$ for which $T^b(x) = T^{a_{i'+1}(x')-c}(x')$, contradicting the $T^{\leq 28(k-1)+8}$-independence of $D_{i'}$. ⊠

As D is $T^{\leq 14(k-1)+4}$-independent, conditions (5) and (6) imply that if $x \in \bigcup_{a>0} \bigcup_{b>a+1} D_{a,b,i+1}$, then $(T^{a_{i+1}}(x), T^{b_{i+1}}(x)]_T \subseteq (x, T_{D_{i+1}}(x))_T$. As the intervals of the latter form are pairwise disjoint, those of the former form are not only pairwise disjoint, but are not adjacent to one another. Lemma 3.12 therefore ensures that the functions

$$P_{i+1} = P_i \cup \bigcup_{q \in Q_{i+1}} \bigcup_{j \in (c^q, d^q] \setminus I^q} T^{\pi^{v^q}(j)-j} \restriction T^j(D_{i+1,q})$$

and

$$S_{i+1} = S_i \cup \bigcup_{q \in Q_{i+1}} \bigcup_{j \in (c^q, d^q]} T^{\sigma^{v^q}(j)-j} \restriction T^j(D_{i+1,q})$$

are well-defined and as desired. ⊠

To obtain further results of this form, we need several preliminaries.

Proposition 3.13. *Suppose that* $\{1\} \subseteq K \subseteq \mathbb{Z}^+$, $K \nsubseteq \{1,2\}$, X *is a Borel space, and* $T \colon X \to X$ *is an aperiodic smooth Borel automorphism. Then there exist* $R, S \in [T]$, *whose orbits all have cardinality in* K *and for which each possibility occurs infinitely often on every* T*-orbit, such that* $T = S \circ R$.

Proof. To see that it is sufficient to establish the special case of the proposition where $X = \mathbb{Z}$ and $T = \mathrm{Succ}$, suppose that ρ and σ are permutations of $\mathbb{Z}$, whose orbits all have cardinality in K and for which each possibility occurs infinitely often, such that $\mathrm{Succ} = \sigma \circ \rho$. Fix a Borel transversal $B \subseteq X$ of E_T^X and observe that the functions $R = \bigcup_{n \in \mathbb{Z}} T^{\rho(n)-n} \restriction T^n(B)$ and $S = \bigcup_{n \in \mathbb{Z}} T^{\sigma(n)-n} \restriction T^n(B)$ are as desired.

It remains to establish the special case. For all $k \geq 2$, let τ_k be the permutation of $[0, 2k-2]$ given by $\tau_k = (0\ k\ k+1\ \cdots\ 2k-2)$. Then

$$c_{\tau_k}(i) = \begin{cases} 0 & \text{if } i < k \text{ and} \\ i & \text{otherwise} \end{cases}$$

for all $i \in [0, 2k-2)$. It follows that every τ_k-orbit and non-empty c_{τ_k}-preimage of a singleton has cardinality 1 or k. Moreover, every possibility occurs with the sole exception that there is no singleton whose c_{τ_k}-preimage has cardinality 1 when $k = 2$. Fix an enumeration $(k_n)_{n \in \mathbb{N}}$ of $K \setminus \{1\}$ in which every element of $K \setminus \{1\}$ appears infinitely often, set $i_0 = 1$, and let σ_0 be the trivial permutation of $[1, 1]$. Given $n \in \mathbb{N}$, $i_n > 0$, and a permutation σ_n of $[1-n, i_n]$, define $i_{n+1} = i_n + 2k_n - 3$ and appeal to Proposition 2.4 to see that the extension of σ_n to $[-n, i_{n+1}]$, given by $\sigma_{n+1} = (\mathrm{Succ}^{-n} \circ \tau_{k_n} \circ \mathrm{Succ}^n) *_{1-n} \sigma_n$, is covering and non-crossing. It follows that the function $\sigma = \bigcup_{n \in \mathbb{N}} \sigma_n$ is Succ-covering and Succ-non-crossing. As it is also periodic, Proposition 2.3 ensures that it is Succ-oriented. A straightforward induction reveals that the cardinality of every σ-orbit and non-empty c_σ-preimage of a singleton is in K. To see that every possibility in $K \setminus \{1\}$ occurs infinitely often, note that σ_{n+1} and $c_{\sigma_{n+1}}$ have one more such orbit and preimage than σ_n and c_{σ_n} whenever $k_n = k$. To see that 1 also occurs infinitely often, note that σ_{n+1} and $c_{\sigma_{n+1}}$ have one more such orbit and preimage than σ_n and c_{σ_n} whenever $k_n \geq 3$ (hence the requirement that $\{1\} \subseteq K$ and $K \nsubseteq \{1, 2\}$). Proposition 1.10 therefore implies that σ and $\rho = \sigma^{-1} \circ \mathrm{Succ}$ are as desired. ⊠

Remark 3.14. In the special case that K is finite, the proof of Theorem 3 can be modified to show that smoothness can be weakened to separability in the statement of Proposition 3.13.

A *partial transversal* of an equivalence relation E on a set X is a set $Y \subseteq X$ that intersects each E-class in at most one point.

Proposition 3.15. *Suppose that X is a Borel space and Γ is a countable group of Borel automorphisms of X. Then there is a Borel transversal $B \subseteq X$ of E_Γ^X if and only if there is a cover $(B_n)_{n \in \mathbb{N}}$ of X by Borel partial transversals of E_Γ^X.*

Proof. If $B \subseteq X$ is a Borel transversal of E_Γ^X, then $(\gamma B)_{\gamma \in \Gamma}$ is a cover of X by Borel transversals of E_Γ^X. Conversely, if there is a cover of X by countably-many Borel partial transversals of E_Γ^X, then E_Γ^X is separable and the graph $G = E_\Gamma^X \setminus \Delta(X)$ has a Borel $\mathbb{N}$-coloring. As $G = \bigcup_{\gamma \in \Gamma} \mathrm{graph}(\gamma \restriction \mathrm{supp}(\gamma))$, Proposition 1.8 ensures that it is Γ-decomposable, so Proposition 1.15 yields a Borel transversal of E_Γ^X. ⊠

A countable group Γ of Borel automorphisms of a Borel space X is *smooth* if E_Γ^X admits a Borel transversal.

Proposition 3.16. *Suppose that $k \leq \aleph_0$, X is a Borel space, Γ is a smooth countable group of Borel automorphisms of X, and $B \subseteq X$ is a Borel set whose intersection with each Γ-orbit has cardinality k. Then there is a partition of B into k Borel transversals of $E_\Gamma^X \restriction B$.*

Proof. We first handle the case that k is finite. By the pigeonhole principle, it is sufficient to recursively construct a sequence $(B_n)_{n<k}$ of pairwise disjoint Borel transversals of $E_\Gamma^X \restriction B$. Suppose that $n < k$ and we have already found $(B_m)_{m<n}$. Then Proposition 3.15 ensures that the graph $G_n = (E_\Gamma^X \setminus \Delta(X)) \cap ((B \setminus \bigcup_{m<n} B_m) \times (B \setminus \bigcup_{m<n} B_m)$

has a Borel $\mathbb{N}$-coloring. As

$$G_n = \bigcup_{\gamma \in \Gamma} \operatorname{graph}(\gamma \upharpoonright \operatorname{supp}(\gamma)) \cap ((B \setminus \bigcup_{m<n} B_m) \times (B \setminus \bigcup_{m<n} B_m)),$$

Proposition 1.8 implies that it is Γ-decomposable. Proposition 1.15 therefore yields a Borel maximal G_n-independent set $B'_n \subseteq X$, so the set $B_n = B'_n \cap (B \setminus \bigcup_{m<n} B_m)$ is a Borel transversal of $E_\Gamma^X \upharpoonright B$, which completes the recursive construction.

In order to handle the case that k is infinite, extra care must be taken to ensure that the resulting transversals cover X, since the pigeonhole principle no longer suffices to yield this conclusion. We nevertheless proceed in essentially the same fashion, but first appeal to Proposition 3.15 to obtain a cover $(A_n)_{n \in \mathbb{N}}$ of B by Borel partial transversals of $E_\Gamma^X \upharpoonright B$, and require that $A_n \subseteq B'_n$ when applying Proposition 1.15 in the recursive construction, which ensures that $\bigcup_{m \leq n} A_m \subseteq \bigcup_{m \leq n} B_m$. In particular, it follows that $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} B_n$, so the fact that $B = \bigcup_{n \in \mathbb{N}} A_n$ implies that $B = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

Proposition 3.17. *Suppose that X is a Borel space, Γ is a smooth countable group of Borel automorphisms of X, $A, B \subseteq X$ are Borel, and $|A \cap \Gamma x| = |B \cap \Gamma x|$ for all $x \in X$. Then there is a Γ-decomposable bijection $\pi \colon A \to B$.*

Proof. Proposition 1.9 ensures that, by partitioning X into countably-many Γ-invariant Borel sets, we can assume that there exists $k \leq \aleph_0$ such that $|A \cap \Gamma x| = |B \cap \Gamma x| = k$ for all $x \in X$. Proposition 3.16 then yields partitions $(A_n)_{n<k}$ and $(B_n)_{n<k}$ of A and B into Borel transversals of E_Γ^X, so the function $\pi = \bigcup_{\gamma \in \Gamma} \gamma \upharpoonright (\bigcup_{n<k} A_n \cap \gamma^{-1} B_n)$ is as desired. ⊠

For each $k \leq \aleph_0$, the *period k part* of a bijection $T \colon X \to X$ is given by $\operatorname{Per}_k(T) = \{x \in X \mid |[x]_T| = k\}$.

Proposition 3.18. *Suppose that X is a Borel space, Γ is a smooth countable group of Borel automorphisms of X, and $S, T \in [\Gamma]$ have the property that $|(\operatorname{Per}_k(S) \cap \Gamma x)/E_S^X| = |(\operatorname{Per}_k(T) \cap \Gamma x)/E_T^X|$ for all $1 \leq k \leq \aleph_0$ and $x \in X$. Then S is conjugate to T in $[\Gamma]$.*

Proof. By Proposition 3.15, there are Borel transversals $A \subseteq X$ and $B \subseteq X$ of E_S^X and E_T^X. For all $1 \leq k \leq \aleph_0$, define $A_k = A \cap \operatorname{Per}_k(S)$ and $B_k = B \cap \operatorname{Per}_k(T)$, appeal to Proposition 3.17 to obtain a Γ-decomposable bijection $\pi_k \colon A_k \to B_k$, and set $\phi_k = \bigcup_{n \in \mathbb{Z}} T^n \circ \pi_k \circ S^{-n}$. Then the function $\phi = \bigcup_{1 \leq k \leq \aleph_0} \phi_k$ is as desired. ⊠

For each equivalence relation E on a set X and $n \in \mathbb{N}$, let $[X]^n_E$ denote the family of all sets $S \in [X]^n$ such that $S \times S \subseteq E$, and define

$[X]^{<\aleph_0} = \bigcup_{n\in\mathbb{N}} [X]^n$ and $[X]^{<\aleph_0}_E = \bigcup_{n\in\mathbb{N}} [X]^n_E$. Given a countable group Γ of permutations of a set X and a set $Y \subseteq X$, let $[Y]^n_\Gamma$ and $[Y]^{<\aleph_0}_\Gamma$ denote $[Y]^n_{E^X_\Gamma \restriction Y}$ and $[Y]^{<\aleph_0}_{E^X_\Gamma \restriction Y}$. When X is a Borel space and Γ is a countable group of Borel automorphisms of X, we say that a family $\mathcal{S} \subseteq [X]^{<\aleph_0}_\Gamma$ is Γ-*decomposable* if there is a sequence $(B_\Delta)_{\Delta\in[\Gamma]^{<\aleph_0}}$ of Borel subsets of X with the property that $\mathcal{S} = \bigcup_{\Delta\in[\Gamma]^{<\aleph_0}} \{\Delta x \mid x \in B_\Delta\}$.

Proposition 3.19. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, $B \subseteq X$ is Borel, and F is a finite subequivalence relation of $E^X_\Gamma \restriction B$ for which B/F is Γ-decomposable. Then F is Γ-decomposable.*

Proof. Fix Borel sets $B_\Delta \subseteq X$ with $B/F = \bigcup_{\Delta\in[\Gamma]^{<\aleph_0}} \{\Delta x \mid x \in B_\Delta\}$. Then $F = \bigcup_{\Delta\in[\Gamma]^{<\aleph_0}} \bigcup_{\gamma,\delta\in\Delta} \operatorname{graph}(\gamma\delta^{-1} \restriction \delta B_\Delta)$. ⊠

Proposition 3.20. *Suppose that X is a Borel space, Γ is a countable group of separable Borel automorphisms of X, and $\mathcal{S} \subseteq [X]^{<\aleph_0}_\Gamma$ is Γ-decomposable. Then there is a Γ-decomposable maximal family $\mathcal{R} \subseteq \mathcal{S}$ of pairwise disjoint sets.*

Proof. Fix Borel sets $B_\Delta \subseteq X$ for which $\mathcal{S} = \bigcup_{\Delta\in[\Gamma]^{<\aleph_0}} \{\Delta x \mid x \in B_\Delta\}$. For all $\Delta \in [\Gamma]^{<\aleph_0}$, let G_Δ be the graph on X with respect to which x and y are neighbors if and only if $x \neq y$, $x, y \in B_\Delta$, and $\Delta x \cap \Delta y \neq \emptyset$. Then $G_\Delta = \bigcup_{\gamma,\delta\in\Delta} \operatorname{graph}(\delta^{-1}\gamma \restriction (B_\Delta \cap \gamma^{-1}\delta B_\Delta \cap \operatorname{supp}(\delta^{-1}\gamma)))$, so Proposition 1.8 ensures that G_Δ is Γ-decomposable. As G_Δ has finite vertical sections, Proposition 1.14 yields a Borel $\mathbb{N}$-coloring of G_Δ. Fix an enumeration $(\Delta_n)_{n\in\mathbb{N}}$ of $[\Gamma]^{<\aleph_0}$.

We will recursively find Borel sets $B_n \subseteq B_{\Delta_n}$ for which the sets in the family $\mathcal{R}_n = \bigcup_{m<n} \{\Delta_m x \mid x \in B_m\}$ are pairwise disjoint. Given $n \in \mathbb{N}$ for which we have already found $(B_m)_{m<n}$, define

$$\begin{aligned} C_n &= \{x \in B_{\Delta_n} \mid \Delta_n x \cap \textstyle\bigcup \mathcal{R}_n = \emptyset\} \\ &= B_{\Delta_n} \setminus \textstyle\bigcup_{m<n} \bigcup_{\delta_m\in\Delta_m} \bigcup_{\delta_n\in\Delta_n} \delta_n^{-1}\delta_m B_m. \end{aligned}$$

Then the graph $G_n = G_{\Delta_n} \cap (C_n \times C_n)$ is Γ-decomposable, so Proposition 1.15 yields a Borel maximal G_n-independent set $D_n \subseteq X$. Define $B_n = C_n \cap D_n$ and observe that the sets in the corresponding family $\mathcal{R}_{n+1}$ are pairwise disjoint, which completes the recursive construction.

Note that the sets in the family $\mathcal{R}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ are pairwise disjoint. To see that it is a maximal family of pairwise disjoint sets in $\mathcal{S}$, suppose that $S \in \mathcal{S}$, fix $n \in \mathbb{N}$ and $x \in B_{\Delta_n}$ for which $S = \Delta_n x$, and observe that if $S \cap \bigcup \mathcal{R}_n = \emptyset$, then $x \in C_n$, so there exists $y \in C_n \cap D_n$ for which $\Delta_n x \cap \Delta_n y \neq \emptyset$, thus $S \cap \bigcup \mathcal{R}_{n+1} \neq \emptyset$. ⊠

Proposition 3.21. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X whose supports are Borel, R is a Γ-decomposable binary relation on X, and $\gamma, \delta \in \Gamma$. Then the corresponding set $B = \{x \in X \mid \gamma \cdot x \mathrel{R} \delta \cdot x\}$ is Borel.*

Proof. Fix Borel sets $B_\lambda \subseteq X$ for which $R = \bigcup_{\lambda \in \Gamma} \mathrm{graph}(\lambda \restriction B_\lambda)$ and observe that $B = \bigcup_{\lambda \in \Gamma} \{x \in X \mid \gamma \cdot x \in B_\lambda \text{ and } \lambda\gamma \cdot x = \delta \cdot x\} = \bigcup_{\lambda \in \Gamma} \gamma^{-1} B_\lambda \setminus \mathrm{supp}(\delta^{-1}\lambda\gamma)$. ⊠

Proposition 3.22. *Suppose that $k \in \mathbb{Z}^+$, X is a Borel space, Γ is a countable group of Borel automorphisms of X whose supports are Borel, $B \subseteq X$ is Borel, and G is a Γ-decomposable graph on X. Then the set $\mathcal{S} = \{S \in [B]^k_\Gamma \mid S \text{ is } G\text{-independent}\}$ is Γ-decomposable.*

Proof. For all $\Delta \in [\Gamma]^k$, define

$$\begin{aligned} B_\Delta = \{x \in X \mid \forall \delta \in \Delta\ \delta \cdot x \in B \text{ and } \\ \forall \delta \in \Delta \forall \gamma \in \Delta \setminus \{\delta\}\ (\gamma \cdot x \neq \delta \cdot x \text{ and } \neg \gamma \cdot x \mathrel{G} \delta \cdot x)\} \\ = \textstyle\bigcap_{\delta \in \Delta} \delta^{-1} B \cap \bigcap_{\delta \in \Delta} \bigcap_{\gamma \in \Delta \setminus \{\delta\}} \{x \in \mathrm{supp}(\delta^{-1}\gamma) \mid \neg \gamma \cdot x \mathrel{G} \delta \cdot x\}. \end{aligned}$$

Then $\mathcal{S} = \bigcup_{\Delta \in [\Gamma]^k} \{\Delta x \mid x \in B_\Delta\}$, so Proposition 3.21 ensures that it is Γ-decomposable. ⊠

Proposition 3.23. *Suppose that $k \in \mathbb{Z}^+$, X is a Borel space, Γ is a countable group of separable Borel automorphisms of X, $B \subseteq X$ is a Borel set whose intersection with each Γ-orbit is infinite, and G is a Γ-decomposable graph on X whose vertical sections are finite. Then there is a Γ-decomposable equivalence relation F on B whose classes are G-independent and have cardinality k.*

Proof. We first establish the special case where Γ is smooth. Fix enumerations $(\gamma_n)_{n \in \mathbb{N}}$ of Γ and $(\Delta_n)_{n \in \mathbb{N}}$ of $[\Gamma]^k$. For all $s \in \mathbb{N}^{<\mathbb{N}}$, Propositions 1.8 and 3.21 ensure that the set A_s of $x \in X$ such that

(1) $\forall i < |s| \forall \delta \in \Delta_{s(i)}\ \delta \cdot x \in B$,
(2) $\forall i < |s| \forall \delta \in \Delta_{s(i)} \forall \gamma \in \Delta_{s(i)} \setminus \{\delta\}\ (\gamma \cdot x \neq \delta \cdot x \text{ and } \neg \gamma \cdot x \mathrel{G} \delta \cdot x)$,
(3) $\forall i < |s|\ (\gamma_i \cdot x \in B \implies \exists j < |s|\ \gamma_i \cdot x \in \Delta_{s(j)} x)$, and
(4) $\forall i < j < |s|\ \Delta_{s(i)} x \cap \Delta_{s(j)} x = \emptyset$

is Borel. Fix a Borel transversal $B_\emptyset \subseteq X$ of E^X_Γ and recursively define $B_{s \frown (n)} = (A_{s \frown (n)} \setminus \bigcup_{m < n} A_{s \frown (m)}) \cap B_s$ for all $n \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$. Then the family $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \bigcup_{s \in \mathbb{N}^{n+1}} \{\Delta_{s(n)} x \mid x \in B_s\}$ partitions X and Proposition 3.19 ensures that the equivalence relation F on X, given by $X/F = \mathcal{S}$, is as desired.

To establish the general case, appeal to Propositions 3.20 and 3.22 to obtain a Γ-decomposable maximal family $\mathcal{S} \subseteq [X]^k_\Gamma$ of pairwise disjoint

G-independent subsets of B of cardinality k. Then the Borel set $B = {\sim}\bigcup \mathcal{S}$ intersects every Γ-orbit in a finite set, so Proposition 1.16 ensures that $E_\Gamma^X \upharpoonright \Gamma B$ is smooth. By the previous paragraph, we can therefore assume that $X = \bigcup \mathcal{S}$, in which case Proposition 3.19 implies that the equivalence relation F on X, given by $X/F = \mathcal{S}$, is as desired. ⊠

Proposition 3.24. *Suppose that X is a Borel space and $T\colon X \to X$ is a separable Borel automorphism of a Borel space whose orbits all have finite odd cardinality. Then there are involutions $I_1, I_2 \in [T]$, conjugate in $[T]$, such that $T = I_2 \circ I_1$.*

Proof. As Proposition 1.16 ensures that T is smooth, Remark 1.6 yields involutions $I_1, I_2 \in [T]$, each having exactly one fixed point on every T-orbit, such that $T = I_2 \circ I_1$. But Proposition 3.17 implies that I_1 and I_2 are conjugate in $[T]$. ⊠

For each set K of countable cardinals, the *period K part* of a bijection $T\colon X \to X$ is given by $\mathrm{Per}_K(T) = \bigcup_{k \in K} \mathrm{Per}_k(T)$. We can now give the following:

Proof of Theorem 5. We first handle the special case of the proposition where $\Gamma x \cap \mathrm{Per}_{2\mathbb{N}}(T)$ is finite but non-empty for all $x \in X$, in which case Proposition 1.16 ensures that Γ is smooth. We will consider three subcases; the desired special case will then follow from Proposition 1.9, since it ensures that X can be partitioned into three Γ-invariant Borel sets, each falling into at least one of these cases.

Suppose first that $\Gamma x \setminus \mathrm{supp}(T)$ is infinite for all $x \in X$, appeal to Proposition 3.16 to obtain a partition $(B_n)_{n \in \mathbb{N}}$ of ${\sim}\mathrm{supp}(T)$ into Borel transversals of $E_\Gamma^X \upharpoonright {\sim}\mathrm{supp}(T)$, fix an involution ι of $\mathbb{N}$ whose support is infinite and co-infinite, and define $I = \bigcup_{\gamma \in \Gamma} \bigcup_{n \in \mathbb{N}} \gamma \upharpoonright (B_n \cap \gamma^{-1} B_{\iota(n)})$. As Proposition 3.15 ensures that $T \upharpoonright \mathrm{supp}(T)$ is smooth, Proposition 1.1 yields involutions $I_1, I_2 \in [T \upharpoonright \mathrm{supp}(T)]$ with the property that $T \upharpoonright \mathrm{supp}(T) = I_2 \circ I_1$, in which case Proposition 3.18 implies that the involutions $S_k = I \cup I_k$, for $1 \leq k \leq 2$, are as desired.

Suppose next that $\Gamma x \cap \mathrm{Per}_{2\mathbb{N}+3}(T)$ is infinite for all $x \in X$. As Proposition 3.15 ensures that T is smooth, Remark 1.6 yields involutions $I_1, I_2 \in [T \upharpoonright \mathrm{Per}_{2\mathbb{N}+3}(T)]$, each having exactly one fixed point on every $(T \upharpoonright \mathrm{Per}_{2\mathbb{N}+3}(T))$-orbit, for which $T \upharpoonright \mathrm{Per}_{2\mathbb{N}+3}(T) = I_2 \circ I_1$, and Proposition 1.1 yields involutions $J_1, J_2 \in [T \upharpoonright {\sim}\mathrm{Per}_{2\mathbb{N}+3}(T)]$ for which $T \upharpoonright {\sim}\mathrm{Per}_{2\mathbb{N}+3}(T) = J_2 \circ J_1$, in which case Proposition 3.18 implies that the involutions $S_k = I_k \cup J_k$, for $1 \leq k \leq 2$, are as desired.

Suppose finally that the *aperiodic part* of T, given by $\mathrm{Aper}(T) = {\sim}\mathrm{Per}(T)$, is Γ-complete. As Proposition 3.15 ensures that $T \upharpoonright \mathrm{Aper}(T)$

is smooth, it follows from Proposition 3.13 that there are automorphisms $R_1, R_2 \in [T \restriction \mathrm{Aper}(T)]$, whose orbits are all of cardinality 1, 2, or n and for which each possibility occurs infinitely often on every orbit of $T \restriction \mathrm{Aper}(T)$, with the property that $T \restriction \mathrm{Aper}(T) = R_2 \circ R_1$. As Proposition 3.15 also ensures that $T \restriction \mathrm{Per}(T)$ is smooth, Proposition 1.1 yields involutions $I_1, I_2 \in [T \restriction \mathrm{Per}(T)]$ for which $T \restriction \mathrm{Per}(T) = I_2 \circ I_1$, in which case Proposition 3.18 implies that the automorphisms $S_k = I_k \cup R_k$, for $1 \leq k \leq 2$, are as desired.

We now consider the general case. As Proposition 1.9 ensures that $\{x \in X \mid \Gamma x \cap \mathrm{Per}_{2\mathbb{N}}(T) \text{ is finite but non-empty}\}$ is Borel, by throwing out this set, we can assume that $\Gamma x \cap \mathrm{Per}_{2\mathbb{N}}(T)$ is empty or infinite for all $x \in X$. As Theorem 4 yields the case that T is aperiodic and Proposition 3.24 yields the case that every T-orbit has finite odd cardinality even without the assumption that Γ is aperiodic, Proposition 1.9 allows us to assume that $X = \mathrm{Per}_{2\mathbb{N}}(T)$.

Appeal to Proposition 1.16 to obtain a Borel transversal $B \subseteq X$ of T. By Proposition 3.23, there is a Γ-decomposable equivalence relation F on B whose classes are all of cardinality 2. As the involution H generating F is Γ-decomposable and therefore Borel, another application of Proposition 1.16 yields a Borel transversal $A \subseteq B$ of F. By Remark 1.6, there are involutions $I_1, I_2 \in [T \restriction [A]_T]$ with the property that I_2 has exactly two fixed points on every orbit of $T \restriction [A]_T$, I_1 has no fixed points, and $T \restriction [A]_T = I_2 \circ I_1$, as well as involutions $J_1, J_2 \in [T \restriction [B \setminus A]_T]$ such that J_2 has no fixed points, J_1 has exactly two fixed points on every orbit of $T \restriction [B \setminus A]_T$, and $T \restriction [B \setminus A]_T = J_2 \circ J_1$. As the group Δ generated by H and T is smooth, Proposition 3.18 implies that the involutions $S_k = I_k \cup J_k$, for $1 \leq k \leq 2$, are as desired. ⊠

4. Normal closures

We begin this section with a variant of Proposition 3.18:

Proposition 4.1. *Suppose that X and Y are standard Borel spaces and $S \colon X \to X$ and $T \colon Y \to Y$ are smooth Borel automorphisms with the property that $|\mathrm{Per}_k(S)/E_S^X| = |\mathrm{Per}_k(T)/E_T^Y|$ for all $1 \leq k \leq \aleph_0$. Then S and T are Borel isomorphic.*

Proof. Fix Borel transversals $A \subseteq X$ and $B \subseteq Y$ of S and T. For all $1 \leq k \leq \aleph_0$, appeal to Proposition 1.9 and the isomorphism theorem for Borel subsets of standard Borel spaces (see, for example, [Kec95, Corollary 13.4 and Theorem 15.6]) to obtain a Borel isomorphism $\pi_k \colon A \cap \mathrm{Per}_k(S) \to B \cap \mathrm{Per}_k(T)$, and define $\phi_k = \bigcup_{n \in \mathbb{Z}} T^n \circ \pi_k \circ S^{-n}$. Then the function $\phi = \bigcup_{1 \leq k \leq \aleph_0} \phi_k$ is as desired. ⊠

The *commutator* of g and h is given by $[g,h] = ghg^{-1}h^{-1}$.

Proposition 4.2. *Suppose that X is a standard Borel space, $\{1\} \subseteq K \subseteq \mathbb{Z}^+$, and $T \colon X \to X$ is a Borel automorphism whose support is uncountable. Then there is a Borel automorphism $S \colon X \to X$ for which the cardinalities of the orbits of $[S,T]$ are in K and each possibility occurs uncountably often.*

Proof. By Proposition 1.15, there is a T-independent Borel set $B \subseteq X$ that intersects every non-trivial T-orbit. As B is uncountable, the isomorphism theorem for Borel subsets of standard Borel spaces ensures that it is Borel isomorphic to $\mathbb{R} \times K$, so there is a partition $(B_k)_{k \in K}$ of B into uncountable Borel sets. As the isomorphism theorem for Borel subsets of standard Borel spaces (or the fact that $\mathbb{R}$ is Borel isomorphic to $\mathbb{R} \times k$) also implies that each B_k is Borel isomorphic to $\mathbb{R} \times k$, there is a Borel automorphism $S_k \colon B_k \to B_k$ whose orbits have cardinality k for all $k \in K$. Define $S = (\bigcup_{k \in K} S_k) \cup (\mathrm{id}_{\sim B})$. If $k \in K$, then

$$\begin{aligned}(S \circ T \circ S^{-1} \circ T^{-1}) \restriction B_k &= S \circ T \circ (S^{-1} \restriction T^{-1}(B_k)) \circ T^{-1} \\ &= S \circ T \circ (\mathrm{id}_{T^{-1}(B_k)}) \circ T^{-1} \\ &= (S \circ T \circ \mathrm{id} \circ T^{-1}) \restriction B_k \\ &= S \restriction B_k \\ &= S_k\end{aligned}$$

and

$$\begin{aligned}(S \circ T \circ S^{-1} \circ T^{-1}) \restriction T(B_k) &= S \circ T \circ (S^{-1} \restriction B_k) \circ T^{-1} \\ &= (S \restriction T(B_k)) \circ T \circ S_k^{-1} \circ T^{-1} \\ &= (\mathrm{id}_{T(B_k)}) \circ T \circ S_k^{-1} \circ T^{-1} \\ &= T \circ S_k^{-1} \circ T^{-1},\end{aligned}$$

so $|[x]_{[S,T]}| = k$ for all $x \in B_k \cup T(B_k)$. But

$$\begin{aligned}&(S \circ T \circ S^{-1} \circ T^{-1}) \restriction \sim(B \cup T(B)) \\ &= S \circ T \circ (S^{-1} \restriction \sim(T^{-1}(B) \cup B)) \circ T^{-1} \\ &= S \circ T \circ (\mathrm{id}_{\sim(T^{-1}(B) \cup B)}) \circ T^{-1} \\ &= (S \circ T \circ \mathrm{id} \circ T^{-1}) \restriction \sim(B \cup T(B)) \\ &= S \restriction \sim(B \cup T(B)) \\ &= \mathrm{id}_{\sim(B \cup T(B))},\end{aligned}$$

so $|[x]_{[S,T]}| = 1$ for all $x \in \sim(B \cup T(B))$. ⊠

Proposition 4.3. *Suppose that X is an uncountable standard Borel space and $T\colon X \to X$ is a smooth Borel automorphism. Then there are Borel involutions $I, J\colon X \to X$, with uncountable co-uncountable supports, for which $T = I \circ J$.*

Proof. We first handle the special case that $T = \mathrm{id}$. As the isomorphism theorem for standard Borel spaces ensures that X is Borel isomorphic to $\mathbb{R} \times 2$, there is an uncountable co-uncountable Borel set $A \subseteq X$. As the isomorphism theorem for Borel subsets of standard Borel spaces (or the fact that $\mathbb{R}$ is Borel isomorphic to $\mathbb{R} \times 2$) also implies that A is Borel isomorphic to $\mathbb{R} \times 2$, there is an uncountable Borel set $B \subseteq A$ for which there is a Borel isomorphism $\pi\colon B \to A \setminus B$, in which case the functions $I = J = \pi^{\pm 1} \cup \mathrm{id}_{\sim A}$ are as desired.

We next handle the special case that T has no fixed points. Fix a Borel transversal $A \subseteq X$ of T. As A is uncountable, the isomorphism theorem for Borel subsets of standard Borel spaces ensures that A is Borel isomorphic to $\mathbb{R} \times 2$, so there is an uncountable Borel set $B \subseteq A$ with the property that $A \setminus B$ is also uncountable. Set $C = [B]_T$ and appeal to Remark 1.6 to obtain Borel involutions $I', J' \in [T \upharpoonright C]$ for which I' has a fixed point on every orbit of $T \upharpoonright C$ and $T \upharpoonright C = I' \circ J'$, as well as Borel involutions $I'', J'' \in [T \upharpoonright {\sim}C]$ for which J'' has a fixed point on every orbit of $T \upharpoonright {\sim}C$ and $T \upharpoonright {\sim}C = I'' \circ J''$. Then the functions $I = I' \cup I''$ and $J = J' \cup J''$ are as desired.

We now handle the general case. Fix $1 \leq k \leq \aleph_0$ for which $\mathrm{Per}_k(T)$ is uncountable. As Proposition 1.9 ensures that this set is Borel, and therefore standard Borel by [Kec95, Corollary 13.4], the special cases yield Borel involutions $I', J'\colon \mathrm{Per}_k(T) \to \mathrm{Per}_k(T)$, with uncountable co-uncountable supports, for which $T \upharpoonright \mathrm{Per}_k(T) = I' \circ J'$. By Proposition 1.1, there are Borel involutions $I'', J''\colon {\sim}\mathrm{Per}_k(T) \to {\sim}\mathrm{Per}_k(T)$ with the property that $T \upharpoonright {\sim}\mathrm{Per}_k(T) = I'' \circ J''$. But then the functions $I = I' \cup I''$ and $J = J' \cup J''$ are as desired. ⊠

Proposition 4.4. *Suppose that X is a standard Borel space, $\{1, 2\} \subsetneq K \subseteq \mathbb{Z}^+$, and $T\colon X \to X$ is a non-smooth Borel automorphism. Then there exist $R, S \in [T]$, whose orbits all have cardinality in K and for which each possibility occurs uncountably often, such that $T = R \circ S$.*

Proof. As Proposition 1.9 ensures that $\mathrm{Per}(T)$ is Borel and Proposition 1.16 implies that $T \upharpoonright \mathrm{Per}(T)$ is smooth, Proposition 1.1 yields Borel involutions $I', J' \in [T \upharpoonright \mathrm{Per}(T)]$ for which $T \upharpoonright \mathrm{Per}(T) = I' \circ J'$. By Silver's perfect set theorem (see, for example, [Sil80], although the special case we need is far simpler to prove), there is an uncountable T-invariant Borel set $B \subseteq \mathrm{Aper}(T)$ for which $T \upharpoonright B$ is smooth. By Proposition 3.13, there exist $R'', S'' \in [T \upharpoonright B]$, whose orbits all have cardinality in K and for which each possibility occurs on every orbit of $T \upharpoonright B$, such that $T \upharpoonright B = R'' \circ S''$. Fix $k \in K \setminus \{1, 2\}$, and appeal to Theorem 3 to obtain $R''', S''' \in [T \upharpoonright (\mathrm{Aper}(T) \setminus B)]$, whose orbits all have cardinality 1 or k, such that $T \upharpoonright (\mathrm{Aper}(T) \setminus B) = R''' \circ S'''$. Then the functions $R = I' \cup R'' \cup R'''$ and $S = J' \cup S'' \cup S'''$ are as desired. $\boxtimes$

Remark 4.5. In the special case that K is finite, the need for Silver's theorem can be eliminated by replacing the use of Proposition 3.13 with that of Remark 3.14.

We can now give the following:

Proof of Theorem 6. Note first that if $I \colon X \to X$ is a Borel automorphism for which there is a Borel automorphism $R \colon X \to X$ such that I is Borel isomorphic to $[R, T]$ as witnessed by $P \colon X \to X$, then

$$\begin{aligned} I &= P^{-1} \circ [R, T] \circ P \\ &= P^{-1} \circ R \circ T \circ R^{-1} \circ T^{-1} \circ P \\ &= (P^{-1} \circ R \circ T \circ R^{-1} \circ P) \circ (P^{-1} \circ T^{-1} \circ P), \end{aligned}$$

so I is a composition of two conjugates of $T^{\pm 1}$. In particular, it is sufficient to show that S is a composition of two such automorphisms.

We first handle the case that S is smooth. By Proposition 4.2, there is a Borel automorphism $R \colon X \to X$ for which $[R, T]$ is an involution with uncountable co-uncountable support. By Proposition 4.3, there are Borel involutions $I, J \colon X \to X$, with uncountable co-uncountable supports, for which $S = I \circ J$. But Proposition 4.1 ensures that I, J, and $[R, T]$ are Borel isomorphic.

We now consider the case that S is not smooth. By Proposition 4.2, there is a Borel automorphism $R \colon X \to X$ for which every orbit of $[R, T]$ has cardinality 1, 2, or 3, and each possibility occurs uncountably often. By Proposition 4.4, there are Borel automorphisms $I, J \colon X \to X$, whose orbits all have cardinality 1, 2, or 3 and for which each possibility occurs uncountably often, such that $S = I \circ J$. But Proposition 4.1 ensures that I, J, and $[R, T]$ are Borel isomorphic. $\boxtimes$

5. Bergman's property

The *saturation* of a set $Y \subseteq X$ with respect to an equivalence relation E on X is given by $[Y]_E = \{x \in X \mid \exists y \in Y \; x \mathrel{E} y\}$.

Proposition 5.1. *Suppose that X is a Borel space, $T \colon X \to X$ is a Borel automorphism whose orbits all have the same cardinality, and E is a T-decomposable equivalence relation on X. Then there exists $S \in [T]$ for which $E = E^X_S$.*

Proof. Fix Borel sets $B_n \subseteq X$ such that $E = \bigcup_{n\in\mathbb{Z}} \operatorname{graph}(T^n \restriction B_n)$.

Suppose first that every T-orbit has cardinality $k \in \mathbb{Z}^+$. For all $1 \leq j \leq k$, define $A_j = \bigcup_{i\in j+k\mathbb{Z}} B_i$. Then the function $S\colon X \to X$, given by $S = \bigcup_{1\leq j\leq k} T^j \restriction (A_j \setminus \bigcup_{1\leq i<j} A_i)$, is as desired. So we can assume that T is aperiodic.

For all $N \in [\mathbb{N}]^{<\aleph_0}$, fix a transitive permutation σ_N of N and define $B_N = (\bigcap_{n\in N} B_n) \setminus (\bigcup_{n\in\mathbb{Z}\setminus N} B_n)$. Then the map $S_N\colon [B_N]_E \to [B_N]_E$, given by $S_N = \bigcup_{n\in N} T^{\sigma_N(n)-n} \restriction T^n(B_N)$, generates $E \restriction [B_N]_E$, so the function $S_{<\infty} = \bigcup_{N\in[\mathbb{N}]^{<\aleph_0}} S_N$ generates $E \restriction \bigcup_{N\in[\mathbb{N}]^{<\aleph_0}} [B_N]_E$.

Fix a transitive permutation σ of $\mathbb{Z}^+$. For all $N \in [\mathbb{N}]^{<\aleph_0}$, define

$$B_N^- = (\textstyle\bigcap_{n\in N} B_n \setminus \bigcup_{n\in\mathbb{N}\setminus N} B_n) \cap (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} B_{-m})$$

and

$$B_N^+ = (\textstyle\bigcap_{n\in N} B_{-n} \setminus \bigcup_{n\in\mathbb{N}\setminus N} B_{-n}) \cap (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} B_m).$$

For all $k \in \mathbb{Z}^+$ and $* \in \{-,+\}$, set $B_k^* = \bigcup_{N\in[\mathbb{N}]^k} B_N^*$ and define $S_k^*\colon B_k^* \to B_{\sigma(k)}^*$ by $S_k^* = \bigcup_{n\in\mathbb{Z}} T^n \restriction (B_k^* \cap T^{-n}(B_{\sigma(k)}^*))$. Then the function $S^* = \bigcup_{k\in\mathbb{Z}^+} S_k^*$ generates $E \restriction \bigcup_{k\in\mathbb{Z}^+} B_k^*$ for all $* \in \{-,+\}$.

Set $B_\mathbb{Z} = (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} B_{-m}) \cap (\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} B_m)$ and note that the map $S_\mathbb{Z}\colon B_\mathbb{Z} \to B_\mathbb{Z}$, given by $S_\mathbb{Z} = \bigcup_{n\in\mathbb{Z}^+} T^n \restriction (B_\mathbb{Z} \cap (B_n \setminus \bigcup_{1\leq m<n} B_m))$, generates $E \restriction B_\mathbb{Z}$, so the map $S = S_{<\infty} \cup S^- \cup S^+ \cup S_\mathbb{Z}$ is as desired. ⊠

In the special case where T is separable, the hypothesis on the cardinalities of the T-orbits is unnecessary:

Proposition 5.2. *Suppose that X is a Borel space, $T\colon X \to X$ is a Borel automorphism of X whose powers have Borel supports, and E is a T-decomposable equivalence relation on X. Then there exists $S \in [T]$ for which $E = E_S^X$.*

Proof. By Propositions 1.8 and 5.1, there exists $S_k \in [T \restriction \operatorname{Per}_k(T)]$ such that $E \restriction \operatorname{Per}_k(T) = E_{S_k}^{\operatorname{Per}_k(T)}$ for all $1 \leq k \leq \aleph_0$. Then the function $S = \bigcup_{1\leq k\leq\aleph_0} S_k$ is as desired. ⊠

To extend this result further, we need one more basic observation:

Proposition 5.3. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X whose supports are Borel, Δ is a countable subgroup of Γ, and R is a Γ-decomposable binary relation on X. Then $E_\Delta^X \cap R$ is Δ-decomposable.*

Proof. By Proposition 3.21, the set $B_\delta = \{x \in X \mid x \mathrel{R} \delta \cdot x\}$ is Borel for all $\delta \in \Delta$. But $E_\Delta^X \cap R = \bigcup_{\delta\in\Delta} \operatorname{graph}(\delta \restriction B_\delta)$. ⊠

Finally, we have the following:

Proposition 5.4. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X whose supports are Borel, and E is a Γ-decomposable equivalence relation on X. Then there exists $\Delta \leq [\Gamma]$ for which $E = E^X_\Delta$.*

Proof. For all $\gamma \in \Gamma$, Proposition 5.3 ensures that $E \cap E^X_\gamma$ is γ-decomposable, so Proposition 5.2 yields $T_\gamma \in [\gamma]$ for which $E \cap E^X_\gamma = E^X_{T_\gamma}$. Then the group Δ generated by $\{T_\gamma \mid \gamma \in \Gamma\}$ is as desired. ⊠

Proposition 5.5. *Suppose that X is a Borel space, Γ is a countable group of separable Borel automorphisms of X, $Y \subseteq X$ is Γ-invariant, and $T \in [\Gamma \upharpoonright Y]$. Then there exists $S \in [\Gamma]$ such that $S \upharpoonright Y = T$.*

Proof. Fix Borel sets $B_\gamma \subseteq X$ such that $T = \bigcup_{\gamma \in \Gamma} \gamma \upharpoonright (B_\gamma \cap Y)$. We can clearly assume that the sets B_γ are pairwise disjoint. Then the Γ-invariant set

$$\begin{aligned} B &= \{x \in X \mid (B_\gamma \cap \Gamma x)_{\gamma \in \Gamma} \text{ and } (\gamma B_\gamma \cap \Gamma x)_{\gamma \in \Gamma} \text{ both partition } \Gamma x\} \\ &= \{x \in X \mid \forall \gamma \in \Gamma \forall i < 2 \exists \delta \in \Delta \ \gamma \cdot x \in \delta^i B_\delta \setminus \textstyle\bigcup_{\lambda \in \Gamma \setminus \{\delta\}} \lambda^i B_\lambda\} \\ &= \textstyle\bigcap_{\gamma \in \Gamma} \bigcap_{i<2} \bigcup_{\delta \in \Gamma} \gamma^{-1}(\delta^i B_\delta \setminus \bigcup_{\lambda \in \Gamma \setminus \{\delta\}} \lambda^i B_\lambda) \end{aligned}$$

is Borel and the function $S = \mathrm{id}_{\sim B} \cup \bigcup_{\gamma \in \Gamma} \gamma \upharpoonright (B \cap B_\gamma)$ is the desired extension of T to an element of $[\Gamma]$. ⊠

Given a countable group Γ of Borel automorphisms of a Borel space X, we say that a set $Y \subseteq X$ is Γ-*large* if there is a finite set $\Delta \subseteq [\Gamma]$ for which $X = \Delta Y$.

Proposition 5.6. *Suppose that X is a Borel space and Γ is an aperiodic countable group of separable Borel automorphisms of X. Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint Γ-large Borel subsets of X.*

Proof. It is sufficient to show that every Γ-large Borel set $A \subseteq X$ can be partitioned into two Γ-large Borel subsets. Towards this end, note that $|A \cap \Gamma x| = \aleph_0$ for all $x \in X$, so Proposition 3.23 yields a Γ-decomposable equivalence relation F on A whose classes have cardinality 2. As the involution I generating F is Γ-decomposable and therefore Borel, Proposition 1.16 yields a Borel transversal $B \subseteq A$ of F. Set $C = A \setminus B$. As $I \cup \mathrm{id}_{\sim A} \in [\Gamma]$, the fact that $A = B \cup C = B \cup I(B) = C \cup I(C)$ implies that B and C are Γ-large. ⊠

For all sets $\Delta \subseteq \Gamma$ and $Y \subseteq X$, define $\Delta_{\{Y\}} = \{\delta \in \Delta \mid Y = \delta Y\}$ and $\Delta \upharpoonright Y = \{\delta \upharpoonright Y \mid \delta \in \Delta\}$.

Proposition 5.7. *Suppose that X is a Borel space, Γ is a group of separable Borel automorphisms of X that is closed under countable decomposition, Δ is an aperiodic countable subgroup of Γ, and $(\Gamma_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of Γ. Then there exist a Δ-large Borel set $B \subseteq X$ and $n \in \mathbb{N}$ for which $\Gamma_{\{B\}} \restriction B \subseteq \Gamma_n \restriction B$ and $[\Delta]_{\{B\}} \restriction B \subseteq (\Gamma_n \cap [\Delta]) \restriction B$.*

Proof. Appeal to Proposition 5.6 to obtain a sequence $(B_n)_{n\in\mathbb{N}}$ of pairwise disjoint Δ-large Borel subsets of X. It is sufficient to show that if Λ is a subgroup of Γ containing Δ that is closed under countable decomposition, then $\Lambda_{\{B_n\}} \restriction B_n \subseteq (\Gamma_n \cap \Lambda) \restriction B_n$ for all but finitely many $n \in \mathbb{N}$. Suppose, towards a contradiction, that there is an infinite set $N \subseteq \mathbb{N}$ such that there exists $\lambda_n \in (\Lambda_{\{B_n\}} \restriction B_n) \setminus ((\Gamma_n \cap \Lambda) \restriction B_n)$ for all $n \in N$. As Λ is closed under countable decomposition, it contains the automorphism $\lambda = (\bigcup_{n\in N} \lambda_n) \cup (\mathrm{id} \restriction {\sim}\bigcup_{n\in N} B_n)$. As the latter is in Γ_n for all but finitely many $n \in \mathbb{N}$, and therefore for some $n \in N$, this contradicts the fact that $\lambda_n = \lambda \restriction B_n$ is not in $(\Gamma_n \cap \Lambda) \restriction B_n$. ⊠

Define $\Delta_Y = \{\delta \in \Delta \mid \forall y \in Y \ y = \delta \cdot y\}$.

Proposition 5.8. *Suppose that X is a Borel space, Γ is a group of separable Borel automorphisms of X that is closed under countable decomposition, Δ is an aperiodic countable subgroup of Γ, and $(\Gamma_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of Γ. Then there exist a Δ-large Borel set $B \subseteq X$ and $n \in \mathbb{N}$ for which $\Gamma_{\{B\}} \restriction B \subseteq \Gamma_n \restriction B$, $[\Delta]_{\{B\}} \restriction B \subseteq (\Gamma_n \cap [\Delta]) \restriction B$, and $\Gamma_{\sim B} \subseteq (\Gamma_n)^4$.*

Proof. By replacing each Γ_n with $\Gamma_n \cap \Gamma_n^{-1}$, we can assume that each Γ_n is symmetric. By Proposition 5.7, there exist a Δ-large Borel set $X' \subseteq X$ and $n' \in \mathbb{N}$ with the property that $\Gamma_{\{X'\}} \restriction X' \subseteq \Gamma_{n'} \restriction X'$ and $[\Delta]_{\{X'\}} \restriction X' \subseteq (\Gamma_{n'} \cap [\Delta]) \restriction X'$, so $\Gamma_{\{B\}} \restriction B \subseteq \Gamma_{n'} \restriction B$ and $[\Delta]_{\{B\}} \restriction B \subseteq (\Gamma_{n'} \cap [\Delta]) \restriction B$ for all Borel sets $B \subseteq X'$. As the diagonal on X is the graph of the identity function on X and $E_\Delta^X \restriction X' = \bigcup_{\gamma\in\Gamma} \mathrm{graph}(\gamma \restriction (\mathrm{supp}(\gamma) \cap X' \cap \gamma^{-1}X'))$, it follows from Proposition 1.8 that the union of these two equivalence relations is Γ-decomposable, so Proposition 5.4 yields a countable subgroup Δ' of $[\Delta]$ for which $E_{\Delta'}^X$ is the aforementioned union. Set $\Gamma' = \Gamma_{\sim X'}$ and $\Gamma'_n = (\Gamma_n)_{\sim X'}$ for all $n \in \mathbb{N}$. As Δ is aperiodic and X' is Δ-large, it follows that $\Delta' \restriction X'$ is aperiodic, so Proposition 5.7 yields a $(\Delta' \restriction X')$-large Borel set $B \subseteq X'$ and $n \geq n'$ for which $(\Gamma' \restriction X')_{\{B\}} \restriction B \subseteq (\Gamma'_n \restriction X') \restriction B$. Then $\Gamma'_{\{B\}} \restriction B = (\Gamma' \restriction X')_{\{B\}} \restriction B \subseteq (\Gamma'_n \restriction X') \restriction B = \Gamma'_n \restriction B$ and B is Δ-large. As the diagonal on X is the graph of the identity function on X and $E_\Delta^X \restriction B = \bigcup_{\gamma\in\Gamma} \mathrm{graph}(\gamma \restriction (\mathrm{supp}(\gamma) \cap B \cap \gamma^{-1}B))$, it follows from Proposition 1.8 that the union of these two equivalence relations is Γ-decomposable, so Proposition 5.4 yields a countable subgroup Δ'' of $[\Delta]$ for which $E^X_{\Delta''}$ is the aforementioned union.

It remains to show that if $\gamma \in \Gamma_{\sim B}$, then $\gamma \in (\Gamma_n)^4$. By Theorem 5, there exist $\delta, \lambda \in [\langle \{\gamma \restriction B\} \cup (\Delta'' \restriction B)\rangle]$ for which $\gamma \restriction B = [\delta, \lambda]$. Then $\delta \cup \mathrm{id}_{\sim B} \in \Gamma$ and $\lambda \cup \mathrm{id}_{\sim B} \in \Gamma'$, so there are extensions $\delta' \in \Gamma_{n'}$ and $\lambda' \in \Gamma'_n$ of $\delta \cup \mathrm{id}_{X' \setminus B}$ and $\lambda \cup \mathrm{id}_{\sim X'}$. Then

$$\begin{aligned}[\delta', \lambda'] \restriction B &= \delta' \lambda' (\delta')^{-1} (\lambda')^{-1} \restriction B \\ &= \delta \lambda \delta^{-1} \lambda^{-1} \\ &= [\delta, \lambda] \\ &= \gamma \restriction B,\end{aligned}$$

$$\begin{aligned}[\delta', \lambda'] \restriction (X' \setminus B) &= \delta' \lambda' (\delta')^{-1} (\lambda')^{-1} \restriction (X' \setminus B) \\ &= \delta' \lambda' ((\delta')^{-1} \restriction (X' \setminus B)) (\lambda')^{-1} \\ &= \delta' (\lambda' \restriction (X' \setminus B)) (\lambda')^{-1} \\ &= \delta' \restriction (X' \setminus B) \\ &= \mathrm{id}_{X' \setminus B},\end{aligned}$$

and

$$\begin{aligned}[\delta', \lambda'] \restriction {\sim} X' &= \delta' \lambda' (\delta')^{-1} (\lambda')^{-1} \restriction {\sim} X' \\ &= \delta' \lambda' (\delta')^{-1} \restriction {\sim} X' \\ &= \delta' (\lambda' \restriction {\sim} X') (\delta')^{-1} \\ &= (\delta' \restriction {\sim} X') (\delta')^{-1} \\ &= \mathrm{id}_{\sim X'},\end{aligned}$$

so $\gamma = [\delta', \lambda'] \in (\Gamma_n)^4$. ⊠

We can now give the following:

Proof of Theorem 7. By Proposition 5.8, there exist $n \in \mathbb{N}$ and a Δ-large Borel set $A \subseteq X$ for which $\Gamma_{\sim A} \subseteq (\Gamma_n)^4$. As in the proof of Proposition 5.6, there exist a Borel set $B \subseteq A$ and an involution $\iota_A \in [\Delta]$ such that $\iota_A B = A \setminus B$. Fix $k \in \mathbb{N}$ and $\delta_i \in [\Delta]$ with the property that $X = \bigcup_{i<k} \delta_i B$. Without loss of generality, we can assume that $\delta_0 = \mathrm{id}$ and $\delta_1 = \iota_A$. Set $B_j = \delta_j B \setminus \bigcup_{i<j} \delta_i B$ for all $j < k$.

Lemma 5.9. *Suppose that $i \leq j < k$. Then there is an involution $\iota_{i,j} \in [\Delta]$ for which $\iota_{i,j}(B_i \cup B_j) \subseteq A$.*

Proof. We can assume that $j \geq 2$, since otherwise the identity function is as desired. If $i = 0$, then $(\iota_A \delta_j^{-1} \restriction B_j)^{\pm 1} \cup \mathrm{id}_{\sim (B_j \cup \iota_A \delta_j^{-1} B_j)}$ is as desired. If $i = 1$, then $(\delta_j^{-1} \restriction B_j)^{\pm 1} \cup \mathrm{id}_{\sim (B_j \cup \delta_j^{-1} B_j)}$ is as desired.

If $i = j$, then either of the last two functions is as desired. And $(\delta_i^{-1} \restriction B_i)^{\pm 1} \cup (\iota_A \delta_j^{-1} \restriction B_j)^{\pm 1} \cup \mathrm{id}_{\sim(B_i \cup B_j \cup \delta_i^{-1} B_i \cup \iota_A \delta_j^{-1} B_j)}$ is as desired otherwise. ⊠

By increasing n, we can assume that $\iota_{i,j} \in \Gamma_n$ for all $i \leq j < k$. We will show that $\Gamma \subseteq (\Gamma_n)^{9k(k+1)}$. By Theorem 1, it is sufficient to show that every involution $\iota \in \Gamma$ is in $(\Gamma_n)^{3k(k+1)}$. Towards this end, set $B_{i,j} = (B_i \cap \iota B_j) \cup (\iota B_i \cap B_j)$ and $\iota'_{i,j} = \iota_{i,j}(\iota \restriction B_{i,j})\iota_{i,j} \cup \mathrm{id}_{\sim \iota_{i,j} B_{i,j}}$ for all $i \leq j < k$. Then $\iota = \prod_{i \leq j < k} \iota_{i,j} \iota'_{i,j} \iota_{i,j} \in (\Gamma_n \Gamma_{\sim A} \Gamma_n)^{k(k+1)/2} \subseteq (\Gamma_n)^{6k(k+1)/2} = (\Gamma_n)^{3k(k+1)}$. ⊠

In order to establish a similar result concerning the k-Bergman property, we again need several preliminaries.

Proposition 5.10. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, and $A, B \subseteq X$ are Borel. Then there is a Γ-invariant Borel set $Y \subseteq X$ for which there are Γ-decomposable injections $\phi \colon A \cap Y \to B \cap Y$ and $\psi \colon B \setminus Y \to A \setminus Y$.*

Proof. Fix an enumeration $(\gamma_n)_{n \in \mathbb{N}}$ of Γ and recursively define sets $A_n = (A \setminus \bigcup_{m<n} A_m) \cap \gamma_n^{-1}(B \setminus \bigcup_{m<n} \gamma_m A_m)$. Set $A_\infty = \bigcup_{n \in \mathbb{N}} A_n$ and $B_\infty = \bigcup_{n \in \mathbb{N}} \gamma_n A_n$, and observe that the function $\pi = \bigcup_{n \in \mathbb{N}} \gamma_n \restriction A_n$ is a bijection of A_∞ with B_∞.

Lemma 5.11. *If $x \in X$, then $A \cap \Gamma x \subseteq A_\infty$ or $B \cap \Gamma x \subseteq B_\infty$.*

Proof. Suppose that there exists $y \in (A \cap \Gamma x) \setminus A_\infty$. Given $z \in B \cap \Gamma x$, fix $n \in \mathbb{N}$ for which $\gamma_n \cdot y = z$, and observe that the facts that $y \notin A_n$ and $y \in A \setminus \bigcup_{m<n} A_m$ ensure that $y \notin \gamma_n^{-1}(B \setminus \bigcup_{m<n} \gamma_m A_m)$, in which case $z \notin B \setminus \bigcup_{m<n} \gamma_m A_m$, so $z \in B_\infty$, thus $B \cap \Gamma x \subseteq B_\infty$. ⊠

It follows that the set $Y = \{x \in X \mid A \cap \Gamma x \subseteq A_\infty\} = {\sim}\Gamma(A \setminus A_\infty)$ and the maps $\phi = \pi \restriction (A \cap Y)$ and $\psi = \pi^{-1} \restriction (B \setminus Y)$ are as desired. ⊠

A *Borel embedding* of a Borel space X into a Borel space Y is a Borel injection $\phi \colon X \to Y$ sending Borel sets to Borel sets.

Proposition 5.12 (Schröder–Bernstein). *Suppose that X and Y are Borel spaces and $\phi \colon X \to Y$ and $\psi \colon Y \to X$ are Borel embeddings. Then there is a Borel set $B \subseteq \psi(Y)$ for which $(\phi \restriction {\sim}B) \cup (\psi^{-1} \restriction B)$ is a Borel isomorphism of X with Y.*

Proof. Endow the set $Z = X \amalg Y$ with the smallest Borel structure containing the Borel subsets of X and Y. Then the function $T \colon Z \to Z$, given by $T = \phi \cup \psi$, is a Borel embedding. Define $C = \bigcap_{n \in \mathbb{N}} T^n(Z)$.

If $z \in C$, then the orbit of z under T is finite or of type $\mathbb{Z}$, so both ϕ and ψ^{-1} induce bijections of $[z]_T \cap X$ with $[z]_T \cap Y$. If $z \in {\sim}C$, then there is a unique point $w \in [z]_T \setminus T(Z)$, in which case ϕ induces a bijection of $[z]_T \cap X$ with $[z]_T \cap Y$ if $w \in X$, and ψ^{-1} induces a bijection of $[z]_T \cap X$ with $[z]_T \cap Y$ if $w \in Y$. In particular, it follows that the set $B = [Y \setminus \phi(X)]_T \cap X$ is as desired. ⊠

Given a countable group Γ of Borel automorphisms of a Borel space X, a Γ-*compression* of a Borel set $B \subseteq X$ is a Γ-decomposable injection $\phi \colon B \to B$ for which $B \subseteq \Gamma(B \setminus \phi(B))$. A set $B \subseteq X$ is Γ-*compressible* if there is a Γ-compression of B.

Proposition 5.13. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, and $B \subseteq X$ is a Γ-compressible Borel set. Then there is a Γ-decomposable bijection $\pi \colon \Gamma B \to B$.*

Proof. By Proposition 5.12, it is sufficient to produce a Γ-decomposable injection $\pi \colon \Gamma B \to B$. Towards this end, fix an enumeration $(\gamma_n)_{n \in \mathbb{N}}$ of Γ, as well as a Γ-compression $\phi \colon B \to B$ of B, and set $A = B \setminus \phi(B)$. For all $n \in \mathbb{N}$, define $A_n = \gamma_n^{-1} A \setminus \bigcup_{m<n} \gamma_m^{-1} A$ and $\pi = \bigcup_{n \in \mathbb{N}} (\phi^n \circ \gamma_n) \restriction A_n$. As the sets of the form $\phi^n(A)$, for $n \in \mathbb{N}$, are pairwise disjoint and contained in B, it follows that π is as desired. ⊠

Proposition 5.14. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, and $B \subseteq X$ is a Γ-compressible Borel set. Then there is an involution $I \in [\Gamma]$ for which $\Gamma B = B \cup I(B)$.*

Proof. Fix a Γ-compression $\phi \colon B \to B$ of B and set $A = B \setminus \phi(B)$. Then the sets $C = (\Gamma B \setminus B) \cup \bigcup_{n \in \mathbb{N}} \phi^{2n}(A)$ and $D = \bigcup_{n \in \mathbb{N}} \phi^{2n+1}(A)$ are Γ-compressible and $\Gamma B = \Gamma C = \Gamma D$, so two applications of Proposition 5.13 yield a Γ-decomposable bijection $\pi \colon C \to D$, in which case the function $I = \pi^{\pm 1} \cup \mathrm{id}_{\sim(C \cup D)}$ is an involution and the fact that $\Gamma B \setminus B \subseteq C = \pi^{-1}(D) = I(D) \subseteq I(B)$ ensures that $\Gamma B = B \cup I(B)$. ⊠

Proposition 5.15. *Suppose that X is a Borel space, Γ is a countable group of Borel automorphisms of X, $A \subseteq X$ is a Borel set, $T \in [\Gamma]$, and $A \cup T(A)$ is Γ-compressible. Then A is Γ-compressible.*

Proof. Note that if $D \subseteq X$ is a Borel set and $\phi \colon D \to D$ is a Γ-compression of D, then the definition of the induced transformation ϕ_C from the proof of Proposition 1.1 makes sense for any Borel set $C \subseteq D$ (even though ϕ is not surjective). Setting $C' = C \cap \bigcup_{n \in \mathbb{N}} \phi^n(C \setminus \phi(C)) \cap \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \phi^{-m}(C)$, it follows that $\phi_{C'}$ is a Γ-compression of C'.

Define $B = T(A)$, fix a Γ-compression ϕ of the set $D = A \cup B$, and observe that the functions $\psi = \phi_{A'} \cup \mathrm{id}_{A \cap (\Gamma A' \setminus A')}$ and $\psi' = (\phi_{B'} \restriction (B' \setminus \Gamma A')) \cup \mathrm{id}_{B \cap (\Gamma B' \setminus (B' \cup \Gamma A'))}$ are Γ-compressions of $A \cap \Gamma A'$ and $B \cap (\Gamma B' \setminus \Gamma A')$, so $T^{-1} \circ \psi' \circ T$ is a Γ-compression of $A \cap (\Gamma B' \setminus \Gamma A')$, thus

$\psi \cup (T^{-1} \circ \psi' \circ T)$ is a Γ-compression of $A \cap (\Gamma A' \cup \Gamma B')$. To see that A is Γ-compressible, it therefore only remains to show that $A \subseteq \Gamma A' \cup \Gamma B'$.

Suppose that $x \in A$. Then there exists $y \in (D \setminus \phi(D)) \cap \Gamma x$. Fix $C \in \{A, B\}$ for which $y \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \phi^{-m}(C)$, let n be the least natural number such that the point $z = \phi^n(y)$ is in C, and observe that $z \in C \setminus \phi(C)$, so $z \in C'$, thus $x \in \Gamma z \subseteq \Gamma A' \cup \Gamma B'$. ⊠

A countable group Γ of Borel automorphisms of a Borel space X is *compressible* if X is Γ-compressible.

Proposition 5.16. *Suppose that X is a Borel space, Γ is a compressible countable group of Borel automorphisms of X, and $A \subseteq X$ is a Γ-complete Borel set. Then the following are equivalent:*

1. *The set A is Γ-compressible.*
2. *There is an involution $I \in [\Gamma]$ for which $X = A \cup I(A)$.*
3. *The set A is Γ-large.*

Proof. As (1) $\implies$ (2) follows from Proposition 5.14 and (2) $\implies$ (3) is trivial, it is sufficient to show that (3) $\implies$ (1). By the obvious induction, we need only show that if $n \in \mathbb{Z}^+$ and $(T_i)_{i \leq n}$ is a sequence of elements of $[\Gamma]$ for which $A \cup \bigcup_{i \leq n} T_i(A)$ is Γ-compressible, then $A \cup \bigcup_{i<n} T_i(A)$ is Γ-compressible. But if $B = A \cup \bigcup_{i<n} T_i(A)$, then $A \cup \bigcup_{i \leq n} T_n(A) \subseteq B \cup T_n(B)$, so $B \cup T_n(B)$ is Γ-compressible, thus Proposition 5.15 ensures that B is Γ-compressible. ⊠

We can now give the following:

Proof of Theorem 8. By Proposition 5.8, there exist a Δ-large Borel set $B \subseteq X$ and $n \in \mathbb{N}$ with the property that $\Gamma_{\{B\}} \restriction B \subseteq \Gamma_n \restriction B$, $[\Delta]_{\{B\}} \restriction B \subseteq (\Gamma_n \cap [\Delta]) \restriction B$, and $\Gamma_{\sim B} \subseteq (\Gamma_n)^4$. As Proposition 5.6 ensures that B can be partitioned into two Δ-large Borel sets, by replacing B with one of these sets we can assume that the set $A = {\sim}B$ is also Δ-large. Another application of Proposition 5.6 yields a partition of this set B into Δ-large Borel sets $C, D \subseteq X$.

Lemma 5.17. *There is an involution $\iota \in [\Delta]$ for which $\iota B = A \cup C$.*

Proof. By Propositions 5.13 and 5.16, there is a Δ-decomposable bijection $\pi \colon D \to A$. Then the function $\iota = \pi^{\pm 1} \cup \mathrm{id}_C$ is as desired. ⊠

By increasing n, we can assume that $\iota \in \Gamma_n$.

Lemma 5.18. *The sets $\Gamma_{\{A \cup C\}} \restriction (A \cup C)$ and $[\Delta]_{\{A \cup C\}} \restriction (A \cup C)$ are contained in $(\Gamma_n)^3 \restriction (A \cup C)$ and $(\Gamma_n \cap [\Delta])^3 \restriction (A \cup C)$.*

Proof. If $\gamma \in \Gamma_{\{A \cup C\}}$, then $\iota\gamma\iota \in \Gamma_{\{B\}}$, so there exists $\delta \in \Gamma_n$ for which $\iota\gamma\iota \upharpoonright B = \delta \upharpoonright B$, thus

$$\begin{aligned}\gamma \upharpoonright (A \cup C) &= \iota\iota\gamma\iota\iota \upharpoonright (A \cup C)\\ &= \iota(\iota\gamma\iota \upharpoonright B)\iota\\ &= \iota(\delta \upharpoonright B)\iota\\ &= \iota\delta\iota \upharpoonright (A \cup C),\end{aligned}$$

hence $\gamma \upharpoonright (A \cup C) \in (\Gamma_n)^3 \upharpoonright (A \cup C)$. Moreover, if $\gamma \in [\Delta]$, then there is such a δ in $[\Delta]$, so $\gamma \upharpoonright (A \cup C) \in (\Gamma_n \cap [\Delta])^3 \upharpoonright (A \cup C)$. ⊠

As $\Gamma_{\{A\}} \upharpoonright A \subseteq \Gamma_{\{A \cup C\}} \upharpoonright A$ and $[\Delta]_{\{A\}} \upharpoonright A \subseteq [\Delta]_{\{A \cup C\}} \upharpoonright A$, Lemma 5.18 implies the analogous fact in which $A \cup C$ is replaced with A.

It remains to show that if $\gamma \in \Gamma$, then $\gamma \in (\Gamma_n)^{14}$.

Lemma 5.19. *There is a Borel automorphism $T \in (\Gamma_n \cap [\Delta])^3$ for which $B \setminus T^{-1}(\gamma A)$ is $\langle \Delta \cup \{\gamma\}\rangle$-large.*

Proof. By Proposition 5.10, there is a partition of X into $\langle \Delta \cup \{\gamma\}\rangle$-invariant Borel sets $Y, Z \subseteq X$ for which there are $\langle \Delta \cup \{\gamma\}\rangle$-decomposable injections $\phi\colon (A \setminus \gamma A) \cap Y \to (B \setminus \gamma A) \cap Y$ and $\psi\colon (B \setminus \gamma A) \cap Z \to (A \setminus \gamma A) \cap Z$. Then the extension of $\phi^{\pm 1}$ to X supported by the union of $(A \setminus \gamma A) \cap Y$ and its image under ϕ is in $[\langle \Delta \cup \{\gamma\}\rangle]$, as is the extension of $\psi^{\pm 1}$ to X supported by the union of $(B \setminus \gamma A) \cap Z$ and its image under ψ. As B is Δ-large, it follows that $\gamma B = {\sim}\gamma A = (A \setminus \gamma A) \cup (B \setminus \gamma A)$ is $\langle \Delta \cup \{\gamma\}\rangle$-large, thus $(B \setminus \gamma A) \cap Y$ is $(\langle \Delta \cup \{\gamma\}\rangle \upharpoonright Y)$-large and $(A \setminus \gamma A) \cap Z$ is $(\langle \Delta \cup \{\gamma\}\rangle \upharpoonright Z)$-large. As A and C are Δ-large, Propositions 5.13 and 5.16 yield a Δ-decomposable bijection $\pi\colon A \cap Z \to C \cap Z$. Then $\pi^{\pm 1} \cup \mathrm{id}_{(D \cup Y)} \in [\Delta]$, so Lemma 5.18 yields $T \in (\Gamma_n \cap [\Delta])^3$ for which $T \upharpoonright (A \cup C) = \pi^{\pm 1} \cup \mathrm{id}_{(A \cup C) \cap Y}$. The fact that

$$\begin{aligned}T^{-1}((B \setminus \gamma A) \cap Y) &= T^{-1}(B \cap Y) \setminus T^{-1}(\gamma A)\\ &= (Y \setminus T^{-1}(A \cap Y)) \setminus T^{-1}(\gamma A)\\ &= (Y \setminus (A \cap Y)) \setminus T^{-1}(\gamma A)\\ &= (B \cap Y) \setminus T^{-1}(\gamma A)\\ &= (B \setminus T^{-1}(\gamma A)) \cap Y\end{aligned}$$

and

$$\begin{aligned}T^{-1}((A \setminus \gamma A) \cap Z) &= T^{-1}(A \cap Z) \setminus T^{-1}(\gamma A)\\ &= (C \cap Z) \setminus T^{-1}(\gamma A)\\ &\subseteq (B \cap Z) \setminus T^{-1}(\gamma A)\\ &= (B \setminus T^{-1}(\gamma A)) \cap Z\end{aligned}$$

ensures both that $(B \setminus T^{-1}(\gamma A)) \cap Y$ is $(\langle \Delta \cup \{\gamma\} \rangle \restriction Y)$-large and $(B \setminus T^{-1}(\gamma A)) \cap Z$ is $(\langle \Delta \cup \{\gamma\} \rangle \restriction Z)$-large, thus $B \setminus T^{-1}(\gamma A)$ is $\langle \Delta \cup \{\gamma\} \rangle$-large. ⊠

Lemma 5.20. *There is a Borel automorphism $S \in \Gamma_n$ for which $(S^{-1} \circ T^{-1})(\gamma A) \subseteq A \cup C$ and $(A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$ is $\langle \Delta \cup \{\gamma, S\} \rangle$-large.*

Proof. By Proposition 5.6, there is a partition of $B \setminus T^{-1}(\gamma A)$ into $\langle \Delta \cup \{\gamma\} \rangle$-large Borel sets $C', D' \subseteq X$. By Propositions 5.13 and 5.16, there are $\langle \Delta \cup \{\gamma\} \rangle$-decomposable bijections $\phi\colon C \to (B \cap T^{-1}(\gamma A)) \cup C'$ and $\psi\colon D \to D'$. Then $\mathrm{id}_A \cup \phi \cup \psi \in [\langle \Delta \cup \{\gamma\} \rangle] \leq \Gamma$, so there exists $S \in \Gamma_n$ for which $S \restriction B = \phi \cup \psi$, in which case

$$(S^{-1} \circ T^{-1})(\gamma A) = S^{-1}(A \cap T^{-1}(\gamma A)) \cup S^{-1}(B \cap T^{-1}(\gamma A)) \subseteq A \cup C$$

and $C' \subseteq S(C) \setminus T^{-1}(\gamma A)$, so $S(A \cup C) \setminus T^{-1}(\gamma A)$ is $\langle \Delta \cup \{\gamma\} \rangle$-large, thus $(A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$ is $\langle \Delta \cup \{\gamma, S\} \rangle$-large. ⊠

Lemma 5.21. *There exists $R \in (\Gamma_n)^3$ with $(R^{-1} \circ S^{-1} \circ T^{-1})(\gamma A) = A$.*

Proof. By Propositions 5.13 and 5.16, there are $\langle \Delta \cup \{\gamma, S\} \rangle$-decomposable bijections $\phi\colon A \to (S^{-1} \circ T^{-1})(\gamma A)$ and $\psi\colon C \to (A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$. Then $\phi \cup \psi \cup \mathrm{id}_D \in [\langle \Delta \cup \{\gamma, S\} \rangle] \leq \Gamma$, so Lemma 5.18 yields $R \in (\Gamma_n)^3$ for which $R \restriction A = \phi$, and clearly any such automorphism is as desired. ⊠

By the comment immediately following the proof of Lemma 5.18, there exists $Q \in (\Gamma_n)^3$ for which $Q \restriction A = (R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma) \restriction A$. Then $\mathrm{supp}(Q^{-1} \circ R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma) \subseteq B$, so $Q^{-1} \circ R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma \in (\Gamma_n)^4$, thus $\gamma \in (T \circ S \circ R \circ Q)(\Gamma_n)^4 \subseteq (\Gamma_n)^3 \Gamma_n (\Gamma_n)^3 (\Gamma_n)^3 (\Gamma_n)^4 = (\Gamma_n)^{14}$. ⊠

Given a set X, a partial function $d\colon X \times X \rightharpoonup \mathbb{R}$, $\epsilon > 0$, and a set $Y \subseteq X$, define $\mathcal{B}_d(Y, \epsilon) = \{x \in X \mid \exists y \in Y \ d(x, y) < \epsilon\}$. Given a binary relation R on X, define $R^{-1} = \{(y, x) \in X \times X \mid x \ R \ y\}$ and $R^{\pm 1} = R \cup R^{-1}$. Given a Borel measure μ on a Borel space X and Borel automorphisms $S, T\colon X \to X$ for which $\mathrm{supp}(S^{-1}T)$ is Borel, define

$$d_\mu(S, T) = \mu(\mathrm{supp}(S^{-1}T)) = \mu(\{x \in X \mid S(x) \neq T(x)\}).$$

Proposition 5.22. *Suppose that X is a Borel space, Γ is an aperiodic countable group of separable Borel automorphisms of X, there is a Γ-invariant Borel probability measure μ on X, $\epsilon < 1$, and $k \in \mathbb{Z}^+$. Then there is an exhaustive increasing sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of subsets of $[\Gamma]$ with the property that $\forall n \in \mathbb{N} \ [\Gamma] \neq \mathcal{B}_{d_\mu}((\Gamma_n)^k, \epsilon)$.*

Proof. As Γ is countable, there is an exhaustive increasing sequence $(\Delta_n)_{n \in \mathbb{N}}$ of finite subsets of Γ. For all $n \in \mathbb{N}$, define

$$\Gamma_n = \{\gamma \in [\Gamma] \mid \mu(\{x \in X \mid \gamma \cdot x \notin \Delta_n x\}) \leq (1 - \epsilon)/k\}.$$

The fact that $(\Delta_n)_{n \in \mathbb{N}}$ is increasing ensures that so too is $(\Gamma_n)_{n \in \mathbb{N}}$. The fact that $\mu(X) < \infty$ implies that $[\Gamma] = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

Lemma 5.23. *If $\gamma \in (\Gamma_n)^k$, then $\mu(\{x \in X \mid \gamma \cdot x \notin (\Delta_n)^k x\}) \leq 1 - \epsilon$.*

Proof. Fix $\gamma_i \in \Gamma_n$ with the property that $\gamma = \prod_{i<k} \gamma_i$. For all $j < k$, set $B_j = \{x \in X \mid (\prod_{i \leq j} \gamma_i) \cdot x \notin \Delta_n(\prod_{i<j} \gamma_i) \cdot x\}$ and note that $\mu(B_j) = \mu((\prod_{i<j} \gamma_i) B_j) = \mu(\{x \in X \mid \gamma_j \cdot x \notin \Delta_n x\}) \leq (1-\epsilon)/k$ since μ is Γ-invariant. As $\{x \in X \mid \gamma \cdot x \notin (\Delta_n)^k x\} \subseteq \bigcup_{j<k} B_j$, it follows that $\mu(\{x \in X \mid \gamma \cdot x \notin (\Delta_n)^k x\}) \leq \sum_{j<k} \mu(B_j) \leq 1 - \epsilon$. ⊠

Let G be the digraph on X with respect to which distinct points x and y are related if and only if $y \in (\Delta_n)^k x$. Then the vertical sections of $G^{\pm 1}$ are finite, so Propositions 1.8, 3.22, and 3.23 yield a Γ-decomposable equivalence relation F on X whose classes are G-independent sets of cardinality two. Let ι be the involution whose graph is $F \setminus \Delta(X)$, and appeal to Lemma 5.23 to see that if $\gamma \in (\Gamma_n)^k$, then $d_\mu(\gamma, \iota) \geq \mu(\{x \in X \mid \gamma \cdot x \in (\Delta_n)^k\}) \geq \epsilon$. ⊠

Finally, we can give the following:

Proof of Theorem 9. By Proposition 5.22, we need only show $\neg(1) \implies (2)$. As Becker–Kechris's generalization of Nadkarni's theorem (see [Nad90] and [BK96, Theorem 4.3.1]) ensures that Γ is compressible, this follows from Theorem 8. ⊠

Proof of Theorem 10. Again by Proposition 5.22, we need only show $\neg(1) \implies (2)$. As the generalization of Hopf's theorem analogous to Becker–Kechris's generalization of Nadkarni's theorem (see [Hop32] and [Nad98, §10]) yields a Γ-invariant μ-conull Borel set $B \subseteq X$ for which the corresponding group $\Gamma \restriction B$ compressible, Theorem 8 ensures that $[\Gamma \restriction B]$ has the k-Bergman property, thus so too does $[\Gamma]/\mu$. ⊠

6. Boolean algebras

For each Boolean algebra $\mathfrak{B}$, set $\mathfrak{B}_+ = \mathfrak{B} \setminus \{\mathbb{0}\}$, let $\mathbb{E}_0(\mathfrak{B})$ denote the equivalence relation on $\mathrm{Dec}(\mathfrak{B}_+^{\mathbb{N}}) = \{x \in \mathfrak{B}_+^{\mathbb{N}} \mid \forall n \in \mathbb{N}\ x(n+1) \leq x(n)\}$ given by $x \mathrel{\mathbb{E}_0(\mathfrak{B})} y \iff \exists n \in \mathbb{N} \forall m \geq n\ x(m) = y(m)$, and define $X_{\mathfrak{B}} = \mathrm{Dec}(\mathfrak{B}_+^{\mathbb{N}})/\mathbb{E}_0(\mathfrak{B})$. Note that if $b \in \mathfrak{B}$, then the set

$$\begin{aligned}\widetilde{\mathcal{N}_b} &= \{x \in \mathrm{Dec}(\mathfrak{B}_+^{\mathbb{N}}) \mid \exists n \in \mathbb{N}\ x(n) \leq b\} \\ &= \{x \in \mathrm{Dec}(\mathfrak{B}_+^{\mathbb{N}}) \mid \exists n \in \mathbb{N} \forall m \geq n\ x(m) \leq b\}\end{aligned}$$

is $\mathbb{E}_0(\mathfrak{B})$-invariant. Define $\mathcal{N}_b = \widetilde{\mathcal{N}_b}/\mathbb{E}_0(\mathfrak{B})$ for all $b \in \mathfrak{B}$.

Proposition 6.1. *Suppose that $\mathfrak{B}$ is a Boolean algebra and $b \in \mathfrak{B}$. Then $b = \mathbb{0} \iff \mathcal{N}_b = \emptyset$.*

Proof. Clearly $\mathcal{N}_{\mathbb{0}} = \emptyset$. But if $b \neq \mathbb{0}$, then the $\mathbb{E}_0(\mathfrak{B})$-class of the $\mathbb{N}$-sequence with constant value b is in $\mathcal{N}_b$, so $\mathcal{N}_b \neq \emptyset$. ⊠

Proposition 6.2. *Suppose that $\mathfrak{B}$ is a Boolean algebra and $a, b \in \mathfrak{B}$. Then $\mathcal{N}_{a \cdot b} = \mathcal{N}_a \cap \mathcal{N}_b$.*

Proof. If $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{a \cdot b}$, then there exists $n \in \mathbb{N}$ for which $x(n) \leq a \cdot b$, so $x(n) \leq a$ and $x(n) \leq b$, thus $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_a \cap \mathcal{N}_b$. Conversely, if $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_a \cap \mathcal{N}_b$, then there exist $m, n \in \mathbb{N}$ for which $x(m) \leq a$ and $x(n) \leq b$, so $x(\max\{m, n\}) \leq a \cdot b$, thus $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{a \cdot b}$. ⊠

Proposition 6.2 ensures that the sets of the form $\mathcal{N}_b$ are closed under finite intersections. Endow $X_{\mathfrak{B}}$ with the topology they generate.

Proposition 6.3. *Suppose that $\mathfrak{B}$ is a Boolean algebra and $B \subseteq \mathfrak{B}$. Then $\Sigma B = \mathbb{1}$ if and only if $\bigcup_{b \in B} \mathcal{N}_b$ is dense.*

Proof. Note first that if $a, b \in \mathfrak{B}$, then Propositions 6.1 and 6.2 ensure that $b \leq -a \iff a \cdot b = \mathbb{0} \iff \mathcal{N}_{a \cdot b} = \emptyset \iff \mathcal{N}_a \cap \mathcal{N}_b = \emptyset$. But $\Sigma B \neq \mathbb{1}$ if and only if there is a non-zero element a of $\mathfrak{B}$ such that $\forall b \in B \ b \leq -a$, whereas Propositions 6.1 and 6.2 imply that $\bigcup_{b \in B} \mathcal{N}_b$ is not dense if and only if there is a non-zero element a of $\mathfrak{B}$ such that $\forall b \in B \ \mathcal{N}_a \cap \mathcal{N}_b = \emptyset$. ⊠

We say that a topological space is a *Baire space* if countable intersections of dense open sets are dense.

Proposition 6.4. *Suppose that $\mathfrak{B}$ is a Boolean algebra. Then $X_{\mathfrak{B}}$ is a Baire space.*

Proof. By Propositions 6.1 and 6.2, it is sufficient to show that if b is a non-zero element of $\mathfrak{B}$ and $(U_n)_{n \in \mathbb{N}}$ is a sequence of dense open subsets of $X_{\mathfrak{B}}$, then $\mathcal{N}_b \cap \bigcap_{n \in \mathbb{N}} U_n \neq \emptyset$. Towards this end, set $b_0 = b$. Given $n \in \mathbb{N}$ and a non-zero element $b_n \leq b$ of $\mathfrak{B}$ for which $\mathcal{N}_{b_n} \subseteq \bigcap_{m<n} U_m$, note that $\mathcal{N}_{b_n} \cap U_n$ is a non-empty open set by Proposition 6.1, so Propositions 6.1 and 6.2 yield a non-zero element b_{n+1} of $\mathfrak{B}$ such that $\mathcal{N}_{b_{n+1}} \subseteq \mathcal{N}_{b_n} \cap U_n$, in which case $\mathcal{N}_{b_{n+1} - b_n} \subseteq \mathcal{N}_{b_{n+1}} \setminus \mathcal{N}_{b_n} = \emptyset$, thus Proposition 6.1 ensures that $b_{n+1} - b_n = \mathbb{0}$, hence $b_{n+1} \leq b_n$. It only remains to observe that $[(b_n)_{n \in \mathbb{N}}]_{\mathbb{E}_0(\mathfrak{B})} \in \bigcap_{n \in \mathbb{N}} \mathcal{N}_{b_n} \subseteq \mathcal{N}_b \cap \bigcap_{n \in \mathbb{N}} U_n$. ⊠

So as to avoid confusion with the Borel structure generated by the underlying topology, we use the term *σ-Borel* to refer to the Borel structure on each set $X \subseteq X_{\mathfrak{B}}$ generated by the sets of the form $\mathcal{N}_b \cap X$.

Proposition 6.5. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and $B \subseteq X_{\mathfrak{B}}$ is σ-Borel. Then there exists $b \in \mathfrak{B}$ with the property that $B \bigtriangleup \mathcal{N}_b$ is meager.*

Proof. By induction on the construction of B. If there exists $b \in \mathfrak{B}$ for which $B = \mathcal{N}_b$, then $B \bigtriangleup \mathcal{N}_b = \emptyset$. If there exists $a \in \mathfrak{B}$ for which $({\sim}B) \bigtriangleup \mathcal{N}_a$ is meager, then set $b = -a$ and note that $\mathcal{N}_a \subseteq {\sim}\mathcal{N}_b$, so

$$\begin{aligned} B \bigtriangleup \mathcal{N}_b &= ({\sim}B) \bigtriangleup ({\sim}\mathcal{N}_b) \\ &\subseteq (({\sim}B) \bigtriangleup \mathcal{N}_a) \cup (\mathcal{N}_a \bigtriangleup ({\sim}\mathcal{N}_b)) \\ &= (({\sim}B) \bigtriangleup \mathcal{N}_a) \cup (({\sim}\mathcal{N}_b) \setminus \mathcal{N}_a) \\ &= (({\sim}B) \bigtriangleup \mathcal{N}_a) \cup (({\sim}\mathcal{N}_b) \cap ({\sim}\mathcal{N}_a)) \\ &= (({\sim}B) \bigtriangleup \mathcal{N}_a) \cup {\sim}(\mathcal{N}_b \cup \mathcal{N}_a), \end{aligned}$$

which is meager by Proposition 6.3. If there exist σ-Borel sets $B_n \subseteq X$ and $b_n \in \mathfrak{B}$ such that $B = \bigcup_{n\in\mathbb{N}} B_n$ and $B_n \bigtriangleup \mathcal{N}_{b_n}$ is meager for all $n \in \mathbb{N}$, then set $b = \sum_{n\in\mathbb{N}} b_n$ and note that $\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n} \subseteq \mathcal{N}_b \subseteq {\sim}\mathcal{N}_{-b}$, so

$$\begin{aligned} B \bigtriangleup \mathcal{N}_b &\subseteq (B \bigtriangleup (\textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n})) \cup ((\textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n}) \bigtriangleup \mathcal{N}_b) \\ &= ((\textstyle\bigcup_{n\in\mathbb{N}} B_n) \bigtriangleup (\textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n})) \cup (\mathcal{N}_b \setminus \textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n}) \\ &\subseteq (\textstyle\bigcup_{n\in\mathbb{N}} B_n \bigtriangleup \mathcal{N}_{b_n}) \cup (\mathcal{N}_b \cap ({\sim}\textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n})) \\ &\subseteq (\textstyle\bigcup_{n\in\mathbb{N}} B_n \bigtriangleup \mathcal{N}_{b_n}) \cup (({\sim}\mathcal{N}_{-b}) \cap ({\sim}\textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n})) \\ &= (\textstyle\bigcup_{n\in\mathbb{N}} B_n \bigtriangleup \mathcal{N}_{b_n}) \cup {\sim}(\mathcal{N}_{-b} \cup \textstyle\bigcup_{n\in\mathbb{N}} \mathcal{N}_{b_n}), \end{aligned}$$

which is meager by Proposition 6.3. ⊠

Recall that a subset of a topological space is G_δ if it is a countable intersection of open sets.

Proposition 6.6. *Suppose that X is a Baire space, Γ is a countable group of homeomorphisms of X, and $Y \subseteq X$ is comeager. Then there is a Γ-invariant dense G_δ set $Z \subseteq Y$.*

Proof. Fix dense open sets $U_n \subseteq X$ for which $\bigcap_{n\in\mathbb{N}} U_n \subseteq Y$. Then the fact that γ is a homeomorphism ensures that γU_n is dense and open for all $\gamma \in \Gamma$ and $n \in \mathbb{N}$, so the Γ-invariant set $Z = \bigcap_{\gamma\in\Gamma} \bigcap_{n\in\mathbb{N}} \gamma U_n$ is G_δ and the fact that X is a Baire space implies that Z is dense. ⊠

For each order homomorphism $\phi \colon \mathfrak{A} \to \mathfrak{B}$ between Boolean algebras, define $\bar{\phi} \colon X_{\mathfrak{A}} \to X_{\mathfrak{B}}$ by $\bar{\phi}([x]_{\mathbb{E}_0(\mathfrak{A})}) = [\phi \circ x]_{\mathbb{E}_0(\mathfrak{B})}$. Given a set Φ of such homomorphisms, define $\bar{\Phi} = \{\bar{\phi} \mid \phi \in \Phi\}$.

Proposition 6.7. *Suppose that $\mathfrak{A}$ is a Boolean algebra. Then* $\overline{\mathrm{id}_{\mathfrak{A}}} = \mathrm{id}_{X_{\mathfrak{A}}}$.

Proof. If $x \in \mathrm{Dec}(\mathfrak{A}_+^{\mathbb{N}})$, then $\overline{\mathrm{id}_{\mathfrak{A}}}([x]_{\mathbb{E}_0(\mathfrak{A})}) = [\mathrm{id}_{\mathfrak{A}} \circ x]_{\mathbb{E}_0(\mathfrak{A})} = [x]_{\mathbb{E}_0(\mathfrak{A})}$. ⊠

Proposition 6.8. *Suppose that $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are Boolean algebras and $\phi \colon \mathfrak{A} \to \mathfrak{B}$ and $\psi \colon \mathfrak{B} \to \mathfrak{C}$ are order homomorphisms. Then $\overline{\psi \circ \phi} = \bar{\psi} \circ \bar{\phi}$.*

Proof. If $x \in \mathrm{Dec}(\mathfrak{A}^{\mathbb{N}}_{+})$, then $\overline{\psi \circ \phi}([x]_{\mathbb{E}_0(\mathfrak{A})}) = [\psi \circ \phi \circ x]_{\mathbb{E}_0(\mathfrak{C})}$ whereas $(\bar{\psi} \circ \bar{\phi})([x]_{\mathbb{E}_0(\mathfrak{A})}) = \bar{\psi}([\phi \circ x]_{\mathbb{E}_0(\mathfrak{B})}) = [\psi \circ \phi \circ x]_{\mathbb{E}_0(\mathfrak{C})}$. ⊠

Proposition 6.9. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and $\pi \colon \mathfrak{A} \to \mathfrak{B}$ is an isomorphism. Then $\bar{\pi}^{-1} = \overline{\pi^{-1}}$ and $\bar{\pi}(\mathcal{N}_a) = \mathcal{N}_{\pi(a)}$ for all $a \in \mathfrak{A}$, so $\bar{\pi}$ is both a homeomorphism and a σ-Borel isomorphism.*

Proof. Note first that $\bar{\pi}^{-1} = \overline{\pi^{-1}}$, since $\overline{\pi^{-1}} \circ \bar{\pi} = \overline{\pi^{-1} \circ \pi} = \overline{\mathrm{id}_{\mathfrak{A}}} = \mathrm{id}_{X_{\mathfrak{A}}}$ and $\bar{\pi} \circ \overline{\pi^{-1}} = \overline{\pi \circ \pi^{-1}} = \overline{\mathrm{id}_{\mathfrak{B}}} = \mathrm{id}_{X_{\mathfrak{B}}}$ by Propositions 6.7 and 6.8. It follows that if $a \in \mathfrak{A}$ and $y \in \mathrm{Dec}(\mathfrak{B}^{\mathbb{N}}_{+})$, then

$$\begin{aligned}
[y]_{\mathbb{E}_0(\mathfrak{B})} \in \bar{\pi}(\mathcal{N}_a) &\iff \overline{\pi^{-1}}([y]_{\mathbb{E}_0(\mathfrak{B})}) \in \mathcal{N}_a \\
&\iff [\pi^{-1} \circ y]_{\mathbb{E}_0(\mathfrak{A})} \in \mathcal{N}_a \\
&\iff \exists n \in \mathbb{N}\ (\pi^{-1} \circ y)(n) \leq a \\
&\iff \exists n \in \mathbb{N}\ y(n) \leq \pi(a) \\
&\iff [y]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{\pi(a)},
\end{aligned}$$

so $\bar{\pi}(\mathcal{N}_a) = \mathcal{N}_{\pi(a)}$ for all $a \in \mathfrak{A}$. This easily implies that $\bar{\pi}$ sends open sets to open sets. As $\bar{\pi}$ is bijective, it also ensures that $\bar{\pi}$ sends σ-Borel sets to σ-Borel sets. But the analogous fact holds of $\overline{\pi^{-1}}$ and therefore of $\bar{\pi}^{-1}$, so $\bar{\pi}$ is a homeomorphism and a σ-Borel isomorphism. ⊠

Given a countable group Γ of automorphisms of a σ-complete Boolean algebra $\mathfrak{B}$, we say that a function $\phi \colon \mathfrak{B} \to \mathfrak{B}$ is Γ-*decomposable* if there is a partition $(b_\gamma)_{\gamma \in \Gamma}$ of $\mathbb{1}$ such that $\forall b \in \mathfrak{B}\ \phi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot b_\gamma)$. Clearly every such function is an order homomorphism. As Propositions 6.7–6.9 ensure that $\bar{\Gamma}$ is a countable group of σ-Borel automorphisms of $X_{\mathfrak{B}}$, we can also consider our earlier notion of $\bar{\Gamma}$-decomposability for σ-Borel partial functions on $X_{\mathfrak{B}}$.

Proposition 6.10. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, and $\phi \colon \mathfrak{B} \to \mathfrak{B}$ is Γ-decomposable. Then there is a dense open set $U \subseteq X_{\mathfrak{B}}$ with the property that $\bar{\phi} \restriction U$ is $\bar{\Gamma}$-decomposable.*

Proof. Fix a partition $(b_\gamma)_{\gamma \in \Gamma}$ of $\mathbb{1}$ with the property that $\forall b \in \mathfrak{B}\ \phi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot b_\gamma)$. By Proposition 6.3, the open set $U = \bigcup_{\gamma \in \Gamma} \mathcal{N}_{b_\gamma}$ is dense. To see that $\bar{\phi} \restriction U = \bigcup_{\gamma \in \Gamma} \bar{\gamma} \restriction \mathcal{N}_{b_\gamma}$, note that if $\gamma \in \Gamma$ and $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{b_\gamma}$, then $(\phi \circ x)(n) = (\gamma \circ x)(n)$ for all $n \in \mathbb{N}$ sufficiently large that $x(n) \leq b_\gamma$, so $\phi \circ x\ \mathbb{E}_0(\mathfrak{B})\ \gamma \circ x$, thus $\bar{\phi}([x]_{\mathbb{E}_0(\mathfrak{B})}) = \bar{\gamma} \cdot [x]_{\mathbb{E}_0(\mathfrak{B})}$. ⊠

For all $b \in \mathfrak{B}$, endow the set $\mathfrak{B}_b = \{a \in \mathfrak{B} \mid a \leq b\}$ with the Boolean algebra structure it inherits from $\mathfrak{B}$.

Proposition 6.11. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, $(b_\gamma)_{\gamma \in \Gamma}$ is a partition of $\mathbb{1}$, and $\phi \colon \mathfrak{B} \to \mathfrak{B}$ is given by $\phi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot b_\gamma)$ for all $b \in \mathfrak{B}$. Then the following are equivalent:*

1. *The function ϕ is an isomorphism of $\mathfrak{B}$ with $\mathfrak{B}_{\phi(\mathbb{1})}$.*
2. *The function ϕ is injective.*
3. *The sequence $(\gamma b_\gamma)_{\gamma \in \Gamma}$ is a partition of $\phi(\mathbb{1})$.*

Proof. Clearly (1) $\implies$ (2). To see $\neg$(3) $\implies$ $\neg$(2), note that $\phi(\mathbb{1}) = \sum_{\gamma \in \Gamma} \gamma b_\gamma$, so if $(\gamma b_\gamma)_{\gamma \in \Gamma}$ is not a partition of $\phi(\mathbb{1})$, then there are distinct $\gamma, \delta \in \Gamma$ for which the element of $\mathfrak{B}$ given by $b = \prod_{\lambda \in \{\gamma, \delta\}} \lambda b_\lambda$ is not zero, and since $\lambda^{-1} b \leq b_\lambda$ for all $\lambda \in \{\gamma, \delta\}$, it follows that $\gamma^{-1} b$ and $\delta^{-1} b$ are distinct non-zero elements of $\mathfrak{B}$ but $\phi(\lambda^{-1} b) = \lambda \lambda^{-1} b = b$ for all $\lambda \in \{\gamma, \delta\}$, so ϕ is not injective. To see (3) $\implies$ (1), note that $\phi \restriction \mathfrak{B}_{b_\gamma} = \gamma \restriction \mathfrak{B}_{b_\gamma}$ is an isomorphism of $\mathfrak{B}_{b_\gamma}$ with $\mathfrak{B}_{\gamma b_\gamma}$ for all $\gamma \in \Gamma$, so if $(b_\gamma)_{\gamma \in \Gamma}$ and $(\gamma b_\gamma)_{\gamma \in \Gamma}$ are partitions of $\mathbb{1}$ and $\phi(\mathbb{1})$, then ϕ is an isomorphism of $\mathfrak{B}$ with $\mathfrak{B}_{\phi(\mathbb{1})}$. ⊠

The following two observations ensure that if Γ is a countable group of automorphisms of a σ-complete Boolean algebra $\mathfrak{B}$, then the set of Γ-decomposable automorphisms of $\mathfrak{B}$ is also a group.

Proposition 6.12. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, and π is a Γ-decomposable automorphism of $\mathfrak{B}$. Then π^{-1} is Γ-decomposable.*

Proof. Fix a partition $(a_\gamma)_{\gamma \in \Gamma}$ of $\mathbb{1}$ with the property that $\forall b \in \mathfrak{B}\ \pi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot a_\gamma)$ and define $b_\gamma = \gamma^{-1} a_{\gamma^{-1}}$ for all $\gamma \in \Gamma$. Proposition 6.11 ensures that $(b_\gamma)_{\gamma \in \Gamma}$ is a partition of $\mathbb{1}$. Moreover, if $b \leq b_\gamma$, then $\gamma b \leq a_{\gamma^{-1}}$, so $\pi(\gamma b) = \gamma^{-1} \gamma b = b$, thus $\pi^{-1}(b) = \gamma b$, hence if $b \in \mathfrak{B}$, then $\pi^{-1}(b) = \sum_{\gamma \in \Gamma} \pi^{-1}(b \cdot b_\gamma) = \sum_{\gamma \in \Gamma} \gamma(b \cdot b_\gamma)$. ⊠

Proposition 6.13. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, and $\phi, \psi \colon \mathfrak{B} \to \mathfrak{B}$ are Γ-decomposable. Then $\phi \circ \psi$ is Γ-decomposable.*

Proof. Fix partitions $(c_\gamma)_{\gamma \in \Gamma}$ and $(d_\delta)_{\delta \in \Gamma}$ of $\mathbb{1}$ with the property that $\forall b \in \mathfrak{B}\ \phi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot c_\gamma)$ and $\forall b \in \mathfrak{B}\ \psi(b) = \sum_{\delta \in \Gamma} \delta(b \cdot d_\delta)$ and define $b_{\gamma,\delta} = (\delta^{-1} c_\gamma) \cdot d_\delta$ for all $\gamma, \delta \in \Gamma$ and $b_\lambda = \sum_{\gamma \in \Gamma} b_{\gamma, \gamma^{-1}\lambda}$ for all $\lambda \in \Gamma$. As $(c_\gamma)_{\gamma \in \Gamma}$ is a partition of $\mathbb{1}$, so too is $(\delta^{-1} c_\gamma)_{\gamma \in \Gamma}$ for all $\delta \in \Gamma$, thus $(b_{\gamma,\delta})_{\gamma \in \Gamma}$ is a partition of d_δ for all $\delta \in \Gamma$. As $(d_\delta)_{\delta \in \Gamma}$ is a partition of $\mathbb{1}$, so too is $(b_{\gamma,\delta})_{\gamma,\delta \in \Gamma}$. As $(\{(\gamma, \gamma^{-1}\lambda) \mid \gamma \in \Gamma\})_{\lambda \in \Gamma}$ is a partition of $\Gamma \times \Gamma$, it follows that $(b_\lambda)_{\lambda\in\Gamma}$ is a partition of $\mathbb{1}$. Moreover, if $b \leq b_{\gamma,\delta}$, then $\delta b \leq c_\gamma$ and $b \leq d_\delta$, hence $(\phi\circ\psi)(b) = \phi(\delta b) = \gamma\delta b$, thus if $b \leq b_\lambda$, then

$$\begin{aligned}(\phi\circ\psi)(b) &= \textstyle\sum_{\gamma\in\Gamma}(\phi\circ\psi)(b\cdot b_{\gamma,\gamma^{-1}\lambda})\\ &= \textstyle\sum_{\gamma\in\Gamma}\gamma\gamma^{-1}\lambda(b\cdot b_{\gamma,\gamma^{-1}\lambda})\\ &= \lambda(b\cdot b_\lambda),\end{aligned}$$

so if $b \in \mathfrak{B}$, then $(\phi\circ\psi)(b) = \sum_{\lambda\in\Gamma}(\phi\circ\psi)(b\cdot b_\lambda) = \sum_{\lambda\in\Gamma}\lambda(b\cdot b_\lambda)$. ⊠

The *full group* of a countable group Γ of automorphisms of a σ-complete Boolean algebra $\mathfrak{B}$ is the group $[\Gamma]$ of Γ-decomposable automorphisms of $\mathfrak{B}$. In the special case that there is a single automorphism π of $\mathfrak{B}$ that generates Γ, we define the *full group* of π to be the full group of Γ. We also use $[\pi]$ to denote $[\Gamma]$.

Proposition 6.14. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, and $\pi \in [\Gamma]$. Then there is a $\bar{\Gamma}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\pi} \restriction X \in [\bar{\Gamma} \restriction X]$.*

Proof. By Proposition 6.10, there is a dense open set $U \subseteq X_{\mathfrak{B}}$ for which $\bar{\pi} \restriction U$ is $\bar{\Gamma}$-decomposable. As Proposition 6.9 ensures that each element of $\bar{\Gamma}$ is a homeomorphism, Proposition 6.6 yields a $\bar{\Gamma}$-invariant dense G_δ set $X \subseteq U$. As X is $\bar{\pi}$-invariant and Proposition 6.9 also implies that $\bar{\pi}$ is a σ-Borel automorphism, it follows that $\bar{\pi} \restriction X \in [\bar{\Gamma} \restriction X]$. ⊠

In order to generalize the converse of Proposition 6.14, we will need:

Proposition 6.15. *Suppose that $\mathfrak{B}$ is a Boolean algebra, a and b are elements of $\mathfrak{B}$, A and B are subsets of $X_{\mathfrak{B}}$ for which $A \mathbin{\triangle} \mathcal{N}_a$ and $B \mathbin{\triangle} \mathcal{N}_b$ are meager, and ϕ and ψ are automorphisms of $\mathfrak{B}$.*

(1) *If $\phi(a)\cdot\psi(b) = \mathbb{0}$, then $\bar{\phi}(\mathcal{N}_a) \cap \bar{\psi}(\mathcal{N}_b) = \emptyset$.*
(2) *If $\bar{\phi}(A) \cap \bar{\psi}(B) = \emptyset$, then $\phi(a)\cdot\psi(b) = \mathbb{0}$.*

Proof. To establish (1), appeal to Propositions 6.1, 6.2, and 6.9 to see that $\bar{\phi}(\mathcal{N}_a) \cap \bar{\psi}(\mathcal{N}_b) = \mathcal{N}_{\phi(a)} \cap \mathcal{N}_{\psi(b)} = \mathcal{N}_{\phi(a)\cdot\psi(b)} = \mathcal{N}_{\mathbb{0}} = \emptyset$.

To establish (2), first apply Propositions 6.2 and 6.9 to see that

$$\begin{aligned}\mathcal{N}_{\phi(a)\cdot\psi(b)} &= \mathcal{N}_{\phi(a)} \cap \mathcal{N}_{\psi(b)}\\ &= \bar{\phi}(\mathcal{N}_a) \cap \bar{\psi}(\mathcal{N}_b)\\ &\subseteq (\bar{\phi}(A) \mathbin{\triangle} \bar{\phi}(\mathcal{N}_a)) \cup (\bar{\psi}(B) \mathbin{\triangle} \bar{\psi}(\mathcal{N}_b))\\ &= \bar{\phi}(A \mathbin{\triangle} \mathcal{N}_a) \cup \bar{\psi}(B \mathbin{\triangle} \mathcal{N}_b).\end{aligned}$$

As Proposition 6.9 ensures that $\bar{\phi}$ and $\bar{\psi}$ are homeomorphisms, it follows that the last set is meager, so Proposition 6.4 implies that the first is empty, thus Proposition 6.1 ensures that $\phi(a)\cdot\psi(b) = \mathbb{0}$. ⊠

Given a property P of elements of a topological space X, we write $\forall^* x \in X\ P(x)$ to indicate that $\{x \in X \mid P(x)\}$ is comeager.

Proposition 6.16. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, $X \subseteq X_{\mathfrak{B}}$ is a $\bar{\Gamma}$-invariant comeager set, and $T \in [\bar{\Gamma} \upharpoonright X]$. Then there exists $\pi \in [\Gamma]$ such that $\forall^* x \in X_{\mathfrak{B}}\ \bar{\pi}(x) = T(x)$.*

Proof. Fix a partition $(B_\gamma)_{\gamma \in \Gamma}$ of X into σ-Borel subsets of X for which $T = \bigcup_{\gamma \in \Gamma} \bar{\gamma} \upharpoonright B_\gamma$. For each $\gamma \in \Gamma$, Proposition 6.5 yields $b_\gamma \in \mathfrak{B}$ such that $B_\gamma \mathbin{\triangle} \mathcal{N}_{b_\gamma}$ is meager. As Proposition 6.7 ensures that $\overline{\mathrm{id}_{\mathfrak{B}}} = \mathrm{id}_{X_{\mathfrak{B}}}$, Proposition 6.15 implies that $(b_\gamma)_{\gamma \in \Gamma}$ and $(\gamma b_\gamma)_{\gamma \in \Gamma}$ are sequences of pairwise disjoint elements of $\mathfrak{B}$. As $X = \bigcup_{\gamma \in \Gamma} B_\gamma = \bigcup_{\gamma \in \Gamma} \bar{\gamma} B_\gamma$ and Proposition 6.9 ensures that each element of $\bar{\Gamma}$ is a homeomorphism, it follows that $\bigcup_{\gamma \in \Gamma} \mathcal{N}_{b_\gamma}$ and $\bigcup_{\gamma \in \Gamma} \bar{\gamma} \mathcal{N}_{b_\gamma}$ are comeager, thus so too is $\bigcup_{\gamma \in \Gamma} \mathcal{N}_{\gamma b_\gamma}$ by another application of Proposition 6.9. Proposition 6.4 therefore implies that $\bigcup_{\gamma \in \Gamma} \mathcal{N}_{b_\gamma}$ and $\bigcup_{\gamma \in \Gamma} \mathcal{N}_{\gamma b_\gamma}$ are dense, so Proposition 6.3 ensures that $(b_\gamma)_{\gamma \in \Gamma}$ and $(\gamma b_\gamma)_{\gamma \in \Gamma}$ are partitions of $\mathbb{1}$, thus Proposition 6.11 implies that we obtain an element of $[\Gamma]$ by setting $\pi(b) = \sum_{\gamma \in \Gamma} \gamma(b \cdot b_\gamma)$ for all $b \in \mathfrak{B}$. But the set $Y = \bigcup_{\gamma \in \Gamma} B_\gamma \cap \mathcal{N}_{b_\gamma}$ is comeager and if $[y]_{\mathbb{E}_0(\mathfrak{B})} \in Y$, then there exists $\gamma \in \Gamma$ for which $[y]_{\mathbb{E}_0(\mathfrak{B})} \in B_\gamma \cap \mathcal{N}_{b_\gamma}$, and if $n \in \mathbb{N}$ is sufficiently large that $y(n) \leq b_\gamma$, then $(\pi \circ y)(n) = (\gamma \circ y)(n)$, so $\pi \circ y\ \mathbb{E}_0(\mathfrak{B})\ \gamma \circ y$, in which case $\bar{\pi}([y]_{\mathbb{E}_0(\mathfrak{B})}) = \bar{\gamma} \cdot [y]_{\mathbb{E}_0(\mathfrak{B})} = T([y]_{\mathbb{E}_0(\mathfrak{B})})$, thus $\bar{\pi} \upharpoonright Y = T \upharpoonright Y$. ⊠

We next turn to a basic observation concerning Boolean algebras:

Proposition 6.17. *Suppose that $\mathfrak{B}$ is a Boolean algebra, $b \in \mathfrak{B}$, and ϕ and ψ are automorphisms of $\mathfrak{B}$ for which $\phi(b) \neq \psi(b)$. Then there is a non-zero element $a \leq b$ of $\mathfrak{B}$ with the property that $\phi(a) \cdot \psi(a) = \mathbb{0}$.*

Proof. As at least one of $\phi(b) - \psi(b)$ and $\psi(b) - \phi(b)$ is not zero, the same holds of the elements of $\mathfrak{B}$ given by

$$c = \phi^{-1}(\phi(b) - \psi(b)) = b - (\phi^{-1} \circ \psi)(b)$$

and

$$d = \psi^{-1}(\psi(b) - \phi(b)) = b - (\psi^{-1} \circ \phi)(b),$$

so it only remains to note that $\phi(c) \cdot \psi(c) \leq (\phi(b) - \psi(b)) \cdot \psi(b) = \mathbb{0}$ and $\phi(d) \cdot \psi(d) \leq \phi(b) \cdot (\psi(b) - \phi(b)) = \mathbb{0}$, thus c or d is as desired. ⊠

The following corollary ensures that the automorphism satisfying the conclusion of Proposition 6.16 is uniquely determined:

Proposition 6.18. *Suppose that $\mathfrak{B}$ is a Boolean algebra and ϕ and ψ are automorphisms of $\mathfrak{B}$ with the property that $\forall^* x \in X_{\mathfrak{B}}\ \bar{\phi}(x) = \bar{\psi}(x)$. Then $\phi = \psi$.*

Proof. If there exists $b \in \mathfrak{B}$ for which $\phi(b) \neq \psi(b)$, then Proposition 6.17 yields a non-zero element $a \leq b$ of $\mathfrak{B}$ with the property that $\phi(a) \cdot \psi(a) = \mathbb{0}$, so Proposition 6.15 ensures that $\bar{\phi}(\mathcal{N}_a) \cap \bar{\psi}(\mathcal{N}_a) = \emptyset$, thus $\bar{\phi}(x) \neq \bar{\psi}(x)$ for all $x \in \mathcal{N}_a$, contradicting Proposition 6.4. ⊠

Given an automorphism π of $\mathfrak{B}$, we say that an element b of $\mathfrak{B}$ is *π-independent* if $b \cdot \pi(b) = \mathbb{0}$. We say that an automorphism π of a σ-complete Boolean algebra $\mathfrak{B}$ is *separable* if there are countable sets $B_k \subseteq \mathfrak{B}$ of π^k-independent elements of $\mathfrak{B}$ with the property that $\forall k \in \mathbb{Z}^+ \forall a \leq -\Sigma B_k \ a = \pi^k(a)$.

Proposition 6.19. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and π is a separable automorphism of $\mathfrak{B}$. Then there is a $\bar{\pi}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\pi} \restriction X$ is a separable σ-Borel automorphism.*

Proof. Proposition 6.9 ensures that $\bar{\pi}$ is a σ-Borel automorphism, thus so too is its restriction to any $\bar{\pi}$-invariant set. For all $k \in \mathbb{Z}^+$, fix a countable set $B_k \subseteq \mathfrak{B}$ of π^k-independent elements of $\mathfrak{B}$ with the property that $a = \pi^k(a)$ for all $a \leq -\Sigma B_k$. As Proposition 6.7 implies that $\overline{\mathrm{id}_{\mathfrak{B}}} = \mathrm{id}_{X_{\mathfrak{B}}}$, Proposition 6.15 ensures that each set in the family $\mathcal{B}_k = \{\mathcal{N}_b \mid b \in B_k\}$ is $\bar{\pi}^k$-independent. As Proposition 6.3 implies that the open set $U_k = \mathcal{N}_{-\Sigma B_k} \cup \bigcup \mathcal{B}_k$ is dense, it follows that $\bigcap_{k \in \mathbb{Z}^+} U_k$ is comeager. As Proposition 6.9 ensures that $\bar{\pi}$ is a homeomorphism, Proposition 6.6 yields a $\bar{\pi}$-invariant dense G_δ set $X \subseteq \bigcap_{k \in \mathbb{Z}^+} U_k$. But if $k \in \mathbb{Z}^+$ and $[x]_{\mathbb{E}_0(\mathfrak{B})} \in X \setminus \bigcup \mathcal{B}_k$, then $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{-\Sigma B_k}$, and if $n \in \mathbb{N}$ is sufficiently large that $x(n) \leq -\Sigma B_k$, then $x(n) = (\pi^k \circ x)(n)$, so $x \ \mathbb{E}_0(\mathfrak{B}) \ \pi^k \circ x$, thus $[x]_{\mathbb{E}_0(\mathfrak{B})} = \bar{\pi}^k([x]_{\mathbb{E}_0(\mathfrak{B})})$. ⊠

Proposition 6.20. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, π is an automorphism of $\mathfrak{B}$, and there is a $\bar{\pi}$-invariant comeager set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\pi} \restriction X$ is a separable σ-Borel automorphism. Then π is separable.*

Proof. For all $k \in \mathbb{Z}^+$, Proposition 1.8 ensures that the support of $\bar{\pi}^k \restriction X$ is a σ-Borel subset of X, so Proposition 1.14 implies that it is the union of a countable set $\mathcal{B}_k$ of $\bar{\pi}^k$-independent σ-Borel subsets of X. By Proposition 6.5, there is a function $\phi_k \colon \mathcal{B}_k \to \mathfrak{B}$ with the property that $B \mathbin{\triangle} \mathcal{N}_{\phi_k(B)}$ is meager for all $B \in \mathcal{B}_k$. As Proposition 6.7 ensures that $\overline{\mathrm{id}_{\mathfrak{B}}} = \mathrm{id}_{X_{\mathfrak{B}}}$, Proposition 6.15 implies that the elements of the set $B_k = \phi_k(\mathcal{B}_k)$ are π^k-independent. Suppose, towards a contradiction, that there is a non-zero element $c \leq -\Sigma B_k$ of $\mathfrak{B}$ for which $c \neq \pi^k(c)$. Proposition 6.17 then yields a non-zero π^k-independent element $d \leq c$ of $\mathfrak{B}$. As $\overline{\mathrm{id}_{\mathfrak{B}}} = \mathrm{id}_{X_{\mathfrak{B}}}$, Proposition 6.15 ensures that $\mathcal{N}_d$

is $\bar{\pi}^k$-independent and therefore contained in the support of $\bar{\pi}^k$. As Proposition 6.4 implies that $\mathcal{N}_d$ is not meager, there exists $B \in \mathcal{B}_k$ for which $B \cap \mathcal{N}_d$ is not meager. Setting $b = \phi_k(B)$, it follows that $\mathcal{N}_b \cap \mathcal{N}_d$ is not meager, so Proposition 6.2 ensures that $\mathcal{N}_{b \cdot d}$ is not meager, thus Proposition 6.1 implies that $b \cdot d$ is not zero, contradicting the fact that $d \leq -\Sigma B_k \leq -b$. ⊠

Given a non-zero element b of $\mathfrak{B}$ and a property P of elements of $\mathfrak{B}$, we write $\forall^{\mathfrak{B}} a \leq b\ P(a)$ to indicate that $\forall \mathbb{0} < a' \leq b \exists \mathbb{0} < a \leq a'\ P(a)$.

Proposition 6.21. *Suppose that $\mathfrak{B}$ is a Boolean algebra, b is a non-zero element of $\mathfrak{B}$, $k \in \mathbb{Z}^+$, and $(\pi_i)_{i<k}$ is a sequence of automorphisms of $\mathfrak{B}$ such that $\forall i < j < k \forall^{\mathfrak{B}} a \leq b\ \pi_i(a) \neq \pi_j(a)$. Then there is a non-zero element $a \leq b$ of $\mathfrak{B}$ such that $\forall i < j < k\ \pi_i(a) \cdot \pi_j(a) = \mathbb{0}$.*

Proof. Fix an enumeration $(i_n, j_n)_{n<k(k-1)/2}$ of $\{(i,j) \in k \times k \mid i < j\}$, set $a'_0 = b$, and given $n < k(k-1)/2$ and a non-zero element $a'_n \leq b$ of $\mathfrak{B}$, fix a non-zero element $a_n \leq a'_n$ of $\mathfrak{B}$ for which $\pi_{i_n}(a_n) \neq \pi_{j_n}(a_n)$ and appeal to Proposition 6.17 to obtain a non-zero element $a'_{n+1} \leq a_n$ of $\mathfrak{B}$ such that $\pi_{i_n}(a'_{n+1}) \cdot \pi_{j_n}(a'_{n+1}) = \mathbb{0}$. Then $a'_{k(k-1)/2}$ is as desired. ⊠

We say that a countable group Γ of automorphisms of $\mathfrak{B}$ is *aperiodic* on a non-zero element b of $\mathfrak{B}$ if

$$\forall k \in \mathbb{Z}^+ \forall^{\mathfrak{B}} c \leq b \exists (\gamma_i)_{i<k} \in \Gamma^k \forall i < j < k \forall^{\mathfrak{B}} d \leq c\ \gamma_i d \neq \gamma_j d.$$

When $b = \mathbb{1}$, we also say that Γ is *aperiodic*. For all $k \in \mathbb{N}$, the *period $\geq k$ part* of a countable group Γ of permutations of a set X is given by $\mathrm{Per}_{\geq k}(\Gamma) = \{x \in X \mid |\Gamma x| \geq k\}$.

Proposition 6.22. *Suppose that $\mathfrak{B}$ is a Boolean algebra, b is a non-zero element of $\mathfrak{B}$, and Γ is a countable group of automorphisms of $\mathfrak{B}$ that is aperiodic on b. Then the aperiodic part of $\bar{\Gamma}$ is comeager in $\mathcal{N}_b$.*

Proof. It is sufficient to show that if $k \in \mathbb{Z}^+$, then the period $\geq k$ part of $\bar{\Gamma}$ contains a dense open subset of $\mathcal{N}_b$. But if $c' \leq b$ is a non-zero element of $\mathfrak{B}$, then the aperiodicity of Γ on b yields a non-zero element $c \leq c'$ of $\mathfrak{B}$ and $(\gamma_i)_{i<k} \in \Gamma^k$ such that $\forall i < j < k \forall^{\mathfrak{B}} d \leq c\ \gamma_i d \neq \gamma_j d$, so Proposition 6.21 gives rise to a non-zero element $d \leq c$ of $\mathfrak{B}$ such that $\forall i < j < k\ (\gamma_i d) \cdot (\gamma_j d) = \mathbb{0}$, and if $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_d$ and $n \in \mathbb{N}$ is sufficiently large that $x(n) \leq d$, then $(\gamma_i \circ x)(n) \cdot (\gamma_j \circ x)(n) = \mathbb{0}$ for all $i < j < k$, so $\neg \gamma_i \circ x\ \mathbb{E}_0(\mathfrak{B})\ \gamma_j \circ x$ for all $i < j < k$, thus $\overline{\gamma_i} \cdot [x]_{\mathbb{E}_0(\mathfrak{B})} \neq \overline{\gamma_j} \cdot [x]_{\mathbb{E}_0(\mathfrak{B})}$ for all $i < j < k$, hence $\mathcal{N}_d \subseteq \mathrm{Per}_{\geq k}(\bar{\Gamma})$. ⊠

We say that an automorphism π of $\mathfrak{B}$ is *aperiodic* on a non-zero element b of $\mathfrak{B}$ if $\forall k \in \mathbb{Z}^+ \forall^{\mathfrak{B}} a \leq b\ a \neq \pi^k(a)$. When $b = \mathbb{1}$, we also say that π is *aperiodic*.

Proposition 6.23. *Suppose that $\mathfrak{B}$ is a Boolean algebra, b is a non-zero element of $\mathfrak{B}$, and π is an automorphism of $\mathfrak{B}$. Then π is aperiodic on b if and only if $\langle \pi \rangle$ is aperiodic on b.*

Proof. Suppose first that π is aperiodic on b. To see that $\langle \pi \rangle$ is aperiodic on b, suppose that $k \in \mathbb{Z}^+$ and $c' \leq b$ is a non-zero element of $\mathfrak{B}$. Then the aperiodicity of π on b ensures that $\forall^{\mathfrak{B}} c \leq c' \; c \neq \pi^i(c)$ for all $0 < i < k$, so $\forall^{\mathfrak{B}} c \leq c' \; \pi^i(c) \neq \pi^j(c)$ for all $i < j < k$, thus Proposition 6.21 yields a non-zero element $c \leq c'$ of $\mathfrak{B}$ such that $\forall i < j < k \; \pi^i(c) \cdot \pi^j(c) = \mathbb{0}$. Setting $\gamma_i = \pi^i$ for all $i < k$, it follows that $\gamma_i d \neq \gamma_j d$ for all non-zero elements $d \leq c$ of $\mathfrak{B}$ and $i < j < k$.

Suppose now that $\langle \pi \rangle$ is aperiodic on b. To see that π is aperiodic on b, suppose that $k \in \mathbb{Z}^+$ and $a' \leq b$ is a non-zero element of $\mathfrak{B}$, appeal to the aperiodicity of $\langle \pi \rangle$ on b to obtain a non-zero element $a'' \leq a'$ of $\mathfrak{B}$ and $(n_i)_{i<k+1} \in \mathbb{Z}^{k+1}$ such that $\forall i < j < k+1 \forall^{\mathfrak{B}} a \leq a'' \; \pi^{n_i}(a) \neq \pi^{n_j}(a)$, and fix $i < j < k+1$ for which $n_j - n_i$ is divisible by n, as well as an element $a \leq a''$ of $\mathfrak{B}$ with the property that $\pi^{n_i}(a) \neq \pi^{n_j}(a)$. Then $a \neq \pi^{n_j - n_i}(a)$, so $a \neq \pi^n(a)$. ⊠

Given a non-zero element b of $\mathfrak{B}$ and a property P of elements of $\mathfrak{B}$, we write $\exists^{\mathfrak{B}} a \leq b \; P(a)$ to indicate that $\exists \mathbb{0} < a' \leq b \forall \mathbb{0} < a \leq a' \; P(a)$. We say that π is *periodic* if $\forall b > \mathbb{0} \exists k \in \mathbb{Z}^+ \exists^{\mathfrak{B}} a \leq b \; a = \pi^k(a)$, or equivalently, if π is not aperiodic on any non-zero element of $\mathfrak{B}$.

Proposition 6.24. *Suppose that $\mathfrak{B}$ is a Boolean algebra and π is a periodic automorphism of $\mathfrak{B}$. Then the periodic part of $\bar{\pi}$ contains a dense open set.*

Proof. It is sufficient to note that if b is a non-zero element of $\mathfrak{B}$, then the periodicity of π yields $k \in \mathbb{Z}^+$ and a non-zero element $a' \leq b$ of $\mathfrak{B}$ such that $a = \pi^k(a)$ for all $a \leq a'$, and if $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{a'}$ and $n \in \mathbb{N}$ is sufficiently large that $x(n) \leq a'$, then $x(n) = (\pi^k \circ x)(n)$, so $x \; \mathbb{E}_0(\mathfrak{B}) \; \pi^k \circ x$, thus $[x]_{\mathbb{E}_0(\mathfrak{B})} = \bar{\pi}^k([x]_{\mathbb{E}_0(\mathfrak{B})})$, hence $\mathcal{N}_{a'} \subseteq \mathrm{Per}(\bar{\pi})$. ⊠

Suppose now that $K \subseteq \mathbb{Z}^+$. We say that π has *strict period* K if $\forall b > \mathbb{0} \exists k \in K \exists^{\mathfrak{B}} a \leq b \; k = \min\{i \in \mathbb{Z}^+ \mid a = \pi^i(a)\}$.

Proposition 6.25. *Suppose that $\mathfrak{B}$ is a Boolean algebra, $K \subseteq \mathbb{Z}^+$ is finite, and π is an automorphism of $\mathfrak{B}$ that has strict period K. Then $\pi^{\mathrm{lcm}(K)} = \mathrm{id}_{\mathfrak{B}}$.*

Proof. If there exists $b \in \mathfrak{B}$ for which $b \neq \pi^{\mathrm{lcm}(K)}(b)$, then Proposition 6.17 yields a non-zero $\pi^{\mathrm{lcm}(K)}$-independent element $c \leq b$ of $\mathfrak{B}$, but the fact that π has strict period K yields a non-zero element $d \leq c$ of $\mathfrak{B}$ for which $d = \pi^{\mathrm{lcm}(K)}(d)$, contradicting the fact that d is also $\pi^{\mathrm{lcm}(K)}$-independent. ⊠

Proposition 6.26. *Suppose that $\mathfrak{B}$ is a Boolean algebra and π is a periodic automorphism of $\mathfrak{B}$. Then π has strict period $\mathbb{Z}^+$.*

Proof. If b is a non-zero element of $\mathfrak{B}$, then the periodicity of π yields $k \in \mathbb{Z}^+$ for which there is a non-zero element $a' \leq b$ of $\mathfrak{B}$ such that $a = \pi^k(a)$ for all $a \leq a'$. Let k be the least such positive integer and fix such an a'. Then $\forall^{\mathfrak{B}} a \leq a'\ a \neq \pi^i(a)$ for all $0 < i < k$, so $\forall^{\mathfrak{B}} a \leq a'\ \pi^i(a) \neq \pi^j(a)$ for all $i < j < k$, thus Proposition 6.21 yields a non-zero element $a'' \leq a'$ of $\mathfrak{B}$ such that $\forall i < j < k\ \pi^i(a'') \cdot \pi^j(a'') = \mathbb{0}$, in which case $k = \min\{i \in \mathbb{Z}^+ \mid a = \pi^i(a)\}$ for all non-zero elements $a \leq a''$ of $\mathfrak{B}$. ⊠

Proposition 6.27. *Suppose that $\mathfrak{B}$ is a Boolean algebra, $K \subseteq \mathbb{Z}^+$, and π is an automorphism of $\mathfrak{B}$ for which the period K part of $\bar{\pi}$ is comeager. Then π has strict period K.*

Proof. Note first that π cannot be aperiodic on any non-zero element b of $\mathfrak{B}$, since otherwise Propositions 6.22 and 6.23 ensure that the aperiodic part of $\bar{\pi}$ is comeager in $\mathcal{N}_b$, contradicting Proposition 6.4. Proposition 6.26 therefore implies that π has strict period $\mathbb{Z}^+$, so we need only show that if a' is a non-zero element of $\mathfrak{B}$ and $k \in \mathbb{Z}^+$ has the property that $\forall \mathbb{0} < a \leq a'\ k = \min\{i \in \mathbb{Z}^+ \mid a = \pi^i(a)\}$, then $k \in K$. But if $[x]_{\mathbb{E}_0(\mathfrak{B})} \in \mathcal{N}_{a'}$ and $n \in \mathbb{N}$ is sufficiently large that $x(n) \leq a'$, then $k = \min\{i \in \mathbb{Z}^+ \mid x(n) = (\pi^i \circ x)(n)\}$, so

$$\begin{aligned} k &= \min\{i \in \mathbb{Z}^+ \mid x\ \mathbb{E}_0(\mathfrak{B})\ \pi^i \circ x\} \\ &= \min\{i \in \mathbb{Z}^+ \mid [x]_{\mathbb{E}_0(\mathfrak{B})} = \bar{\pi}^i([x]_{\mathbb{E}_0(\mathfrak{B})})\}, \end{aligned}$$

thus $\mathcal{N}_{a'} \subseteq \mathrm{Per}_k(\bar{\pi})$, hence Proposition 6.4 ensures that $k \in K$. ⊠

Given a countable group Γ of automorphisms of a σ-complete Boolean algebra $\mathfrak{B}$, we say that an element b of $\mathfrak{B}$ is Γ-*complete* if $\mathbb{1} = \sum_{\gamma \in \Gamma} \gamma b$. A Γ-*compression* is a Γ-decomposable injection $\phi \colon \mathfrak{B} \to \mathfrak{B}$ for which $-\phi(\mathbb{1})$ is Γ-complete. We say that Γ is *compressible* if there is such a Γ-compression.

Proposition 6.28. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is a countable group of automorphisms of $\mathfrak{B}$, and ϕ is a Γ-compression. Then there is a $\bar{\Gamma}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\phi} \restriction X$ is a $\bar{\Gamma}$-compression of X.*

Proof. By Proposition 6.10, there is a dense open set $U \subseteq X_{\mathfrak{B}}$ for which $\bar{\phi} \restriction U$ is Γ-decomposable. As $\bar{\phi}(X_{\mathfrak{B}}) \subseteq \mathcal{N}_{\phi(\mathbb{1})} \subseteq {\sim}\mathcal{N}_{-\phi(\mathbb{1})}$, Proposition 6.9 ensures that $\bigcup_{\gamma \in \Gamma} \mathcal{N}_{\gamma(-\phi(\mathbb{1}))} = \bigcup_{\gamma \in \Gamma} \bar{\gamma} \mathcal{N}_{-\phi(\mathbb{1})} \subseteq \bigcup_{\gamma \in \Gamma} \bar{\gamma}({\sim}\bar{\phi}(X_{\mathfrak{B}}))$. As Proposition 6.3 implies that the first of these sets is dense, the last is comeager. As Proposition 6.9 ensures that each element of $\bar{\Gamma}$ is a homeomorphism, Proposition 6.6 yields a $\bar{\Gamma}$-invariant dense G_δ set $X \subseteq U \cap \bigcup_{\gamma \in \Gamma} \bar{\gamma}(\sim\bar{\phi}(X_{\mathfrak{B}}))$. As Proposition 6.11 implies that ϕ is an isomorphism of $\mathfrak{B}$ with $\mathfrak{B}_{\phi(\mathbb{1})}$, Proposition 6.9 ensures that $\bar{\phi}$ is injective, so to see that $\bar{\phi} \restriction X$ is a $\bar{\Gamma}$-compression of X, it only remains to note that $X \subseteq \bigcup_{\gamma \in \Gamma} \bar{\gamma}(\sim\bar{\phi}(X_{\mathfrak{B}})) \subseteq \bigcup_{\gamma \in \Gamma} \bar{\gamma}(\sim\bar{\phi}(X))$. ⊠

An *involution* of $\mathfrak{B}$ is an automorphism π of $\mathfrak{B}$ for which $\pi^2 = \mathrm{id}_{\mathfrak{B}}$.

Theorem 6.29. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and π is a separable automorphism of $\mathfrak{B}$. Then there are involutions $\iota_1, \iota_2, \iota_3 \in [\pi]$ for which $\pi = \iota_3 \circ \iota_2 \circ \iota_1$.*

Proof. By Proposition 6.19, there is a $\bar{\pi}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\pi} \restriction X$ is a separable σ-Borel automorphism. By Theorem 1, there are involutions $I_1, I_2, I_3 \in [\bar{\pi} \restriction X]$ with the property that $\bar{\pi} \restriction X = I_3 \circ I_2 \circ I_1$. By Propositions 6.6, 6.9, and 6.16, there exist $\iota_1, \iota_2, \iota_3 \in [\pi]$ and a $\bar{\pi}$-invariant dense G_δ set $Y \subseteq X$ with the property that $\forall 1 \leq i \leq 3$ $\overline{\iota_i} \restriction Y = I_i \restriction Y$. Then $\overline{\iota_i}^2(y) = I_i^2(y) = y$ and $\bar{\pi}(y) = (I_3 \circ I_2 \circ I_1)(y) = (\overline{\iota_3} \circ \overline{\iota_2} \circ \overline{\iota_1})(y)$ for all $1 \leq i \leq 3$ and $y \in Y$, so ι_i is an involution for all $1 \leq i \leq 3$ and $\pi = \iota_3 \circ \iota_2 \circ \iota_1$ by Propositions 6.8 and 6.18. ⊠

Theorem 6.30. *Suppose that $k \geq 2$, $\mathfrak{B}$ is a σ-complete Boolean algebra, π is an aperiodic automorphism of $\mathfrak{B}$, and there are periodic automorphisms $\phi_1, \ldots, \phi_k \in [\pi]$ with the property that $\pi = \phi_k \circ \cdots \circ \phi_1$. Then π is separable.*

Proof. By Propositions 6.6, 6.9, 6.14, 6.22, 6.23, and 6.24, there is a $\bar{\pi}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ such that $\bar{\pi} \restriction X$ is an aperiodic σ-Borel automorphism and $\forall 1 \leq i \leq k$ $\overline{\phi_i} \restriction X$ is a periodic element of $[\pi \restriction X]$. As Proposition 6.8 ensures that $\bar{\pi} = \overline{\phi_k} \circ \cdots \circ \overline{\phi_1}$, Theorem 2 implies that $\bar{\pi} \restriction X$ is separable, thus so too is π by Proposition 6.20. ⊠

Theorem 6.31. *Suppose that $k_1 \geq 2$, $k_2 \geq 3$, $\mathfrak{B}$ is a σ-complete Boolean algebra, and π is an aperiodic separable automorphism of $\mathfrak{B}$. Then there exist $\phi_1, \phi_2 \in [\pi]$ such that ϕ_1 has strict period $\{k_1\}$, ϕ_2 has strict period $\{1, k_2\}$, and $\pi = \phi_2 \circ \phi_1$.*

Proof. By Propositions 6.6, 6.9, 6.19, 6.22, and 6.23, there is a $\bar{\pi}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ such that $\bar{\pi} \restriction X$ is an aperiodic separable σ-Borel automorphism. By Theorem 3, there exist $S_1, S_2 \in [\bar{\pi} \restriction X]$ for which every orbit of S_1 has cardinality k_1, every orbit of S_2 has cardinality 1 or k_2, and $\bar{\pi} \restriction X = S_2 \circ S_1$. By Propositions 6.6, 6.9, and 6.16, there exist $\phi_1, \phi_2 \in [\pi]$ and a $\bar{\pi}$-invariant dense G_δ set $Y \subseteq X$ such that $\forall 1 \leq i \leq 2$ $\overline{\phi_i} \restriction Y = S_i \restriction Y$. Then $|[y]_{\overline{\phi_1}}| = |[y]_{S_1}| = k_1$,

$|[y]_{\overline{\phi_2}}| = |[y]_{S_2}| \in \{1, k_2\}$, and $\bar{\pi}(y) = (S_2 \circ S_1)(y) = (\overline{\phi_2} \circ \overline{\phi_1})(y)$ for all $y \in Y$, so ϕ_1 and ϕ_2 have strict periods $\{k_1\}$ and $\{1, k_2\}$ by Proposition 6.27 and $\pi = \phi_2 \circ \phi_1$ by Propositions 6.8 and 6.18. ⊠

Theorem 6.32. *Suppose that $k \geq 3$, $\mathfrak{B}$ is a σ-complete Boolean algebra, and π is an aperiodic separable automorphism of $\mathfrak{B}$. Then there exist $\phi_1, \phi_2 \in [\pi]$ such that ϕ_1^{-1} and ϕ_2 are conjugate in $[\pi]$, ϕ_1 and ϕ_2 have strict period $\{1, k\}$, and $\pi = \phi_2 \circ \phi_1$.*

Proof. By Propositions 6.6, 6.9, 6.19, 6.22, and 6.23, there is a $\bar{\pi}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ with the property that $\bar{\pi} \restriction X$ is an aperiodic separable σ-Borel automorphism. By Theorem 4, there exist $S, S_1, S_2 \in [\bar{\pi} \restriction X]$ with the property that every orbit of S_1 and S_2 has cardinality 1 or k, $S_1^{-1} = S \circ S_2 \circ S^{-1}$, and $\bar{\pi} \restriction X = S_2 \circ S_1$. By Propositions 6.6, 6.9, and 6.16, there exist $\phi, \phi_1, \phi_2 \in [\pi]$ and a $\bar{\pi}$-invariant dense G_δ set $Y \subseteq X$ such that $\bar{\phi} \restriction Y = S \restriction Y$ and $\forall 1 \leq i \leq 2\ \overline{\phi_i} \restriction Y = S_i \restriction Y$. Then $|[y]_{\overline{\phi_i}}| = |[y]_{S_i}| \in \{1, k\}$ for all $1 \leq i \leq 2$, $\overline{\phi_1}^{-1}(y) = S_1^{-1}(y) = (S \circ S_2 \circ S^{-1})(y) = (\bar{\phi} \circ \overline{\phi_2} \circ \bar{\phi}^{-1})(y)$, and $\bar{\pi}(y) = (S_2 \circ S_1)(y) = (\overline{\phi_2} \circ \overline{\phi_1})(y)$ for all $y \in Y$, so ϕ_i has strict period $\{1, k\}$ for all $1 \leq i \leq 2$ by Proposition 6.27 and $\phi_1^{-1} = \phi \circ \phi_2 \circ \phi^{-1}$ and $\pi = \phi_2 \circ \phi_1$ by Propositions 6.8 and 6.18. ⊠

Theorem 6.33. *Suppose that $k \geq 3$, $\mathfrak{B}$ is a σ-complete Boolean algebra, Γ is an aperiodic countable group of separable automorphisms of $\mathfrak{B}$, and $\pi \in [\Gamma]$. Then there exist $\phi_1, \phi_2 \in [\pi]$ such that ϕ_1^{-1} and ϕ_2 are conjugate in $[\Gamma]$, ϕ_1 and ϕ_2 have strict period $\{1, 2, k\}$, and $\pi = \phi_2 \circ \phi_1$.*

Proof. By Propositions 6.6, 6.9, 6.14, 6.19, and 6.22, there is a $\bar{\Gamma}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ such that $\bar{\Gamma} \restriction X$ is an aperiodic group of separable σ-Borel automorphisms and $\bar{\pi} \restriction X \in [\bar{\Gamma} \restriction X]$. Theorem 4 yields $S \in [\bar{\Gamma} \restriction X]$ and $S_1, S_2 \in [\bar{\pi} \restriction X]$ for which every orbit of S_1 and S_2 has cardinality 1, 2, or k, $S_1^{-1} = S \circ S_2 \circ S^{-1}$, and $\bar{\pi} \restriction X = S_2 \circ S_1$. By Propositions 6.6, 6.9, and 6.16, there exist $\phi \in [\Gamma]$, $\phi_1, \phi_2 \in [\pi]$, and a $\bar{\Gamma}$-invariant dense G_δ set $Y \subseteq X$ with the property that $\bar{\phi} \restriction Y = S \restriction Y$ and $\forall 1 \leq i \leq 2\ \overline{\phi_i} \restriction Y = S_i \restriction Y$. Then $|[y]_{\overline{\phi_i}}| = |[y]_{S_i}| \in \{1, 2, k\}$ for all $1 \leq i \leq 2$, $\overline{\phi_1}^{-1}(y) = S_1^{-1}(y) = (S \circ S_2 \circ S^{-1})(y) = (\bar{\phi} \circ \overline{\phi_2} \circ \bar{\phi}^{-1})(y)$, and $\bar{\pi}(y) = (S_2 \circ S_1)(y) = (\overline{\phi_2} \circ \overline{\phi_1})(y)$ for all $y \in Y$, so ϕ_i has strict period $\{1, 2, k\}$ for all $1 \leq i \leq 2$ by Proposition 6.27 and $\phi_1^{-1} = \phi \circ \phi_2 \circ \phi^{-1}$ and $\pi = \phi_2 \circ \phi_1$ by Propositions 6.8 and 6.18. ⊠

We say that a group Γ of automorphisms of a σ-complete Boolean algebra is *closed under countable decomposition* if $[\Delta] \subseteq \Gamma$ for every countable subgroup Δ of Γ.

Theorem 6.34. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and Γ is a group of separable automorphisms of $\mathfrak{B}$ that is closed under countable decomposition and has an aperiodic countable subgroup Δ. Then Γ has the Bergman property.*

Proof. We first reduce the theorem to the special case where $\Gamma = [\Delta]$. Suppose, towards a contradiction, that Γ does not have the Bergman property. Then there is an exhaustive increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of Γ with the property that there exists $\gamma_n \in \Gamma \setminus (\Gamma_n)^n$ for all $n \in \mathbb{N}$. Let Δ' be the group generated by $\{\gamma_n \mid n \in \mathbb{N}\} \cup \Delta$ and define $\Gamma' = [\Delta']$ and $\Gamma'_n = \Gamma' \cap \Gamma_n$ for all $n \in \mathbb{N}$. Then $\gamma_n \in \Gamma' \setminus (\Gamma'_n)^n$ for all $n \in \mathbb{N}$, so Γ' does not have the Bergman property, contradicting the aforementioned special case of the theorem.

To establish the special case, suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of Γ and appeal to Propositions 6.6, 6.9, 6.19, and 6.22 to obtain a $\bar{\Delta}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\Delta} \restriction X$ is an aperiodic group of separable σ-Borel automorphisms. For all $n \in \mathbb{N}$, let Γ'_n be the set of σ-Borel automorphisms of X that agree with an element of $\overline{\Gamma_n}$ on a comeager set. Proposition 6.16 ensures that $(\Gamma'_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of $[\bar{\Delta} \restriction X]$, so Theorem 7 yields $k \in \mathbb{N}$ for which $[\bar{\Delta} \restriction X] = (\Gamma'_k)^k$. To see that $\Gamma = (\Gamma_k)^k$, suppose that $\gamma \in \Gamma$, apply Proposition 6.14 to obtain $T \in [\bar{\Delta} \restriction X]$ with the property that $\forall^* x \in X_{\mathfrak{B}}\ \bar{\gamma} \cdot x = T(x)$, and fix $T_1, \ldots, T_k \in \Gamma'_k$ for which $T = T_k \circ \cdots \circ T_1$, as well as $\gamma_1, \ldots, \gamma_k \in \Gamma_k$ with the property that $\forall^* x \in X_{\mathfrak{B}}\ \overline{\gamma_i} \cdot x = T_i(x)$ for all $1 \le i \le k$. By Propositions 6.6 and 6.9, there is a $\bar{\Delta}$-invariant dense G_δ set $Y \subseteq X$ such that $\bar{\gamma} \restriction Y = T \restriction Y$ and $\forall 1 \le i \le k\ \overline{\gamma_i} \restriction Y = T_i \restriction Y$, so $\bar{\gamma} \restriction Y = T \restriction Y = (T_k \circ \cdots \circ T_1) \restriction Y = (\overline{\gamma_k} \cdots \overline{\gamma_1}) \restriction Y$, thus $\gamma = \gamma_k \cdots \gamma_1 \in (\Gamma_k)^k$ by Propositions 6.8 and 6.18. ⊠

Theorem 6.35. *Suppose that $\mathfrak{B}$ is a σ-complete Boolean algebra and Γ is a group of separable automorphisms of $\mathfrak{B}$ that is closed under countable decomposition and has a compressible countable subgroup Δ. Then Γ has the 14-Bergman property.*

Proof. We first reduce the theorem to the special case where $\Gamma = [\Delta]$. Suppose, towards a contradiction, that Γ does not have the 14-Bergman property. Then there is an exhaustive increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of Γ with the property that there exists $\gamma_n \in \Gamma \setminus (\Gamma_n)^{14}$ for all $n \in \mathbb{N}$. Let Δ' be the group generated by $\{\gamma_n \mid n \in \mathbb{N}\} \cup \Delta$ and define $\Gamma' = [\Delta']$ and $\Gamma'_n = \Gamma' \cap \Gamma_n$ for all $n \in \mathbb{N}$. Then $\gamma_n \in \Gamma' \setminus (\Gamma'_n)^{14}$ for all $n \in \mathbb{N}$, so Γ' does not have the 14-Bergman property, contradicting the aforementioned special case of the theorem.

To establish the special case, suppose that $(\Gamma_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of Γ and appeal to Propositions 6.6, 6.9, 6.19, and 6.28 to obtain a $\bar{\Delta}$-invariant dense G_δ set $X \subseteq X_{\mathfrak{B}}$ for which $\bar{\Delta} \restriction X$ is an aperiodic group of separable σ-Borel automorphisms. For all $n \in \mathbb{N}$, let Γ'_n be the set of σ-Borel automorphisms of X that agree with an element of $\overline{\Gamma_n}$ on a comeager set. Proposition 6.16 ensures that $(\Gamma'_n)_{n\in\mathbb{N}}$ is an exhaustive increasing sequence of subsets of $[\bar{\Delta} \restriction X]$, so Theorem 8 yields $n \in \mathbb{N}$ for which $[\bar{\Delta} \restriction X] = (\Gamma'_n)^{14}$. To see that $\Gamma = (\Gamma_n)^{14}$, suppose that $\gamma \in \Gamma$, apply Proposition 6.14 to obtain $T \in [\bar{\Delta} \restriction X]$ with the property that $\forall^* x \in X_{\mathfrak{B}}\ \bar{\gamma} \cdot x = T(x)$, and fix $T_1, \ldots, T_{14} \in \Gamma'_n$ for which $T = T_{14} \circ \cdots \circ T_1$, as well as $\gamma_1, \ldots, \gamma_{14} \in \Gamma_n$ with the property that $\forall^* x \in X_{\mathfrak{B}}\ \overline{\gamma_i} \cdot x = T_i(x)$ for all $1 \leq i \leq 14$. By Propositions 6.6 and 6.9, there is a $\bar{\Delta}$-invariant dense G_δ set $Y \subseteq X$ such that $\bar{\gamma} \restriction Y = T \restriction Y$ and $\forall 1 \leq i \leq 14\ \overline{\gamma_i} \restriction Y = T_i \restriction Y$, so $\bar{\gamma} \restriction Y = T \restriction Y = (T_{14} \circ \cdots \circ T_1) \restriction Y = (\overline{\gamma_{14}} \cdots \overline{\gamma_1}) \restriction Y$, thus $\gamma = \gamma_{14} \cdots \gamma_1 \in (\Gamma_n)^{14}$ by Propositions 6.8 and 6.18. ⊠

We close this section with the observation from [Fre04, §382M] that all automorphisms of complete Boolean algebras are separable in the presence of full choice, thereby eliminating the need for separability in the hypotheses of the special cases of Theorems 6.29 and 6.31–6.35 for complete Boolean algebras under `AC`. The first of these simplified special cases was originally established in [Ryz93].

Proposition 6.36 (`AC`)**.** *Suppose that* $\mathfrak{B}$ *is a complete Boolean algebra. Then every automorphism of* $\mathfrak{B}$ *is separable.*

Proof. We will establish the ostensibly stronger fact that if π is an automorphism of $\mathfrak{B}$, then there is a π-independent element b of $\mathfrak{B}$ with the property that $a = \pi(a)$ for all elements $a \leq -(\pi^{-1}(b) + b + \pi(b))$ of $\mathfrak{B}$. Towards this end, note that if $(b_\alpha)_{\alpha<\gamma}$ is a strictly increasing sequence of π-independent elements of $\mathfrak{B}$, then the element of $\mathfrak{B}$ given by $b = \sum_{\alpha<\gamma} b_\alpha$ is itself π-independent, for if $b \cdot \pi(b) \neq \mathbb{0}$, then there exist $\alpha < \gamma$ such that $b_\alpha \cdot \pi(b) \neq \mathbb{0}$ and $\beta < \gamma$ such that $b_\alpha \cdot \pi(b_\beta) \neq \mathbb{0}$, contradicting the fact that $b_{\max\{\alpha,\beta\}}$ is π-independent. It follows that there is a maximal such sequence, in which case the corresponding ordinal γ is a successor, so the corresponding element b of $\mathfrak{B}$ is $b_{\gamma-1}$. But if there is an element $c \leq -(\pi^{-1}(b) + b + \pi(b))$ of $\mathfrak{B}$ for which $c \neq \pi(c)$, then Proposition 6.17 yields a non-zero π-independent element $d \leq c$ of $\mathfrak{B}$, and since $b \cdot \pi(d) = \pi(\pi^{-1}(b) \cdot d)$, it follows that $(b+d) \cdot \pi(b+d) = (b \cdot \pi(b)) + (b \cdot \pi(d)) + (d \cdot \pi(b)) + (d \cdot \pi(d)) = \mathbb{0}$, so $b+d$ is π-independent, contradicting the maximality of $(b_\alpha)_{\alpha<\gamma}$. ⊠

Acknowledgements. I would like to thank David Fremlin for introducing me to the problem of writing automorphisms as compositions of three involutions (well before either of us were aware of [Ryz93]). As the work presented here originated in my dissertation, I would also like to thank my PhD advisors, Alexander Kechris and John Steel.

References

[Ber06] G. M. Bergman, *Generating infinite symmetric groups*, Bull. London Math. Soc. **38** (2006), no. 3, 429–440. MR 2239037

[BK96] H. Becker and A. S. Kechris, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996. MR 1425877

[DG05] M. Droste and R. Göbel, *Uncountable cofinalities of permutation groups*, J. London Math. Soc. (2) **71** (2005), no. 2, 335–344. MR 2122432

[Fre04] D. H. Fremlin, *Measure theory. Vol. 3*, Torres Fremlin, Colchester, 2004, Measure algebras, Corrected second printing of the 2002 original. MR 2459668

[Hop32] E. Hopf, *Theory of measure and invariant integrals*, Trans. Amer. Math. Soc. **34** (1932), no. 2, 373–393. MR 1501643

[Kec95] A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)

[KST99] A. S. Kechris, S. J. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Advances in Mathematics **141** (1999), no. 1, 1–44. MR 1667145

[Mil04] B. D. Miller, *Full groups, classification, and equivalence relations*, ProQuest LLC, Ann Arbor, MI, 2004, Thesis (Ph.D.)–University of California, Berkeley. MR 2706636

[Nad90] M. G. Nadkarni, *On the existence of a finite invariant measure*, Proc. Indian Acad. Sci. Math. Sci. **100** (1990), no. 3, 203–220. MR 1081705

[Nad98] ———, *Basic ergodic theory*, second ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 1998. MR 1725389

[Ryz85] V. V. Ryzhikov, *Representation of transformations preserving the Lebesgue measure, in the form of a product of periodic transformations*, Mat. Zametki **38** (1985), no. 6, 860–865, 957. MR 823423

[Ryz93] ———, *Factorization of an automorphism of a full Boolean algebra into the product of three involutions*, Mat. Zametki **54** (1993), no. 2, 79–84, 159. MR 1244984

[Sho90] R. M. Shortt, *Normal subgroups of measurable automorphisms*, Fund. Math. **135** (1990), no. 3, 177–187. MR 1077509

[Sil80] J. H. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, Ann. Math. Logic **18** (1980), no. 1, 1–28.

Benjamin D. Miller, 10520 Boardwalk Loop #702, Lakewood Ranch, FL 34202

Email address: `glimmeffros@gmail.com`

URL: `https://glimmeffros.github.io`