A CHARACTERIZATION OF THE EXISTENCE OF INVARIANT MEASURES, II: FROM INVOLUTIONS TO AUTOMORPHISMS

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To Alexander Kechris, on the occasion of his formal retirement from Caltech.

ABSTRACT. We show that an aperiodic countable equivalence relation fails to admit an invariant probability measure exactly when every conjugacy class generating its full group does so in boundedly many steps.

INTRODUCTION

This paper is a continuation of [Milb], to which we refer the reader for basic definitions and notation. Our primary goal is to show:

Theorem 1. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and $n \geq 18$. Then exactly one of the following holds:*

1. *There is an E-invariant Borel probability measure.*
2. $\forall T \in [E]\ ([E] = \langle \mathrm{Cl}_{[E]}(T) \rangle \implies [E] = \mathrm{Cl}_{[E]}(T)^n)$.

Theorem 2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-conservative E-quasi-invariant Borel probability measure on X, and $n \geq 12$. Then exactly one of the following holds:*

1. *There is an E-invariant Borel probability measure $\nu \ll \mu$.*
2. $\forall T \in [E]\ ([E] = [\langle \mathrm{Cl}_{[E]}(T) \rangle]_{\equiv_\mu} \implies [E] = [\mathrm{Cl}_{[E]}(T)^n]_{\equiv_\mu})$.

We obtain Theorems 1 and 2 from known results and:

Theorem 3. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and the support of $T \in [E]$ is E-large. Then there is an involution $I \in \mathrm{Cl}_{[E]}(T)^2$ whose support is E-large.*

In §1, we prove a pair of representation results for periodic automorphisms. In §2, we establish Theorems 1–3.

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1. Periodic automorphisms as products

We begin this section by providing a proof of the following well-known fact for the reader's convenience:

Proposition 1. *Suppose that $n > 0$. Then there are involutions $\iota_1, \iota_2 \colon \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ with the property that $\iota_2(0) \equiv 0 \pmod{n}$ and $(\iota_1 \circ \iota_2)(k) \equiv 1 + k \pmod{n}$ for all $k \in \mathbb{Z}/n\mathbb{Z}$.*

Proof. For all $k \in \mathbb{Z}/n\mathbb{Z}$, define

$$\iota_1(k) \equiv 1 - k \pmod{n} \text{ and } \iota_2(k) \equiv -k \pmod{n}.$$

Clearly $\iota_2(0) \equiv -0 \pmod{n} \equiv 0 \pmod{n}$. To see that ι_1 and ι_2 are involutions, note that $\iota_1^2(k) \equiv \iota_1(1-k) \pmod{n} \equiv 1-(1-k) \pmod{n} \equiv k \pmod{n}$ and $\iota_2^2(k) \equiv \iota_2(-k) \pmod{n} \equiv -(-k) \pmod{n} \equiv k \pmod{n}$. So it only remains to observe that $(\iota_1 \circ \iota_2)(k) \equiv \iota_1(-k) \pmod{n} \equiv 1 - (-k) \pmod{n} \equiv 1 + k \pmod{n}$. □

In particular, this yields the following corollary:

Proposition 2. *Suppose that $n > m > 0$. Then there are involutions $\iota_1 \colon n \to n$ and $\iota_2 \colon n-m \to n-m$ with the property that $(0 \ \cdots \ n-1) = \iota_1 \circ (\iota_2 \cup (n-m \ \cdots \ n-1))$.*

Proof. By Proposition 1, there are involutions $\iota_1', \iota_2' \colon \{0, \ldots, n-m\} \to \{0, \ldots, n-m\}$ such that $\iota_2'(n-m) = n-m$ and $\iota_1' \circ \iota_2' = (0 \ \cdots \ n-m)$. Set $\iota_1 = \iota_1' \cup \mathrm{id}_{\{n-m+1,\ldots,n-1\}}$, $\iota_2 = \iota_2' \restriction (n-m)$, and $\sigma = \iota_2 \cup (n-m \ \cdots \ n-1)$. Then $k < n-m \implies (\iota_1 \circ \sigma)(k) = (\iota_1' \circ \iota_2')(k) = k+1$, $n-m \leq k < n-1 \implies (\iota_1 \circ \sigma)(k) = \mathrm{id}_{\{n-m+1,\ldots,n-1\}}(k+1) = k+1$, and $(\iota_1 \circ \sigma)(n-1) = \iota_1(n-m) = (\iota_1' \circ \iota_2')(n-m) = 0$. □

We now turn towards Borel automorphisms:

Proposition 3. *Suppose that $n > m > 0$, X is a standard Borel space, and $T \colon X \to X$ is a Borel automorphism whose orbits all have cardinality n. Then there exist $I, S \in [T]$ such that I is an involution, the restriction of S to each orbit of T is the disjoint union of an involution and an m-cycle, and $T = I \circ S$.*

Proof. By Proposition 2, there are permutations $\iota, \sigma \colon n \to n$ such that ι is an involution, σ is the disjoint union of an involution and an m-cycle, and $(0 \ \cdots \ n-1) = \iota \circ \sigma$.

Lemma 4. *There is a Borel function $\phi \colon X \to n$ such that $\phi \restriction [x]_T$ is a conjugacy of $T \restriction [x]_T$ with $(0 \ \cdots \ n-1)$ for all $x \in X$.*

Proof. Fix a Borel transversal $B \subseteq X$ of T and define $\phi(T^i(x)) = i$ for all $i < n$ and $x \in X$. □

Define $R(x) = ((\phi \restriction [x]_T)^{-1} \circ \rho \circ \phi)(x)$ for all $(R, \rho) \in \{(I, \iota), (S, \sigma)\}$ and $x \in X$. Then $\phi \restriction [x]_T$ is a conjugacy of $I \restriction [x]_T$, $S \restriction [x]_T$, and $T \restriction [x]_T$ with ι, σ, and $(0 \ \cdots \ n-1)$, so $I \restriction [x]_T$ is an involution, $S \restriction [x]_T$ is the disjoint union of an involution and an m-cycle, and $T \restriction [x]_T = (I \circ S) \restriction [x]_T$. □

By passing to an aperiodic equivalence relation, we obtain greater control over the cardinalities of the orbits of the composants:

Proposition 5. *Suppose that X is standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $k_i \geq 2$ for all $i < 3$, and every orbit of $T_0 \in [E]$ has cardinality k_0. Then there exist $T_1, T_2 \in [E]$ such that every orbit of T_i has cardinality k_i for all $i \in \{1, 2\}$ and $T_0 = T_1 \circ T_2$.*

Proof. By [Mil00], there are a finite group G and $g_0, g_1, g_2 \in G$ such that g_i has order k_i for all $i < 3$ and $g_0 = g_1 g_2$. Define $U_i \colon G \to G$ by $U_i(g) = g_i g$ for all $g \in G$ and $i < 3$. By [KM04, Proposition 7.4], there is a finite Borel subequivalence relation F of E whose classes are all T_0-invariant and have cardinality $|G|$.

Lemma 6. *There is a Borel function $\phi \colon X \to G$ such that $\phi \restriction [x]_F$ is a conjugacy of $T_0 \restriction [x]_F$ with U_0 for all $x \in X$.*

Proof. Fix Borel transversals $A \subseteq G$ of U_0 and $B \subseteq X$ of F. Then there is a Borel function $\psi \colon A \times B \to X$ such that $x \mathrel{F} \psi(g, x)$ for all $g \in A$ and $x \in B$ and $\psi(A \times B)$ is a transversal of T_0. Define $\phi \colon X \to G$ by $(\phi \circ T_0^k \circ \psi)(g, x) = U_0^k(g)$ for all $g \in A$, $x \in B$, and $k < k_0$. □

For all $i \in \{1, 2\}$ and $x \in X$, define $T_i(x) = ((\phi \restriction [x]_F)^{-1} \circ U_i \circ \phi)(x)$. Then $\phi \restriction [x]_F$ is a conjugacy of $T_i \restriction [x]_F$ with U_i, so every orbit of $T_i \restriction [x]_F$ has cardinality k_i and $T_0 \restriction [x]_F = (T_1 \circ T_2) \restriction [x]_F$. □

2. Main results

The *aperiodic part* of an equivalence relation E on X is given by

$$\mathrm{Aper}(E) = \{x \in X \mid [x]_E \text{ is infinite}\}.$$

We say that a set $Y \subseteq X$ is E-*aperiodic* if $Y = \mathrm{Aper}(E \restriction Y)$.

Theorem 7. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and the support of $T \in [E]$ is E-aperiodic. Then there is an involution $I \in [E]$ for which $(I \circ T)^2$ is an involution and* $\mathrm{supp}(T) \preccurlyeq_E 2\mathrm{supp}((I \circ T)^2)$.

Proof. For all $n \geq 2$, define $B_n = \{x \in X \mid |[x]_T| = n\}$ and $A_n = \mathrm{Aper}(E \upharpoonright B_n)$. By Proposition 5, there is an involution $I_n \in [E \upharpoonright A_n]$ such that every orbit of $I_n \circ (T \upharpoonright A_n)$ has cardinality four.

Define $A_\infty = \mathrm{Aper}(E_T^X)$. By [Mila, Theorem 3], there is an involution $I_\infty \in [E \upharpoonright A_\infty]$ such that every orbit of $I_\infty \circ (T \upharpoonright A_\infty)$ has cardinality one or four. As the support of $I_\infty \circ (T \upharpoonright A_\infty)$ intersects every I_∞-orbit, it follows that $\mathrm{supp}(T \upharpoonright A_\infty) \preccurlyeq_E 2\mathrm{supp}(I_\infty \circ (T \upharpoonright A_\infty))$.

We now handle the case that E is smooth, for which it is sufficient to find an involution $I \in [E]$ such that $(I \circ T)^2$ is an involution whose support is E-aperiodic and has the same E-saturation as the support of T. For all $n \geq 5$, Proposition 3 yields an involution $I_n' \in [E \upharpoonright B_n]$ such that the restriction of $I_n' \circ (T \upharpoonright B_n)$ to each orbit of $T \upharpoonright B_n$ is a disjoint union of an involution and a four-cycle. Set $B = {\sim}(\bigcup_{2 \leq n \leq 4} A_n \cup \bigcup_{n \geq 5} B_n \cup A_\infty)$. Then [Mila, Proposition 1.1] gives rise to an involution $I' \in [E \upharpoonright B]$ for which $I' \circ (T \upharpoonright B)$ is an involution, in which case the involution $I = \bigcup_{2 \leq n \leq 4} I_n \cup \bigcup_{n \geq 5} I_n' \cup I_\infty \cup I'$ is as desired.

For the general case, note that the above special case allows us to assume that $A_n = B_n$ for all $n \geq 2$. Then the trivial extension I of $\bigcup_{n \geq 2} I_n \cup I_\infty$ to X is as desired. □

Remark 8. For all $k \geq 3$, the same idea can be used to produce an involution $I \in [E]$ for which every orbit of $(I \circ T)^2$ has cardinality 1 or k and $\mathrm{supp}(T) \preccurlyeq_E 2\mathrm{supp}((I \circ T)^2)$.

Remark 9. For all $J \in [E]$ whose orbits all have cardinality 1 or k and for which $2\mathrm{supp}(J) \preccurlyeq_E \mathrm{supp}(T)$, a technical modification of the proof that every aperiodic Borel automorphism is the composition of an involution and an automorphism whose orbits all have cardinality 1 or $2k$ (see [Mila, Theorem 3]) can be used to ensure that $(I \circ T)^2$ is conjugate to J off of an E-invariant Borel set on which E is smooth.

Remark 10. One can simplify the above proof and obtain the generalization where $\mathrm{supp}(J) \preccurlyeq_E \mathrm{supp}(T)$ by employing the (unpublished) fact that if T is aperiodic, then there exist $I, S \in [T]$ such that I is an involution, every orbit of S has cardinality $2k$, and $T = I \circ S$.

The following fact and [Milb, Theorem 1] easily yield Theorem 1:

Theorem 11. *Suppose that X is a standard Borel space, E is a compressible countable Borel equivalence relation on X, and the support of $T \in [E]$ is E-large. Then $[E] \subseteq \mathrm{Cl}_{[E]}(T)^{18}$.*

Proof. By Theorem 7, there is an involution $I \in [E]$ for which $(I \circ T)^2$ is an involution whose support is E-large. As the proof of [Mila, Theorem 14] ensures that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}((I \circ T)^2)^3 \subseteq \mathrm{Cl}_{[E]}(T)^6$, the fact that every Borel automorphism is the composition of three involutions in its full group (see, for example, [Mila, Theorem 1]) allows us to conclude that $[E] \subseteq \mathrm{Inv}([E])^3 \subseteq \mathrm{Cl}_{[E]}(T)^{18}$. □

The following fact and [Milb, Theorem 2] easily yield Theorem 2:

Theorem 12. *Suppose that $n \geq 1$, X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $S, T \in [E]$, $\mathrm{supp}(S) \preccurlyeq_E n\mathrm{supp}(T)$, and the support of T is E-aperiodic. Then $S \in [\mathrm{Cl}_{[E]}(T)^{12n}]_{\equiv_E}$.*

Proof. By Theorem 7, there is an involution $I \in [E]$ with the property that $(I \circ T)^2$ is an involution and $\mathrm{supp}(T) \preccurlyeq_E 2\mathrm{supp}((I \circ T)^2)$, so $\mathrm{supp}(S) \preccurlyeq_E 2n\mathrm{supp}((I \circ T)^2)$. By [Mila, Theorem 1], there are involutions $I_1, I_2, I_3 \in [T]$ for which $S = I_1 \circ I_2 \circ I_3$. As $\mathrm{supp}(I_m) \subseteq \mathrm{supp}(S)$ for all $m \in \{1, 2, 3\}$, it follows from [Milb, Theorem 12] that $I_1, I_2, I_3 \in [\mathrm{Cl}_{[E]}((I \circ T)^2)^{2n}]_{\equiv_E} \subseteq [\mathrm{Cl}_{[E]}(T)^{4n}]_{\equiv_E}$, thus $S \in [\mathrm{Cl}_{[E]}(T)^{12n}]_{\equiv_E}$. □

Remark 13. One can show that $\mathrm{Cl}_{[E]}(T)^4 = [\mathrm{Cl}_{[E]}(T)^4]_{\equiv_E}$ by combining the fact that every permutation of $\mathbb{N}$ is a product of four conjugates of any permutation with infinite support (see [Ber73]) with the (unpublished) generalization of [Milb, Proposition 9] to automorphisms. This allows us to conclude that $S \in \mathrm{Cl}_{[E]}(T)^{12n}$ in Theorem 12, which yields the generalization of Theorem 1 where $n \geq 12$.

Remark 14. By employing Remark 8 and using the fact that every aperiodic Borel automorphism is the composition of two automorphisms in its full group whose orbits all have cardinality 1 or k in place of the fact that every Borel automorphism is the composition of three involutions in its full group and the generalization of [Milb, Theorem 12] to automorphisms whose orbits all have cardinality 1 or k, one can further strengthen the conclusion of Theorem 12 to $S \in \mathrm{Cl}_{[E]}(T)^{8n}$, which yields the generalizations of Theorems 1 and 2 where $n \geq 8$.

Remark 15. By employing Remark 9, one can further strengthen the conclusion of Theorem 12 to $S \in \mathrm{Cl}_{[E]}(T)^{4n}$, which yields the further generalizations of Theorems 1 and 2 where $n \geq 4$.

Remark 16. By employing Remark 10, one can further strengthen the conclusion of Theorem 12 to $S \in \mathrm{Cl}_{[E]}(T)^{2n}$ when $n \geq 2$.

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