A GENERALIZATION OF THE DYE–KRIEGER THEOREM

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Abstract. We show that if a countable Borel equivalence relation is in the closure of the class of all smooth Borel equivalence relations under countable increasing union and countable intersection, then it is measure hyperfinite.

Partially order the set $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ by extension and let $s \frown t$ denote the concatenation of sequences $s, t \in \mathbb{N}^{<\mathbb{N}}$. A *tree* on $\mathbb{N}$ is a non-empty set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ that is closed under restriction. For all partial functions $f \colon T \rightharpoonup \mathbb{N}$, let T_f denote the tree on $\mathbb{N}$ consisting of all $t \in T$ for which $t(i) = f(t \upharpoonright i)$ whenever $i \in \mathrm{dom}(t)$ and $t \upharpoonright i \in \mathrm{dom}(f)$.

A tree T on $\mathbb{N}$ is *well-founded* if every element of $\mathbb{N}^{\mathbb{N}}$ has a maximal restriction in T. Let ∂T denote the set of all maximal elements of T. We say that T is *fully branching* if $t \frown (n) \in T$ for all $n \in \mathbb{N}$ and $t \in T \setminus \partial T$. The *pruning rank* of such a tree is defined via transfinite recursion by $\rho(\{\emptyset\}) = 0$ and $\rho(T) = \sup\{\rho(T_n)+1 \mid n \in \mathbb{N}\}$ if $T \neq \{\emptyset\}$, where $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid (n) \frown t \in T\}$ for all $n \in \mathbb{N}$.

Let $\bigcup^{\uparrow}$ denote increasing union and $\mathcal{P}(X)$ the family of all subsets of a set X. For our purposes here, a *parse tree* on X is a triple (F, O, T), where T is a well-founded fully branching tree on $\mathbb{N}$, $F \colon T \to \mathcal{P}(X)$, $O \colon T \setminus \partial T \to \{\bigcup^{\uparrow}, \bigcap\}$, and $F(t) = O(t)_{n \in \mathbb{N}} F(t \frown (n))$ for all $t \in T \setminus \partial T$.

Proposition 1. *Suppose that X is a set, (F, O, T) is a parse tree on X, and $f \colon O^{-1}(\bigcup^{\uparrow}) \to \mathbb{N}$. Then $\bigcap_{t \in \partial T_f} F(t) \subseteq F(\emptyset)$.*

Proof. By transfinite induction on the pruning rank of T. □

Given a family $\mathcal{F} \subseteq \mathcal{P}(X)$, we say that a set $Y \subseteq X$ is *hyper* $\mathcal{F}$ if it is a countable increasing union of elements of $\mathcal{F}$. Let $\mathrm{cl}(\mathcal{F})$ denote the closure of $\mathcal{F}$ under countable increasing union and countable intersection. A *witness* to the membership of a set $Y \subseteq X$ in $\mathrm{cl}(\mathcal{F})$ is a parse tree (F, O, T) on X for which $F(\partial T) \subseteq \mathcal{F}$ and $Y = F(\emptyset)$.

Proposition 2. *Suppose that X is a set and $\mathcal{F}$ is a family of subsets of X. Then every set in $\mathrm{cl}(\mathcal{F})$ admits a witness to membership in $\mathrm{cl}(\mathcal{F})$.*

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Proof. Simply observe that the family of subsets of X that admit witnesses to membership in $\mathrm{cl}(\mathcal{F})$ contains $\mathcal{F}$ and is closed under countable increasing union and countable intersection. $\square$

A topological space is *Polish* if it is separable and admits a compatible complete metric. A *Borel space* is a set equipped with a distinguished σ-algebra of *Borel subsets*. Such a space is *standard Borel* if these sets coincide with the σ-algebra generated by a Polish topology on the set. A function between Borel spaces is *Borel* if preimages of Borel sets are Borel. A *Borel probability measure* on a Borel space is a probability measure on its Borel subsets.

Following the usual abuse of language, we say that an equivalence relation is *countable* if all of its equivalence classes are countable and *finite* if all of its equivalence classes are finite.

Theorem 3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\mathcal{F}$ is a family of Borel equivalence relations on X that is closed under countable intersection and restriction to Borel subsets of X, and μ is a Borel probability measure on X. If $E \in \mathrm{cl}(\mathcal{F})$, then there is a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is hyper $\mathcal{F}$.*

Proof. Proposition 2 yields a witness (F, O, T) to the membership of E in $\mathrm{cl}(\mathcal{F})$. Fix $\epsilon_{n,t} > 0$ for which $\sum\{\epsilon_{n,t} \mid n \in \mathbb{N} \text{ and } t \in O^{-1}(\bigcup\!\!\uparrow)\} < \infty$. By the Lusin–Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]), there are Borel functions $\phi_n \colon X \to X$ with $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$. For all $n \in \mathbb{N}$, fix a function $f_n \colon O^{-1}(\bigcup\!\!\uparrow) \to \mathbb{N}$ with the property that, for all $t \in O^{-1}(\bigcup\!\!\uparrow)$, the μ-measure of the set

$$B_{n,t} = \{x \in X \mid \forall i < n \ (x \ F(t) \ \phi_i(x) \implies x \ F(t \frown (f_n(t))) \ \phi_i(x))\}$$

is at least $1 - \epsilon_{n,t}$. For all $n \in \mathbb{N}$, set $\epsilon_n = \sum\{\epsilon_{n,t} \mid t \in O^{-1}(\bigcup\!\!\uparrow)\}$ and observe that the μ-measure of the set $B_n = \bigcap\{B_{n,t} \mid t \in O^{-1}(\bigcup\!\!\uparrow)\}$ is at least $1 - \epsilon_n$, so the μ-measure of the set $C_n = \bigcap_{m \geq n} B_m$ is at least $1 - \sum_{m \geq n} \epsilon_m$. As $\sum_{m \geq n} \epsilon_m \to 0$, the set $C = \bigcup_{n \in \mathbb{N}} C_n$ is μ-conull.

For all $n \in \mathbb{N}$, define $F_n = \bigcap_{t \in \partial T_{f_n}} F(t)$ and $E_n = \bigcap_{m \geq n} F_m$. As $\mathcal{F}$ is closed under countable intersection and $(E_n)_{n \in \mathbb{N}}$ is increasing, the equivalence relation $E_\infty = \bigcup_{n \in \mathbb{N}} E_n$ is hyper $\mathcal{F}$. Proposition 1 ensures that $E_\infty \subseteq E$. To see that $E \restriction C \subseteq E_\infty$, it is sufficient to show that if $i < n$ are natural numbers and $x \in C_n$, then $x \ E_n \ \phi_i(x)$. But if $m \geq n$, then $x \in B_m$, so $x \ F(t) \ \phi_i(x)$ for all $t \in T_{f_m}$ by a straightforward induction on the length of t, thus $x \ F_m \ \phi_i(x)$. $\square$

A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a map $\pi \colon X \to Y$ such that $x \ E \ y \iff \pi(x) \ F \ \pi(y)$

for all $x, y \in X$. A Borel equivalence relation on a standard Borel space is *smooth* if it admits a Borel reduction to equality on $2^{\mathbb{N}}$. A Borel equivalence relation on a standard Borel space is *hyperfinite* if it is a countable increasing union of finite Borel subequivalence relations.

Theorem 4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\mathcal{F}$ is the family of smooth Borel equivalence relations on X, and μ is a Borel probability measure on X. If $E \in \mathrm{cl}(\mathcal{F})$, then there is a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is hyperfinite.*

Proof. As $\mathcal{F}$ is clearly closed under Borel restriction and countable intersection, Theorem 3 yields a μ-conull Borel set $C \subseteq X$ for which $E \restriction C$ is hyper $\mathcal{F}$, thus hyperfinite by [DJK94, Theorem 7.1]. ☐

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References

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