ESSENTIAL VALUES OF COCYCLES AND THE BOREL STRUCTURE OF $\mathbb{R}/\mathbb{Q}$

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Abstract. We introduce essential values of Borel cocycles from analytic equivalence relations to countable discrete groups, establish a Glimm–Effros-style characterization of the circumstances under which such cocycles have a given non-trivial essential value, and obtain a Dougherty–Jackson–Kechris-style embedding theorem for such cocyles with hyperfinite domains. We then use these results to classify suitably Borel finite equivalence relations and free actions of finite groups on $\mathbb{R}/\mathbb{Q}$. Assuming that $(\mathbb{Z} * \mathbb{Z})$-orderable Borel equivalence relations are hyperfinite, we also show that every suitably Borel automorphism of $\mathbb{R}/\mathbb{Q}$ is both a product of three involutions and a commutator, and that the group of all such automorphisms has exactly four proper normal subgroups and the 12-Bergman property.

Introduction

Endow $\mathbb{N}$ with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets and *co-analytic* if its complement is analytic. Souslin's theorem ensures that a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, the proof of [Kec95, 14.11]). A function between topological spaces is *Borel* if preimages of open sets are Borel.

The *diagonal* on X is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$ and the *restriction* of a binary relation R on X to a set $Y \subseteq X$ is given by $R \restriction Y = R \cap (Y \times Y)$. A *partial transversal* of an equivalence relation E on a set X is a set $Y \subseteq X$ for which $E \restriction Y = \Delta(Y)$. Following the usual abuse of language, we say that E is *countable* if all of its equivalence classes are countable, and *finite* if all of its equivalence classes are finite.

2010 *Mathematics Subject Classification.* Primary 03E15, 28A05, 37B05.
Key words and phrases. Automorphism, Borel, cocycle, quotient.
The author was partially supported by FWF grant P29999.

A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\phi\colon X \to Y$ such that $(\phi \times \phi)(R) \subseteq S$. A *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, and an *embedding* of R into S is an injective reduction of R to S.

We say that a countable analytic equivalence relation E on a Hausdorff space X is *smooth* if X is a countable union of Borel partial transversals of E. Well-known examples of non-smooth equivalence relations include the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m)$ and the orbit equivalence relation $\mathbb{E}_V$ induced by the action of $\mathbb{Q}$ on $\mathbb{R}$ by addition. One form of the *Glimm–Effros dichotomy* ensures that a countable analytic equivalence relation E on a Hausdorff space is smooth if and only if there is no continuous embedding of $\mathbb{E}_0$ into E (see, for example, [Mil12, Theorem 14]).

Given an equivalence relation E on a set X and a group Γ, we say that a function $\rho\colon E \to \Gamma$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ for all $x \mathrel{E} y \mathrel{E} z$. Given an equivalence relation F on a set Y, a *homomorphism* from a cocycle $\rho\colon E \to \Gamma$ to a cocycle $\sigma\colon F \to \Gamma$ is a homomorphism $\phi\colon X \to Y$ from E to F for which $\rho = \sigma \circ (\phi \times \phi)$. An *embedding* of ρ into σ is a homomorphism from ρ to σ that is also an embedding of E into F.

When X is a Hausdorff space, E is analytic, and ρ is Borel, we say that a set $\Lambda \subseteq \Gamma$ is an *essential value* of ρ if $\Lambda \neq \emptyset$ and X is not a countable union of Borel sets $B \subseteq X$ with the property that $\Lambda \nsubseteq \rho((E \restriction B) \setminus \Delta(B))$. It is easy to see that if E is countable, $\Lambda = \{1_\Gamma\}$, and ρ is constant, then Λ is an essential value of ρ if and only if E is not smooth.

We say that a sequence $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ is a *redundant enumeration* of Λ if $\Lambda \subseteq \boldsymbol{\lambda}(\mathbb{N} \setminus n)$ for all $n \in \mathbb{N}$. Let $\frown$ denote concatenation of sequences and define $\boldsymbol{\lambda}^s = \prod_{i<|s|} \boldsymbol{\lambda}(i)^{s(i)}$ for all $s \in 2^{<\mathbb{N}}$ and $\wp_{\boldsymbol{\lambda}}\colon \mathbb{E}_0 \to \Gamma$ by $\wp_{\boldsymbol{\lambda}}(s \frown c, t \frown c) = \boldsymbol{\lambda}^s(\boldsymbol{\lambda}^t)^{-1}$ for all $c \in 2^{\mathbb{N}}$ and $(s, t) \in \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$. If E is countable, $\Lambda = \{1_\Gamma\}$, and ρ is constant, then the Glimm–Effros dichotomy ensures that Λ is an essential value of ρ if and only if there is a continuous embedding of $\wp_{\boldsymbol{\lambda}}$ into ρ.

In §1, we extend this result to non-trivial values of Λ:

Theorem 1. *Suppose that $\Lambda \leq \Gamma$ are countable discrete non-trivial groups, $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ is a redundant enumeration of Λ, X is a Hausdorff space, E is an analytic equivalence relation on X, and $\rho\colon E \to \Gamma$ is a Borel cocycle. Then the following are equivalent:*

1. *The set Λ is an essential value of ρ.*
2. *There is a continuous embedding $\pi\colon 2^{\mathbb{N}} \to X$ of $\wp_{\boldsymbol{\lambda}}$ into ρ.*

A *Polish* space is a second-countable topological space that admits a compatible complete metric. Every such space is analytic (see, for example, [Kec95, Theorem 7.9]). A *Borel space* is a set equipped with a distinguished σ-algebra. Such a space is *standard* if the latter is the smallest σ-algebra containing a Polish topology on the former.

A Borel equivalence relation on a standard Borel space is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite Borel subequivalence relations. Examples of such equivalence relations include $\mathbb{E}_0$ and $\mathbb{E}_V$. By the *Dougherty–Jackson–Kechris embedding theorem* (see [DJK94, Theorem 7.1]), every such equivalence relation admits a Borel embedding into $\mathbb{E}_0$. When $\Lambda = \{1_\Gamma\}$, it follows that the constant cocycle on every such equivalence relation is Borel embeddable into $\wp_{\boldsymbol{\lambda}}$.

In §2, we generalize this result to non-trivial values of Λ:

Theorem 2. *Suppose that Γ is a countable discrete group, $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$ is a redundant enumeration of Γ, X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, and $\rho\colon E \to \Gamma$ is a Borel cocycle. Then there is a Borel embedding $\pi\colon X \to 2^{\mathbb{N}}$ of ρ into $\wp_{\boldsymbol{\gamma}}$.*

Given an equivalence relation E on a topological space X, we say that a set $B \subseteq X/E$ is *Borel* if $\bigcup B$ is Borel. More generally, given a countable set I and equivalence relations E_i on topological spaces X_i for all $i \in I$, we say that a set $R \subseteq \prod_{i\in I} X_i/E_i$ is *weakly Borel* if the *lifting* $\tilde{R} = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid ([x_i]_{E_i})_{i\in I} \in R\}$ is Borel. Given an equivalence relation F on a topological space Y, we say that a function $\phi\colon X/E \to Y/F$ is *strongly Borel* if its graph is weakly Borel.

The Glimm–Effros dichotomy and the Lusin–Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]) easily imply that the quotient of a standard Borel space by a countable Borel equivalence relation is standard if and only if the latter is smooth. A *hyperfinite quotient* is the quotient of a standard Borel space by a hyperfinite Borel equivalence relation. The Glimm–Effros dichotomy, the Dougherty–Jackson–Kechris embedding theorem, and the proof of the Schröder–Bernstein theorem ensure that there is only one non-standard such quotient up to strong Borel isomorphism. The *isomorphism theorem for standard Borel spaces* (see, for example, [Kec95, Theorem 15.6]) therefore implies that 0, 1, 2, …, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}/\mathbb{Q}$ are the only hyperfinite quotients up to strong Borel isomorphism. We refer to this fact as the *isomorphism theorem for hyperfinite quotients*.

Given a group action $\Gamma \curvearrowright X$, we say that a set $Y \subseteq X$ is Γ-*complete* if it intersects every Γ-orbit. When X is a Borel space, we say that an action $\Gamma \curvearrowright X$ by Borel automorphisms is *orbit ergodic* if there is no Γ-complete Borel set $B \subseteq X$ whose complement is also Γ-complete.

An *embedding* of an action $\Gamma \curvearrowright X$ into an action $\Gamma \curvearrowright Y$ is an injection $\pi \colon X \to Y$ such that $\pi(\gamma \cdot x) = \gamma \cdot \pi(x)$ for all $\gamma \in \Gamma$ and $x \in X$. An *isomorphism* is a surjective embedding.

For each cocycle $\rho \colon E \to \Gamma$, define $E_\rho = \rho^{-1}(\{1_\Gamma\})$. For each redundant enumeration $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$ of Γ, set $X_{\boldsymbol{\gamma}} = \{c \in 2^{\mathbb{N}} \mid \wp_{\boldsymbol{\gamma}}(\{c\} \times [c]_{\mathbb{E}_0}) = \Gamma\}$ and $E_{\boldsymbol{\gamma}} = E_{\wp_{\boldsymbol{\gamma}}} \restriction X_{\boldsymbol{\gamma}}$ and define $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ by $[c]_{E_{\boldsymbol{\gamma}}} = \gamma \cdot [d]_{E_{\boldsymbol{\gamma}}} \iff \wp_{\boldsymbol{\gamma}}(c, d) = \gamma$ for all $(c, d) \in \mathbb{E}_0 \restriction X_{\boldsymbol{\gamma}}$ and $\gamma \in \Gamma$.

In §3, we use Theorems 1 and 2 to show that these are both minimal orbit-ergodic actions and essentially the only orbit-ergodic free actions for which the lifting E_Γ^X of the corresponding orbit equivalence relation $E_\Gamma^{X/E}$ is hyperfinite:

Theorem 3. *Suppose that Γ is a non-trivial countable discrete group, $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$ is a redundant enumeration of Γ, X is an analytic Hausdorff space, E is a Borel equivalence relation on X, and $\Gamma \curvearrowright X/E$ is a free action by strongly Borel automorphisms. Then the following are equivalent:*

1. *The action $\Gamma \curvearrowright X/E$ is orbit ergodic.*
2. *There is a strongly Borel embedding of $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ into $\Gamma \curvearrowright X/E$.*

Moreover, if X is standard Borel and E_Γ^X is hyperfinite, then these conditions are equivalent to:

3. *There is a strongly Borel isomorphism of $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ with $\Gamma \curvearrowright X/E$.*

The *orbit cocycle* associated with a free action $\Gamma \curvearrowright X/E$ is the cocycle $\rho_\Gamma^X \colon E_\Gamma^X \to \Gamma$ given by $\rho_\Gamma^X(x, y) = \gamma \iff [x]_E = \gamma \cdot [y]_E$. For all $k \in \mathbb{Z}^+$, let S_k denote the *symmetric group* of all permutations of k. Given a Polish space X, a Borel equivalence relation F on X, and a weakly Borel equivalence relation E on X/F whose classes all have cardinality k, we say that a cocycle $\rho \colon \tilde{E} \to S_k$ is an *index cocycle* for E if there are Borel functions $\phi_i \colon X \to X$ with the property that $\phi_0 = \mathrm{id}$, $[x]_{\tilde{E}} = \bigcup_{i<k} [\phi_i(x)]_F$ for all $x \in X$, and $\rho(x, y) = \sigma \iff \forall i < k \ \phi_{\sigma(i)}(x) \ F \ \phi_i(y)$ for all $\sigma \in S_k$ and $x \ \tilde{E} \ y$. If F is the orbit equivalence relation induced by a Borel action of a Polish group, then [dRM, Theorem 2.12] ensures the existence of such cocycles.

In §4, we use Theorem 3 to show that, on the non-standard hyperfinite quotient, there are essentially only finitely-many free actions of any finite group by strongly Borel automorphisms and weakly Borel equivalence relations whose classes have any given finite cardinality:

Theorem 4. *Suppose that Γ is a finite group. Then the set of essential values of the corresponding orbit cocycle is a complete invariant* *for strong Borel isomorphism of free actions of Γ by strongly Borel automorphisms of the non-standard hyperfinite quotient.*

Theorem 5. *Suppose that $k \in \mathbb{Z}^+$. Then the set of essential values of any corresponding index cocycle is a complete invariant for strongly Borel isomorphism of weakly Borel equivalence relations whose classes have cardinality k on the non-standard hyperfinite quotient.*

A function between Borel spaces is *Borel* if preimages of Borel sets are Borel. A *Borel automorphism* of a Borel space X is a Borel bijection $T \colon X \to X$ whose inverse is also Borel.

Given a binary relation R on X, we say that a family of subsets of X *separates R-related points* if any two distinct R-related points in X are separated by a set in the family. We say that R is *separable* if there is a countable family of Borel subsets of X that separates R-related points. We say that a Borel automorphism $T \colon X \to X$ is *separable* if the graphs of its powers are separable.

An *isomorphism* of bijections $S \colon X \to X$ and $T \colon Y \to Y$ is a bijection $\pi \colon X \to Y$ such that $\pi \circ S = T \circ \pi$. Given a positive integer h, the *tower of constant height h* over a bijection $T \colon X \to X$ is the bijection $T * h \colon X \times \{1, \ldots, h\} \to X \times \{1, \ldots, h\}$ given by

$$(T * h)(x, i) = \begin{cases} (x, i+1) & \text{if } i < h \text{ and} \\ (T(x), 1) & \text{otherwise.} \end{cases}$$

Theorem 3 ensures that if $2 \leq k \leq \aleph_0$, then there is essentially a unique strongly Borel automorphism $\mathbb{T}_k$ of a hyperfinite quotient for which every orbit has cardinality k, the corresponding free action of $\mathbb{Z}/k\mathbb{Z}$ is orbit ergodic, and the lifting of the corresponding orbit equivalence relation is hyperfinite.

We say that a strongly Borel automorphism $T \colon X/E \to X/E$ is *decomposable* if there is a T-invariant Borel set $B \subseteq X/E$ such that $T \upharpoonright B$ is separable and $T \upharpoonright {\sim}B$ is strongly Borel isomorphic to a disjoint union of countably-many automorphisms of the form $\mathbb{T}_k * h$.

In §5, we use Theorem 3 to show that if the lifting of the orbit equivalence relation induced by every strongly Borel automorphism of a hyperfinite quotient is hyperfinite—an assumption easily seen to be equivalent to the long-standing conjecture that every $(\mathbb{Z} * \mathbb{Z})$-orderable Borel equivalence relation is hyperfinite—then every strongly Borel automorphism of a hyperfinite quotient is decomposable:

Theorem 6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $T \colon X/E \to X/E$ is a strongly Borel automorphism, and E_T^X is hyperfinite. Then T is decomposable.*

Remark 7. We say that a bijection is *periodic* if all of its orbits are finite. If X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, and $T \colon X/E \to X/E$ is a periodic strongly Borel automorphism, then E_T^X is hyperfinite by [JKL02, Proposition 1.3 (vii)], so Theorem 6 ensures that T is decomposable.

Remark 8. If X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, $T \colon X/E \to X/E$ is a strongly Borel automorphism, and μ is a Borel probability measure on X, then there is an E_T^X-invariant μ-conull Borel set $C \subseteq X$ on which E_T^X is hyperfinite by [JKL02, Proposition 2.15 (ix)] and the main result of [CFW81], so Theorem 6 ensures that $T \upharpoonright C$ is decomposable.

Remark 9. If X is a Polish space, E is a countable Borel equivalence relation on X, and $T \colon X/E \to X/E$ is a strongly Borel automorphism, then there is an E_T^X-invariant comeager Borel set $C \subseteq X$ on which E_T^X is hyperfinite by [HK96, Theorem 6.2], so Theorem 6 ensures that $T \upharpoonright C$ is decomposable.

An element γ of a group Γ is an *involution* if $\gamma^2 = 1_\Gamma$. The *commutator* of group elements γ and δ is given by $[\gamma, \delta] = \gamma\delta\gamma^{-1}\delta^{-1}$.

In §6, we show that decomposable strongly Borel automorphisms of hyperfinite quotients are both products of involutions and special kinds of commutators:

Theorem 10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $T \colon X/E \to X/E$ is a decomposable strongly Borel automorphism. Then there are strongly Borel involutions $I, J, K \colon X/E \to X/E$ for which $T = I \circ J \circ K$.*

Theorem 11. *Suppose that $n \geq 3$, X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X with infinitely-many classes, and $T \colon X/E \to X/E$ is a decomposable strongly Borel automorphism. Then there are strongly Borel automorphisms $R, S \colon X/E \to X/E$, whose orbits have cardinality 1, 2, or n, such that R^{-1} and S are strongly Borel isomorphic and $T = S \circ R$.*

Recall that the *support* of an automorphism $T \colon X \to X$ is given by $\mathrm{supp}(T) = \{x \in X \mid x \neq T(x)\}$.

In §7, we prove the following fact, which—when combined with the analogous fact for uncountable standard Borel spaces (see [Mil, Theorem 6]) and the assumption that every strongly Borel automorphism of a hyperfinite quotient is decomposable—implies that the group of strongly Borel automorphisms of the non-standard hyperfinite quotient has exactly four proper normal subgroups:

Theorem 12. *Suppose that X is a standard Borel space, E is a non-smooth hyperfinite Borel equivalence relation on X, $S\colon X/E \to X/E$ is a decomposable strongly Borel automorphism, and $T\colon X/E \to X/E$ is a strongly Borel automorphism whose support is not standard. Then S is a product of four conjugates of $T^{\pm 1}$.*

For each $k \in \mathbb{N}$, we say that a group Γ has the *k-Bergman property* if, for every exhaustive increasing sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of subsets of Γ, there exists $n \in \mathbb{N}$ such that $\Gamma = (\Gamma_n)^k$. Let $\mathrm{Aut}_{\mathrm{sB}}(X/E)$ denote the group of all strongly Borel automorphisms of X/E.

In §8, we establish the following fact, whose proof also yields the analogous facts for the *symmetric group* of all permutations of $\mathbb{N}$ and the group of all Borel automorphisms of $\mathbb{R}$, slightly strengthening [Ber06, Theorem 6] and [DG05, Theorem 3.4]:

Theorem 13. *Suppose that X is a standard Borel space, E is a non-smooth hyperfinite Borel equivalence relation on X, and every strongly Borel automorphism of X/E is decomposable. Then* $\mathrm{Aut}_{\mathrm{sB}}(X/E)$ *has the* 12*-Bergman property.*

1. Minimality

A *digraph* on a set X is an irreflexive set $G \subseteq X \times X$. We say that a set $Y \subseteq X$ is *G-independent* if $G \upharpoonright Y = \emptyset$. It is well-known that if X is Hausdorff and G is analytic, then every G-independent analytic set $A \subseteq X$ is contained in a G-independent Borel set $B \subseteq X$ (see, for example, [dRM, Proposition 1.2]).

The following observation ensures that if a Borel cocycle $\rho\colon E \to \Gamma$ from an analytic equivalence relation on a Hausdorff space to a countable discrete group has any essential values at all, then $\{1_\Gamma\}$ is an essential value:

Proposition 1.1. *Suppose that Γ is a countable discrete group, X is a Hausdorff space, E is an analytic equivalence relation on X, and $\rho\colon E \to \Gamma$ is a Borel cocycle for which $E_\rho = \Delta(X)$. Then ρ has no essential values.*

Proof. It is sufficient to show that there is no $\gamma \in \Gamma$ for which $\{\gamma\}$ is an essential value of ρ. As $1_\Gamma \notin \rho(E \setminus \Delta(X))$, we can assume that $\gamma \neq 1_\Gamma$. As the analyticity of E yields that of X and every analytic Hausdorff space admits a countable separating family of Borel sets, there is a sequence $(D_n)_{n\in\mathbb{N}}$ of Borel sets separating E-related points.

Lemma 1.2. *The set $A_n = \{x \in D_n \mid \exists y \in {\sim}D_n \ \gamma = \rho(x,y)\}$ is $\rho^{-1}(\{\gamma\})$-independent for all $n \in \mathbb{N}$.*

Proof. Suppose, towards a contradiction, that there exist $x, y \in A_n$ for which $\gamma = \rho(x, y)$. Fix $z \in {\sim}D_n$ such that $\gamma = \rho(x, z)$. Then $\rho(y, z) = \rho(y, x)\rho(x, z) = \gamma^{-1}\gamma = 1_\Gamma$, so $y = z$, contradicting the fact that $y \in A_n \subseteq D_n$ and $z \notin D_n$. ⊠

It follows that there is a $\rho^{-1}(\{\gamma\})$-independent Borel set $B_n \subseteq X$ containing A_n for all $n \in \mathbb{N}$, so we need only show that ${\sim}\bigcup_{n\in\mathbb{N}} B_n$ is $\rho^{-1}(\{\gamma\})$-independent. Towards this end, simply note that if $x \in X$, $y \in {\sim}\bigcup_{n\in\mathbb{N}} B_n$, and $\gamma = \rho(x, y)$, then there exists $n \in \mathbb{N}$ such that $x \in D_n$ and $y \notin D_n$, so $x \in A_n \subseteq B_n \subseteq \bigcup_{n\in\mathbb{N}} B_n$. ⊠

In order to show that the group $\langle\Lambda\rangle$ generated by an essential value Λ is also an essential value, we will need:

Proposition 1.3. *Suppose that Γ is a countable discrete group, X is a Hausdorff space, E is an analytic equivalence relation on X, $\rho \colon E \to \Gamma$ is a Borel cocycle, Λ is an essential value of ρ, $\lambda \in \Lambda$, and*

$$A = \{x \in X \mid \exists y \in [x]_E \ \lambda = \rho(x, y)\}.$$

Then Λ is an essential value of $\rho \restriction (E \restriction A)$.

Proof. We can assume that $\lambda \neq 1_\Gamma$, since otherwise $A = X$. Suppose, towards a contradiction, that there is a cover $(A_n)_{n\in\mathbb{N}}$ of A by Borel subsets of A such that $\Lambda \nsubseteq \rho((E \restriction A_n) \setminus \Delta(A_n))$ for all $n \in \mathbb{N}$. Proposition 1.1 ensures that, by subdividing each A_n into countably many Borel subsets of A, we can assume that there exists $\lambda_n \in \Lambda \setminus \{1_\Gamma\}$ such that A_n is $\rho^{-1}(\{\lambda_n\})$-independent for all $n \in \mathbb{N}$. Fix a $\rho^{-1}(\{\lambda_n\})$-independent Borel set $B_n \subseteq X$ containing A_n for all $n \in \mathbb{N}$. Then ${\sim}\bigcup_{n\in\mathbb{N}} B_n$ is disjoint from A and therefore $\rho^{-1}(\{\lambda\})$-independent, so Λ is not an essential value of ρ, a contradiction. ⊠

As promised, we now obtain:

Proposition 1.4. *Suppose that Γ is a countable discrete group, X is a Hausdorff space, E is an analytic equivalence relation on X, $\rho \colon E \to \Gamma$ is a Borel cocycle, and Λ is an essential value of ρ. Then so too is $\langle\Lambda\rangle$.*

Proof. By replacing Λ with $\Lambda^{\pm 1}$, we can assume that Λ is symmetric. By Proposition 1.1, we can assume that $1_\Gamma \in \Lambda$. As $\langle\{1_\Gamma\}\rangle = \{1_\Gamma\}$, we can also assume that $\Lambda \neq \{1_\Gamma\}$. By another application of Proposition 1.1, it is sufficient to show that $\langle\Lambda\rangle \setminus \{1_\Gamma\}$ is an essential value of ρ. As any cover of X by countably many Borel sets includes a Borel set $B \subseteq X$ for which Λ is an essential value of $\rho \restriction (E \restriction B)$, it is sufficient to show that $\langle\Lambda\rangle \setminus \{1_\Gamma\} \subseteq \rho(E)$.

Given $\lambda \in \langle \Lambda \rangle \setminus \{1_\Gamma\}$, fix $n \in \mathbb{Z}^+$ and $(\lambda_i)_{i<n} \in \Lambda^n$ with the property that $\lambda = \prod_{i<n} \lambda_i$. Set $A_0 = X$ and recursively define

$$A_{i+1} = \{x \in A_i \mid \exists y \in A_i \ \rho(x, y) = \lambda_{n-1-i}\}$$

for all $i < n$. By Proposition 1.3, there exists $x_n \in A_n$. By reverse recursion, there exists $x_i \in A_i$ such that $\rho(x_{i+1}, x_i) = \lambda_{n-1-i}$ for all $i < n$. Then $\rho(x_n, x_0) = \rho(x_n, x_{n-1}) \cdots \rho(x_1, x_0) = \lambda_0 \cdots \lambda_{n-1} = \lambda$. ⊠

We next turn our attention to cocycles of the form $\wp_{\boldsymbol{\lambda}}$:

Proposition 1.5. *Suppose that $B \subseteq 2^\mathbb{N}$ is a non-meager set with the Baire property, Γ is a group, $\Lambda \subseteq \Gamma$ is conjugation invariant, and $\boldsymbol{\lambda} \in \Lambda^\mathbb{N}$ is a redundant enumeration of Λ. Then $\Lambda \subseteq \wp_{\boldsymbol{\lambda}}((\mathbb{E}_0 \upharpoonright B) \setminus \Delta(B))$.*

Proof. Suppose that $\lambda \in \Lambda$ and fix $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s$ (see, for example, [Kec95, Proposition 8.26]). Then there exists $n \geq |s|$ for which $\boldsymbol{\lambda}(n) = (\boldsymbol{\lambda}^s)^{-1} \lambda \boldsymbol{\lambda}^s$. Let ι be the isometry of $2^\mathbb{N}$ that flips the n^{th} coordinate of its input. As B is comeager in $\mathcal{N}_{s \frown (0)^{n-|s|} \frown (i)}$ for all $i < 2$, there exists $c \in B \cap \mathcal{N}_{s \frown (0)^{n-|s|} \frown (0)} \cap \iota(B \cap \mathcal{N}_{s \frown (0)^{n-|s|} \frown (1)})$ (see, for example, [Kec95, Exercise 8.45]), in which case $c, \iota(c) \in B$ and $\wp_{\boldsymbol{\lambda}}(\iota(c), c) = \boldsymbol{\lambda}^s \boldsymbol{\lambda}(n) (\boldsymbol{\lambda}^s)^{-1} = \lambda$. ⊠

For all $s \in 2^{<\mathbb{N}}$, let G_s denote the digraph on $2^\mathbb{N}$ consisting of all pairs of the form $(s \frown (1) \frown c, s \frown (0) \frown c)$ where $c \in 2^\mathbb{N}$.

Proposition 1.6. *Suppose that Γ is a group, $\boldsymbol{\gamma} \in \Gamma^\mathbb{N}$, X is a set, E is an equivalence relation on X, $\rho \colon E \to \Gamma$ is a cocycle, and $\phi \colon 2^\mathbb{N} \to X$ is a homomorphism from $\mathbb{E}_0$ to E such that ρ has constant value $\boldsymbol{\gamma}(n)$ on $G_{(0)^n}$ for all $n \in \mathbb{N}$. Then ϕ is a homomorphism from $\wp_{\boldsymbol{\gamma}}$ to ρ.*

Proof. We will show that if $n \in \mathbb{N}$, then $\boldsymbol{\gamma}^s = \rho(\phi(s \frown c), \phi((0)^n \frown c))$ for all $c \in 2^\mathbb{N}$ and $s \in 2^n$, as this implies that if $t \in 2^n$, then

$$\begin{aligned}
&\rho(\phi(s \frown c), \phi(t \frown c)) \\
&\quad = \rho(\phi(s \frown c), \phi((0)^n \frown c)) \rho(\phi((0)^n \frown c), \phi(t \frown c)) \\
&\quad = \rho(\phi(s \frown c), \phi((0)^n \frown c)) \rho(\phi(t \frown c), \phi((0)^n \frown c))^{-1} \\
&\quad = \boldsymbol{\gamma}^s (\boldsymbol{\gamma}^t)^{-1} \\
&\quad = \wp_{\boldsymbol{\gamma}}(s \frown c, t \frown c).
\end{aligned}$$

Granting the desired result at n, note that if $c \in 2^\mathbb{N}$ and $s \in 2^n$, then

$$\begin{aligned}
&\rho(\phi(s \frown (1) \frown c), \phi((0)^n \frown (0) \frown c)) \\
&\quad = \rho(\phi(s \frown (1) \frown c), \phi((0)^n \frown (1) \frown c)) \\
&\qquad \rho(\phi((0)^n \frown (1) \frown c), \phi((0)^n \frown (0) \frown c)) \\
&\quad = \boldsymbol{\gamma}^s \boldsymbol{\gamma}(n) = \boldsymbol{\gamma}^{s \frown (1)},
\end{aligned}$$

so the desired result holds at $n+1$. ⊠

For all sets $R \subseteq X \times Y$, define $R^{-1} = \{(y,x) \in Y \times X \mid x \mathrel{R} y\}$. A *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on a set X to a sequence $(S_i)_{i \in I}$ of binary relations on a set Y is a function $\phi \colon X \to Y$ that is a homomorphism from R_i to S_i for all $i \in I$.

Proposition 1.7. *Suppose that Γ is a group, $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$ is a redundant enumeration of Γ, R is a nowhere dense binary relation on $2^{\mathbb{N}}$, and S is a meager binary relation on $2^{\mathbb{N}}$. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $({\sim}\Delta(2^{\mathbb{N}}), {\sim}\mathbb{E}_0, \wp_{\boldsymbol{\gamma}})$ to $({\sim}R, {\sim}S, \wp_{\boldsymbol{\gamma}})$.*

Proof. Fix dense open sets $U_n \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ for which $S \cap \bigcap_{n \in \mathbb{N}} U_n = \emptyset$. By replacing each U_n with $U_n \setminus R$, we can assume that R is disjoint from each U_n. By replacing each U_n with $\bigcap_{m \le n} U_m$, we can assume that $(U_n)_{n \in \mathbb{N}}$ is decreasing. By replacing each U_n with $U_n \cap U_n^{-1}$, we can assume that each U_n is symmetric.

Lemma 1.8. *Suppose that $n \in \mathbb{N}$ and $\phi \colon 2^n \to 2^{<\mathbb{N}}$. Then there is a pair $(t_0, t_1) \in \bigcup_{k \in \mathbb{Z}^+} 2^k \times 2^k$ such that:*

1. *$\forall s_0, s_1 \in 2^n \; \prod_{i<2} \mathcal{N}_{\phi(s_i) \frown t_i} \subseteq U_n$.*
2. *$\boldsymbol{\gamma}(n) = \boldsymbol{\gamma}^{\phi((0)^n) \frown t_1} (\boldsymbol{\gamma}^{\phi((0)^n) \frown t_0})^{-1}$.*

Proof. Fix an enumeration $(s_{0,m}, s_{1,m})_{m < 4^n}$ of $2^n \times 2^n$, define $t_{0,0} = t_{1,0} = \emptyset$, and recursively find $t_{i,m+1} \sqsupseteq t_{i,m}$ with the property that $\prod_{i<2} \mathcal{N}_{\phi(s_{i,m}) \frown t_{i,m+1}} \subseteq U_n$ for all $m < 4^n$. By extending $t_{0,4^n}$ or $t_{1,4^n}$ if necessary, we can assume that they have the same length. Define $\ell = |\phi((0)^n) \frown t_{0,4^n}| = |\phi((0)^n) \frown t_{1,4^n}|$, fix $m \ge \ell$ for which $\boldsymbol{\gamma}(m) = (\boldsymbol{\gamma}^{\phi((0)^n) \frown t_{1,4^n}})^{-1} \boldsymbol{\gamma}(n) \boldsymbol{\gamma}^{\phi((0)^n) \frown t_{0,4^n}}$, and set $t_i = t_{i,4^n} \frown (0)^{m-\ell} \frown (i)$ for all $i < 2$. It only remains to observe that $\boldsymbol{\gamma}^{\phi((0)^n) \frown t_1} (\boldsymbol{\gamma}^{\phi((0)^n) \frown t_0})^{-1} = \boldsymbol{\gamma}^{\phi((0)^n) \frown t_{1,4^n}} \boldsymbol{\gamma}(m) (\boldsymbol{\gamma}^{\phi((0)^n) \frown t_{0,4^n}})^{-1} = \boldsymbol{\gamma}(n)$. ⊠

Fix the unique function $\phi_0 \colon 2^0 \to 2^0$ and recursively appeal to Lemma 1.8 to find pairs $(t_{0,n}, t_{1,n}) \in \bigcup_{k \in \mathbb{Z}^+} 2^k \times 2^k$ such that

1. $\forall s_0, s_1 \in 2^n \; \prod_{i<2} \mathcal{N}_{\phi_n(s_i) \frown t_{i,n}} \subseteq U_n$ and
2. $\boldsymbol{\gamma}(n) = \boldsymbol{\gamma}^{\phi_n((0)^n) \frown t_{1,n}} (\boldsymbol{\gamma}^{\phi_n((0)^n) \frown t_{0,n}})^{-1}$,

where $\phi_n \colon 2^n \to 2^{<\mathbb{N}}$ is given by $\phi_n(s) = \bigoplus_{m<n} t_{s(m),m}$ (and $\bigoplus_{m<n} t_m$ denotes the concatenation of $t_0, t_1, \ldots, t_{n-1}$). Define $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \restriction n)$ for all $c \in 2^{\mathbb{N}}$.

To see that ϕ is a homomorphism from $({\sim}\Delta(2^{\mathbb{N}}), {\sim}\mathbb{E}_0)$ to $({\sim}R, {\sim}S)$, it is sufficient to observe that if $c, d \in 2^{\mathbb{N}}$, $n \in \mathbb{N}$, and $c(n) \neq d(n)$, then $(\phi(c), \phi(d)) \in \mathcal{N}_{\phi_{n+1}(c \restriction (n+1))} \times \mathcal{N}_{\phi_{n+1}(d \restriction (n+1))} \subseteq U_n$.

To see that ϕ is a homomorphism from $\mathbb{E}_0$ to $\mathbb{E}_0$, note that if $c \mathbin{\mathbb{E}_0} d$, then there exists $n \in \mathbb{N}$ such that $\forall m \geq n\ c(m) = d(m)$, in which case $\forall m \geq \sum_{k<n} |t_{0,k}|\ \phi(c)(m) = \phi(d)(m)$.

To see that ϕ is a homomorphism from $\wp_{\boldsymbol{\gamma}}$ to $\wp_{\boldsymbol{\gamma}}$, observe that if $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$, then $\wp_{\boldsymbol{\gamma}}(\phi((0)^n \frown (1) \frown c), \phi((0)^n \frown (0) \frown c)) = \boldsymbol{\gamma}^{\phi_n((0)^n) \frown t_{1,n}} (\boldsymbol{\gamma}^{\phi_n((0)^n) \frown t_{0,n}})^{-1} = \boldsymbol{\gamma}(n)$, and appeal to Proposition 1.6. ⊠

An N-*coloring* of an I-indexed sequence $\boldsymbol{G}$ of digraphs on a set X is a function $c \colon X \to N$ with the property that

$$\forall n \in N \exists i \in I\ c^{-1}(\{n\}) \text{ is } \boldsymbol{G}(i)\text{-independent.}$$

We can now give the following:

Proof of Theorem 1. As any sequence of sets witnessing that Λ is not an essential value of ρ can be pulled back through any Borel homomorphism from $\wp_{\boldsymbol{\lambda}}$ to ρ to obtain a sequence of sets witnessing that Λ is not an essential value of $\wp_{\boldsymbol{\lambda}}$, Proposition 1.5 ensures that if there is a Borel homomorphism from $\wp_{\boldsymbol{\lambda}}$ to ρ, then Λ is an essential value of ρ.

Conversely, suppose that Λ is an essential value of ρ and define $\boldsymbol{G}(n) = \rho^{-1}(\{\boldsymbol{\lambda}(n)\}) \setminus \Delta(X)$ for all $n \in \mathbb{N}$. Then there is no Borel $\mathbb{N}$-coloring of $\boldsymbol{G}$, so the straightforward generalization of the $\mathbb{G}_0$ dichotomy to $\mathbb{N}$-indexed sequences of digraphs (see [Mil12, Theorem 21]) yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(G_{(0)^n})_{n \in \mathbb{N}}$ to $\boldsymbol{G}$. Proposition 1.6 ensures that ϕ is a homomorphism from $\wp_{\boldsymbol{\lambda}}$ to ρ. Let (D', E', ρ') be the pullback of $(\Delta(X), E, \rho)$ through ϕ.

Lemma 1.9. *The equivalence relation $E_{\rho'}$ is meager.*

Proof. Suppose, towards a contradiction, that $E_{\rho'}$ is not meager. By the Kuratowski–Ulam theorem (see, for example, [Kec95, Theorem 8.41]), there exists $c \in 2^{\mathbb{N}}$ for which $[c]_{E_{\rho'}}$ is not meager. Fix $\lambda \in \Lambda \setminus \{1_\Gamma\}$. Then Proposition 1.5 yields a pair $(a, b) \in \mathbb{E}_0 \upharpoonright [c]_{E_{\rho'}}$ for which $\wp_{\boldsymbol{\lambda}}(a, b) = \lambda$, so $\rho'(a, b) = \lambda$, contradicting the fact that $a \mathbin{E_{\rho'}} b$. ⊠

Lemma 1.10. *The equivalence relation E' is meager.*

Proof. By the Kuratowski–Ulam theorem, every $E_{\rho'}$-class is meager. But every E'-class is a countable union of $E_{\rho'}$-classes, so every E'-class is meager, thus another application of the Kuratowski–Ulam theorem ensures that E' is meager. ⊠

Proposition 1.7 now yields a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $({\sim}\Delta(2^{\mathbb{N}}), {\sim}\mathbb{E}_0, \wp_{\boldsymbol{\lambda}})$ to $({\sim}D', {\sim}E', \wp_{\boldsymbol{\lambda}})$, in which case $\phi \circ \psi$ is a continuous embedding of $\wp_{\boldsymbol{\lambda}}$ into ρ. ⊠

2. Maximality

We associate with each sequence $(f_n)_{n \in \mathbb{N}}$ of partial functions on a set X the partial functions $f^0 = \mathrm{id} \restriction X$ and $f^n = f_{n-1} \circ \cdots \circ f_0$ for all $n \in \mathbb{Z}^+$. Given a Borel cocycle $\rho \colon E \to \Gamma$ from a countable Borel equivalence relation on a standard Borel space X to a countable discrete group and $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$, a $(\boldsymbol{\gamma}, \rho)$-*cascade* is an $\mathbb{N}$-sequence of (≤ 2)-to-1 Borel retractions $f_n \colon B_n \to B_{n+1}$, where $(B_n)_{n \in \mathbb{N}}$ is a decreasing sequence of Borel subsets of X, such that:

1. $B_0 = X$.
2. $\forall n \in \mathbb{N} \forall x \in B_n \setminus B_{n+1}\ \boldsymbol{\gamma}(n) = \rho(f_n(x), x)$.
3. $\forall x\ E\ y \exists n \in \mathbb{N}\ f^n(x) = f^n(y)$.

Proposition 2.1. *Suppose that* Γ *is a countable discrete group,* X *is a standard Borel space,* E *is a hyperfinite Borel equivalence relation on* X*,* $\rho \colon E \to \Gamma$ *is a Borel cocycle, and* $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$ *is a redundant enumeration of* Γ*. Then there is a* $(\boldsymbol{\gamma}, \rho)$*-cascade.*

Proof. Fix an exhaustive increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E for which $F_0 = \Delta(X)$. Set $B_0 = X$.

Lemma 2.2. *There is a Borel linear ordering* $\leq_E$ *of* E*.*

Proof. Fix a sequence $(R_n)_{n \in \mathbb{N}}$ of Borel subsets of E that separates points. Define $\pi \colon E \to 2^{\mathbb{N}}$ by $\pi(e)(n) = 1 \iff e \in R_n$ for all $n \in \mathbb{N}$. Then π is injective, so the pullback $\leq_E$ of the lexicographical ordering of $2^{\mathbb{N}}$ through π is a linear order. As each of the sets R_n is Borel, it follows that π is Borel, so $\leq_E$ is Borel. ⊠

Suppose now that $n \in \mathbb{N}$ and $B_n \subseteq X$ is Borel. For all $k \in \mathbb{Z}^+$, let $\mathcal{C}_{k,n}$ be the family of $(F_k \restriction B_n)$-classes C of cardinality at least two that are partial transversals of F_{k-1}, and let $R_{k,n}$ be the set of pairs $(x, y) \in {\sim}\Delta(X)$ for which there exists $C \in \mathcal{C}_{k,n}$ with the property that (x, y) is the $\leq_E$-minimal pair of distinct elements of C such that $\boldsymbol{\gamma}(n) = \rho(y, x)$. Set $R_n = \bigcup_{k \in \mathbb{Z}^+} R_{k,n}$ and $B_{n+1} = B_n \setminus \mathrm{proj}_0(R_n)$. As the sets in the family $\mathcal{C}_n = \bigcup_{k \in \mathbb{Z}^+} \mathcal{C}_{k,n}$ are pairwise disjoint, we obtain a retraction $f_n \colon B_n \to B_{n+1}$ by setting $f_n(x) = y \iff x\ R_n\ y$ for all $x \in B_n \setminus B_{n+1}$. The Lusin–Novikov uniformization theorem easily implies that B_{n+1} and f_n are Borel.

It only remains to show that $(f_n)_{n \in \mathbb{N}}$ is the desired $(\boldsymbol{\gamma}, \rho)$-cascade. Suppose, towards a contradiction, that there exist $x\ E\ y$ such that $f^n(x) \neq f^n(y)$ for all $n \in \mathbb{N}$. Then there is a least $k \in \mathbb{Z}^+$ for which there is an F_k-class C containing points x' and y' with the property that $f^n(x') \neq f^n(y')$ for all $n \in \mathbb{N}$.

Lemma 2.3. *Suppose that* $n \in \mathbb{N}$*. Then* $f^n(C) \subseteq C$*.*

Proof. If $n \in \mathbb{N}$ has the property that $f^n(C) \subseteq C$, then $f^n(x')$ and $f^n(y')$ are distinct elements of $B_n \cap C$, so $B_n \cap C$ is not a partial transversal of F_k, thus $f^{n+1}(C) \subseteq f_n(f^n(C)) \subseteq f_n(B_n \cap C) \subseteq C$. The obvious induction therefore yields the desired result. ⊠

Fix $n \in \mathbb{N}$ sufficiently large that $\forall m \geq n \ B_m \cap C = B_n \cap C$. Let (x'', y'') be the $\leq_E$-minimal pair of distinct points in $B_n \cap C$. By increasing n if necessary, we can assume that $\boldsymbol{\gamma}(n) = \rho(y'', x'')$, in which case $x'' \notin B_{n+1}$, the desired contradiction. ⊠

For all groups Γ, $\boldsymbol{\gamma} \in \Gamma^{\mathbb{N}}$, and $n \in \mathbb{N}$, define $\mathrm{IP}^n(\boldsymbol{\gamma}) = \{\boldsymbol{\gamma}^s \mid s \in 2^n\}$. We can now give the following:

Proof of Theorem 2. By trivially extending X, E, and ρ, we can assume that X is uncountable, so the isomorphism theorem for standard Borel spaces allows us to assume that $X = 2^{\mathbb{N}}$. Appeal to Proposition 2.1 to obtain a $(\boldsymbol{\gamma}, \rho)$-cascade $(f_n)_{n \in \mathbb{N}}$ and define $\beta \colon X \to 2^{\mathbb{N}}$ by $\beta(x)(n) = 0 \iff f^n(x) = f^{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$. As f_n is (≤ 2)-to-1 for all $n \in \mathbb{N}$, a straightforward induction ensures that if $n \in \mathbb{N}$ and $x, y \in X$ have the property that $f^n(x) = f^n(y)$ and $\beta(x) \upharpoonright n = \beta(y) \upharpoonright n$, then $x = y$. We can therefore define functions $u_s \colon X \to 2^n$, for all $n \in \mathbb{N}$ and $s \in 2^n$, by setting

$$u_s(x) = \begin{cases} y \upharpoonright n & \text{if } f^n(x) = f^n(y) \text{ and } s = \beta(y) \upharpoonright n, \text{ and} \\ (0)^n & \text{if no such } y \text{ exists.} \end{cases}$$

Fix an enumeration $(s_{k,n})_{k<2^n}$ of 2^n and define $u_n \colon X \to 2^{n \cdot 2^n}$ by setting $u_n(x) = \bigoplus_{k<2^n} u_{s_{k,n}}(x)$.

We will now recursively define $k_n \in \mathbb{N}$ and $\pi_n \colon X \to 2^{k_n}$. We begin by setting $k_0 = 0$ and $\pi_0(x) = \emptyset$ for all $x \in X$. Suppose now that $n \in \mathbb{N}$ and we have defined k_n and π_n. Set $\ell_n = k_n + n \cdot 2^n + 1$ and $k_{n+1} = \min\{k \in \mathbb{N} \mid \mathrm{IP}^{\ell_n}(\boldsymbol{\gamma})^{-1}\boldsymbol{\gamma}(n)^{-1}\mathrm{IP}^{\ell_n}(\boldsymbol{\gamma}) \subseteq \boldsymbol{\gamma}(k \setminus \ell_n)\}$. Define $\pi'_n \colon X \to 2^{\ell_n}$ by $\pi'_n(x) = \pi_n(x) \frown u_n(x) \frown (\beta(x)(n))$, $m_n \colon X \to k_{n+1} \setminus \ell_n$ by

$$m_n(x) = \min\{m \geq \ell_n \mid \boldsymbol{\gamma}(m) = (\boldsymbol{\gamma}^{\pi'_n(f^n(x))})^{-1}\boldsymbol{\gamma}(n)^{-1}\boldsymbol{\gamma}^{\pi'_n(f^{n+1}(x))}\},$$

$v_n \colon X \to 2^{k_{n+1}-\ell_n}$ by

$$v_n(x) = \begin{cases} (0)^{k_{n+1}-\ell_n} & \text{if } \beta(x)(n) = 0 \text{ and} \\ (0)^{m_n(x)-\ell_n} \frown (1) \frown (0)^{k_{n+1}-m_n(x)-1} & \text{otherwise,} \end{cases}$$

and $\pi_{n+1} \colon X \to 2^{k_{n+1}}$ by $\pi_{n+1}(x) = \pi'_n(x) \frown v_n(x)$. We will show that the function $\pi(x) = \bigcup_{n \in \mathbb{N}} \pi_n(x)$ is as desired.

The Lusin–Novikov uniformization theorem easily implies that all of the functions mentioned thus far are Borel.

To see that π is injective, note that if $\pi(x) = \pi(y)$, then $x \upharpoonright n = u_{\beta(x) \upharpoonright n}(x) = u_{\beta(y) \upharpoonright n}(y) = y \upharpoonright n$ for all $n \in \mathbb{N}$, so $x = y$.

To see that π is a homomorphism from E to $\mathbb{E}_0$, note that if $x \mathrel{E} y$, then there exists $n \in \mathbb{N}$ for which $f^n(x) = f^n(y)$, so $\beta(x)(k) = \beta(y)(k)$, $u_k(x) = u_k(y)$, and $m_k(x) = m_k(y)$ for all $k \geq n$, thus $\pi'_k(x) = \pi'_k(y)$ and $v_k(x) = v_k(y)$ for all $k \geq n$, hence $\pi(x)(k) = \pi(y)(k)$ for all $k \geq k_n$.

To see that π is a homomorphism from $\sim E$ to $\sim \mathbb{E}_0$, suppose that $\pi(x) \mathrel{\mathbb{E}_0} \pi(y)$, in which case there exists $n \in \mathbb{N}$ with the property that $u_m(x) \frown (\beta(x)(m)) \frown v_m(x) = u_m(y) \frown (\beta(y)(m)) \frown v_m(y)$ for all $m \geq n$. But $\beta(f^n(x)) \upharpoonright n = \beta(f^n(y)) \upharpoonright n = (0)^n$ and if $m \geq n$, then $\beta(f^n(x))(m) = \beta(x)(m) = \beta(y)(m) = \beta(f^n(y))(m)$. It follows that $f^n(x) \upharpoonright m = u_{\beta(f^n(x)) \upharpoonright m}(x) = u_{\beta(f^n(y)) \upharpoonright m}(y) = f^n(y) \upharpoonright m$ for all $m \in \mathbb{N}$, so $x \mathrel{E} f^n(x) = f^n(y) \mathrel{E} y$.

Finally, to see that π is a homomorphism from ρ to $\wp_{\boldsymbol{\gamma}}$, suppose that $x \mathrel{E} y$, so there exists $n \in \mathbb{N}$ for which $f^{n+1}(x) = f^{n+1}(y)$, thus $\rho(x, y)$ is the product of $\rho(f^0(x), f^1(x)) \cdots \rho(f^n(x), f^{n+1}(x))$ and $\rho(f^{n+1}(y), f^n(y)) \cdots \rho(f^1(y), f^0(y))$. It remains to show that if $m \leq n$ and $w \in \{x, y\}$, then $\rho(f^{m+1}(w), f^m(w)) = \wp_{\boldsymbol{\gamma}}(\pi(f^{m+1}(w)), \pi(f^m(w)))$. If $\beta(w)(m) = 0$, then $f^{m+1}(w) = f^m(w)$, so $\pi(f^{m+1}(w)) = \pi(f^m(w))$, in which case $\rho(f^{m+1}(w), f^m(w)) = 1_\Gamma = \wp_{\boldsymbol{\gamma}}(\pi(f^{m+1}(w)), \pi(f^m(w)))$. So it only remains to note that if $\beta(w)(m) = 1$, then

$$
\begin{aligned}
&\wp_{\boldsymbol{\gamma}}(\pi(f^{m+1}(w)), \pi(f^m(w))) \\
&\quad = \boldsymbol{\gamma}^{\pi_{m+1}(f^{m+1}(w))} (\boldsymbol{\gamma}^{\pi_{m+1}(f^m(w))})^{-1} \\
&\quad = \boldsymbol{\gamma}^{\pi'_m(f^{m+1}(w))} (\boldsymbol{\gamma}^{(0)^{\ell_m} \frown v_m(f^m(w))})^{-1} (\boldsymbol{\gamma}^{\pi'_m(f^m(w))})^{-1} \\
&\quad = \boldsymbol{\gamma}^{\pi'_m(f^{m+1}(w))} ((\boldsymbol{\gamma}^{\pi'_m(f^m(w))})^{-1} \boldsymbol{\gamma}(m)^{-1} \boldsymbol{\gamma}^{\pi'_m(f^{m+1}(w))})^{-1} (\boldsymbol{\gamma}^{\pi'_m(f^m(w))})^{-1} \\
&\quad = \boldsymbol{\gamma}^{\pi'_m(f^{m+1}(w))} (\boldsymbol{\gamma}^{\pi'_m(f^{m+1}(w))})^{-1} \boldsymbol{\gamma}(m) \boldsymbol{\gamma}^{\pi'_m(f^m(w))} (\boldsymbol{\gamma}^{\pi'_m(f^m(w))})^{-1} \\
&\quad = \boldsymbol{\gamma}(m) \\
&\quad = \rho(f^{m+1}(w), f^m(w)),
\end{aligned}
$$

since $(f_n)_{n \in \mathbb{N}}$ is a $(\boldsymbol{\gamma}, \rho)$-cascade. ⊠

Combining Theorems 1 and 2, we obtain:

Theorem 2.4. *Suppose that $\Lambda \leq \Gamma$ are non-trivial countable discrete groups, X is a standard Borel space, Y is a Hausdorff space, E is a hyperfinite Borel equivalence relation on X, F is an analytic equivalence relation on Y, $\rho \colon E \to \Lambda$ and $\sigma \colon F \to \Gamma$ are Borel cocycles, and Λ is an essential value of σ. Then there is a Borel embedding $\pi \colon X \to Y$ of ρ into σ.*

Proof. Fix a redundant enumeration $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ of Λ. Then Theorem 1 yields a a continuous embedding $\phi\colon 2^{\mathbb{N}} \to Y$ of $\wp_{\boldsymbol{\lambda}}$ into σ, and Theorem 2 yields a Borel embedding $\psi\colon X \to 2^{\mathbb{N}}$ of ρ into $\wp_{\boldsymbol{\lambda}}$, so the function $\pi = \phi \circ \psi$ is as desired. ⊠

3. Orbit ergodicity

We begin this section by noting several basic properties of weakly Borel sets.

Proposition 3.1. *Suppose that X and Y are Hausdorff spaces, E and F are analytic equivalence relations on X and Y, and $R \subseteq X/E \times Y/F$ is weakly Borel. Then R^{-1} is weakly Borel.*

Proof. Define $I\colon X \times Y \to Y \times X$ by $I(x, y) = (y, x)$. Then I is a homeomorphism, so $I(\tilde{R})$ is Borel. But $I(\tilde{R})$ is the lifting of R^{-1}, thus R^{-1} is weakly Borel. ⊠

Proposition 3.2. *Suppose that X, Y, and Z are Hausdorff spaces, D, E, and F are analytic equivalence relations on X, Y, and Z, and $S\colon Y/E \to Z/F$ and $T\colon X/D \to Y/E$ are strongly Borel. Then $S \circ T$ is strongly Borel.*

Proof. Observe that if $x \in X$ and $z \in Z$, then

$$(S \circ T)([x]_D) = [z]_F \iff \exists y \in Y\ (T([x]_D) = [y]_E \text{ and } S([y]_E) = [z]_F)$$

and

$$(S \circ T)([x]_D) \neq [z]_F \iff \exists y \in Y\ (T([x]_D) = [y]_E \text{ and } S([y]_E) \neq [z]_F).$$

It follows that the lifting of graph$(S \circ T)$ is both analytic and co-analytic, and therefore Borel, so graph$(S \circ T)$ is weakly Borel, thus $S \circ T$ is strongly Borel. ⊠

Proposition 3.3. *Suppose that X is an analytic Hausdorff space and E is a Borel equivalence relation on X. Then $\mathrm{Aut}_{\mathrm{sB}}(X/E)$ is a group.*

Proof. Note that the identity function on X/E is strongly Borel, since the lifting of its graph is E. But Propositions 3.1 and 3.2 ensure that $\mathrm{Aut}_{\mathrm{sB}}(X/E)$ is closed under inversion and composition. ⊠

Proposition 3.4. *Suppose that X and Y are Hausdorff spaces, E and F are analytic equivalence relations on X and Y, and $T\colon X/E \to Y/F$ is strongly Borel. Then T is Borel.*

Proof. Set $G = \mathrm{graph}(T)$. Given a Borel set $B \subseteq Y/F$, note that $\bigcup T^{-1}(B) = \mathrm{proj}_X(\tilde{G} \cap (X \times \bigcup B))$ and the latter set is analytic, so $\sim\bigcup T^{-1}(B) = \bigcup T^{-1}(\sim B)$ is analytic, thus $\bigcup T^{-1}(B)$ is Borel, hence $T^{-1}(B)$ is Borel. ⊠

The *product* of equivalence relations E and F on sets X and Y is the equivalence relation $E \times F$ on $X \times Y$ given by

$$(x, y)\ (E \times F)\ (x', y') \iff (x\ E\ x' \text{ and } y\ F\ y').$$

If X is a Hausdorff space, E is analytic, G is an $(E \times E)$-invariant analytic graph on X, and $B \subseteq X$ is a G-independent Borel set, then there is an E-invariant G-independent Borel set $C \subseteq X$ containing B (see, for example, [dRM, Proposition 2.1]).

Proposition 3.5. *Suppose that Γ is a non-trivial countable discrete group, X is a Hausdorff space, E is a Borel equivalence relation on X, and $\Gamma \curvearrowright X/E$ is a free action by strongly Borel automorphisms. Then Γ is an essential value of $\rho_\Gamma^X \iff \Gamma \curvearrowright X/E$ is orbit ergodic.*

Proof. Suppose first that Γ is not an essential value of ρ_Γ^X. Then Proposition 1.1 ensures that $\Gamma \setminus \{1_\Gamma\}$ is not an essential value of ρ_Γ^X, so there is a cover $(B_n)_{n\in\mathbb{N}}$ of X by Borel sets with the property that there is a non-identity element $\gamma_n \in {\sim}\rho_\Gamma^X(E_\Gamma^X \upharpoonright B_n)$ for all $n \in \mathbb{N}$. Fix E-invariant $(\rho_\Gamma^X)^{-1}(\{\gamma_n\})$-independent Borel sets $C_n \subseteq X$ containing B_n. Then the sets $D_n = C_n/E$ do not contain $E_\Gamma^{X/E}$-classes and cover X/E, and Proposition 3.4 ensures that the sets $D'_n = D_n \setminus \bigcup_{m<n} \Gamma D_m$ are Borel. As $\bigcup_{n\in\mathbb{N}} D'_n$ is Γ- and co-Γ-complete, it follows that $\Gamma \curvearrowright X/E$ is not orbit ergodic.

Conversely, suppose that $\Gamma \curvearrowright X/E$ is not orbit ergodic. Then there is a Γ- and co-Γ-complete Borel set $B \subseteq X/E$. Note that the sets of the form $B_\gamma = B \setminus \gamma^{-1}B$ and $B'_\gamma = \gamma B \setminus B$, where $\gamma \in \Gamma \setminus \{1_\Gamma\}$, cover X/E. Proposition 3.4 ensures that these sets are Borel. As $\bigcup B_\gamma$ and $\bigcup B'_\gamma$ are $(\rho_\Gamma^X)^{-1}(\{\gamma\})$-independent for all $\gamma \in \Gamma$, it follows that Γ is not an essential value of ρ_Γ^X. ⊠

We can now give the following:

Proof of Theorem 3. If $\Gamma \curvearrowright X/E$ is orbit ergodic, then Proposition 3.5 ensures that Γ is an essential value of ρ_Γ^X, so Theorem 1 yields a continuous embedding $\pi \colon 2^{\mathbb{N}} \to X$ of $\wp_{\boldsymbol{\gamma}}$ into ρ_Γ^X, and any such embedding factors to a strongly Borel embedding of $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ into $\Gamma \curvearrowright X/E$. To see the converse, appeal to Proposition 3.4 to see that the failure of orbit ergodicity is closed downward under strong Borel embeddability, so if $\Gamma \curvearrowright X/E$ is not orbit ergodic, then Propositions 1.5 and 3.5 ensure that there is no strongly Borel embedding of $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ into $\Gamma \curvearrowright X/E$.

To see that these actions are strongly Borel isomorphic when X is standard Borel, E is Borel, E_Γ^X is hyperfinite, and $\Gamma \curvearrowright X/E$ is orbit ergodic, note that Γ is an essential value of $\wp_{\boldsymbol{\gamma}} \upharpoonright (\mathbb{E}_0 \upharpoonright X_{\boldsymbol{\gamma}})$ and ρ_Γ^X

by Propositions 1.5 and 3.5, so Theorem 2.4 yields Borel embeddings $\phi\colon X_{\boldsymbol{\gamma}} \to X$ of $\mathbb{p}_{\boldsymbol{\gamma}} \restriction (\mathbb{E}_0 \restriction X_{\boldsymbol{\gamma}})$ into ρ_Γ^X and $\psi\colon X \to X_{\boldsymbol{\gamma}}$ of ρ_Γ^X into $\mathbb{p}_{\boldsymbol{\gamma}} \restriction (\mathbb{E}_0 \restriction X_{\boldsymbol{\gamma}})$. Let ϕ' and ψ' be the corresponding strongly Borel embeddings of $X_{\boldsymbol{\gamma}}/\mathbb{E}_0$ into X/E_Γ^X and X/E_Γ^X into $X_{\boldsymbol{\gamma}}/\mathbb{E}_0$, and appeal to the proof of the Schröder–Bernstein theorem (see, for example, [Mil, Proposition 5.10]) to obtain a Borel set $B \subseteq X_{\boldsymbol{\gamma}}/\mathbb{E}_0$ with the property that $(\phi' \restriction {\sim}B) \cup ((\psi')^{-1} \restriction B)$ is a strongly Borel isomorphism of $X_{\boldsymbol{\gamma}}/\mathbb{E}_0$ with X/E_Γ^X. Then $(\phi \restriction {\sim}\bigcup B) \cup (\psi^{-1} \restriction \bigcup B)$ induces a strongly Borel isomorphism of $\Gamma \curvearrowright X_{\boldsymbol{\gamma}}/E_{\boldsymbol{\gamma}}$ with $\Gamma \curvearrowright X/E$. ⊠

4. Finite equivalence relations and group actions

A *transversal* of an equivalence relation E on a set X is a partial transversal $Y \subseteq X$ of E for which $X = [Y]_E$.

Proposition 4.1. *Suppose that Γ is a discrete finite group, $n \in \mathbb{Z}^+$, X is a Hausdorff space, E is an analytic equivalence relation on X, $\rho\colon E \to \Gamma$ is a Borel cocycle, the E-saturation of every E_ρ-invariant Borel set is Borel, and $(\Lambda_i)_{i<n}$ is an injective enumeration of a transversal of the conjugacy equivalence relation on the set of maximal essential values of ρ. Then there is a partition $(B_i)_{i<n}$ of X into E-invariant Borel sets for which there are $(E \restriction B_i)$-complete Borel sets $A_i \subseteq B_i$ such that $\rho(E \restriction A_i) \subseteq \Lambda_i$ and Λ_i is an essential value of $\rho \restriction (E \restriction A_i)$ for all $i < n$.*

Proof. Let $\mathcal{D}$ be the family of non-empty subsets of Γ that are not essential values of ρ. For each $\Delta \in \mathcal{D}$, fix a partition $\mathcal{B}_\Delta$ of X into countably-many E_ρ-invariant Borel sets with the property that $\Delta \nsubseteq \rho((E \restriction B) \setminus \Delta(B))$ for all $B \in \mathcal{B}_\Delta$. Fix an enumeration $(A'_k)_{k\in\mathbb{N}}$ of the set of atoms of the Boolean algebra generated by $\bigcup_{\Delta\in\mathcal{D}} \mathcal{B}_\Delta$. For all $k \in \mathbb{N}$, fix $\gamma_k \in \Gamma$ and $i_k < n$ such that $\rho(E \restriction A'_k) \subseteq \gamma_k^{-1}\Delta_{i_k}\gamma_k$ and set $A''_k = \gamma_k A'_k \setminus \bigcup_{j<k}[A'_j]_E$. Then the sets $A_i = \bigcup\{A''_k \mid i = i_k\}$ and $B_i = [A_i]_E$ are as desired. ⊠

The *complete equivalence relation* on X is given by $I(X) = X \times X$. For all equivalence relations E on X, groups Γ, and cocycles $\rho\colon E \to \Gamma$, set $X_\Gamma = \Gamma \times X$ and $E_\Gamma = I(\Gamma) \times E$, and define $\rho_\Gamma\colon E_\Gamma \to \Gamma$ by $\rho_\Gamma((\gamma, x), (\delta, y)) = \gamma\rho(x, y)\delta^{-1}$ for all $\gamma, \delta \in \Gamma$ and $x \mathrel{E} y$, as well as $\Gamma \curvearrowright X_\Gamma/E_{\rho_\Gamma}$ by $[(\gamma, x)]_{E_{\rho_\Gamma}} = \lambda \cdot [(\delta, y)]_{E_{\rho_\Gamma}} \iff \rho_\Gamma((\gamma, x), (\delta, y)) = \lambda$.

Theorem 4.2. *Suppose that $\Lambda \leq \Gamma$ are countable discrete groups, $\boldsymbol{\lambda} \in \Lambda^{\mathbb{N}}$ is a redundant enumeration of Λ, X is a standard Borel space, E is a Borel equivalence relation on X, $\Gamma \curvearrowright X/E$ is a free action by strongly Borel automorphisms for which E_Γ^X is hyperfinite, and there* *is an E^X_Γ-complete Borel set $B \subseteq X$ for which $\rho^X_\Gamma(E^X_\Gamma \upharpoonright B) \subseteq \Lambda$ and Λ is an essential value of $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright B)$. Then $\Gamma \curvearrowright (2^\mathbb{N})_\Gamma / E_{(\wp_{\boldsymbol{\lambda}})_\Gamma}$ and $\Gamma \curvearrowright X/E$ are strongly Borel isomorphic.*

Proof. By the Glimm–Effros dichotomy when Λ is trivial and Theorem 1 otherwise, there is a continuous embedding $\phi \colon 2^\mathbb{N} \to B$ of $\wp_{\boldsymbol{\lambda}}$ into $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright B)$. By Theorem 2, there is a Borel embedding $\psi \colon B \to 2^\mathbb{N}$ of $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright B)$ into $\wp_{\boldsymbol{\lambda}}$. Let ϕ' and ψ' be the corresponding strongly Borel embeddings of $2^\mathbb{N}/\mathbb{E}_0$ into X/E^X_Γ and X/E^X_Γ into $2^\mathbb{N}/\mathbb{E}_0$. Then the proof of the Schröder–Bernstein theorem yields a Borel set $C \subseteq 2^\mathbb{N}/\mathbb{E}_0$ for which $(\phi' \upharpoonright {\sim}C) \cup ((\psi')^{-1} \upharpoonright C)$ is a strongly Borel isomorphism of $2^\mathbb{N}/\mathbb{E}_0$ with X/E^X_Γ, in which case $(\phi \upharpoonright {\sim}\bigcup C) \cup (\psi^{-1} \upharpoonright \bigcup C)$ induces a strongly Borel isomorphism of $\Gamma \curvearrowright (2^\mathbb{N})_\Gamma / E_{(\wp_{\boldsymbol{\lambda}})_\Gamma}$ with $\Gamma \curvearrowright X/E$. ⊠

We can now give the following:

Proof of Theorem 4. Let $\mathcal{E}$ be the family of maximal essential values of ρ^X_Γ. By Proposition 1.4, every element of $\mathcal{E}$ is a subgroup of Γ. Fix an injective enumeration $(\Lambda_i)_{i<n}$ of a transversal of the conjugacy equivalence relation on $\mathcal{E}$, as well as redundant enumerations $\boldsymbol{\lambda}_i \in \Lambda_i^\mathbb{N}$ of Λ_i for all $i < n$. It is sufficient to show that $\Gamma \curvearrowright X/E$ is strongly Borel isomorphic to the disjoint union of $\Gamma \curvearrowright (X_{\boldsymbol{\lambda}_i})_\Gamma / E_{(\wp_{\boldsymbol{\lambda}_i})_\Gamma}$ for $i < n$. As Proposition 4.1 yields a partition $(B_i)_{i<n}$ of X into E^X_Γ-invariant Borel sets with the property that there is an $(E^X_\Gamma \upharpoonright B_i)$-complete Borel set $A_i \subseteq B_i$ such that $\rho^X_\Gamma(E^X_\Gamma \upharpoonright A_i) \subseteq \Lambda_i$ and Λ_i is an essential value of $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright A_i)$ for all $i < n$, Theorem 4.2 yields the desired result. ⊠

For all equivalence relations E on a set X, $k \in \mathbb{Z}^+$, and cocycles $\rho \colon E \to S_k$, let $E_{k,\rho}$ be the subequivalence relation of $E_k = I(k) \times E$ given by $(i, x) \mathrel{E_{k,\rho}} (j, y) \iff i = \rho(x, y)(j)$.

Theorem 4.3. *Suppose that $k \in \mathbb{Z}^+$, $\Lambda \leq S_k$, $\boldsymbol{\lambda} \in \Lambda^\mathbb{N}$ is a redundant enumeration of Λ, X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X, F is an index k Borel subequivalence relation of E, $\rho \colon E \to S_k$ is an index cocycle, and there is an E-complete Borel set $B \subseteq X$ for which $\rho(E \upharpoonright B) \subseteq \Lambda$ and Λ is an essential value of $\rho \upharpoonright (E \upharpoonright B)$. Then $(\mathbb{E}_0)_k/(\mathbb{E}_0)_{k,\wp_{\boldsymbol{\lambda}}}$ and E/F are strongly Borel isomorphic.*

Proof. By the Glimm–Effros dichotomy when Λ is trivial and Theorem 1 when Λ is not trivial, there is a continuous embedding $\phi \colon 2^\mathbb{N} \to B$ of $\wp_{\boldsymbol{\lambda}}$ into $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright B)$. By Theorem 2, there is a Borel embedding $\psi \colon B \to 2^\mathbb{N}$ of $\rho^X_\Gamma \upharpoonright (E^X_\Gamma \upharpoonright B)$ into $\wp_{\boldsymbol{\lambda}}$. Let ϕ' and ψ' be the corresponding strongly Borel embeddings of $2^\mathbb{N}/\mathbb{E}_0$ into X/E and X/E into $2^\mathbb{N}/\mathbb{E}_0$. Then the proof of the Schröder–Bernstein theorem yields a Borel set

$C \subseteq 2^{\mathbb{N}}/\mathbb{E}_0$ for which $(\phi' \upharpoonright {\sim}C) \cup ((\psi')^{-1} \upharpoonright C)$ is a strongly Borel isomorphism of $2^{\mathbb{N}}/\mathbb{E}_0$ with X/E, in which case $(\phi \upharpoonright {\sim}\bigcup C) \cup (\psi^{-1} \upharpoonright \bigcup C)$ induces a strongly Borel isomorphism of $(\mathbb{E}_0)_k/(\mathbb{E}_0)_{k,\wp_{\boldsymbol{\lambda}}}$ with E/F. ⊠

We can now give the following:

Proof of Theorem 5. Fix an index cocycle ρ and let $\mathcal{E}$ be the family of maximal essential values of ρ. By Proposition 1.4, every element of $\mathcal{E}$ is a subgroup of Γ. Fix an injective enumeration $(\Lambda_i)_{i<n}$ of a transversal of the conjugacy equivalence relation on $\mathcal{E}$, as well as redundant enumerations $\boldsymbol{\lambda}_i \in \Lambda_i^{\mathbb{N}}$ of Λ_i for all $i < n$. It is sufficient to show that E is strongly Borel isomorphic to the disjoint union of $(\mathbb{E}_0)_k/(\mathbb{E}_0)_{k,\wp_{\boldsymbol{\lambda}_i}}$ for $i < n$. As Proposition 4.1 yields a partition $(B_i)_{i<n}$ of X into E-invariant Borel sets with the property that there is an $(\tilde{E} \upharpoonright B_i)$-complete Borel set $A_i \subseteq B_i$ such that $\rho(\tilde{E} \upharpoonright A_i) \subseteq \Lambda_i$ and Λ_i is an essential value of $\rho \upharpoonright (\tilde{E} \upharpoonright A_i)$ for all $i < n$, Theorem 4.3 yields the desired result. ⊠

5. Decomposability

We say that a Borel automorphism $T \colon X \to X$ of a Borel space is *orbit ergodic* if the corresponding $\mathbb{Z}$-action is orbit ergodic. We use $[Y]_T$ to denote the E_T^X-saturation of a set $Y \subseteq X$, and we say that a set $Y \subseteq X$ is *T-complete* if $X = [Y]_T$.

Proposition 5.1. *Suppose that X is a Borel space and $T \colon X \to X$ is a fixed-point-free Borel automorphism. Then exactly one of the following holds:*

1. *The graph of T is separable.*
2. *The automorphism T is orbit ergodic.*

Proof. Suppose first that T is not orbit ergodic. Then there is a T-complete co-T-complete Borel set $B \subseteq X$, in which case the family $\{T^n(B) \mid n \in \mathbb{Z}\} \cup \{{\sim}T^n(B) \mid n \in \mathbb{Z}\}$ separates graph(T)-related points, so graph(T) is separable.

Conversely, suppose that graph(T) is separable. Then there is a sequence $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X separating graph(T)-related points. Note that the sets of the form $C_n = B_n \setminus T^{-1}(B_n)$ are graph(T)-independent and cover X, and the sets of the form $C'_n = C_n \setminus \bigcup_{m<n} [C_m]_T$ are Borel. As $\bigcup_{n\in\mathbb{N}} C'_n$ is T-complete and co-T-complete, it follows that T is not orbit ergodic. ⊠

Given $k \in \mathbb{Z}^+$, we say that a set $Y \subseteq X$ is *$T^{<k}$-independent* if $Y \cap \bigcup_{0<i<k} T^{-i}(Y) = \emptyset$.

Proposition 5.2. *Suppose that $k \in \mathbb{Z}^+$, X is a Borel space, $T\colon X \to X$ is a Borel automorphism whose orbits all have cardinality at least k, and $\mathrm{graph}(T^j)$ is separable for all $j < k$. Then there is a $T^{<k}$-independent T^k-invariant Borel set $B \subseteq X$ with the property that $\mathrm{graph}(T^k \restriction {\sim}[B]_T)$ is separable.*

Proof. As the digraph $G = \bigcup_{0<j<k} \mathrm{graph}(T^j)$ is separable and has finite horizontal and vertical sections, [Mil, Propositions 1.12 and 1.13] yield a Borel maximal $T^{<k}$-independent set $C \subseteq X$. Set $D = C \triangle T^k(C)$ and observe that $\{T^n(C) \mid n \in \mathbb{Z}\} \cup \{{\sim}T^n(C) \mid n \in \mathbb{Z}\}$ separates $\mathrm{graph}(T^k \restriction [D]_T)$-related points, the set $B = C \setminus [D]_T$ is T^k-invariant, and ${\sim}[B]_T = [D]_T$ (as the maximality of C ensures that it is T-complete). ⊠

We say that a cardinal κ *divides* a cardinal λ, or $\kappa \mid \lambda$, if there is a cardinal μ for which $\lambda = \kappa \times \mu$, where $\times$ denotes cardinal multiplication.

Proposition 5.3. *Suppose that $2 \leq k \leq \aleph_0$, X is a Borel space, and $T\colon X \to X$ is a Borel automorphism whose orbits all have cardinality k. Then there are $T^{<j}$-independent T^j-invariant Borel sets $B_j \subseteq X$, with pairwise disjoint T-saturations, such that:*

1. *$\forall 0 < j < k$ ($B_j \neq \emptyset \implies T^j \restriction B_j$ is orbit ergodic).*
2. *$T \restriction {\sim}\bigcup\{[B_j]_T \mid 0 < j < k$ and $j \mid k\}$ is separable.*

Proof. Set $X_1 = X$. Suppose that $0 < j < k$ and $X_j \subseteq X$ is a T-invariant Borel set with the property that $\mathrm{graph}(T^i \restriction X_j)$ is separable for all $i < j$. If $\mathrm{graph}(T^j \restriction X_j)$ is separable, then set $B_j = \emptyset$. Otherwise, Proposition 5.2 yields a $T^{<j}$-independent T^j-invariant Borel set $B_j \subseteq X_j$ with the property that $\mathrm{graph}(T^j \restriction (X_j \setminus [B_j]_T))$ is separable. Define $X_{j+1} = X_j \setminus [B_j]_T$.

To see (2), note that if $0 < j < k$ and $B_j \neq \emptyset$, then the facts that B_j is $T^{<j}$-independent and T^j-invariant and every orbit of T has cardinality k ensure that $j \mid k$. To see (1), observe that if $T^j \restriction B_j$ is not orbit ergodic, then Proposition 5.1 yields a sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets of B_j separating $\mathrm{graph}(T^j \restriction B_j)$-related points, so $(T^i(A_n))_{(i,n)\in j\times\mathbb{N}}$ separates $\mathrm{graph}(T^j \restriction [B_j]_T)$-related points, contradicting the fact that $\mathrm{graph}(T^j \restriction X_j)$ is not separable. ⊠

Proposition 5.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $T\colon X/E \to X/E$ is a strongly Borel automorphism. Then $\mathrm{supp}(T)$ is Borel.*

Proof. Define $G = \mathrm{graph}(T)$. Then the vertical sections of $\tilde{G}$ are countable and $\bigcup \mathrm{supp}(T) = \mathrm{proj}_0(\tilde{G} \setminus E)$, so the Lusin–Novikov uniformization theorem yields that $\bigcup \mathrm{supp}(T)$ is Borel, thus so too is $\mathrm{supp}(T)$. ⊠

For all $k \in \mathbb{Z}^+$, the *period k part* of an automorphism $T\colon X \to X$ is given by $\mathrm{Per}_k(T) = (\bigcap_{0<i<k} \mathrm{supp}(T^i)) \setminus \mathrm{supp}(T^k)$. The *periodic part* of T is given by $\mathrm{Per}(T) = \bigcup_{k \in \mathbb{Z}^+} \mathrm{Per}_k(T)$. The *aperiodic part* of T is given by $\mathrm{Aper}(T) = \mathrm{Per}_{\aleph_0}(T) = {\sim}\mathrm{Per}(T)$. Propositions 3.2 and 5.4 ensure that if X is a standard Borel space, E is a countable Borel equivalence relation on X, and $T\colon X/E \to X/E$ is a strongly Borel automorphism, then all of these sets are Borel.

We can now give the following:

Proof of Theorem 6. By Proposition 5.3, it is enough to show that if $h \mid k$ and there is a $T^{<h}$-independent T^h-invariant T-complete Borel set $B \subseteq X/E$ for which $T^h \restriction B$ is orbit ergodic, then there is a strongly Borel isomorphism of T with $\mathbb{T}_{k/h} * h$. But Theorem 3 yields a strongly Borel isomorphism $\pi\colon B \to X/E$ of $T^h \restriction B$ with $\mathbb{T}_{k/h}$, in which case the function $T^i([x]_E) \mapsto (\pi([x]_E), i)$, where $1 \leq i \leq h$ and $x \in \bigcup B$, is as desired. ⊠

Remark 5.5. The hyperfiniteness of E_T^X in the statement of Theorem 6 can be replaced with the hypothesis that $E_T^{X/E}$ is *hyperfinite* in the sense that it is the union of an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of finite weakly Borel subequivalence relations. To establish this, appeal first to the fact that the class of hyperfinite Borel equivalence relations on standard Borel spaces is closed under passage to finite-index Borel superequivalence relations (see [JKL02, Proposition 1.5]) to see that $E_T^{\bigcup \mathrm{Per}(T)}$ is hyperfinite, so Theorem 6 implies that $T \restriction \mathrm{Per}(T)$ is decomposable. But the fact that $E_T^{X/E}$ is hyperfinite easily implies that $T \restriction \mathrm{Aper}(T)$ is separable, thus T is decomposable.

6. Products of periodic automorphisms

We begin this section by noting the following fact, which ensures that writing an automorphism as a product of two involutions is essentially the same as finding an anti-commuting involution:

Proposition 6.1. *Suppose that X is a set, $T\colon X \to X$ is a bijection, and $I\colon X \to X$ is an involution. Then $I \circ T = T^{-1} \circ I$ if and only if $I \circ T$ is an involution.*

Proof. If $I \circ T = T^{-1} \circ I$, then $(I \circ T)^2 = (I \circ T) \circ (T^{-1} \circ I) = I^2 = \mathrm{id}$. Conversely, if $(I \circ T)^2 = \mathrm{id}$, then $I \circ T = (I \circ T)^{-1} = T^{-1} \circ I$. ⊠

By combining Proposition 6.1 with the following observation, one can show that if an automorphism is a composition of two involutions, then so too is any constant height tower over it:

Proposition 6.2. *Suppose that $h \geq 2$, X is a set, $T\colon X \to X$ is a bijection, $I\colon X \to X$ is an involution for which $I \circ T = T^{-1} \circ I$, and $J\colon X \times \{1, \ldots, h\} \to X \times \{1, \ldots, h\}$ is given by*

$$J(x, i) = \begin{cases} (I(x), 1) & \text{if } i = 1 \text{ and} \\ ((I \circ T)(x), h - i + 2) & \text{otherwise.} \end{cases}$$

Then J is an involution and $J \circ (T * h) = (T * h)^{-1} \circ J$.

Proof. To see that J is an involution, note that $J \upharpoonright (X \times \{1\})$ is a copy of I and $J^2(x, i) = J((I \circ T)(x), h - i + 2) = (x, h - (h - i + 2) + 2) = (x, i)$ for all $2 \leq i \leq h$ and $x \in X$ by Proposition 6.1.

To see that $J \circ (T * h) = (T * h)^{-1} \circ J$, note that if $x \in X$, then

$$\begin{aligned} (J \circ (T * h))(x, 1) &= J(x, 2) \\ &= ((I \circ T)(x), h) \\ &= ((T^{-1} \circ I)(x), h) \\ &= (T * h)^{-1}(I(x), 1) \\ &= ((T * h)^{-1} \circ J)(x, 1), \end{aligned}$$

$$\begin{aligned} (J \circ (T * h))(x, h) &= J(T(x), 1) \\ &= ((I \circ T)(x), 1) \\ &= (T * h)^{-1}((I \circ T)(x), 2) \\ &= ((T * h)^{-1} \circ J)(x, h), \end{aligned}$$

and

$$\begin{aligned} (J \circ (T * h))(x, i) &= J(x, i + 1) \\ &= ((I \circ T)(x), h - i + 1) \\ &= (T * h)^{-1}((I \circ T)(x), h - i + 2) \\ &= ((T * h)^{-1} \circ J)(x, i) \end{aligned}$$

for all $1 < i < h$. ⊠

We now give an explicit description of $\mathbb{T}_k$ for all $2 \leq k \leq \aleph_0$. We use $\forall^\infty n \in \mathbb{N}\ \phi(n)$ as shorthand for $\exists N \in \mathbb{N} \forall n \geq N\ \phi(n)$. We say that natural numbers m and n are *congruent* modulo k, or $m \equiv n \pmod{k}$, if $k \mid |m - n|$. Let $\mathbb{F}_k$ denote the subequivalence relation of $\mathbb{E}_0$ given by $c\ \mathbb{F}_k\ d \iff \forall^\infty n \in \mathbb{N}\ \sum_{m<n} c(m) \equiv \sum_{m<n} d(m) \pmod{k}$. Let $\mathbb{T}_k$ denote the strongly Borel automorphism of the quotient of the set of non-eventually-constant sequences in $2^{\mathbb{N}}$ by $\mathbb{F}_k$ given by $\mathbb{T}_k([c]_{\mathbb{F}_k}) = [d]_{\mathbb{F}_k} \iff \forall^\infty n \in \mathbb{N}\ 1 + \sum_{m<n} c(m) \equiv \sum_{m<n} d(m) \pmod{k}$.

Proposition 6.3. *Suppose that $2 \leq k \leq \aleph_0$. Then $\mathbb{T}_k$ is orbit ergodic.*

Proof. Set $\boldsymbol{g} = (1_{\mathbb{Z}/k\mathbb{Z}})^\infty$. Then the orbit cocycle associated with the action generated by $\mathbb{T}_k$ agrees with $\wp_{\boldsymbol{g}}$ on the restriction of $G_{(0)^n}$ to the non-eventually constant sequences for all $n \in \mathbb{N}$, and therefore with the restriction of $\mathbb{E}_0$ to the non-eventually constant sequences by Proposition 1.6, so Proposition 1.5 ensures that $\{1_{\mathbb{Z}/k\mathbb{Z}}\}$ is an essential value of the corresponding orbit cocycle, thus Proposition 1.4 implies that so too is $\mathbb{Z}/k\mathbb{Z}$, hence $\mathbb{T}_k$ is orbit ergodic by Proposition 3.5. ⊠

For all $c \in 2^{\mathbb{N}}$, let $\overline{c}$ be the element of $2^{\mathbb{N}}$ given by $\overline{c}(n) = 1 - c(n)$ for all $n \in \mathbb{N}$. For all $2 \leq k \leq \aleph_0$, let $\mathbb{I}_k$ be the involution of the quotient of the set of non-eventually-constant sequences in $2^{\mathbb{N}}$ by $\mathbb{F}_k$ given by $\mathbb{I}_k([c]_{\mathbb{F}_k}) = [\overline{c}]_{\mathbb{F}_k}$ for all non-eventually constant sequences $c \in 2^{\mathbb{N}}$.

Proposition 6.4. *Suppose that* $2 \leq k \leq \aleph_0$. *Then* $\mathbb{I}_k \circ \mathbb{T}_k = \mathbb{T}_k^{-1} \circ \mathbb{I}_k$.

Proof. Note that if $c \in 2^{\mathbb{N}}$ is not eventually constant and $n \in \mathbb{N}$, then

$$\begin{aligned}(\mathbb{I}_k \circ \mathbb{T}_k)([(1)^n \frown (0) \frown c]_{\mathbb{F}_k}) &= \mathbb{I}_k([(1)^{n+1} \frown c]_{\mathbb{F}_k}) \\ &= [(0)^{n+1} \frown \overline{c}]_{\mathbb{F}_k} \\ &= \mathbb{T}_k^{-1}([(0)^n \frown (1) \frown \overline{c}]_{\mathbb{F}_k}) \\ &= (\mathbb{T}_k^{-1} \circ \mathbb{I}_k)([(1)^n \frown (0) \frown c]_{\mathbb{F}_k}).\end{aligned}$$

But if $d \in 2^{\mathbb{N}}$ is not eventually constant, then there exist a non-eventually-constant $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ for which $d = (1)^n \frown (0) \frown c$. ⊠

A *(partial) transversal* of a bijection $T \colon X \to X$ is a (partial) transversal of E_T^X.

We can now give the following:

Proof of Theorem 10. By [Mil, Theorem 1], it is sufficient to handle the case that T is of the form $\mathbb{T}_k * h$. But this follows from Propositions 6.1, 6.2, and 6.4. ⊠

Proof of Theorem 11. As the automorphisms of the form $\mathbb{T}_k \amalg \mathbb{T}_k^{-1}$ are orbit ergodic, it is sufficient to handle the case that T is separable or of the form $(\mathbb{T}_k \amalg \mathbb{T}_k^{-1}) * h$ by Theorem 3. For the latter, appeal to Propositions 6.1, 6.2, and 6.4 to obtain strongly Borel involutions I and J for which $\mathbb{T}_k * h = I \circ J$, and note that $(\mathbb{T}_k \amalg \mathbb{T}_k^{-1}) * h$ and $(I \amalg J) \circ (J \amalg I)$ are strongly Borel isomorphic. For the former, we need only show that T is contained in an aperiodic countable group Γ of separable strongly Borel automorphisms of X/E by [Mil, Theorem 5].

If the periodic part of T is finite, then fix $x \in X$ whose E-class is in the aperiodic part of T, and a transversal $B \subseteq X/E$ of the restriction of T to its periodic part. Then the group Γ generated by T and the transpositions of the form $([x]_E \ [y]_E)$, where $[y]_E \in B$, is as desired.

We can therefore assume that T is periodic. By [Mil, Proposition 1.15], there is a Borel transversal $B \subseteq X/E$ of T. It is sufficient to show that there is an aperiodic separable strongly Borel automorphism $S \colon B \to B$, since the $E_T^{X/E}$-saturations of sets in separating families for the powers of such an S together with separating families for the powers of T yield a separating family for the equivalence relation generated by S and T, so the group generated by $S \cup (\mathrm{id} \upharpoonright {\sim}B)$ and T is as desired. If B is countable, then the existence of such an S follows from the fact that B is strongly Borel isomorphic to $\mathbb{Z}$ (where the latter is endowed with the power set Borel structure). If B is uncountable but standard, then the existence of such an S follows from the fact that B is strongly Borel isomorphic to $\mathbb{R} \times \mathbb{Z}$ (by the isomorphism theorem for standard Borel spaces). And if B is not standard, then the existence of such an S follows from the fact that B is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times \mathbb{Z}$ (by the isomorphism theorem for hyperfinite quotients). ⊠

7. Normalizers

Let $[x]_T$ denote the orbit of a point $x \in X$ under a bijection $T \colon X \to X$. We say that a strongly Borel automorphism is *smooth* if it admits a Borel transversal.

Proposition 7.1. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y, and $S \colon X/E \to X/E$ and $T \colon Y/F \to Y/F$ are smooth strongly Borel automorphisms with the property that there is a strongly Borel automorphism $\phi_k \colon \bigcup \mathrm{Per}_k(S)/E_S^X \to \bigcup \mathrm{Per}_k(T)/E_T^Y$ for all $1 \leq k \leq \aleph_0$. Then S and T are strongly Borel isomorphic.*

Proof. Fix Borel transversals $A \subseteq X/E$ and $B \subseteq Y/F$ of S and T. For all $1 \leq k \leq \aleph_0$, define $\psi_k \colon A \cap \mathrm{Per}_k(S) \to B \cap \mathrm{Per}_k(T)$ by

$$\psi_k([x]_E) = [y]_F \iff (x \in \bigcup A,\ y \in \bigcup B, \text{ and } \phi_k([x]_S) = [y]_T)$$

and set $\pi_k = \bigcup_{n \in \mathbb{Z}} T^n \circ \psi_k \circ S^{-n}$. Then the function $\pi = \bigcup_{1 \leq k \leq \aleph_0} \pi_k$ is the desired isomorphism. ⊠

Proposition 7.2. *Suppose that $\{1\} \subseteq K \subseteq \mathbb{Z}^+$, X is a standard Borel space, E is a countable Borel equivalence relation on X, and $T \colon X/E \to X/E$ is a strongly Borel automorphism whose support is not standard. Then there is a strongly Borel automorphism $S \colon X/E \to X/E$ for which the cardinalities of the orbits of $[S, T]$ are in K and each possibility occurs on a non-standard Borel set.*

Proof. As the fact that $\mathrm{supp}(T)$ is not standard ensures that $E \upharpoonright \bigcup \mathrm{supp}(T)$ is not smooth, [Mil11, Theorem 4] yields a Borel partial transversal $B \subseteq \operatorname{supp}(T)$ of T that is a non-standard hyperfinite quotient. As the isomorphism theorem for hyperfinite quotients ensures that B is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times K$, there is a partition $(B_k)_{k \in K}$ of B into non-standard Borel sets. For all $k \in K$, the isomorphism theorem for hyperfinite quotients also implies that B_k is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times k$, so there is a Borel automorphism $S_k \colon B_k \to B_k$ whose orbits have cardinality k. Define $S = (\bigcup_{k \in K} S_k) \cup (\mathrm{id} \upharpoonright {\sim}B)$. Note that if $k \in K$, then

$$\begin{aligned}(S \circ T \circ S^{-1} \circ T^{-1}) \upharpoonright B_k &= S \circ T \circ (S^{-1} \upharpoonright T^{-1}(B_k)) \circ T^{-1} \\ &= S \circ T \circ (\mathrm{id} \upharpoonright T^{-1}(B_k)) \circ T^{-1} \\ &= (S \circ T \circ \mathrm{id} \circ T^{-1}) \upharpoonright B_k \\ &= S \upharpoonright B_k \\ &= S_k\end{aligned}$$

and

$$\begin{aligned}(S \circ T \circ S^{-1} \circ T^{-1}) \upharpoonright T(B_k) &= S \circ T \circ (S^{-1} \upharpoonright B_k) \circ T^{-1} \\ &= (S \upharpoonright T(B_k)) \circ T \circ S_k^{-1} \circ T^{-1} \\ &= (\mathrm{id} \upharpoonright T(B_k)) \circ T \circ S_k^{-1} \circ T^{-1} \\ &= T \circ S_k^{-1} \circ T^{-1},\end{aligned}$$

so $|[x]_{[S,T]}| = k$ for all $x \in B_k \cup T(B_k)$. But

$$\begin{aligned}&(S \circ T \circ S^{-1} \circ T^{-1}) \upharpoonright {\sim}(B \cup T(B)) \\ &= S \circ T \circ (S^{-1} \upharpoonright {\sim}(T^{-1}(B) \cup B)) \circ T^{-1} \\ &= S \circ T \circ (\mathrm{id} \upharpoonright {\sim}(T^{-1}(B) \cup B)) \circ T^{-1} \\ &= (S \circ T \circ \mathrm{id} \circ T^{-1}) \upharpoonright {\sim}(B \cup T(B)) \\ &= S \upharpoonright {\sim}(B \cup T(B)) \\ &= \mathrm{id} \upharpoonright {\sim}(B \cup T(B)),\end{aligned}$$

thus $|[x]_{[S,T]}| = 1$ for all $x \in {\sim}(B \cup T(B))$. ⊠

Given a Borel equivalence relation E on a standard Borel space X, the *full group* of a strongly Borel automorphism $T \colon X/E \to X/E$ is the group $[T]$ of strongly Borel automorphisms of the form $\bigcup_{n \in \mathbb{Z}} T^n \upharpoonright B_n$, where $(B_n)_{n \in \mathbb{Z}}$ is a sequence of Borel subsets of X/E. In what follows, we often implicitly use the simple observation that the class of standard Borel spaces is closed under countable unions.

Proposition 7.3. *Suppose that X is a standard Borel space, E is a non-smooth countable Borel equivalence relation on X, and $T \colon X/E \to$*

X/E is a smooth strongly Borel automorphism. Then T is a composition of two strongly Borel involutions with non-standard co-non-standard supports.

Proof. We first handle the special case that $T = \mathrm{id}$. By the Glimm–Effros dichotomy, there is a non-standard Borel set $A \subseteq X/E$ that is a hyperfinite quotient. As the isomorphism theorem for hyperfinite quotients ensures that A is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times 3$, there is a partition of A into strongly Borel isomorphic Borel sets B, C, and D. Fix a strongly Borel isomorphism $\pi \colon B \to C$ and observe that the involutions $I = J = \pi^{\pm 1} \cup (\mathrm{id} \restriction ((\sim A) \cup D))$ are as desired.

We next handle the special case that T has no fixed points. Fix a Borel transversal $A \subseteq X/E$ of T. By the Glimm–Effros dichotomy, there is a non-standard Borel set $B \subseteq A$ that is a hyperfinite quotient. As the isomorphism theorem for hyperfinite quotients ensures that B is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times 2$, by thinning it down, we can assume that neither B nor $A \setminus B$ are standard. Set $C = [B]_T$. Then [Mil, Remark 1.4] yields Borel involutions $I', J' \in [T \restriction C]$, for which I' has a fixed point on every orbit of $T \restriction C$, such that $T \restriction C = I' \circ J'$. It also yields Borel involutions $I'', J'' \in [T \restriction \sim C]$, for which J'' has a fixed point on every orbit of $T \restriction \sim C$, such that $T \restriction \sim C = I'' \circ J''$. Then the involutions $I = I' \cup I''$ and $J = J' \cup J''$ are as desired.

For the general case, fix $1 \leq k \leq \aleph_0$ for which $\mathrm{Per}_k(T)$ is not standard, as well as Borel involutions $I', J' \colon \mathrm{Per}_k(T) \to \mathrm{Per}_k(T)$, with non-standard co-non-standard supports, for which $T \restriction \mathrm{Per}_k(T) = I' \circ J'$. By [Mil, Proposition 1.1], there are Borel involutions $I'', J'' \colon \sim\mathrm{Per}_k(T) \to \sim\mathrm{Per}_k(T)$ with the property that $T \restriction \sim\mathrm{Per}_k(T) = I'' \circ J''$. Then the involutions $I = I' \cup I''$ and $J = J' \cup J''$ are as desired. ⊠

Proposition 7.4. *Suppose that $h \in \mathbb{Z}^+$ and $k \geq 2$. Then $\mathbb{T}_k * h$ is a composition of two strongly Borel involutions with non-standard co-non-standard supports.*

Proof. By [Mil11, Theorem 4], there is a $\mathbb{T}_k$-invariant non-standard Borel set $B \subseteq \{c \in 2^{\mathbb{N}} \mid c \text{ is not eventually constant}\}/\mathbb{F}_k$ on which $\mathbb{T}_k$ is smooth, so $C = B \times \{1, \ldots, h\}$ is a $(\mathbb{T}_k * h)$-invariant non-standard Borel set on which $\mathbb{T}_k * h$ is smooth. Proposition 7.3 then ensures that $(\mathbb{T}_k * h) \restriction C$ is the composition of two strongly Borel involutions I' and J' with non-standard co-non-standard supports. Proposition 1.5 implies that B is meager, so Propositions 1.5 and 3.5 ensure that $\mathbb{T}_k \restriction \sim B$ is orbit ergodic, thus Theorem 3 implies that it is strongly Borel isomorphic to $\mathbb{T}_k$, hence $(\mathbb{T}_k * h) \restriction \sim C$ is strongly Borel isomorphic to $\mathbb{T}_k * h$. Propositions 6.1, 6.2, and 6.4 therefore imply that $(\mathbb{T}_k * h) \restriction \sim C$

is the composition of two strongly Borel involutions I'' and J'', in which case the involutions $I = I' \cup I''$ and $J = J' \cup J''$ are as desired. ⊠

Proposition 7.5. *Suppose that* $\{1, 2\} \subsetneq K \subseteq \mathbb{Z}^+$, *$X$ is a standard Borel space, E is a countable Borel equivalence relation on X, and* $T : X/E \to X/E$ *is a separable strongly Borel automorphism whose aperiodic part is not standard. Then there exist* $R, S \in [T]$, *whose orbits all have cardinality in K and for which each possibility occurs on a non-standard Borel set, such that* $T = R \circ S$.

Proof. As [Mil, Proposition 1.15] ensures that $T \restriction \mathrm{Per}(T)$ is smooth, [Mil, Proposition 1.1] yields Borel involutions $I', J' \in [T \restriction \mathrm{Per}(T)]$ for which $T \restriction \mathrm{Per}(T) = I' \circ J'$. By [Mil11, Theorem 4], there is a T-invariant non-standard Borel set $B \subseteq \mathrm{Aper}(T)$ for which $T \restriction B$ is smooth. By [Mil, Proposition 3.16], there exist $R'', S'' \in [T \restriction B]$, whose orbits all have cardinality in K and for which each cardinality occurs on every orbit of $T \restriction B$, such that $T \restriction B = R'' \circ S''$. Fix $k \in K \setminus \{1, 2\}$ and appeal to [Mil, Theorem 3] to obtain $R''', S''' \in [T \restriction (\mathrm{Aper}(T) \setminus B)]$, whose orbits all have cardinality 1 or k, such that $T \restriction (\mathrm{Aper}(T) \setminus B) = R''' \circ S'''$. Then the automorphisms $R = I' \cup R'' \cup R'''$ and $S = J' \cup S'' \cup S'''$ are as desired. ⊠

We can now give the following:

Proof of Theorem 12. Note first that if $I : X/E \to X/E$ is a strongly Borel automorphism for which there is a strongly Borel automorphism $R : X/E \to X/E$ such that I is strongly Borel isomorphic to $[R, T]$ as witnessed by $P : X/E \to X/E$, then

$$\begin{aligned} I &= P^{-1} \circ [R, T] \circ P \\ &= P^{-1} \circ R \circ T \circ R^{-1} \circ T^{-1} \circ P \\ &= (P^{-1} \circ R \circ T \circ R^{-1} \circ P) \circ (P^{-1} \circ T^{-1} \circ P), \end{aligned}$$

so I is a composition of two conjugates of $T^{\pm 1}$. In particular, it is sufficient to show that S is a composition of two such automorphisms.

As S is decomposable, there is an S-invariant Borel set $A \subseteq X/E$, on which S is separable, such that $S \restriction {\sim}A$ is strongly Borel isomorphic to a disjoint union of countably-many automorphisms of the form $\mathbb{T}_k * h$. Note that $S \restriction (A \cap \mathrm{Per}(S))$ is smooth by [Mil, Proposition 1.15].

We first consider the special case that $S \restriction (A \cap \mathrm{Aper}(S))$ is smooth, or equivalently, that $S \restriction A$ is smooth. If $A = X/E$, then Proposition 7.3 yields strongly Borel involutions $I, J : X/E \to X/E$, with non-standard co-non-standard supports, for which $S = I \circ J$. Otherwise, [Mil, Proposition 1.1] and Proposition 7.4 give rise to such involutions. By Proposition 7.2, there is a strongly Borel automorphism

$R\colon X/E \to X/E$ with the property that $[R,T]$ is an involution with non-standard co-non-standard support. The fact that the class of non-smooth hyperfinite Borel equivalence relations on standard Borel spaces is closed under passage to finite-index Borel superequivalence relations, the isomorphism theorem for hyperfinite quotients, and Proposition 7.1 therefore ensure that I, J, and $[R,T]$ are strongly Borel isomorphic.

We next consider the special case that $S \restriction (A \cap \mathrm{Aper}(S))$ is not smooth but $A \cap \mathrm{Aper}(S)$ is standard Borel. By [Mil, Proposition 1.1], there are strongly Borel involutions $I', J'\colon A \cap \mathrm{Per}(S) \to A \cap \mathrm{Per}(S)$ whose composition is $S \restriction (A \cap \mathrm{Per}(S))$; moreover, if $A \cap \mathrm{Per}(S)$ is not standard, then Proposition 7.3 allows us to assume that these involutions have non-standard co-non-standard supports. By [Mil, Proposition 4.4], there are strongly Borel automorphisms $I'', J''\colon A \cap \mathrm{Aper}(S) \to A \cap \mathrm{Aper}(S)$, whose orbits all have cardinality 1, 2, or 3 and for which each possibility occurs on an uncountable Borel set, whose composition is $S \restriction (A \cap \mathrm{Aper}(S))$. By Propositions 6.1, 6.2, and 6.4, there are Borel involutions $I''', J'''\colon {\sim}A \to {\sim}A$ whose composition is $S \restriction {\sim}A$; moreover, if ${\sim}A$ is not standard, then Proposition 7.4 allows us to assume that these involutions have non-standard co-non-standard supports. Then the orbits of the automorphisms $I = I' \cup I'' \cup I'''$ and $J = J' \cup J'' \cup J'''$ have cardinality 1, 2, or 3, the first two possibilities occur on a non-standard Borel set, the last possibility occurs on an uncountable standard Borel set, and $S = I \circ J$. By Silver's perfect set theorem (see [Sil80], although the special case needed here is far simpler to establish), there is a T-invariant uncountable standard Borel set $B \subseteq \mathrm{supp}(T)$. By [Mil, Proposition 4.2], there is a Borel automorphism $R'\colon B \to B$ with the property that the orbits of $[R', T \restriction B]$ have cardinality 1, 2, or 3 and each possibility occurs on an uncountable standard Borel set. By Proposition 7.2, there is a Borel automorphism $R''\colon {\sim}B \to {\sim}B$ with the property that $[R'', T \restriction {\sim}B]$ is an involution with non-standard co-non-standard support. Setting $R = R' \cup R''$, it follows that the orbits of $[R,T]$ have cardinality 1, 2, or 3, the first two possibilities occur on a non-standard Borel set, and the last possibility occurs on an uncountable standard Borel set. The fact that the class of non-smooth hyperfinite Borel equivalence relations on standard Borel spaces is closed under passage to finite-index Borel superequivalence relations, the isomorphism theorem for hyperfinite quotients, and Proposition 7.1 therefore ensure that I, J, and $[R,T]$ are strongly Borel isomorphic.

Finally, we consider the special case that $S \restriction (A \cap \mathrm{Aper}(S))$ is not smooth and $A \cap \mathrm{Aper}(S)$ is not standard Borel (although we will not need the former assumption). By applying [Mil, Proposition 1.1] to

$S \restriction (A \cap \mathrm{Per}(S))$, Proposition 7.5 to $S \restriction (A \cap \mathrm{Aper}(S))$, and Propositions 6.1, 6.2, and 6.4 to $S \restriction {\sim}A$, we obtain Borel automorphisms $I, J \colon X/E \to X/E$, whose orbits all have cardinality 1, 2, or 3 and for which each possibility occurs on a non-standard Borel set, such that $S = I \circ J$. By Proposition 7.2, there is a Borel automorphism $R \colon X/E \to X/E$ for which the orbits of $[R, T]$ have cardinality 1, 2, or 3 and each possibility occurs on a non-standard Borel set. The fact that the class of non-smooth hyperfinite Borel equivalence relations on standard Borel spaces is closed under passage to finite-index Borel superequivalence relations, the isomorphism theorem for hyperfinite quotients, and Proposition 7.1 therefore ensure that I, J, and $[R, T]$ are strongly Borel isomorphic. ⊠

8. The Bergman property

Given a group Γ of permutations of a set X, $\Delta \subseteq \Gamma$, and a set $Y \subseteq X$, define $\Delta_{\{Y\}} = \{\delta \in \Delta \mid Y = \delta Y\}$ and $\Delta \restriction Y = \{\delta \restriction Y \mid \delta \in \Delta\}$.

Proposition 8.1. *Suppose that X is a standard Borel space, E is a non-smooth countable Borel equivalence relation on X, and $(\Gamma_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of subsets of $\mathrm{Aut}_{\mathrm{sB}}(X/E)$. Then there exist a non-standard Borel set $B \subseteq X/E$ and $n \in \mathbb{N}$ with the property that $\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\{B\}} \restriction B \subseteq \Gamma_n \restriction B$.*

Proof. By the Glimm–Effros dichotomy, there is a non-standard Borel set $A \subseteq X/E$ that is a hyperfinite quotient. As the isomorphism theorem for hyperfinite quotients ensures that A is strongly Borel isomorphic to $(\mathbb{R}/\mathbb{Q}) \times \mathbb{N}$, there is a partition $(B_n)_{n \in \mathbb{N}}$ of X/E into non-standard Borel sets. If there is no $n \in \mathbb{N}$ for which B_n and n are as desired, then there exists $\gamma_n \in (\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\{B_n\}} \restriction B_n) \setminus (\Gamma_n \restriction B_n)$ for all $n \in \mathbb{N}$, in which case $\gamma = \bigcup_{n \in \mathbb{N}} \gamma_n$ is a strongly Borel automorphism of X/E, so there exists $n \in \mathbb{N}$ for which $\gamma \in \Gamma_n$, thus $\gamma_n = \gamma \restriction B_n \in \Gamma_n \restriction B_n$, a contradiction. ⊠

Define $\Delta_Y = \bigcap_{y \in Y} \Delta_{\{y\}} = \{\delta \in \Delta \mid \forall y \in Y \ y = \delta \cdot y\}$.

Proposition 8.2. *Suppose that X is a standard Borel space, E is a non-smooth hyperfinite Borel equivalence relation on X, every strongly Borel automorphism of X/E is decomposable, and $(\Gamma_n)_{n \in \mathbb{N}}$ is an exhaustive increasing sequence of subsets of $\mathrm{Aut}_{\mathrm{sB}}(X/E)$. Then there exist a non-standard Borel set $B \subseteq X/E$ and $n \in \mathbb{N}$ with the property that $\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\{B\}} \restriction B \subseteq \Gamma_n \restriction B$ and $\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\sim B} \subseteq (\Gamma_n)^4$.*

Proof. By replacing Γ_n with $\Gamma_n \cap \Gamma_n^{-1}$, we can assume that each Γ_n is symmetric. By Proposition 8.1, there exist a non-standard Borel set

$X' \subseteq X/E$ and $n' \in \mathbb{N}$ such that $\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\{X'\}} \upharpoonright X' \subseteq \Gamma_{n'} \upharpoonright X'$, so $\mathrm{Aut}_{\mathrm{sB}}(X/E)_{\{B\}} \upharpoonright B \subseteq \Gamma_{n'} \upharpoonright B$ for all Borel sets $B \subseteq X'$. Set $\Gamma' = \mathrm{Aut}_{\mathrm{sB}}(X/E)_{\sim X'}$ and $\Gamma'_n = (\Gamma_n)_{\sim X'}$ for all $n \in \mathbb{N}$, and appeal to Proposition 8.1 to obtain a non-standard Borel set $B \subseteq X'$ and $n \geq n'$ with the property that $(\Gamma' \upharpoonright X')_{\{B\}} \upharpoonright B \subseteq (\Gamma'_n \upharpoonright X') \upharpoonright B$, in which case $\Gamma'_{\{B\}} \upharpoonright B = (\Gamma' \upharpoonright X')_{\{B\}} \upharpoonright B \subseteq (\Gamma'_n \upharpoonright X') \upharpoonright B = \Gamma'_n \upharpoonright B$.

It remains to show that if $\gamma \in \mathrm{Aut}_{\mathrm{sB}}(X/E)_{\sim B}$, then $\gamma \in (\Gamma_n)^4$. By Theorem 11, there are strongly Borel automorphisms $\delta, \lambda \colon B \to B$ with the property that $\gamma \upharpoonright B = [\delta, \lambda]$. As $\delta \cup (\mathrm{id} \upharpoonright \sim B) \in \mathrm{Aut}_{\mathrm{sB}}(X/E)$ and $\lambda \cup (\mathrm{id} \upharpoonright \sim B) \in \Gamma'$, there are extensions $\delta' \in \Gamma_m$ and $\lambda' \in \Gamma'_n$ of $\delta \cup (\mathrm{id} \upharpoonright (X' \setminus B))$ and λ. Then

$$\begin{aligned}
[\delta', \lambda'] \upharpoonright B &= \delta'\lambda'(\delta')^{-1}(\lambda')^{-1} \upharpoonright B \\
&= \delta\lambda\delta^{-1}\lambda^{-1} \\
&= [\delta, \lambda] \\
&= \gamma \upharpoonright B,
\end{aligned}$$

$$\begin{aligned}
[\delta', \lambda'] \upharpoonright (X' \setminus B) &= \delta'\lambda'(\delta')^{-1}(\lambda')^{-1} \upharpoonright (X' \setminus B) \\
&= \delta'\lambda'((\delta')^{-1} \upharpoonright (X' \setminus B))(\lambda')^{-1} \\
&= \delta'(\lambda' \upharpoonright (X' \setminus B))(\lambda')^{-1} \\
&= \delta' \upharpoonright (X' \setminus B) \\
&= \mathrm{id} \upharpoonright (X' \setminus B),
\end{aligned}$$

and

$$\begin{aligned}
[\delta', \lambda'] \upharpoonright \sim X' &= \delta'\lambda'(\delta')^{-1}(\lambda')^{-1} \upharpoonright \sim X' \\
&= \delta'\lambda'(\delta')^{-1} \upharpoonright \sim X' \\
&= \delta'(\lambda' \upharpoonright \sim X')(\delta')^{-1} \\
&= (\delta' \upharpoonright \sim X')(\delta')^{-1} \\
&= \mathrm{id} \upharpoonright \sim X',
\end{aligned}$$

so $\gamma = [\delta', \lambda'] \in (\Gamma_n)^4$. ⊠

We can now give the following:

Proof of Theorem 13. By Proposition 8.2, there exist $n \in \mathbb{N}$ and a non-standard Borel set $B \subseteq X/E$ for which $\Gamma_{\{B\}} \upharpoonright B \subseteq \Gamma_n \upharpoonright B$ and $\Gamma_{\sim B} \subseteq (\Gamma_n)^4$. The Glimm–Effros dichotomy ensures that—by thinning down B if necessary—we can assume that the set $A = \sim B$ is not standard. One more application of the Glimm–Effros dichotomy yields a partition of B into non-standard Borel sets $C, D \subseteq X/E$.

Lemma 8.3. *There is a strongly Borel involution* $\iota \colon X/E \to X/E$ *for which* $\iota B = A \cup C$.

Proof. By the isomorphism theorem for hyperfinite quotients, there is a strongly Borel isomorphism $\pi \colon D \to A$, in which case the involution $\iota = \pi^{\pm 1} \cup (\mathrm{id} \restriction C)$ is as desired. ⊠

By the isomorphism theorem for hyperfinite quotients, there is a strongly Borel involution $\iota' \colon X/E \to X/E$ for which $\iota' A = B$. By increasing n if necessary, we can assume that $1_\Gamma, \iota, \iota' \in \Gamma_n$. Then $\Gamma_{\{A \cup C\}} \restriction (A \cup C) \subseteq (\Gamma_n)^3 \restriction (A \cup C)$ by the proof of [Mil, Lemma 5.17]. It remains to show that if $\gamma \in \Gamma$, then $\gamma \in (\Gamma_n)^{12}$.

Lemma 8.4. *There exists* $T \in \Gamma_n$ *with the property that* $B \setminus T^{-1}(\gamma A)$ *is not standard.*

Proof. The fact that B is not standard ensures that γB is not standard. As $\gamma B = {\sim}\gamma A = (A \setminus \gamma A) \cup (B \setminus \gamma A)$, it follows that $A \setminus \gamma A$ or $B \setminus \gamma A$ is not standard. In the latter case, the automorphism $T = 1_\Gamma$ is as desired. In the former, the automorphism $T = \iota'$ is as desired, since $\iota'(A \setminus \gamma A) = B \setminus \iota' \gamma A$. ⊠

Lemma 8.5. *There exists* $S \in \Gamma_n$ *for which* $(S^{-1} \circ T^{-1})(\gamma A) \subseteq A \cup C$ *and* $(A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$ *is not standard.*

Proof. By the Glimm–Effros dichotomy, there is a partition of $B \setminus T^{-1}(\gamma A)$ into non-standard Borel sets $C', D' \subseteq X$. By the isomorphism theorem for hyperfinite quotients, there are strongly Borel isomorphisms $\phi \colon C \to (B \cap T^{-1}(\gamma A)) \cup C'$ and $\psi \colon D \to D'$. Then $(\mathrm{id} \restriction A) \cup \phi \cup \psi$ is a strongly Borel automorphism of X/E, so there exists $S \in \Gamma_n$ for which $S \restriction B = \phi \cup \psi$. Then $(S^{-1} \circ T^{-1})(\gamma A) = S^{-1}(A \cap T^{-1}(\gamma A)) \cup S^{-1}(B \cap T^{-1}(\gamma A)) \subseteq A \cup C$ and $C' \subseteq S(C) \setminus T^{-1}(\gamma A)$, so $S(A \cup C) \setminus T^{-1}(\gamma A)$ is not standard, thus $(A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$ is not standard. ⊠

Lemma 8.6. *There exists* $R \in (\Gamma_n)^3$ *with* $(R^{-1} \circ S^{-1} \circ T^{-1})(\gamma A) = A$.

Proof. By the isomorphism theorem for hyperfinite quotients, there are strongly Borel isomorphisms $\phi \colon A \to (S^{-1} \circ T^{-1})(\gamma A)$ and $\psi \colon C \to (A \cup C) \setminus (S^{-1} \circ T^{-1})(\gamma A)$. Then $\phi \cup \psi \cup (\mathrm{id} \restriction D)$ is a strongly Borel automorphism of X/E, so there exists $R \in (\Gamma_n)^3$ with the property that $R \restriction (A \cup C) = \phi \cup \psi$, in which case $(R^{-1} \circ S^{-1} \circ T^{-1})(\gamma A) = (\phi^{-1} \circ S^{-1} \circ T^{-1})(\gamma A) = A$. ⊠

As in the remark following the proof of [Mil, Lemma 5.17], there exists $Q \in (\Gamma_n)^3$ for which $Q \restriction A = (R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma) \restriction A$. Then

$\text{supp}(Q^{-1} \circ R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma) \subseteq B$, so $Q^{-1} \circ R^{-1} \circ S^{-1} \circ T^{-1} \circ \gamma \in (\Gamma_n)^4$, thus $\gamma \in TSRQ(\Gamma_n)^4 \subseteq \Gamma_n \Gamma_n (\Gamma_n)^3 (\Gamma_n)^3 (\Gamma_n)^4 = (\Gamma_n)^{12}$. ⊠

Acknowledgements. I would like to thank Alexander Kechris for encouraging me to publish Theorem 3 and asking the question that led to Theorem 10.

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