A DICHOTOMY FOR COUNTABLE UNIONS OF SMOOTH BOREL EQUIVALENCE RELATIONS

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Abstract. We show that if an equivalence relation E on a Polish space is a countable union of smooth Borel subequivalence relations, then there is either a Borel reduction of E to a countable Borel equivalence relation on a Polish space or a continuous embedding of $\mathbb{E}_1$ into E. We also establish related results concerning countable unions of more general Borel equivalence relations.

Introduction

A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\varphi\colon X \to Y$ with the property that $(\varphi \times \varphi)(R) \subseteq S$. More generally, a *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on X to a sequence $(S_i)_{i \in I}$ of binary relations on Y is a function $\varphi\colon X \to Y$ that is a homomorphisms from R_i to S_i for all $i \in I$. A *cohomomorphism* is a homomorphism of the corresponding complements, a *reduction* is a homomorphism that is also a cohomorphism, and an *embedding* is an injective reduction.

A *Polish space* is a second countable topological space that admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets. A function between topological spaces is *Borel* if preimages of open sets are Borel.

Following the usual abuse of language, we say that an equivalence relation is *countable* if each of its classes is countable. A Borel equivalence relation E on a Polish space is *smooth* if there is a Borel reduction of E to equality on a Polish space, *hypersmooth* if it is the union of an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of smooth Borel subequivalence relations, and *σ-smooth* if it is a countable union of smooth Borel subequivalence relations. A well-known example of a hypersmooth Borel equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ that is not Borel reducible

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to a countable Borel equivalence relation on a Polish space is given by $c \, \mathbb{E}_1 \, d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m)$.

Our primary result here is the following analog of the Kechris–Louveau dichotomy for hypersmooth Borel equivalence relations on Polish spaces (see [KL97, Theorem 1] or Theorem 6.10):

Theorem 1. *Suppose that E is a σ-smooth Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

(1) *There is a Borel reduction of E to a countable Borel equivalence relation on a Polish space.*
(2) *There is a continuous embedding of $\mathbb{E}_1$ into E.*

A *σ-ideal* on a set X is a family $\mathcal{I}$ of subsets of X that is closed under containment and countable unions. When X is a Polish space, we say that such a σ-ideal is *weakly ccc-on-Borel* if there is no uncountable family of pairwise disjoint Borel subsets of X that are not in $\mathcal{I}$. Given sets X and Y, the *horizontal section* of a set $R \subseteq X \times Y$ at a point y of Y is given by $R^y \coloneqq \{x \in X \mid x \; R \; y\}$, and the *vertical section* of R at point x of X is given by $R_x \coloneqq \{y \in Y \mid x \; R \; y\}$. An assignment $x \mapsto \mathcal{I}_x$, sending each point of X to a σ-ideal on X, is *Borel-on-Borel* if $\{x \in X \mid R_x \in \mathcal{I}_x\}$ is Borel for all Borel sets $R \subseteq X \times X$, and *strongly Borel-on-Borel* if $\{(x, y) \in X \times Y \mid R_{(x,y)} \in \mathcal{I}_x\}$ is Borel for all Polish spaces Y and Borel sets $R \subseteq (X \times Y) \times X$. A Borel equivalence relation E on a Polish space X is *idealistic* if there is an E-invariant Borel-on-Borel assignment $x \mapsto \mathcal{I}_x$ sending each point in X to a σ-ideal on X for which $[x]_E \notin \mathcal{I}_x$. We say that E is *ccc idealistic* if each $\mathcal{I}_x$ can be taken to be weakly ccc-on-Borel, *strongly idealistic* if the assignment $x \mapsto \mathcal{I}_x$ can be taken to be strongly Borel-on-Borel, and *strongly ccc idealistic* if $x \mapsto \mathcal{I}_x$ can be taken to be a strongly Borel-on-Borel assignment of weakly ccc-on-Borel σ-ideals.

Recall that the *orbit equivalence relation* induced by a group action $\Gamma \curvearrowright X$ is the equivalence relation E^X_Γ on X given by $x \; E^X_\Gamma \; y \iff \exists \gamma \in \Gamma \; x = \gamma \cdot y$. The Feldman–Moore theorem ensures that every countable Borel equivalence relation on a Polish space is the orbit equivalence relation induced by a Borel action of a countable discrete group (see [FM77, Theorem 1]), and the proof of [Kec92, §1.II.i] shows that every Borel orbit equivalence relation induced by a Borel action of a Polish group on a Polish space is strongly ccc idealistic. By [KL97, Theorem 4.1], the equivalence relation $\mathbb{E}_1$ is not Borel reducible to a ccc-idealistic Borel equivalence relation on a Polish space.

Much as the Kechris–Louveau dichotomy can be used to show that a hypersmooth Borel equivalence relation on a Polish space is Borel reducible to a ccc-idealistic Borel equivalence relation on a Polish space if and only if it is Borel reducible to the orbit equivalence relation induced by a Borel action of $\mathbb{Z}$ on a Polish space, Theorem 1 yields:

Theorem 2. *Suppose that E is a σ-smooth Borel equivalence relation on a Polish space. Then the following are equivalent:*

1. *There is a Borel reduction of E to a ccc-idealistic Borel equivalence relation on a Polish space.*
2. *There is a Borel reduction of E to the orbit equivalence relation induced by a Borel action of a countable discrete group on a Polish space.*

A subset of a topological space is F_σ if it is a countable union of closed sets, and a binary relation R on a Polish space X is *potentially* F_σ if there is a Polish topology on X, generating the same Borel sets as the given topology, with respect to which R is F_σ. Standard change of topology results and the Lusin–Novikov uniformization theorem (see, for example, [Kec95, §13 and Theorem 18.10]) easily imply that countable Borel equivalence relations on Polish spaces are potentially F_σ.

Kechris–Louveau have asked whether a Borel equivalence relation E on a Polish space is Borel reducible to a ccc-idealistic Borel equivalence relation on a Polish space if and only if there is no continuous embedding of $\mathbb{E}_1$ into E. Much as the Kechris–Louveau dichotomy yields a positive answer to this question in the hypersmooth case, Theorem 1 yields the extension to the σ-smooth case, and the underlying argument can be used to obtain a further generalization:

Theorem 3. *Suppose that E is an equivalence relation on a Polish space that is a countable union of subequivalence relations that are Borel reducible to strongly-ccc-idealistic potentially-F_σ equivalence relations on Polish spaces. Then exactly one of the following holds:*

1. *There is a Borel reduction of E to a ccc-idealistic Borel equivalence relation on a Polish space.*
2. *There is a continuous embedding of $\mathbb{E}_1$ into E.*

As the proof of the Feldman-Moore theorem ensures that countable Borel equivalence relations on Polish spaces are countable unions of finite Borel subequivalence relations, it is not difficult to see that a Borel equivalence relation on a Polish space is σ-smooth if and only if it is a countable union of subequivalence relations that are Borel reducible to countable Borel equivalence relations on Polish spaces, so Theorem 1 also yields:

Theorem 4. *Suppose that E is an equivalence relation on a Polish space that is Borel reducible to a ccc-idealistic Borel equivalence relation* *on a Polish space. If E is a countable union of subequivalence relations that are Borel reducible to countable Borel equivalence relations on Polish spaces, then E is Borel reducible to a countable Borel equivalence relation on a Polish space.*

An equivalence relation E on a set X has *countable index* over a subequivalence relation F if every E-class is a countable union of F-classes. Generalizing Theorem 4 in the spirit of [Kit08, Theorem 1.1], we show:

Theorem 5. *Suppose that E is an equivalence relation on a Polish space that is Borel reducible to a ccc-idealistic Borel equivalence relation on a Polish space and $\mathcal{F}$ is a class of strongly-idealistic potentially-F_σ equivalence relations on Polish spaces that is closed under countable disjoint unions and countable-index Borel superequivalence relations. If E is a countable union of subequivalence relations that are Borel reducible to relations in $\mathcal{F}$, then E is Borel reducible to a relation in $\mathcal{F}$.*

The *saturation* of a set $Y \subseteq X$ with respect to an equivalence relation E on X is given by $[Y]_E \coloneqq \{x \in X \mid \exists y \in Y \; x \; E \; y\}$, the *diagonal* on X is given by $\Delta(X) \coloneqq \{(x, y) \in X \times X \mid x = y\}$, the product of equivalence relations E and F on X and Y is the equivalence relation on $X \times Y$ given by $(x, y) \; E \times F \; (x', y') \iff (x \; E \; x' \text{ and } y \; F \; y')$, the *Friedman–Stanley jump* of E is the equivalence relation on $X^{\mathbb{N}}$ given by $x \; E^+ \; y \iff [x(\mathbb{N})]_E = [y(\mathbb{N})]_E$, and we use $E^\cap$ to denote the binary relation on $X^{\mathbb{N}}$ given by $x \; E^\cap \; y \iff [x(\mathbb{N})]_E \cap [y(\mathbb{N})]_E \neq \emptyset$. Theorem 5 yields:

Theorem 6. *Suppose that E is an equivalence relation on a Polish space that is Borel reducible to a ccc-idealistic Borel equivalence relation on a Polish space and F is a strongly-idealistic potentially-F_σ equivalence relation on a Polish space. Then the following are equivalent:*

1. *The equivalence relation E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta(\mathbb{N})$.*
2. *There is a Borel reduction of E to $(F \times \Delta(\mathbb{N}))^\cap$.*
3. *There is a Borel reduction of E to a countable-index Borel superequivalence relation of $F \times \Delta(\mathbb{N})$.*
4. *There is a Borel homomorphism from $(E, {\sim}E)$ to $((F \times \Delta(\mathbb{N}))^+, {\sim}(F \times \Delta(\mathbb{N}))^\cap)$.*

In particular, if there is a Borel reduction of E to $(F \times \Delta(\mathbb{N}))^\cap$, then there is a Borel reduction of E to $(F \times \Delta(\mathbb{N}))^+$.

In §1, we introduce basic notation and definitions. In §2, we review the compact-open topology. In §3, we establish several preliminaries concerning Borel equivalence relations. In §4, we characterize the existence of small definable cores for appropriate families of finite sets. In §5, we consider approximations to embeddings of $\mathbb{E}_1$ into itself. In §6, we establish a pair of technical dichotomies, and provide an alternate proof of the Kechris–Louveau dichotomy. And in §7, we derive Theorems 1–6 from a single common generalization.

1. Notation and definitions

Given a set Z and a function $f\colon X \times Y \to Z$, define $f^y\colon X \to Z$ and $f_x\colon Y \to Z$ by $f^y(x) \coloneqq f_x(y) \coloneqq f(x, y)$ for all $x \in X$ and $y \in Y$.

The *restriction* of a binary relation R on a set X to a set $Y \subseteq X$ is the binary relation on Y given by $R \upharpoonright Y \coloneqq R \cap (Y \times Y)$. When considering properties of $R \upharpoonright Y$ that depend on the ambient space, it will be understood that this ambient space is Y. For instance, if X is a topological space, then we will say that $R \upharpoonright Y$ is meager, or R is meager *on* Y, if it is meager when viewed as a subset of $Y \times Y$.

Given a topological space X and a property φ of elements of X, we use $\forall^* x \in X\ \varphi(x)$ to indicate that $\{x \in X \mid \varphi(x)\}$ is comeager, and $\exists^* x \in X\ \varphi(x)$ to indicate that $\{x \in X \mid \varphi(x)\}$ is not meager.

The *complete equivalence relation* on X is given by $I(X) = X \times X$.

We refer the reader to [Kec95] for basic descriptive set-theoretic background on Baire category and analytic sets.

2. The compact-open topology

Let X and Y be topological spaces. We use $C(X, Y)$ to denote the set of all continuous mappings $f\colon X \to Y$. For $A \subseteq X$ and $B \subseteq Y$, let $M(A, B) \coloneqq \{f \in C(X, Y) \mid f(A) \subseteq B\}$ (we will denote this set by $M_{C(X,Y)}(A, B)$ in case of potential ambiguity). We endow $C(X, Y)$ with the *compact-open topology*, that is, the topology generated by the the sets of the form $M(K, U)$, where K ranges over compact subsets of X and U ranges over open subsets of Y.

We summarize below some classical properties of the compact-open topology. Most of them can be found, for example, in [Eng89].

Proposition 2.1. *Let X and Z be topological spaces and $Y \subseteq Z$ and view $C(X, Y)$ as a subset of $C(X, Z)$. Then the compact-open topology on $C(X, Y)$ coincides with the topology induced by the compact-open topology on $C(X, Z)$.*

Proof. This immediately follows from the fact that, for $K \subseteq X$ compact and $U \subseteq Z$ open, we have $M_{C(X,Y)}(K, U \cap Y) = M_{C(X,Z)}(K, U) \cap C(X, Y)$. ⊠

Proposition 2.2 (see [Eng89, Theorem 3.4.2])**.** *Let X, Y, and Z be topological spaces, with Y being locally compact. Then the composition mapping $C(Y, Z) \times C(X, Y) \to C(X, Z)$, given by $(f, g) \mapsto f \circ g$, is continuous.*

Proposition 2.3. *Let Y and Z be topological spaces, with Y being locally compact. Then the evaluation mapping $C(Y, Z) \times Y \to Z$, given by $(f, x) \mapsto f(x)$, is continuous.*

Proof. Apply Proposition 2.2 with X being a singleton. ⊠

Proposition 2.4 (see [Eng89, Theorems 3.4.3 and 3.4.8])**.** *Let X, Y, and Z be topological spaces, with X being locally compact and Y being Hausdorff.*

(1) *For every $f \in C(X \times Y, Z)$, the mapping $\Lambda(f) \colon Y \to C(X, Z)$, given by $y \mapsto f^y$, is continuous.*
(2) *The mapping $\Lambda \colon C(X \times Y, Z) \to C(Y, C(X, Z))$ hence defined is a homeomorphism.*

Proposition 2.5 (see [Eng89, Exercise 3.4.B])**.** *Let X_1, X_2, Y_1, and Y_2 be topological spaces, with X_1 and X_2 being Hausdorff. Then the mapping $C(X_1, Y_1) \times C(X_2, Y_2) \to C(X_1 \times X_2, Y_1 \times Y_2)$, given by $(f_1, f_2) \mapsto f_1 \times f_2$, is a homeomorphic embedding.*

Let X be a compact topological space and (Y, d) be a metric space. The *uniform metric* on $C(X, Y)$ associated with d is given by $d_\infty(f, g) \coloneqq \sup_{x \in X} d(f(x), g(x))$.

Proposition 2.6 (see [Eng89, Theorems 4.2.17 and 4.3.13])**.** *Let X be a compact topological space and (Y, d) be a metric space. Then d_∞ is a metric on $C(X, Y)$ that is compatible with the compact-open topology. Moreover, if (Y, d) is complete, then so too is $(C(X, Y), d_\infty)$.*

Proposition 2.7 (see [Eng89, Theorem 3.4.16 and Exercise 4.3.F])**.** *Let X and Y be topological spaces, with X being locally compact and second countable.*

(1) *If Y is second countable, then $C(X, Y)$ is second countable.*
(2) *If Y is completely metrizable, then $C(X, Y)$ is completely metrizable.*

In particular, if Y is Polish, then $C(X, Y)$ is Polish.

Given topological spaces X, Y, and Z and $\mathcal{A} \subseteq C(X, Z)$, we denote by $\mathcal{A}^{\to Y}$ the set of all $f \in C(X \times Y, Z)$ such that, for all $y \in Y$, $f^y \in \mathcal{A}$.

Lemma 2.8. *Let X, Y, and Z be topological spaces, with X being locally compact and Y being compact. Let $\mathcal{U} \subseteq C(X, Z)$ be an open set. Then $\mathcal{U}^{\to Y}$ is an open subset of $C(X \times Y, Z)$.*

Proof. Keeping the notation from the statement of Proposition 2.4, we have $\mathcal{U}^{\to Y} = \Lambda^{-1}(M(Y, \mathcal{U}))$, hence Proposition 2.4 immediately gives the desired result. ⊠

Given a natural number n, a set X, and a sequence $(Y_i)_{i\in\mathbb{N}}$, we identify $(X \times \prod_{i<n} Y_i) \times \prod_{i\geq n} Y_i$ with $X \times \prod_{i\in\mathbb{N}} Y_i$.

Lemma 2.9. *Let X be a locally-compact second-countable topological space, $(Y_n)_{n\in\mathbb{N}}$ be a sequence of compact topological spaces, and Z be a completely metrizable topological space. Then there are compatible complete metrics d_n on $C(X \times \prod_{i<n} Y_i, Z)$ such that $\mathrm{diam}_{\mathbb{N}}(\mathcal{A}^{\to \prod_{i\geq n} Y_i}) \leq \mathrm{diam}_n(\mathcal{A})$ for all $n \in \mathbb{N}$ and $\mathcal{A} \subseteq C(X \times \prod_{i<n} Y_i, Z)$, where diam_n denotes the diameter relative to d_n for all $n \in \mathbb{N} \cup \{\mathbb{N}\}$.*

Proof. We first deal with the special case when X is a singleton. Fix a compatible complete metric d on Z. For all $n \in \mathbb{N} \cup \{\mathbb{N}\}$, identify $X \times \prod_{i<n} Y_i$ with $\prod_{i<n} Y_i$ and let d_n be the uniform metric on $C(\prod_{i<n} Y_i, Z)$. Suppose now that $n \in \mathbb{N}$ and $\mathcal{A} \subseteq C(\prod_{i<n} Y_i, Z)$, and observe that if $f, g \in \mathcal{A}^{\to \prod_{i\geq n} Y_i}$, then

$$\begin{aligned} d_{\mathbb{N}}(f, g) &= \sup_{y\in\prod_{i\in\mathbb{N}} Y_i} d(f(y), g(y)) \\ &= \sup_{v\in\prod_{i\geq n} Y_i} \sup_{u\in\prod_{i<n} Y_i} d(f^v(u), g^v(u)) \\ &= \sup_{v\in\prod_{i\geq n} Y_i} d_n(f^v, g^v) \\ &\leq \mathrm{diam}_n(\mathcal{A}), \end{aligned}$$

so $\mathrm{diam}_{\mathbb{N}}(\mathcal{A}^{\to \prod_{i\geq n} Y_i}) \leq \mathrm{diam}_n(\mathcal{A})$.

We now deal with the general case. Let $Z' = C(X, Z)$. By Proposition 2.4, for every $n \in \mathbb{N} \cup \{\mathbb{N}\}$, the mapping $\Lambda_n \colon C(X \times \prod_{i<n} Y_i, Z) \to C(\prod_{i<n} Y_i, Z')$, defined by $\Lambda_n(f)(y) := f^y$, is a homeomorphism. Moreover, it is easy to verify that, for every $f \in C(X \times \prod_{i\in\mathbb{N}} Y_i, Z)$, $n \in \mathbb{N}$, and $y \in \prod_{i\geq n} Y_i$, we have $\Lambda_n(f^y) = \Lambda_{\mathbb{N}}(f)^y$. It follows that, for every $n \in \mathbb{N}$ and $\mathcal{A} \subseteq C(X \times \prod_{i<n} Y_i, Z)$, we have $\Lambda_{\mathbb{N}}(\mathcal{A}^{\to \prod_{i\geq n} Y_i}) = \Lambda_n(\mathcal{A})^{\to \prod_{i\geq n} Y_i}$.

By Proposition 2.7, Z' is completely metrizable, so we can apply the special case to find, for every $n \in \mathbb{N} \cup \{\mathbb{N}\}$, a complete metric d'_n on $C(\prod_{i<n} Y_i, Z')$ such that, for every $n \in \mathbb{N}$ and $\mathcal{A}' \subseteq C(\prod_{i<n} Y_i, Z')$, we have $\mathrm{diam}_{\mathbb{N}}((\mathcal{A}')^{\to \prod_{i\geq n} Y_i}) \leq \mathrm{diam}_n(\mathcal{A}')$. We define, for every $n \in$

$\mathbb{N} \cup \{\mathbb{N}\}$, the metric d_n on $C(X \times \prod_{i<n} Y_i, Z)$ as the pullback of d'_n through Λ_n. Hence, for $n \in \mathbb{N}$ and $\mathcal{A} \subseteq C(X \times \prod_{i<n} Y_i, Z)$, we have

$$\begin{aligned}\operatorname{diam}_{\mathbb{N}}(\mathcal{A}^{\to \prod_{i \geq n} Y_i}) &= \operatorname{diam}_{\mathbb{N}}(\Lambda_{\mathbb{N}}(\mathcal{A}^{\to \prod_{i \geq n} Y_i})) \\ &= \operatorname{diam}_{\mathbb{N}}(\Lambda_n(\mathcal{A})^{\to \prod_{i \geq n} Y_i}) \\ &\leq \operatorname{diam}_n(\Lambda_n(\mathcal{A})) \\ &= \operatorname{diam}_n(\mathcal{A}),\end{aligned}$$

which completes the proof. ⊠

Lemma 2.10. *Let X be a zero-dimensional compact Polish space, $\Omega \subseteq X$ be an open subset, Y be a topological space, and $\mathcal{B}$ be a basis of open subsets of Y that is closed under finite intersection. Then the sets of the form $\bigcap_{i<n} M(K_i, U_i)$, where $n \in \mathbb{N}$, $(K_i)_{i<n}$ is a sequence of nonempty pairwise disjoint clopen subsets of X that are contained in Ω, and $(U_i)_{i<n}$ is a sequence of nonempty elements of $\mathcal{B}$, form a basis of nonempty open subsets of $C(\Omega, Y)$.*

In the special case when $\Omega = X$, the sets as above, but where we, moreover, require that $(K_i)_{i<n}$ is a partition of X, form a basis of nonempty open subsets of $C(X, Y)$.

Proof. Keeping the notation from the statement of the lemma and taking $y_i \in U_i$ for all $i < n$ when $n > 0$, the mapping $f \colon \Omega \to Y$, defined by $f(x) = y_i$ for all $x \in K_i$ and $f(x) = y_0$ for all $x \notin \bigcup_{i<n} K_i$, is an element of $\bigcap_{i<n} M(K_i, U_i)$, which is hence nonempty. We now show that these sets form a basis of open sets of $C(\Omega, Y)$.

Let $\mathcal{U}$ be an open subset of $C(\Omega, Y)$ and $f \in \mathcal{U}$. We can find $k \in \mathbb{N}$, nonempty compact sets $L_0, \ldots, L_{k-1} \subseteq \Omega$, and open sets $V_0, \ldots, V_{k-1} \subseteq Y$ such that $f \in \bigcap_{i<k} M(L_i, V_i) \subseteq \mathcal{U}$. In particular, for every $i < k$, we have $f(L_i) \subseteq V_i$. Fix $i < k$. For every $x \in L_i$, there exists $W_i^x \in \mathcal{B}$ such that $f(x) \in W_i^x \subseteq V_i$. We can find a clopen subset C_i^x of X, contained in Ω, such that $x \in C_i^x \subseteq f^{-1}(W_i^x)$. Since L_i is compact, we can find $x_i^0, \ldots, x_i^{l_i} \in L_i$ such that $L_i \subseteq \bigcup_{j \leq l_i} C_i^{x_i^j}$. Then

$$f \in \bigcap_{i<k} \bigcap_{j \leq l_i} M(C_i^{x_i^j}, W_i^{x_i^j}) \subseteq \mathcal{U}.$$

This shows that, by shrinking $\mathcal{U}$ if necessary, we can assume that the L_i's are clopen in X and the V_i's are elements of $\mathcal{B}$.

For every set $s \subseteq k$, let $K_s = (\bigcap_{i \in s} L_i) \setminus (\bigcup_{i \in k \setminus s} L_i)$ and $U_s = \bigcap_{i \in s} V_i$. Let $S = \{s \subseteq k \mid K_s \neq \emptyset\} \setminus \{\emptyset\}$. For every $s \in S$, K_s is contained in one of the L_i's, and is therefore a subset of Ω; clearly, the K_s's are pairwise disjoint and clopen. Moreover, we have $\bigcap_{s \in S} M(K_s, U_s) =$

$\bigcap_{i<k} M(L_i, V_i)$. Thus the latter set is an open neighborhood of f of the desired form that is contained in $\mathcal{U}$.

In the special case when $\Omega = X$, a basic open subset $\bigcup_{i<n} M(K_i, U_i)$ of $C(X, Y)$, where $n \in \mathbb{N}$, $(K_i)_{i<n}$ is a sequence of nonempty pairwise disjoint clopen subsets of X, and $(U_i)_{i<n}$ is a sequence of nonempty elements of $\mathcal{B}$, can be rewritten as $\bigcup_{i\leq n} M(K_i, U_i)$, where $K_n = X \setminus \bigcup_{i<n} K_i$ and $U_n = X$. ⊠

Let X and Y be topological spaces and $\mathcal{U}$ be an open subset of $C(X, Y)$. We say that $\mathcal{U}$ is *right stable* when there exists a neighborhood $\mathcal{V}$ of the identity in $C(X, X)$ such that $\mathcal{U} \circ \mathcal{V} = \mathcal{U}$.

Corollary 2.11. *Let X be a zero-dimensional second-countable compact topological space, $\Omega \subseteq X$ be an open set, and Y be a second-countable topological space. Then $C(\Omega, Y)$ admits a countable basis of open subsets whose elements are right stable.*

Proof. Fix a countable basis $\mathcal{B}$ for Y that is closed under finite intersection. Then, by Lemma 2.10, the sets of the form $\mathcal{U} \coloneqq \bigcap_{i<n} M(K_i, U_i)$, where $n \in \mathbb{N}$, $(K_i)_{i<n}$ is a sequence of nonempty pairwise disjoint clopen subsets of X that are contained in Ω, and $(U_i)_{i<n}$ is a sequence of nonempty elements of $\mathcal{B}$, form a basis for $C(\Omega, Y)$. Moreover, this basis is countable. It remains to show that every such set is right stable. Keeping the notation as above, the set $\mathcal{V} \coloneqq \bigcap_{i<n} M(K_i, K_i)$ is an open neighborhood of the identity in $C(\Omega, \Omega)$ and $\mathcal{U} \circ \mathcal{V} = \mathcal{U}$. ⊠

3. Preliminaries

Clearly the family of sets $Y \subseteq X$ on which E has countable index over F is closed under containment and F-saturation.

Lemma 3.1. *Suppose that X and Y are sets, $E \subseteq E'$ are equivalence relations on X, $F \subseteq F'$ are equivalence relations on Y, and $\varphi \colon X \to Y$ is a homomorphism from $(E, \sim E')$ to $(F, \sim F')$. If E' has countable index over E, then F' has countable index over F on $\varphi(X)$.*

Proof. It is sufficient to show that if $\varphi(x) = y$ and $[x]_{E'} = \bigcup_{n\in\mathbb{N}} [x_n]_E$, then $[y]_{F' \upharpoonright \varphi(X)} \subseteq \bigcup_{n\in\mathbb{N}} [\varphi(x_n)]_F$. Towards this end, suppose that $y' \in [y]_{F' \upharpoonright \varphi(X)}$, and fix $x' \in X$ for which $\varphi(x') = y'$. As φ is a cohomomorphism from E' to F', it follows that $x \mathrel{E'} x'$, so there exists $n \in \mathbb{N}$ for which $x_n \mathrel{E} x'$, and since φ is a homomorphism from E to F, it follows that $y' \in [\varphi(x_n)]_F$. ⊠

The following fact will prove useful in complexity calculations:

Proposition 3.2. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, and F is a co-analytic equivalence relation on X. Then exactly one of the following holds:*

1. *The equivalence relation E has countable index over $E \cap F$.*
2. *There is a continuous function $\varphi \colon 2^{\mathbb{N}} \to X$ with the property that $(\varphi \times \varphi)^{-1}(E \setminus F)$ is comeager.*

Proof. To see (2) $\implies$ $\neg$(1), suppose that $R \coloneqq (\varphi \times \varphi)^{-1}(E \setminus F)$ is comeager, appeal to Mycielski's theorem (see, for example, [Kec95, Theorem 19.1]) to obtain a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $\sim\Delta(2^{\mathbb{N}})$ to R, and observe that $\varphi \circ \psi$ is a continuous homomorphism from $\sim\Delta(2^{\mathbb{N}})$ to $E \setminus F$, thus condition (1) fails.

To see $\neg$(1) $\implies$ (2), suppose that there exists $x \in X$ for which $F \restriction [x]_E$ has uncountably many classes, appeal to the straightforward generalization of Silver's perfect set theorem (see [Sil80]) to analytic subsets of Polish spaces to obtain a continuous embedding $\varphi \colon 2^{\mathbb{N}} \to [x]_E$ of $\Delta(2^{\mathbb{N}})$ into $F \restriction [x]_E$, and observe that $(\varphi \times \varphi)^{-1}(E \setminus F) = \sim\Delta(2^{\mathbb{N}})$, thus condition (2) holds. ⊠

Given a binary relation R on a set X, let $\langle R \rangle$ denote the smallest equivalence relation on X containing R.

Proposition 3.3. *Suppose that X and Y are Polish spaces, $R \subseteq (X \times X) \times Y$ is analytic, and F is a co-analytic equivalence relation on X. Then $\{y \in Y \mid \langle R^y \rangle$ has countable index over $\langle R^y \rangle \cap F\}$ is co-analytic.*

Proof. If $y \in Y$, then Proposition 3.2 ensures that $\langle R^y \rangle$ has countable index over $\langle R^y \rangle \cap F$ if and only if there is no continuous function $\varphi \colon 2^{\mathbb{N}} \to X$ for which $(\varphi \times \varphi)^{-1}(\langle R^y \rangle \setminus F)$ is comeager. But Proposition 2.3 and [Kec95, Theorem 29.22] imply that the set of y for which this holds is co-analytic. ⊠

Corollary 3.4. *Suppose that X and Y are Polish spaces, E is an analytic equivalence relation on X, F is a co-analytic equivalence relation on X, and $R \subseteq X \times Y$ is an analytic set. Then $\{y \in Y \mid E$ has countable index over $E \cap F$ on $R^y\}$ is co-analytic.*

Proof. Set $S \coloneqq \{((w, x), y) \in E \times Y \mid w, x \in R^y\}$. If $y \in Y$, then $\langle S^y \rangle = (E \restriction R^y) \cup \Delta(X)$, so $\langle S^y \rangle$ has countable index over $\langle S^y \rangle \cap F$ if and only if E has countable index over $E \cap F$ on R^y, thus Proposition 3.3 yields the desired result. ⊠

Corollary 3.5. *Suppose that X and Y are Polish spaces, F is a co-analytic equivalence relation on X, and $R \subseteq X \times Y$ is analytic. Then $\{y \in Y \mid F$ has only countably many classes on $R^y\}$ is co-analytic.*

Proof. If $y \in Y$, then $I(X)$ has countable index over F on R^y if and only if F has only countably many classes on R^y, so Corollary 3.4 yields the desired result. ⊠

We say that a property Φ of subsets of a Polish space X is $\boldsymbol{\Pi}^1_1$-*on*-$\boldsymbol{\Sigma}^1_1$ if $\{x \in X \mid \Phi(R^y)\}$ is co-analytic for every Polish space Y and analytic set $R \subseteq X \times Y$.

Corollary 3.6. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, F is a co-analytic equivalence relation on X, and $A \subseteq X$ is an analytic set on which E has countable index over $E \cap F$. Then there is a Borel set $B \supseteq A$ on which E has countable index over $E \cap F$.*

Proof. By Corollary 3.4, the property (of A) that $E \restriction A$ has countable index over $(E \cap F) \restriction A$ is $\boldsymbol{\Pi}^1_1$-on-$\boldsymbol{\Sigma}^1_1$, so the first reflection theorem (see, for example, [Kec95, Theorem 35.10]) yields the desired result. ⊠

Proposition 3.7. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X contained in E, and $A \subseteq X$ is an analytic set on which E has countable index over F. Then there is an F-invariant Borel set $B \supseteq A$ on which E has countable index over F.*

Proof. Set $A_0 \coloneqq [A]_F$. Given an analytic set $A_n \subseteq X$ on which E has countable index over F, appeal to Corollary 3.6 to obtain a Borel set $B_n \supseteq A_n$ on which E has countable index over F, and set $A_{n+1} \coloneqq [B_n]_F$. Define $B \coloneqq \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

Proposition 3.8. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, $\varphi \colon X \to Y$ is a Borel reduction of E to F, and E' is a countable-index Borel superequivalence relation of E. Then there is a countable-index Borel superequivalence relation F' of F for which φ is a reduction of E' to F'.*

Proof. Define $F_0 \coloneqq (\varphi \times \varphi)(E')$.

Claim 3.9. *The relation F_0 is transitive.*

Proof. Suppose that $y_1 \mathrel{F_0} y_2 \mathrel{F_0} y_3$. Then there exist $(w_1, w_2), (x_2, x_3) \in E'$ such that $\varphi(w_i) = y_i$ for all $i \in \{1, 2\}$ and $\varphi(x_i) = y_i$ for all $i \in \{2, 3\}$. As φ is a cohomomorphism from E to F, it follows that $w_1 \mathrel{E'} w_2 \mathrel{E} x_2 \mathrel{E'} x_3$, so $w_1 \mathrel{E'} x_3$, thus $y_1 \mathrel{F_0} y_3$. ⊠

It follows that F_0 is an equivalence relation on $\varphi(X)$.

Claim 3.10. *The function φ is a cohomomorphism from E' to F_0.*

Proof. Suppose that $\varphi(x_1)\ F_0\ \varphi(x_2)$. Then there exists $(w_1, w_2) \in E'$ such that $\varphi(w_i) = \varphi(x_i)$ for all $i \in \{1, 2\}$. As φ is a cohomomorphism from E to F, it follows that $x_1\ E\ w_1\ E'\ w_2\ E\ x_2$, so $x_1\ E'\ x_2$. ⊠

Set $F'_0 \coloneqq \langle F \cup F_0 \rangle$, and note that $F_0 = F'_0 \restriction \varphi(X)$, so φ is a cohomomorphism from E' to F'_0. As E' has countable index over E, Proposition 3.1 ensures that F_0 has countable index over $F \restriction \varphi(X)$, and since $[y]_F \neq [y]_{F'_0} \implies [y]_F \cap \varphi(X) \neq \emptyset$ for all $y \in Y$, it follows that F'_0 has countable index over F.

Suppose now that $n \in \mathbb{N}$ and F'_n is an analytic superequivalence relation of F'_0 such that φ is a cohomomorphism from E' to $\langle F'_n \rangle$ and $\langle F'_n \rangle$ has countable index over F. As Proposition 3.3 ensures that this is a $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$ property (of F'_n), the first reflection theorem yields a Borel superrelation R_n of F'_n for which φ is a cohomomorphism from E' to $\langle R_n \rangle$ and $\langle R_n \rangle$ has countable index over F. Set $F'_{n+1} \coloneqq \langle R_n \rangle$.

Clearly $F' \coloneqq \bigcup_{n\in\mathbb{N}} F'_n = \bigcup_{n\in\mathbb{N}} R_n$ is a Borel superequivalence relation of F. To see that φ is a cohomomorphism from E' to F', and therefore a reduction of E' to F', note that if $\varphi(w)\ F'\ \varphi(x)$, then there exists $n \in \mathbb{N}$ for which $\varphi(w)\ F'_n\ \varphi(x)$, so $w\ E'\ x$. To see that F' has countable index over F, note that $[y]_{F'} = \bigcup_{n\in\mathbb{N}} [y]_{F'_n}$ for all $y \in Y$ and F has only countably many classes on $[y]_{F'_n}$ for all $n \in \mathbb{N}$ and $y \in Y$. ⊠

For equivalence relations E and F on sets X and Y, we say that a set $R \subseteq X \times Y$ *induces a partial injection* of X/E into Y/F if $x\ E\ x' \iff y\ F\ y'$ for all $(x, y), (x', y') \in R$.

Proposition 3.11. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and $R \subseteq X \times Y$ is an analytic set inducing a partial injection of X/E into Y/F. Then there is an $(E \times F)$-invariant Borel set $S \supseteq R$ inducing a partial injection of X/E into Y/F.*

Proof. Set $R_0 \coloneqq R$. Given an analytic set $R_n \subseteq X \times Y$ inducing a partial injection of X/E into Y/F, appeal to the first reflection theorem to obtain a Borel set $S_n \supseteq R_n$ inducing a partial injection of X/E into Y/F, and set $R_{n+1} \coloneqq [S_n]_{E\times F}$. Define $S \coloneqq \bigcup_{n\in\mathbb{N}} R_n = \bigcup_{n\in\mathbb{N}} S_n$. ⊠

As a first corollary of this result, we have the following:

Proposition 3.12. *Suppose that X and Y are Polish spaces, E is a Borel equivalence relation on X, F is a strongly-idealistic Borel equivalence relation on Y, and $A \subseteq X$ is an analytic set on which E is Borel reducible to F. Then there is an E-invariant Borel set $B \supseteq A$ on which E is Borel reducible to F.*

Proof. As the graph of any Borel reduction $\pi\colon A \to Y$ of $E \upharpoonright A$ to F is necessarily analytic (see, for example, the proof of [Kec95, Proposition 12.4]), the proposition follows from an application of Proposition 3.11 and [dRM, Proposition 2.8] to the graph of π. ⊠

Proposition 3.13. *Suppose that X and Y are Polish spaces, E is an equivalence relation on X, F is a strongly-idealistic Borel equivalence relation on Y, and there is a Borel reduction $\pi\colon X \to Y$ of E to a countable-index Borel superequivalence relation F' of F. Then there is a Borel homomorphism $\varphi\colon X \to Y^{\mathbb{N}}$ from $(E, {\sim}E)$ to $(F^+, {\sim}F^\cap)$.*

Proof. By [dRM, Theorem 2.12], there are Borel functions $\varphi_n\colon Y \to Y$ such that $[y]_{F'} = \bigcup_{n\in\mathbb{N}}[\varphi_n(y)]_F$ for all $y \in Y$. Then the function $\varphi\colon X \to Y^{\mathbb{N}}$, given by $\varphi(x)(n) = (\varphi_n \circ \pi)(x)$, is as desired. ⊠

Remark 3.14. Proposition 3.8 easily implies the generalization of the converse of Proposition 3.13 in which F need not be strongly idealistic.

We will use Proposition 3.13 in conjunction with the following:

Proposition 3.15. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, and there is a Borel reduction $\pi\colon X \to Y^{\mathbb{N}}$ of E to $F^\cap$. Then E is a countable union of subequivalence relations that are Borel reducible to F.*

Proof. We can assume, without loss of generality, that X is a subset of $2^{\mathbb{N}}$ and that the Borel subsets of X are the Borel subsets of $2^{\mathbb{N}}$ contained in X. Define $\varphi\colon X \to Y^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by

$$\varphi(x)(i,j,k) = \begin{cases} \pi(x)(i) & \text{if } x(k) = 0 \text{ and} \\ \pi(x)(j) & \text{if } x(k) = 1. \end{cases}$$

For all $i, j, k \in \mathbb{N}$, let $E_{i,j,k}$ be the equivalence relation on X given by $x \mathrel{E_{i,j,k}} y \iff \varphi(x)(i,j,k) \mathrel{F} \varphi(y)(i,j,k)$. The fact that π is a cohomomorphism from E to $F^\cap$ ensures that each $E_{i,j,k}$ is a subequivalence relation of E. To see that the union of these equivalence relations is E, suppose that x and y are distinct E-related points of X, and fix $k \in \mathbb{N}$ for which $x(k) \neq y(k)$. By reversing the roles of x and y if necessary, we can assume that $x(k) = 0$. As π is a homomorphism from E to $F^\cap$, there exist $i, j \in \mathbb{N}$ for which $\pi(x)(i) \mathrel{F} \pi(y)(j)$, in which case $\varphi(x)(i,j,k) = \pi(x)(i) \mathrel{F} \pi(y)(j) = \varphi(y)(i,j,k)$, thus $x \mathrel{E_{i,j,k}} y$. ⊠

Remark 3.16. Conversely, a straightforward argument shows that if E is a countable union of subequivalence relations that are Borel reducible to $F \times \Delta(\mathbb{N})$, then E is Borel reducible to $(F \times \Delta(\mathbb{N}))^\cap$.

We also obtain a useful closure property of the class of strongly-idealistic potentially-F_σ Borel equivalence relations on Polish spaces:

Proposition 3.17. *Suppose that X is a Polish space and E is a Borel equivalence relation on X that has a countable-index strongly(-ccc)-idealistic potentially-F_σ subequivalence relation F. Then E is strongly (ccc) idealistic and potentially F_σ.*

Proof. By [dRM, Theorem 2.12], there are Borel functions $\varphi_n \colon X \to X$ with the property that $[x]_E = \bigcup_{n \in \mathbb{N}} [\varphi_n(x)]_F$ for all $x \in X$.

To see that E is potentially F_σ, appeal to standard change of topology results to obtain a Polish topology on X, generating the same Borel sets as the given topology, with respect to which F is F_σ and φ_n is continuous for all $n \in \mathbb{N}$. As $E = \bigcup_{n \in \mathbb{N}} (\varphi_n \times \mathrm{id}_X)^{-1}(F)$, it is F_σ with respect to any such topology.

To see that E is strongly (ccc) idealistic, fix a witness $x \mapsto \mathcal{J}_x$ to the strong (ccc) idealisticity of F.

Claim 3.18. *Suppose that $\varphi \colon X \to X$ is Borel. Then $x \mapsto \mathcal{J}_{\varphi(x)}$ is strongly Borel-on-Borel.*

Proof. Given a Polish space Y and a Borel set $R \subseteq (X \times Y) \times X$, set $S \coloneqq \{((x, (w, y)), z) \in (X \times (X \times Y)) \times X \mid \varphi(w) = x \text{ and } (w, y) \mathrel{R} z\}$, define $\psi \colon X \times Y \to X \times (X \times Y)$ by $\psi(w, y) \coloneqq (\varphi(w), (w, y))$, and observe that

$$\begin{aligned}
&\{(w, y) \in X \times Y \mid R_{(w,y)} \in \mathcal{I}_{\varphi(w)}\} \\
&\quad = \psi^{-1}(\{(\varphi(w), (w, y)) \mid (w, y) \in X \times Y \text{ and } R_{(w,y)} \in \mathcal{I}_{\varphi(w)}\}) \\
&\quad = \psi^{-1}(\{(x, (w, y)) \in X \times (X \times Y) \mid \varphi(w) = x \text{ and } R_{(w,y)} \in \mathcal{I}_x\}) \\
&\quad = \psi^{-1}(\{(x, (w, y)) \in X \times (X \times Y) \mid S_{(x,(w,y))} \in \mathcal{I}_x\}),
\end{aligned}$$

which is Borel. ⊠

If $x \in X$, then $\mathcal{I}_x \coloneqq \bigcap_{n \in \mathbb{N}} \mathcal{J}_{\varphi_n(x)}$ is a (weakly-ccc-on-Borel) σ-ideal for which $[x]_E \notin \mathcal{I}_x$. Clearly $x \mapsto \mathcal{I}_x$ is E-invariant, and Claim 3.18 ensures that it is strongly Borel-on-Borel. ⊠

4. Cores

Given $n \in \mathbb{N}$ and a set X, we use $[X]^n$ to denote the set of all subsets of X of cardinality n. A *partial transversal* of an equivalence relation F on X is a set $Y \subseteq X$ whose intersection with each F-class consists of at most one point. Given a superequivalence relation E of F, we use $[X]^n_{E,F}$ to denote the set of all partial transversals $a \in [X]^n$ of F

that are contained in a single E-class. Define $[X]^{\leq n} \coloneqq \bigcup_{i \leq n} [X]^i$ and $[X]^{\leq n}_{E,F} \coloneqq \bigcup_{i \leq n} [X]^i_{E,F}$.

If X is a standard Borel space, then $[X]^n$ can be viewed as the quotient of the set of injective sequences in X^n by the equivalence relation of enumerating the same set, and equipped with the quotient Borel structure. The latter is standard: If $\prec$ is a Borel strict linear ordering of X, then it is easy to see that the quotient mapping induces a Borel isomorphism between $\{x \in X^n \mid x(0) \prec \cdots \prec x(n-1)\}$ and $[X]^n$. We equip $[X]^{\leq n}$ with the disjoint union Borel structure. Complexities of subsets of $[X]^n_{E,F}$ and $[X]^{\leq n}_{E,F}$ will always be considered relative to the ambient spaces $[X]^n$ and $[X]^{\leq n}$, respectively.

We say that $a, b \in [X]^{\leq n}$ are *F-disjoint* if $[a]_F \cap [b]_F = \emptyset$. We abuse notation by using $[F]^{\leq n}$ to denote the equivalence relation on $[X]^{\leq n}$ given by $a \; [F]^{\leq n} \; b \iff [a]_F = [b]_F$. If F is Borel, then so too is $[F]^{\leq n}$. Finally, we say that a set $C \subseteq X$ is a *core* for a family $\mathcal{A} \subseteq [X]^{\leq n}$ if it intersects every element of $\mathcal{A}$. In this section, we describe the circumstances under which suitable subfamilies of $[X]^{\leq n}$ admit sufficiently small definable cores.

The following result is essentially a special case of [CCCM11, Theorem 12] (although the formalism is quite different and condition (2) appears in a slightly weaker form in [CCCM11, Theorem 12], the proof given there yields the result stated below):

Theorem 4.1 (Caicedo–Clemens–Conley–Miller)**.** *Let $n \geq 1$, X be a standard Borel space, F be a co-analytic equivalence relation on X, and $\mathcal{A} \subseteq [X]^{\leq n}$ be an analytic family of nonempty sets. Then exactly one of the following holds:*

1. *There is an F-invariant core $C \subseteq X$ for $\mathcal{A}$ on which F has only countably many classes.*
2. *There is an uncountable Borel family $\mathcal{P} \subseteq \mathcal{A}$ consisting of pairwise F-disjoint sets.*

The following result is essentially the analog of [CCM16, Proposition 2.3.1] in which bounded finite index is replaced with countable index, and the idea underlying the proof of the former is essentially the same as that underlying the proof of the latter:

Proposition 4.2. *Let $n \geq 1$, X be a standard Borel space, E be an analytic equivalence relation on X, F be a Borel equivalence relation on X contained in E, and $\mathcal{A} \subseteq [X]^{\leq n}_{E,F}$ be an analytic family of nonempty sets such that there is a core for $\mathcal{A} \cap [[x]_E]^{\leq n}$ that intersects only countably many F-classes for all $x \in X$. Then there is an F-invariant Borel core $C \subseteq X$ for $\mathcal{A}$ on which E has countable index over F.*

Proof. We proceed by induction on n. For the case $n = 1$, define $A \coloneqq \{x \in X \mid \{x\} \in \mathcal{A}\}$, note that $E \upharpoonright A$ has countable index over $F \upharpoonright A$, appeal to Proposition 3.7 to obtain an F-invariant Borel set $C \supseteq A$ on which E has countable index over F, and observe that C is a core for $\mathcal{A}$.

We now suppose $n \geq 2$. Given $\mathcal{F} \subseteq [X]^{\leq n}$ and $a \in [X]^{\leq n}$, we let $[a, \mathcal{F}]_F \coloneqq \{[b]_F \mid b \in \mathcal{F}$ and $a \subseteq [b]_F\}$. We build, by reverse recursion, analytic families $\mathcal{A}_k \subseteq [X]^{\leq n}_{E,F}$ and $\mathcal{A}'_k \subseteq [X]^k_{E,F}$ for every $k \leq n$ and an F-invariant Borel set $B_k \subseteq X$ for every $1 \leq k \leq n$ satisfying the following conditions:

(1) For all $k < n$, $\mathcal{A}_k = \{a \in \mathcal{A}_{k+1} \mid a \cap B_{k+1} = \emptyset\}$.
(2) For all $k \leq n$, $\mathcal{A}'_k = \{a \in [X]^k_{E,F} \mid |[a, \mathcal{A}_k]_F| > \aleph_0\}$.
(3) For all $1 \leq k \leq n$, B_k is a core for $\mathcal{A}'_k$ on which E has countable index over F.

We start with $\mathcal{A}_n = \mathcal{A}$. For $k \leq n$, condition (2) uniquely defines $\mathcal{A}'_k$ from $\mathcal{A}_k$. Moreover, since $[X]^k_{E,F}$ is analytic and $\mathcal{A}'_k$ is the set of $a \in [X]^k_{E,F}$ for which $[F]^{\leq n} \upharpoonright \{b \in \mathcal{A}_k \mid a \subseteq [b]_F\}$ has uncountably many classes, Corollary 3.5 ensures that $\mathcal{A}'_k$ is analytic. Similarly, for $k < n$, condition (1) uniquely defines $\mathcal{A}_k$ from $\mathcal{A}_{k+1}$ and B_{k+1}, and ensures that it is analytic. So it only remains to describe the construction of the B_k's. From now on, we fix $1 \leq k \leq n$, assume that the $\mathcal{A}_l$'s and the $\mathcal{A}'_l$'s have been constructed for all $k \leq l \leq n$ and the B_l's have been constructed for $k < l \leq n$, and describe the construction of B_k. In the case when $k = n$, we clearly have $\mathcal{A}'_n = \emptyset$, so we can take $B_n = \emptyset$. Thus we can assume that $1 \leq k < n$. We begin with several preliminary claims. The first of these, Claim 4.3, is also true when $k = 0$, and will later be used in this case.

Claim 4.3. *Let $a \in \mathcal{A}'_k$ and $x \in X$ be such that $a \cup \{x\} \in [X]^{k+1}_{E,F}$. Then $[a \cup \{x\}, \mathcal{A}_k]_F$ is countable.*

Proof. Suppose $[a \cup \{x\}, \mathcal{A}_k]_F \neq \emptyset$. Then there is $b \in \mathcal{A}_k$ such that $a \cup \{x\} \subseteq [b]_F$. By condition (1), we have $b \cap B_{k+1} = \emptyset$. Since B_{k+1} is F-invariant, we have $[b]_F \cap B_{k+1} = \emptyset$, hence $(a \cup \{x\}) \cap B_{k+1} = \emptyset$. Since B_{k+1} is a core for $\mathcal{A}'_{k+1}$, we deduce that $a \cup \{x\} \notin \mathcal{A}'_{k+1}$. Hence, condition (2) ensures that $[a \cup \{x\}, \mathcal{A}_{k+1}]_F$ is countable. Since $\mathcal{A}_k \subseteq \mathcal{A}_{k+1}$, we deduce that $[a \cup \{x\}, \mathcal{A}_k]_F$ is also countable. ⊠

Claim 4.4. *Let $a \in \mathcal{A}'_k$ and $M \subseteq [a]_E$ be a set intersecting only countably many F-classes. Then there exists $b \in \mathcal{A}_k$ such that $a \subseteq [b]_F$ and $M \cap [b]_F \setminus [a]_F = \emptyset$.*

Proof. By Claim 4.3, for all $x \in M \setminus [a]_F$, the set $[a \cup \{x\}, \mathcal{A}_k]_F$ is countable. Since this set only depends on $[x]_F$ and M intersects only countably many F-classes, we deduce that $\bigcup_{x \in M \setminus [a]_F} [a \cup \{x\}, \mathcal{A}_k]_F$ is countable. Since $[a, \mathcal{A}_k]_F$ is uncountable, we deduce that $[a, \mathcal{A}_k]_F \setminus \bigcup_{x \in M \setminus [a]_F} [a \cup \{x\}, \mathcal{A}_k]_F$ is nonempty. An element of this set has the form $[b]_F$, where $b \in \mathcal{A}_k$, $a \subseteq [b]_F$, and $x \notin [b]_F$ for all $x \in M \setminus [a]_F$. From this last condition, we deduce that $M \cap [b]_F \setminus [a]_F = \emptyset$. ⊠

Claim 4.5. *For every $x \in X$, there exists a core for $\mathcal{A}'_k \cap [[x]_E]^k$ that intersects only countably many F-classes.*

Proof. Fix $x \in X$. By the hypotheses of the proposition, there exists a core $D \subseteq X$ for $\mathcal{A} \cap [[x]_E]^{\leq n}$ that intersects only countably many F-classes. Without loss of generality, we can assume that D is F-invariant. We will show that it is also a core for $\mathcal{A}'_k \cap [[x]_E]^k$.

Towards this end, fix $a \in \mathcal{A}'_k \cap [[x]_E]^k$. Then $\mathcal{F} \coloneqq \{b \setminus [a]_F \mid b \in \mathcal{A}_k \text{ and } a \subseteq [b]_F\}$ is an analytic subset of $[X]^{\leq n}$ contained in $[[x]_E]^{\leq n}$. If $\emptyset \in \mathcal{F}$, then there exists $b \in \mathcal{A}_k$ such that $[a]_F = [b]_F$; since $b \in \mathcal{A} \cap [[x]_E]^{\leq n}$, we have $b \cap D \neq \emptyset$, and since D is F-invariant, we also have $a \cap D \neq \emptyset$. So, from now on, we will assume that $\emptyset \notin \mathcal{F}$. Hence, we can apply Theorem 4.1 to $\mathcal{F}$. There are two cases.

Case 1: There is a core $M \subseteq X$ for $\mathcal{F}$ on which F has only countably many classes. We can assume that $M \subseteq [x]_E$. We apply Claim 4.4 to a and M, which yields $b \in \mathcal{A}_k$ such that $a \subseteq [b]_F$ and $M \cap [b]_F \setminus [a]_F = \emptyset$. It follows that $b \setminus [a]_F$ is in $\mathcal{F}$ and does not intersect M, contradicting the fact that M is a core for $\mathcal{F}$.

Case 2: There is an uncountable family $\mathcal{P} \subseteq \mathcal{F}$ consisting of pairwise F-disjoint sets. Since D intersects only countably many F-classes, we can find $c \in \mathcal{P}$ such that $c \cap D = \emptyset$. Fix $b \in \mathcal{A}_k$ such that $a \subseteq [b]_F$ and $c = b \setminus [a]_F$. Then $b \in \mathcal{A} \cap [[x]_E]^{\leq n}$; hence $b \cap D \neq \emptyset$. But we have seen that $b \setminus [a]_F \cap D = \emptyset$, so it follows that $[a]_F \cap D \neq \emptyset$. Since D is F-invariant, we deduce that $a \cap D \neq \emptyset$. ⊠

Claim 4.5 allows us to apply the induction hypothesis to the family $\mathcal{A}'_k$; this gives us the desired Borel set B_k and completes the recursive construction. We now let $A \coloneqq \bigcup \mathcal{A}_0$.

Claim 4.6. *The relation E has countable index over F on A.*

Proof. If $\emptyset \notin \mathcal{A}'_0$, then condition (2) implies that $[\emptyset, \mathcal{A}_0]_F$ is countable. Observe that $[A]_F = \bigcup [\emptyset, \mathcal{A}_0]_F$, so F has only countably many classes on $[A]_F$, and we are done. We can therefore assume that $\emptyset \in \mathcal{A}'_0$.

Fix $x \in X$; we will show that $A \cap [x]_E$ intersects only countably many F-classes. The hypotheses of the proposition give us a core $D \subseteq X$ for $\mathcal{A} \cap [[x]_E]^{\leq n}$ that intersects only countably many F-classes. Since $\emptyset \in \mathcal{A}'_0$, Claim 4.3 ensures that, for every $y \in D$, the set $[\{y\}, \mathcal{A}_0]_F$ is countable, hence the set $\bigcup[\{y\},\mathcal{A}_0]_F$ intersects only countably many F-classes. Since this set only depends on $[y]_F$ and D intersects only countably many F-classes, we deduce that the set $H \coloneqq \bigcup_{y\in D}\bigcup[\{y\},\mathcal{A}_0]_F$ intersects only countably many F-classes.

It only remains to show that $A \cap [x]_E \subseteq H$. Let $z \in A \cap [x]_E$. Then there exists $a \in \mathcal{A}_0 \cap [[x]_E]^{\leq n}$ such that $z \in a$. Since D is a core for $\mathcal{A}_0 \cap [[x]_E]^{\leq n}$, there exists $y \in a \cap D$. Then $[a]_F \in [\{y\},\mathcal{A}_0]_F$, so $z \in \bigcup[\{y\},\mathcal{A}_0]_F$, thus $z \in H$. ⊠

We can now complete the proof of the proposition: By Proposition 3.7, there exists an F-invariant Borel set $B \supseteq A$ on which E has countable index over F. This set is a core for $\mathcal{A}_0$. By condition (1), B_{k+1} is a core for $\mathcal{A}_{k+1} \setminus \mathcal{A}_k$ for every $k < n$. We deduce that $C \coloneqq B \cup \left(\bigcup_{1\leq k\leq n} B_k\right)$ is a core for $\mathcal{A}$, and it is clear that C is a Borel set on which E has countable index over F. ⊠

Combining Proposition 4.2 and Theorem 4.1, we obtain:

Corollary 4.7. *Let $n \geq 1$, X be a standard Borel space, E be an analytic equivalence relation on X, F be a Borel equivalence relation on X contained in E, and $\mathcal{A} \subseteq [X]^{\leq n}_{E,F}$ be an analytic family of nonempty sets. Then exactly one of the following holds:*

1. *There exists an F-invariant Borel core $C \subseteq X$ for $\mathcal{A}$ on which E has countable index over F.*
2. *There exists an uncountable Borel set $\mathcal{P} \subseteq \mathcal{A}$ of pairwise F-disjoint subsets of a single E-class.*

Proof. To see that the two conditions are mutually exclusive, suppose that we are in case (2) and fix a core C for $\mathcal{A}$. Then C contains uncountably many pairwise F-inequivalent elements of $\bigcup\mathcal{P}$. But $\bigcup\mathcal{P}$ is contained in an E-class, so E does not have countable index over F on C.

Suppose now that we are not in case (2). Then, for every $x \in X$, Theorem 4.1 applied to $\mathcal{A}\cap[[x]_E]^{\leq n}$ yields a core for $\mathcal{A}\cap[[x]_E]^{\leq n}$ intersecting only countably many F-classes. Hence, we can apply Proposition 4.2, which ensures that we are in case (1). ⊠

We now state and prove a technical consequence of Corollary 4.7 that will be useful later. Given sets X and Y, an equivalence relation F on $X \times Y$, and $y \in Y$, we use $F^{(y)}$ to denote the equivalence relation on X given by $x \; F^{(y)} \; x' \iff (x, y) \; F \; (x', y)$.

Lemma 4.8. *Let X and Y be Polish spaces, $A \subseteq Y$ be an analytic set, E be an analytic equivalence relation on $X \times Y$, and F be a Borel equivalence relation on $X \times Y$ contained in E such that:*

(a) *For every* $y \in A$, $X \times \{y\}$ *is contained in a single* E*-class.*
(b) *There exists* $n \geq 1$ *such that, for every* $y \in A$, *the equivalence relation* $F^{(y)}$ *has at most* n *non-meager classes.*

Then at least one of the following conditions holds:

(1) *There is an* F*-invariant Borel set* $C \subseteq X \times Y$, *on which* E *has countable index over* F, *such that* C^y *is non-meager for every* $y \in A$.
(2) *There is a continuous mapping* $\psi\colon 2^{\mathbb{N}} \to A$ *such that* $X \times \psi(2^{\mathbb{N}})$ *is contained in a single* E*-class and* $((\mathrm{id}_X \times \psi) \times (\mathrm{id}_X \times \psi))^{-1}(F)$ *is meager.*

Proof. Let $\mathcal{R}$ be the set of $(y, a) \in A \times [X \times Y]^{\leq n}_{E,F}$ with the property that $[a]_F = \{z \in X \times Y \mid [z]^y_F \text{ is non-meager}\}$. The inclusion $[a]_F \subseteq \{z \in X \times Y \mid [z]^y_F \text{ is non-meager}\}$ can be written as

$$\forall z \in a \exists^* x \in X \ (x, y) \ F \ z,$$

so, by [Kec95, Theorem 16.1], the set of $(y, a) \in Y \times [X \times Y]^{\leq n}_{E,F}$ satisfying it is Borel. Similarly, the reverse inclusion can be written as

$$\forall^* x \in X \ (\exists^* x' \in X \ (x, y) \ F \ (x', y) \implies \exists z \in a \ (x, y) \ F \ z),$$

so the set of $(y, a) \in Y \times [X \times Y]^{\leq n}_{E,F}$ satisfying it is also Borel. It follows that $\mathcal{R}$ is an analytic subset of $Y \times [X \times Y]^{\leq n}$, so $\mathcal{A} \coloneqq \mathrm{proj}_{[X \times Y]^{\leq n}}(\mathcal{R})$ is an analytic subset of $[X \times Y]^{\leq n}_{E,F}$.

Suppose that $\emptyset \in \mathcal{A}$. Then there exists $y_0 \in A$ such that $(y_0, \emptyset) \in \mathcal{R}$, that is, $[z]^{y_0}_F$ is meager for all $z \in X \times Y$. Let $\psi\colon 2^{\mathbb{N}} \to A$ be the constant mapping with value y_0 and $F' \coloneqq ((\mathrm{id}_X \times \psi) \times (\mathrm{id}_X \times \psi))^{-1}(F)$. Then, for all $(x, u) \in X \times 2^{\mathbb{N}}$ and $u' \in 2^{\mathbb{N}}$, we have $[(x, u)]^{u'}_{F'} = [(x, y_0)]^{y_0}_F$, so $[(x, u)]^{u'}_{F'}$ is meager; by Kuratowski–Ulam's theorem (see, for example, [Kec95, Theorem 8.41]), it follows that $[(x, u)]_{F'}$ is meager, hence F' is meager. It also follows from condition (a) that $X \times \psi(2^{\mathbb{N}})$ is contained in a single E-class, so ψ witnesses that we are in case (2).

So, from now on, we can assume that $\emptyset \notin \mathcal{A}$, and therefore apply Corollary 4.7 to the family $\mathcal{A}$. There are two cases.

Case 1: There is an F*-invariant Borel core* $C \subseteq X \times Y$ *for* $\mathcal{A}$ *on which* E *has countable index over* F. We will show that C witnesses that we are in case (1). Let $y \in A$; we will show that C^y is non-meager. By condition (b), the equivalence relation $F^{(y)}$ has at most n non-meager classes; we fix a set $\widetilde{a}$ of representatives for these classes and let $a \coloneqq \widetilde{a} \times \{y\}$. Elements of a are clearly pairwise F-inequivalent and pairwise E-equivalent by condition (a), so $a \in [X]^{\leq n}_{E,F}$. Moreover, $[a]_F$ is the union of all F-classes $[z]_F$ for which $[z]^y_F$ is non-meager, so $(y, a) \in \mathcal{R}$, thus $a \in \mathcal{A}$. Since C is a core for $\mathcal{A}$, we can find $x \in X$

such that $(x,y) \in C \cap a$. It follows that $[(x,y)]_F^y$ is non-meager, and since C is F-invariant, we have $[(x,y)]_F \subseteq C$, so C^y is non-meager.

Case 2: There exists an uncountable Borel set $\mathcal{P} \subseteq \mathcal{A}$ of pairwise F-disjoint subsets of a single E-class. Endow $[X \times Y]^{\leq n}$ with any Polish topology compatible with its standard Borel structure. We can assume that $\mathcal{P}$ is homeomorphic to Cantor space. By Jankov–von Neumann's uniformization theorem (see, for example, [Kec95, Theorem 18.1]), we can find a Baire measurable mapping $f\colon \mathcal{P} \to Y$ such that, for all $a \in \mathcal{P}$, $(f(a), a) \in \mathcal{R}$. The mapping f is continuous on a comeager subset of $\mathcal{P}$, so, by shrinking $\mathcal{P}$ if necessary, we can assume that f is continuous on $\mathcal{P}$. Since elements of $\mathcal{P}$ are nonempty, pairwise F-disjoint, and contained in the same E-class, it follows that the sets $\{z \in X \times Y \mid [z]_F^{f(a)} \text{ is non-meager}\}$, for $a \in \mathcal{P}$, are nonempty, pairwise disjoint, and contained in the same E-class. In particular, the mapping f is one-to-one, so $P \coloneqq f(\mathcal{P})$ is homeomorphic to Cantor space. It is clear that $P \subseteq A$; moreover, the sets $\{z \in X \times Y \mid [z]_F^y \text{ is non-meager}\}$, for $y \in P$, are nonempty, pairwise F-disjoint, and contained in the same E-class. Since, for all $y \in P$, the set $\{z \in X \times Y \mid [z]_F^y \text{ is non-meager}\}$ intersects $X \times \{y\}$ and $X \times \{y\}$ is contained in a single E-class, we deduce that $X \times P$ is contained in a single E-class.

We now let $\psi\colon 2^{\mathbb{N}} \to P$ be a homeomorphism and show that ψ witnesses that we are in case (2), or equivalently, that $F \restriction (X \times P)$ is meager. Suppose not. Then, by Kuratowski–Ulam's theorem, there exist $z \in X \times P$ for which $[z]_F$ is non-meager in $X \times P$ and distinct $y_0, y_1 \in P$ for which $[z]_F^{y_0}$ and $[z]_F^{y_1}$ are non-meager, contradicting the fact that the sets of the form $\{z \in X \times Y \mid [z]_F^y \text{ is non-meager}\}$, for $y \in P$, are pairwise disjoint. ⊠

5. Aligned mappings

In this section and the next, we will often work with spaces of the form $(2^m)^n$ where $m, n \in \mathbb{N} \cup \{\mathbb{N}\}$. If $m \leq m'$, $n \leq n'$, $s \in (2^m)^n$, and $t \in (2^{m'})^{n'}$, then we write $s \sqsubseteq t$ to indicate that $s(j)(i) = t(j)(i)$ for all $i < m$ and $j < n$. Let $\mathcal{N}_s \coloneqq \{x \in (2^{\mathbb{N}})^n \mid s \sqsubseteq x\}$ for all $m, n \in \mathbb{N}$ and $s \in (2^m)^n$. The family $\{\mathcal{N}_s \mid m \in \mathbb{N} \text{ and } s \in (2^m)^n\}$ forms a basis of clopen subsets of $(2^{\mathbb{N}})^n$. If $n \in \mathbb{N}$, $m, n' \in \mathbb{N} \cup \{\mathbb{N}\}$, $s \in (2^m)^n$, and $t \in (2^m)^{n'}$, then we denote by $s \frown t$ the *horizontal concatenation* of s and t, that is, the element of $(2^m)^{n+n'}$ given by $(s \frown t)(j)(i) = s(j)(i)$ for $(i,j) \in m \times n$ and $(s \frown t)(n+j)(i) = t(j)(i)$ for $(i,j) \in m \times n'$. When $n' = 1$, we identify $(2^m)^1$ with 2^m, so that, for $s \in (2^m)^n$ and $t \in 2^m$, the horizontal concatenation can be written as $s \frown t$. Similarly, for $m \in \mathbb{N}$, $m', n \in \mathbb{N} \cup \{\mathbb{N}\}$, $s \in (2^m)^n$, and $t \in (2^{m'})^n$, we denote by $s \oplus t$ the *vertical concatenation* of s and t, that is, the element of $(2^{m+m'})^n$ given by $(s \oplus t)(j)(i) = s(j)(i)$ for $(i,j) \in m \times n$ and $(s \oplus t)(j)(m+i) = t(j)(i)$ for $(i,j) \in m' \times n$.

For $f\colon (2^m)^n \to (2^p)^q$ and $g\colon (2^{m'})^{n'} \to (2^{p'})^{q'}$, where $m \leq m'$, $n \leq n'$, $p \leq p'$, and $q \leq q'$, we write $f \sqsubseteq g$ when, for all $s \in (2^m)^n$ and $t \in (2^{m'})^{n'}$, we have $s \sqsubseteq t \implies f(s) \sqsubseteq g(t)$. In the special case when $m, n, p, q \in \mathbb{N}$, $f\colon (2^m)^n \to (2^p)^q$, and $g\colon (2^{\mathbb{N}})^n \to (2^{\mathbb{N}})^q$, we have $f \sqsubseteq g$ if and only if $g(\mathcal{N}_s) \subseteq \mathcal{N}_{f(s)}$ for all $s \in (2^m)^n$.

For $m, n \in \mathbb{N} \cup \{\mathbb{N}\}$ and $k \in \mathbb{N}$, let $\mathbb{F}_k((2^m)^n)$ denote the equivalence relation on $(2^m)^n$ given by $s \mathrel{\mathbb{F}_k((2^m)^n)} t \iff \forall k \leq l < n\ s(l) = t(l)$. Note that this is equality when $k = 0$ and the complete equivalence relation on $(2^m)^n$ when $k \geq n$. In the special case when $m = n = \mathbb{N}$, we use $\mathbb{F}_k$ to denote the corresponding equivalence relation. Observe that $\mathbb{E}_1 = \bigcup_{k \in \mathbb{N}} \mathbb{F}_k$.

For $n \in \mathbb{N}$, an *n-dimensional aligned mapping* is a continuous reduction $\varphi\colon (2^{\mathbb{N}})^n \to (2^{\mathbb{N}})^n$ of $(\mathbb{F}_k((2^{\mathbb{N}})^n))_{k<n}$ to itself. We denote by Al_n the set of all such mappings.

It is clear that Al_n contains the identity and is closed under composition. In the rest of the paper, Al_n will be viewed as a subset of $C((2^{\mathbb{N}})^n, (2^{\mathbb{N}})^n)$ and endowed with the subspace topology.

For all $m, n \in \mathbb{N}$, we let $\mathcal{U}^n_m \coloneqq \{\varphi \in \mathrm{Al}_n \mid \mathrm{id}_{(2^m)^n} \sqsubseteq \varphi\}$. In other words, a mapping $\varphi\colon (2^{\mathbb{N}})^n \to (2^{\mathbb{N}})^n$ belongs to $\mathcal{U}^n_m$ if and only if $\varphi \in \mathrm{Al}_n$ and $\varphi(\mathcal{N}_s) \subseteq \mathcal{N}_s$ for all $s \in (2^m)^n$.

Lemma 5.1. *Fix $n \in \mathbb{N}$. Then $(\mathcal{U}^n_m)_{m \in \mathbb{N}}$ is a basis of neighborhoods of the identity in Al_n and the $\mathcal{U}^n_m$'s are closed under composition.*

Proof. The fact that $\mathcal{U}^n_m$ contains the identity and is closed under composition is clear from the definition. Moreover, we have $\mathcal{U}^n_m = \mathrm{Al}_n \cap \bigcap_{s \in (2^m)^n} M(\mathcal{N}_s, \mathcal{N}_s)$, so $\mathcal{U}^n_m$ is open.

To see that $(\mathcal{U}^n_m)_{m \in \mathbb{N}}$ is a basis of neighborhoods of the identity, take any neighborhood $\mathcal{U}$ of the identity in Al_n. By Lemma 2.10, we can assume that $\mathcal{U} = \mathrm{Al}_n \cap \bigcap_{i \in I} M(K_i, U_i)$, where $(K_i)_{i \in I}$ is a finite partition of $(2^{\mathbb{N}})^n$ into nonempty clopen subsets and $(U_i)_{i \in I}$ is a sequence of open subsets of $(2^{\mathbb{N}})^n$. Subdividing the K_i's if necessary, we can assume that $I = (2^m)^n$ for some $m \in \mathbb{N}$, and $K_s = \mathcal{N}_s$ for all $s \in (2^m)^n$. Since $\mathrm{id}_{(2^{\mathbb{N}})^m} \in \mathcal{U}$, we have $\mathcal{N}_s \subseteq U_s$ for all $s \in (2^m)^n$; it follows that $\mathcal{U}^n_m \subseteq \mathcal{U}$. ⊠

The following lemma is a special case of [CLM14, Proposition 2.6]:

Lemma 5.2. *Let $m, n \in \mathbb{N}$ and Ω be a family of open subsets of $(2^{\mathbb{N}})^n$ which is downwards closed under inclusion and has the property that $\bigcup \Omega$ is dense in $(2^{\mathbb{N}})^n$. Then there exist $m' \geq m$ and a reduction*

$\psi\colon (2^m)^n \to (2^{m'})^n$ *of* $(\mathbb{F}_k((2^m)^n))_{k<n}$ *to* $(\mathbb{F}_k((2^{m'})^n))_{k<n}$*, with* $\mathrm{id}_{(2^m)^n} \sqsubseteq \psi$*, such that* $\mathcal{N}_{\psi(s)} \in \Omega$ *for all* $s \in (2^m)^n$*.*

Corollary 5.3. *Let* $m, n \in \mathbb{N}$ *and* Ω *be a family of open subsets of* $(2^{\mathbb{N}})^n$ *which is downwards closed under inclusion and has the property that* $\bigcup \Omega$ *is dense in* $(2^{\mathbb{N}})^n$*. Then there exists an open mapping* $\varphi \in \mathcal{U}_m^n$ *such that* $\varphi(\mathcal{N}_s) \in \Omega$ *for all* $s \in (2^m)^n$*.*

Proof. Fix m' and ψ as given by Lemma 5.2. Define $\varphi\colon (2^{\mathbb{N}})^n \to (2^{\mathbb{N}})^n$ by $\varphi(s \oplus x) \coloneqq \psi(s) \oplus x$ for all $s \in (2^m)^n$ and $x \in (2^{\mathbb{N}})^n$. Then φ satisfies the desired conditions. ⊠

The next lemma gives an example of a situation where families Ω satisfying the hypotheses of Lemma 5.2 and Corollary 5.3 naturally appear. It will be used in conjunction with the latter in the proof of our first dichotomy.

Lemma 5.4. *Let* X *be a Polish space,* E *be a Baire measurable equivalence relation on* X*, and* Ω *be the set of all open sets* $U \subseteq X$ *for which* $E \upharpoonright U$ *is meager or comeager. Then* $\bigcup \Omega$ *is dense in* X*.*

Proof. Suppose not. Then we can find a nonempty open set $U \subseteq X$ such that no further nonempty subset of U belongs to Ω. In particular, $E \upharpoonright U$ is non-meager, so, by Kuratowski–Ulam's theorem, there exists $x \in U$ such that the class $[x]_E \cap U$ is Baire measurable and non-meager. So we can find a nonempty open subset $V \subseteq U$ such that $[x]_E$ is comeager in V. Hence $V \in \Omega$, a contradiction. ⊠

The following result is a version of Mycielski's theorem for aligned mappings. It is a particular case of [CLM14, Proposition 2.10], although one should note that the statement of the latter is missing the hypothesis that φ is a reduction of $(\mathbb{F}_k((2^m)^n))_{k<n}$ to $(\mathbb{F}_k((2^{m'})^n))_{k<n}$, which holds in the special case we require.

Proposition 5.5. *Let* $m \in \mathbb{N}$*,* $n \geq 1$*, and* R *be a comeager binary relation on* $(2^{\mathbb{N}})^n$*. Then there is a homomorphism* $\varphi \in \mathcal{U}_m^n$ *from* $\sim\mathbb{F}_{n-1}((2^{\mathbb{N}})^n)$ *to* R*.*

6. Two dichotomies

In this section, we prove our two technical dichotomies—Theorems 6.2 and 6.8—and use the second to prove the Kechris–Louveau dichotomy. We start by recalling the following well-known fact:

Proposition 6.1 (see [CLM14, Proposition 2.2])**.** *Fix* $k \in \mathbb{N}$ *and a Baire measurable set* $B \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$*. If* $\mathbb{F}_{k+1}$ *has countable index over* $\mathbb{F}_k$ *on* B*, then* B *is meager.*

Theorem 6.2. *Let X be a Polish space, E be an analytic equivalence relation on X, and $(E_n)_{n\in\mathbb{N}}$ be a sequence of Borel subequivalence relations of E. Then exactly one of the following holds:*

1. *There exists a cover $(B_n)_{n\in\mathbb{N}}$ of X with the property that B_n is an E_n-invariant Borel set on which E has countable index over E_n for all $n \in \mathbb{N}$.*
2. *There is a continuous homomorphism $\varphi\colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{E}_1 \setminus \mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E \setminus \bigcup_{i\leq n} E_i)_{n\in\mathbb{N}}$.*

Proof. We first show that the two conditions are mutually exclusive. Suppose, towards a contradiction, that both hold. For every $n \in \mathbb{N}$, define $B'_n := \varphi^{-1}(B_n)$. Then $(B'_n)_{n\in\mathbb{N}}$ is a covering of $(2^{\mathbb{N}})^{\mathbb{N}}$, so there exists $n_0 \in \mathbb{N}$ such that B'_{n_0} is non-meager. By Proposition 6.1, $\mathbb{F}_{n_0+1}$ does not have countable index over $\mathbb{F}_{n_0}$ on B'_{n_0}, so neither does $\mathbb{E}_1$. As $\varphi \restriction B'_{n_0}$ is a homomorphism from $(\mathbb{E}_1 \setminus \mathbb{F}_{n_0}) \restriction B'_{n_0}$ to $(E \setminus E_{n_0}) \restriction B_{n_0}$, it follows that E does not have countable index over E_{n_0} on B_{n_0}, the desired contradiction.

We now show that at least one of the two conditions holds. We begin by fixing some notation, definitions, and conventions. For every $k \in \mathbb{N}$, let $R_k := \bigcup_{i\leq k} E_i$ and fix a Polish space Y_k and a continuous surjection $\pi_k\colon Y_k \to E \setminus R_k$.

Note that the mapping $((x, x'), y) \mapsto (x \frown y, x' \frown y)$ is a homeomorphism from $\sim\mathbb{F}_k((2^{\mathbb{N}})^{k+1}) \times (2^{\mathbb{N}})^{n-k-1}$ to $\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n)$ for all $n \in \mathbb{N} \cup \{\mathbb{N}\}$ and $k < n$. In the rest of the proof, we will identify these spaces via this homeomorphism. As a consequence, for instance, if $k < m \leq n$ and $g\colon \mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n) \to Y_k$, then it is consistent with our earlier notation to use g^y to denote the mapping $g^y\colon \mathbb{F}_{k+1}((2^{\mathbb{N}})^m) \setminus \mathbb{F}_k((2^{\mathbb{N}})^m) \to Y_k$ given by $g^y(x, x') := g(x \frown y, x' \frown y)$ for all $(x, x') \in \mathbb{F}_{k+1}((2^{\mathbb{N}})^m) \setminus \mathbb{F}_k((2^{\mathbb{N}})^m)$ and $y \in (2^{\mathbb{N}})^{n-m}$.

Suppose that $n \in \mathbb{N} \cup \{\mathbb{N}\}$ and $m \leq n$.

- For $f \in C((2^{\mathbb{N}})^m, X)$, define $f^{\to n} \in C((2^{\mathbb{N}})^n, X)$ by $f^{\to n}(x) := f(x \restriction m)$.
- For $\mathcal{A} \subseteq C((2^{\mathbb{N}})^m, X)$, we abuse notation by using $\mathcal{A}^{\to n}$ to denote the set of all $f \in C((2^{\mathbb{N}})^n, X)$ such that $f^y \in \mathcal{A}$ for all $y \in (2^{\mathbb{N}})^{n-m}$. Note that if $f \in \mathcal{A}$, then $f^{\to n} \in \mathcal{A}^{\to n}$.
- For $k < m$ and $g \in C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^m) \setminus \mathbb{F}_k((2^{\mathbb{N}})^m), Y_k)$, define $g^{\to n} \in C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$ by $g^{\to n}(x, x') := g(x \restriction m, x' \restriction m)$.
- For $k < m$ and $\mathcal{A} \subseteq C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^m) \setminus \mathbb{F}_k((2^{\mathbb{N}})^m), Y_k)$, we abuse notation by using $\mathcal{A}^{\to n}$ to denote the set of $g \in C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$ such that $g^y \in \mathcal{A}$ for all $y \in (2^{\mathbb{N}})^{n-m}$. Note that if $g \in \mathcal{A}$, then $g^{\to n} \in \mathcal{A}^{\to n}$.

By Lemma 2.9, we can find compatible complete metrics d^n on $C((2^{\mathbb{N}})^n, X)$ such that $\operatorname{diam}^{\mathbb{N}}(\mathcal{A}^{\to\mathbb{N}}) \leq \operatorname{diam}^n(\mathcal{A})$ for all $n \in \mathbb{N}$ and $\mathcal{A} \subseteq C((2^{\mathbb{N}})^n, X)$. Similarly, for all $k \in \mathbb{N}$, we can find compatible complete metrics d^n_k on $C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$ such that $\operatorname{diam}^{\mathbb{N}}_k(\mathcal{A}^{\to\mathbb{N}}) \leq \operatorname{diam}^n_k(\mathcal{A})$ for all $n > k$ and $\mathcal{A} \subseteq C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$. As it should not lead to confusion, we will use d to denote these metrics and diam to denote the corresponding diameters.

Using Corollary 2.11 and the fact that $\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n)$ can be identified with an open subset of the compact zero-dimensional Polish space $((2^{\mathbb{N}})^{k+1} \times (2^{\mathbb{N}})^{k+1}) \times (2^{\mathbb{N}})^{n-k-1}$, we can fix countable bases of nonempty open subsets of $C((2^{\mathbb{N}})^n, X)$ and $C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$ whose elements are right stable for all $n \in \mathbb{N}$ and $k < n$. These bases will not be given a name, but we will refer to them by talking about *basic open subsets* of these spaces.

An *approximation* is a triple of the form $a := (n^a, \mathcal{U}^a, (\mathcal{V}^a_k)_{k<n^a})$, where $n^a \in \mathbb{N}$, $\mathcal{U}^a \subseteq C((2^{\mathbb{N}})^{n^a}, X)$ is a basic open set with $\operatorname{diam}(\mathcal{U}^a) \leq 1/n^a$, and $\mathcal{V}^a_k \subseteq C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}), Y_k)$ is a basic open set with $\operatorname{diam}(\mathcal{V}^a_k) \leq 1/n^a$ for all $k < n^a$. Given approximations a and b, we say that b *extends* a if $n^a < n^b$, $\mathcal{U}^b \subseteq (\mathcal{U}^a)^{\to n^b}$, and $\mathcal{V}^b_k \subseteq (\mathcal{V}^a_k)^{\to n^b}$ for all $k < n^a$. We say that b is an *immediate successor* of a if it extends a and $n^b = n^a + 1$.

A *configuration* is a triple of the form $\gamma := (n^\gamma, f^\gamma, (g^\gamma_k)_{k<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $f^\gamma \colon (2^{\mathbb{N}})^{n^\gamma} \to X$ is continuous, and $g^\gamma_k \colon \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}) \to Y_k$ is a continuous mapping such that $(f^\gamma(x), f^\gamma(x')) = \pi_k(g^\gamma_k(x, x'))$ for all $(x, x') \in \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma})$ and $k < n^\gamma$. An immediate consequence of this definition is that if γ is a configuration, then f^γ is a homomorphism from $(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}))_{k<n^\gamma}$ to $(E \setminus R_k)_{k<n^\gamma}$; in particular, it is injective and its range is contained in a single E-class. We say that a configuration γ is *compatible* with an approximation a if $n^\gamma = n^a$, $f^\gamma \in \mathcal{U}^a$, and $g^\gamma_k \in \mathcal{V}^a_k$ for all $k < n^\gamma$. We say that a configuration γ is *generically compatible* with a set $B \subseteq X$ if $(f^\gamma)^{-1}(B)$ is comeager in $(2^{\mathbb{N}})^{n^\gamma}$.

For all $n \in \mathbb{N}$, the set of all configurations γ such that $n^\gamma = n$ can be identified with a subset Conf_n of the space $C((2^{\mathbb{N}})^n, X) \times \prod_{k<n} C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n), Y_k)$. The latter space is Polish by Proposition 2.7, and Proposition 2.3 ensures that Conf_n is closed, thus Polish. For an approximation a and $B \subseteq X$, define $\operatorname{Comp}(a, B)$ as the set of all configurations that are compatible with a and generically compatible with B. By definition, being compatible with an approximation is an open condition. Moreover, if B is Borel, then it follows from [Kec95, Theorem 16.1] and Proposition 2.3 that the set of all

$\gamma \in \mathrm{Conf}_{n^a}$ that are generically compatible with B is Borel. Hence, $\mathrm{Comp}(a, B)$ is a Borel subset of Conf_{n^a}.

For a configuration γ and $\varphi \in \mathrm{Al}_{n^\gamma}$, we abuse notation by using $\gamma \circ \varphi$ to denote the triple $\delta \coloneqq (n^\delta, f^\delta, (g^\delta_k)_{k<n^\delta})$, where $n^\delta \coloneqq n^\gamma$, $f^\delta \coloneqq f^\gamma \circ \varphi$, and $g^\delta_k \coloneqq g^\gamma_k \circ (\varphi \times \varphi) \restriction \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma})$ for all $k < n^\delta$. It follows from the definition of an aligned mapping that $\gamma \circ \varphi$ is a well-defined configuration. Given configurations γ, δ and $m \in \mathbb{N}$, we write $\delta \preccurlyeq_m \gamma$ if there exists $\varphi \in \mathcal{U}^{n^\gamma}_m$ such that $\delta = \gamma \circ \varphi$. It follows from Lemma 5.1 that $\preccurlyeq_m$ is a quasi-ordering of the set of all configurations.

Claim 6.3. *Let a be an approximation. Then there exists $m \in \mathbb{N}$ such that the set $\mathrm{Comp}(a, X)$ of all configurations that are compatible with a is downwards closed under $\preccurlyeq_m$.*

Proof. Since $\mathcal{U}^a$ is a basic open set of $C((2^{\mathbb{N}})^{n^a}, X)$, hence right stable, we can find $p \in \mathbb{N}$ with the property that $\mathcal{U}^a \circ \mathcal{U}^{n^a}_p = \mathcal{U}^a$. Similarly, for every $k < n^a$, since $\mathcal{V}^a_k$ is a right-stable open subset of $C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}), Y_k)$, there is a neighborhood $\mathcal{W}_k$ of the identity in $C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}), \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}))$ such that $\mathcal{V}^a_k \circ \mathcal{W}_k = \mathcal{V}^a_k$. By Propositions 2.5, 2.2, and 2.1, the mapping $\mathrm{Al}_n \to C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}), \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a}))$, given by $\varphi \mapsto (\varphi \times \varphi) \restriction \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a})$, is continuous; hence we can find $m_k \in \mathbb{N}$ such that $(\varphi \times \varphi) \restriction \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a})$ is in $\mathcal{W}_k$ for all $\varphi \in \mathcal{U}^{n^a}_{m_k}$. Then $m \coloneqq \max(p, m_0, \ldots, m_{n^a-1})$ is as desired. ⊠

Claim 6.4. *Let $n \geq 1$ and $\Gamma \colon 2^{\mathbb{N}} \to \mathrm{Conf}_n$ be a continuous mapping and define $F \colon (2^{\mathbb{N}})^{n+1} \to X$ by $F(x \frown y) \coloneqq f^{\Gamma(y)}(x)$ for all $x \in (2^{\mathbb{N}})^n$ and $y \in 2^{\mathbb{N}}$. Then $(F \times F)^{-1}(R_{n-1})$ is meager in $(2^{\mathbb{N}})^{n+1} \times (2^{\mathbb{N}})^{n+1}$.*

Proof. Suppose not. Then there exists $i < n$ such that $(F \times F)^{-1}(E_i)$ is non-meager. By Proposition 2.3, the mapping F is continuous, so $(F \times F)^{-1}(E_i)$ is Borel. Hence, by Kuratowski–Ulam's theorem, we can find $(u, x, z) \in (2^{\mathbb{N}})^{n+1} \times (2^{\mathbb{N}})^{n-1} \times 2^{\mathbb{N}}$ such that the set $A \coloneqq \{y \in 2^{\mathbb{N}} \mid F(u) \mathrel{E_i} F(x \frown y \frown z)\}$ is non-meager in $2^{\mathbb{N}}$. Let $y, y' \in A$ be distinct. Then $\neg x \frown y \mathrel{\mathbb{F}_{n-1}} x \frown y'$, so $f^{\Gamma(z)}(x \frown y) \mathrel{E \setminus R_{n-1}} f^{\Gamma(z)}(x \frown y)$, thus $\neg F(x \frown y \frown z) \mathrel{E_i} F(x \frown y' \frown z)$. Since E_i is an equivalence relation, this contradicts the fact that both $F(x \frown y \frown z)$ and $F(x \frown y' \frown z)$ are E_i-related to $F(u)$. ⊠

We will recursively build a decreasing sequence $(X_\alpha)_{\alpha<\omega_1}$ of Borel subsets of X. We start with $X_0 \coloneqq X$, and for limit ordinals λ, we let $X_\lambda \coloneqq \bigcap_{\alpha<\lambda} X_\alpha$. We now fix $\alpha < \omega_1$ and assume X_α has been constructed; we describe how to construct $X_{\alpha+1}$. We denote by S_α the set of all approximations a for which $\mathrm{Comp}(a, X_\alpha) \neq \emptyset$. We say that an approximation a is *α-terminal* if a has no immediate successor in S_α; we denote by T_α the set of all α-terminal approximations.

Claim 6.5. *Let a be an α-terminal approximation. Then there exists an E_{n^a}-invariant Borel set $B \subseteq X$, on which E has countable index over E_{n^a}, such that* $\mathrm{Comp}(a, X_\alpha \setminus B) = \emptyset$.

Proof. Proposition 2.3 ensures that the function $\mathrm{eval}\colon (2^{\mathbb{N}})^{n^a} \times \mathrm{Conf}_{n^a} \to X$, given by $\mathrm{eval}(x, \gamma) \coloneqq f^\gamma(x)$, is continuous. Fix $m \in \mathbb{N}$ as given by Claim 6.3 applied to the approximation a. Let $\mathcal{B}$ be the set of all $\gamma \in \mathrm{Comp}(a, X_\alpha)$ such that $(f^\gamma \times f^\gamma)^{-1}(E_{n^a}) \restriction \mathcal{N}_s$ is either meager or comeager for all $s \in (2^m)^{n^a}$. The continuity of eval and [Kec95, Theorem 16.1] ensures that $\mathcal{B}$ is a Borel subset of Conf_{n^a}. For all $\gamma \in \mathcal{B}$ and $s \in (2^m)^{n^a}$, Kuratowski–Ulam's theorem implies that each class of $(f^\gamma \times f^\gamma)^{-1}(E_{n^a})$ is either meager or comeager in $\mathcal{N}_s$, so $(f^\gamma \times f^\gamma)^{-1}(E_{n^a})$ has at most 2^{mn^a} non-meager classes. Along with the fact that $f^\gamma((2^{\mathbb{N}})^{n^a})$ is contained in a single E-class for all $\gamma \in \mathcal{B}$, this ensures that we can apply Lemma 4.8 to the spaces $X' \coloneqq (2^{\mathbb{N}})^{n^a}$ and $Y' \coloneqq \mathrm{Conf}_{n^a}$, the set $\mathcal{B} \subseteq Y'$, and the equivalence relations $E' \coloneqq (\mathrm{eval} \times \mathrm{eval})^{-1}(E)$ and $E'_{n^a} \coloneqq (\mathrm{eval} \times \mathrm{eval})^{-1}(E_{n^a})$. There are two cases.

Case 1: There is an E'_{n^a}-invariant Borel set $B' \subseteq (2^{\mathbb{N}})^{n^a} \times \mathrm{Conf}_{n^a}$, on which E' has countable index over E'_{n^a}, such that $(B')^\gamma$ is non-meager in $(2^{\mathbb{N}})^{n^a}$ for all $\gamma \in \mathcal{B}$. Lemma 3.1 ensures that E has countable index over E_{n^a} on $\mathrm{eval}(B')$, and since the latter set is analytic, Proposition 3.7 yields an E_{n^a}-invariant Borel set $B \supseteq \mathrm{eval}(B')$ on which E has countable index over E_{n^a}. Observe that $(B')^\gamma \subseteq (f^\gamma)^{-1}(B)$ for all $\gamma \in \mathcal{B}$, so $(f^\gamma)^{-1}(B)$ is non-meager, thus no $\gamma \in \mathcal{B}$ is generically compatible with $X_\alpha \setminus B$. It remains to show that this holds of every configuration γ that is compatible with a.

We can assume, without loss of generality, that $\gamma \in \mathrm{Comp}(a, X_\alpha)$. Let Ω be the set of all open subsets $U \subseteq (2^{\mathbb{N}})^{n^a}$ with the property that $(f^\gamma \times f^\gamma)^{-1}(E_{n^a}) \restriction U$ is either meager or comeager. By Lemma 5.4, $\bigcup \Omega$ is dense in $(2^{\mathbb{N}})^{n^a}$. Hence, we can apply Corollary 5.3 to find an open mapping $\varphi \in \mathcal{U}_m^{n^a}$ such that $\varphi(\mathcal{N}_s) \in \Omega$ for all $s \in (2^m)^{n^a}$. Now let $\delta \coloneqq \gamma \circ \varphi$. Then $\delta \preccurlyeq_m \gamma$; hence, by the choice of m, δ is compatible with a. Moreover, since γ is generically compatible with X_α and φ is open and one-to-one, it follows that δ is generically compatible with X_α, hence $\delta \in \mathrm{Comp}(a, X_\alpha)$. The conditions on φ imply that if $s \in (2^m)^{n^a}$, then $(f^\gamma \times f^\gamma)^{-1}(E_{n^a}) \restriction \varphi(\mathcal{N}_s)$ is either meager or comeager, hence $(f^\delta \times f^\delta)^{-1}(E_{n^a}) \restriction \mathcal{N}_s$ is either meager or comeager. It follows that $\delta \in \mathcal{B}$. Hence, $(f^\delta)^{-1}(B)$ is non-meager, and since φ is one-to-one and open, we deduce that $(f^\gamma)^{-1}(B)$ is non-meager, thus γ is not generically compatible with $X_\alpha \setminus B$.

Case 2: There exists a continuous mapping $\Gamma\colon 2^\mathbb{N} \to \mathcal{B}$ *such that* $(2^\mathbb{N})^{n^a} \times \Gamma(2^\mathbb{N})$ *is contained in a single E'-class and the equivalence relation* $((\mathrm{id}_{(2^\mathbb{N})^{n^a}} \times \Gamma) \times (\mathrm{id}_{(2^\mathbb{N})^{n^a}} \times \Gamma))^{-1}(E'_{n^a})$ *is meager.* We will show that the approximation a is not α-terminal. Define $F \coloneqq \mathrm{eval} \circ (\mathrm{id}_{(2^\mathbb{N})^{n^a}} \times \Gamma)\colon (2^\mathbb{N})^{n^a+1} \to X$, so that $F(x \frown y) = f^{\Gamma(y)}(x)$ for all $x \in (2^\mathbb{N})^{n^a}$ and $y \in 2^\mathbb{N}$. Our assumptions on Γ imply that $F((2^\mathbb{N})^{n^a+1})$ is contained in a single E-class and $(F \times F)^{-1}(E_{n^a})$ is meager. If $n^a \geq 1$, then Claim 6.4 implies that $(F \times F)^{-1}(R_{n^a-1})$ is meager, in which case $(F \times F)^{-1}(R_{n^a})$ is meager, and this obviously remains true when $n^a = 0$. For all $y \in 2^\mathbb{N}$, $\Gamma(y)$ is generically compatible with X_α, so $\forall y \in 2^\mathbb{N} \forall^* x \in (2^\mathbb{N})^{n^a}\ F(x \frown y) \in X_\alpha$, thus Kuratowski–Ulam's theorem ensures that $F^{-1}(X_\alpha)$ is comeager in $(2^\mathbb{N})^{n^a+1}$.

Let C be a dense G_δ subset of $(2^\mathbb{N})^{n^a+1} \times (2^\mathbb{N})^{n^a+1}$ contained in $(F^{-1}(X_\alpha) \times F^{-1}(X_\alpha)) \setminus (F \times F)^{-1}(R_{n^a})$. For all $(x, x') \in C$, we have $(F(x), F(x')) \in E \setminus R_{n^a}$, so there exists $u \in Y_{n^a}$ such that $\pi_{n^a}(u) = (F(x), F(x'))$. Hence, by Jankov–von Neumann's uniformisation theorem, we can find a Baire measurable mapping $G\colon C \to Y_{n^a}$ such that $\pi_{n^a}(G(x, x')) = (F(x), F(x'))$ for all $(x, x') \in C$. Since Baire measurable mappings are continuous on a comeager set, by shrinking C if necessary, we can assume that G is continuous.

By Proposition 5.5, there exists a homomorphism $\varphi \in \mathcal{U}_m^{n^a+1}$ from $\sim\mathbb{F}_{n^a}((2^\mathbb{N})^{n^a+1})$ to C. Since φ is aligned, it follows that if $y \in 2^\mathbb{N}$, then $\varphi(x \frown y)(n^a)$ does not depend upon x. We denote this value by $\psi(y)$, thereby obtaining a continuous mapping $\psi\colon 2^\mathbb{N} \to 2^\mathbb{N}$. For all $x \in (2^\mathbb{N})^{n^a}$ and $y \in 2^\mathbb{N}$, we hence write $\varphi(x \frown y) = (\varphi^y \upharpoonright n^a)(x) \frown \psi(y)$, where we abuse notation by using $\varphi^y \upharpoonright n^a$ to denote the aligned mapping $(2^\mathbb{N})^{n^a} \to (2^\mathbb{N})^{n^a}$ given by $(\varphi^y \upharpoonright n^a)(x) \coloneqq \varphi(x \frown y) \upharpoonright n^a$. To see that $\varphi^y \upharpoonright n^a \in \mathcal{U}_m^{n^a}$, note that if $s \in (2^m)^{n^a}$ and $x \in \mathcal{N}_s$, then $x \frown y \in \mathcal{N}_{s \frown y \upharpoonright m}$, so $\varphi(x \frown y) \in \mathcal{N}_{s \frown y \upharpoonright m}$, thus $(\varphi^y \upharpoonright n^a)(x) \in \mathcal{N}_s$.

Define $\gamma \coloneqq (n^\gamma, f^\gamma, (g_k^\gamma)_{k<n^\gamma})$, where $n^\gamma \coloneqq n^a + 1$, $f^\gamma\colon (2^\mathbb{N})^{n^\gamma} \to X$ is given by $f^\gamma \coloneqq F \circ \varphi$, $g_k^\gamma\colon \mathbb{F}_{k+1}((2^\mathbb{N})^{n^\gamma}) \setminus \mathbb{F}_k((2^\mathbb{N})^{n^\gamma}) \to Y_k$ is given by $g_k^\gamma(x \frown y, x' \frown y) \coloneqq g_k^{\Gamma(\psi(y))}((\varphi^y \upharpoonright n^a)(x), (\varphi^y \upharpoonright n^a)(x'))$ for all $k < n^a$, $(x, x') \in \mathbb{F}_{k+1}((2^\mathbb{N})^{n^a}) \setminus \mathbb{F}_k((2^\mathbb{N})^{n^a})$, and $y \in 2^\mathbb{N}$, and $g_{n^a}^\gamma \coloneqq G \circ (\varphi \times \varphi) \upharpoonright \sim\mathbb{F}_{n^a}((2^\mathbb{N})^{n^a+1})$. The continuity of F yields that of f^γ, Proposition 2.3 ensures that g_k^γ is continuous for all $k < n^a$, and the continuity of G on $(\varphi \times \varphi)(\sim\mathbb{F}_{n^a}((2^\mathbb{N})^{n^a+1}))$ yields the continuity of $g_{n^a}^\gamma$. Moreover, if

$k < n^a$, $(x, x') \in \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^a}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^a})$, and $y \in 2^{\mathbb{N}}$, then

$$\begin{aligned}
&\pi_k(g_k^\gamma(x \frown y, x' \frown y)) \\
&\quad = \pi_k(g_k^{\Gamma(\psi(y))}((\varphi^y \restriction n^a)(x), (\varphi^y \restriction n^a)(x'))) \\
&\quad = (f^{\Gamma(\psi(y))}((\varphi^y \restriction n^a)(x)), f^{\Gamma(\psi(y))}((\varphi^y \restriction n^a)(x'))) \\
&\quad = (F((\varphi^y \restriction n^a)(x) \frown \psi(y)), F((\varphi^y \restriction n^a)(x') \frown \psi(y))) \\
&\quad = (F(\varphi(x \frown y)), F(\varphi(x' \frown y))) \\
&\quad = (f^\gamma(x \frown y), f^\gamma(x' \frown y)).
\end{aligned}$$

And if $(x, x') \in {\sim}\mathbb{F}_{n^a}((2^{\mathbb{N}})^{n^\gamma})$, then the definition of G ensures that $\pi_{n^a}(G(\varphi(x), \varphi(x'))) = (F(\varphi(x)), F(\varphi(x')))$, that is, $\pi_{n^a}(g_{n^a}^\gamma(x, x')) = (f^\gamma(x), f^\gamma(x'))$. Hence, γ is a configuration. Moreover, since φ is a homomorphism from ${\sim}\mathbb{F}_{n^a}((2^{\mathbb{N}})^{n^a+1})$ to C and $C \subseteq F^{-1}(X_\alpha) \times F^{-1}(X_\alpha)$, it follows that f^γ takes values in X_α; in particular, γ is generically compatible with X_α.

For all $y \in 2^{\mathbb{N}}$, define $\gamma^y := (n^a, (f^\gamma)^y, ((g_k^\gamma)^y)_{k<n^a})$. The formulas defining γ ensure that $\gamma^y = \Gamma(\psi(y)) \circ (\varphi^y \restriction n^a)$; since $\varphi^y \restriction n^a \in \mathcal{U}_m^{n^a}$, it follows that γ^y is a configuration and $\gamma^y \preccurlyeq_m \Gamma(\psi(y))$. Since $\Gamma(\psi(y))$ is compatible with a, our choice of m ensures that γ^y is compatible with a. In particular, if $y \in 2^{\mathbb{N}}$, then $(f^\gamma)^y \in \mathcal{U}^a$, so $f^\gamma \in (\mathcal{U}^a)^{\to n^\gamma}$; similarly, if $k < n^a$ and $y \in 2^{\mathbb{N}}$, then $(g_k^\gamma)^y \in \mathcal{V}_k^a$, so $g_k^\gamma \in (\mathcal{V}_k^a)^{\to n^\gamma}$. Recall that, by Lemma 2.8, $(\mathcal{U}^a)^{\to n^\gamma}$ is an open subset of $C((2^{\mathbb{N}})^{n^\gamma}, X)$ and $(\mathcal{V}_k^a)^{\to n^\gamma}$ is an open subset of $C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}), Y_k)$ for all $k < n^a$. Thus, we can find a basic open set $\mathcal{U}^b \subseteq C((2^{\mathbb{N}})^{n^\gamma}, X)$ with $\mathrm{diam}(\mathcal{U}^b) \leq 1/n^\gamma$ and $f^\gamma \in \mathcal{U}^b \subseteq (\mathcal{U}^a)^{\to n^\gamma}$ and basic open sets $\mathcal{V}_k^b \subseteq C(\mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}), Y_k)$ with $\mathrm{diam}(\mathcal{V}_k^b) \leq 1/n^\gamma$ and $g_k^\gamma \in \mathcal{V}_k^b \subseteq (\mathcal{V}_k^a)^{\to n^\gamma}$ for all $k < n^a$. We can also find a basic open set $\mathcal{V}_{n^a}^b \subseteq C({\sim}\mathbb{F}_{n^a}((2^{\mathbb{N}})^{n^\gamma}), Y_{n^a})$ with $\mathrm{diam}(\mathcal{V}_{n^a}^b) \leq 1/n^\gamma$ and $g_{n^a}^\gamma \in \mathcal{V}_{n^a}^b$. Letting $n^b := n^\gamma$, it follows that $b := (n^b, \mathcal{U}^b, (\mathcal{V}_k^b)_{k<n^b})$ is an approximation which is an immediate successor of a and with which γ is compatible. In particular, $\mathrm{Comp}(b, X_\alpha) \neq \emptyset$, so $b \in S_\alpha$, thus a is not terminal. ⊠

For all $a \in T_\alpha$, let B_α^a be a Borel set as given by Claim 6.5. Define $X_{\alpha+1} := X_\alpha \setminus \bigcup_{a \in T_\alpha} B_\alpha^a$. As there are only countably many approximations, this set is Borel. This completes the inductive construction.

As there are only countably many approximations and $(S_\alpha)_{\alpha<\omega_1}$ is decreasing, there exists $\alpha_0 < \omega_1$ such that $S_{\alpha_0+1} = S_{\alpha_0}$.

Claim 6.6. *Every element of S_{α_0} has an immediate successor in S_{α_0}.*

Proof. Let a be an approximation having no successor in S_{α_0}. Then a is α_0-terminal, so $B_{\alpha_0}^a$ is defined and $\mathrm{Comp}(a, X_{\alpha_0} \setminus B_{\alpha_0}^a) = \emptyset$. Since

$X_{\alpha_0+1} \subseteq X_{\alpha_0} \setminus B^a_{\alpha_0}$, it follows that $\mathrm{Comp}(a, X_{\alpha_0+1}) = \emptyset$, hence $a \notin S_{\alpha_0+1}$. But $S_{\alpha_0+1} = S_{\alpha_0}$, so $\alpha \notin S_{\alpha_0}$. ⊠

Let a^0 denote the unique approximation for which $n^{a^0} \coloneqq 0$ and $\mathcal{U}^{a^0} \coloneqq C((2^{\mathbb{N}})^0, X)$. There are two cases.

Case 1: $a^0 \notin S_{\alpha_0}$. Then no configuration is simultaneously compatible with a^0 and generically compatible with X_{α_0}. Observe that a configuration γ such that $n^\gamma = 0$ essentially consists only of a continuous mapping $f^\gamma \colon (2^{\mathbb{N}})^0 \to X$, and can therefore be identified with a point $x^\gamma \in X$. Every such configuration is compatible with a^0, and such a configuration is generically compatible with X_{α_0} if and only if $x^\gamma \in X_{\alpha_0}$. It follows that $X_{\alpha_0} = \emptyset$, so $X = \bigcup_{\alpha<\alpha_0} X_\alpha \setminus X_{\alpha+1} = \bigcup_{\alpha<\alpha_0} \bigcup_{a \in T_\alpha} B^a_\alpha$. For all $n \in \mathbb{N}$, the set $B_n \coloneqq \bigcup_{\alpha<\alpha_0} \bigcup_{a\in T_\alpha, n^a=n} B^a_\alpha$ is a countable union of E_n-invariant Borel sets on which E has countable index over E_n, and is therefore itself an E_n-invariant Borel set on which E has countable index over E_n. As $X = \bigcup_{n\in\mathbb{N}} B_n$, condition (1) follows.

Case 2: $a^0 \in S_{\alpha_0}$. Then Claim 6.6 yields a sequence $(a^n)_{n\in\mathbb{N}}$ of elements of S_{α_0} such that a^{n+1} is an immediate successor of a^n for all $n \in \mathbb{N}$. It follows that $n^{a^n} = n$ for every $n \in \mathbb{N}$. As $a^n \in S_{\alpha_0}$, there is a configuration γ^n which is compatible with a^n. For all $n \in \mathbb{N}$, define a continuous function $f^n \colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ by $f^n \coloneqq (f^{\gamma^n})^{\to\mathbb{N}}$. Similarly, for all $n \in \mathbb{N}$ and $k < n$, define a continuous function $g^n_k \colon \mathbb{F}_{k+1} \setminus \mathbb{F}_k \to Y_k$ by $g^n_k \coloneqq (g^{\gamma^n}_k)^{\to\mathbb{N}}$. If $m \leq n$, then $f^{\gamma^m} \in \mathcal{U}^{a^m}$ and $f^{\gamma^n} \in \mathcal{U}^{a^n} \subseteq (\mathcal{U}^{a^m})^{\to n}$, so $f^m, f^n \in (\mathcal{U}^{a^m})^{\to\mathbb{N}}$. As our choice of metrics ensures that $\mathrm{diam}((\mathcal{U}^{a^m})^{\to\mathbb{N}}) \leq \mathrm{diam}(\mathcal{U}^{a^m}) \leq 1/m$, it follows that $d(f^m, f^n) \leq 1/m$, so $(f^n)_{n\in\mathbb{N}}$ is a Cauchy sequence and admits a limit $f \in C((2^{\mathbb{N}})^{\mathbb{N}}, X)$. Similarly, for all $k \in \mathbb{N}$, the sequence $(g^n_k)_{n>k}$ is Cauchy, and therefore admits a limit $g_k \in C(\mathbb{F}_{k+1} \setminus \mathbb{F}_k, Y_k)$.

We now show that f satisfies condition (2). It is sufficient to show that if $k \in \mathbb{N}$ and $(x, x') \in \mathbb{E}_1 \setminus \mathbb{F}_k$, then $(f(x), f(x')) \in E \setminus R_k$. As $(R_l)_{l\in\mathbb{N}}$ is increasing and $\mathbb{E}_1 \setminus \mathbb{F}_k = \bigcup_{l\geq k} \mathbb{F}_{l+1} \setminus \mathbb{F}_l$, we can assume that $(x, x') \in \mathbb{F}_{k+1} \setminus \mathbb{F}_k$. Observe that if $n > k$, then

$$\begin{aligned}(f^n(x), f^n(x')) &= (f^{\gamma^n}(x \restriction n), f^{\gamma^n}(x' \restriction n)) \\ &= \pi_k(g^{\gamma^n}_k(x \restriction n, x' \restriction n)) \\ &= \pi_k(g^n_k(x, x')).\end{aligned}$$

By Proposition 2.3, we can take the limit on both sides of the equality to obtain $(f(x), f(x')) = \pi_k(g_k(x, x'))$, so $(f(x), f(x')) \in E \setminus R_k$. ⊠

Before proving our second dichotomy, we note the following consequence of [CLM14, Propositions 2.13 and 2.14]:

Proposition 6.7. *Let $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ be a comeager set and $(R_n)_{n \in \mathbb{N}}$ be a sequence of F_σ binary relations on $(2^{\mathbb{N}})^{\mathbb{N}}$ such that $R_n \cap \mathbb{E}_1 \setminus \mathbb{F}_n = \emptyset$ for all $n \in \mathbb{N}$. Then there is a continuous homomorphism $\varphi \colon (2^{\mathbb{N}})^{\mathbb{N}} \to C$ from $(\mathbb{F}_n, {\sim}\mathbb{F}_n)_{n \in \mathbb{N}}$ to $(\mathbb{F}_n, {\sim}R_n)_{n \in \mathbb{N}}$.*

Our second dichotomy is the following:

Theorem 6.8. *Let X be a Polish space, E be an analytic equivalence relation on X, and $(E_n)_{n \in \mathbb{N}}$ be a sequence of potentially F_σ subequivalence relations of E. Then exactly one of the following holds:*

1. *There is a cover $(B_n)_{n \in \mathbb{N}}$ of X such that B_n is an E_n-invariant Borel set on which E has countable index over E_n for all $n \in \mathbb{N}$.*
2. *There is a continuous homomorphism $\varphi \colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{E}_1, ({\sim}\mathbb{F}_n)_{n \in \mathbb{N}})$ to $(E, ({\sim}\bigcup_{i \leq n} E_i)_{n \in \mathbb{N}})$.*

Moreover, if $E = \bigcup_{n \in \mathbb{N}} E_n$, then the mapping φ in the latter condition is an embedding of $\mathbb{E}_1$ into E.

Proof. The "moreover" part is immediate. Condition (2) in the statement of the theorem is stronger than condition (2) in the statement of Theorem 6.2; hence, as in Theorem 6.2, the two conditions are mutually exclusive. It remains to show that at least one of them holds.

Refining the topology on X if necessary, we can assume that the E_n's are F_σ. We need to show that if condition (2) in the statement of Theorem 6.2 holds, then so too does condition (2) in the statement of Theorem 6.8. Suppose that $\psi \colon (2^{\mathbb{N}})^{\mathbb{N}} \to X$ is a continuous homomorphism from $(\mathbb{E}_1 \setminus \mathbb{F}_n)_{n \in \mathbb{N}}$ to $(E \setminus \bigcup_{i \leq n} E_i)_{n \in \mathbb{N}}$. For all $n \in \mathbb{N}$, define $R'_n \coloneqq (\psi \times \psi)^{-1}(\bigcup_{i \leq n} E_i)$. Then R'_n is F_σ and $R'_n \cap (\mathbb{E}_1 \setminus \mathbb{F}_n) = \emptyset$. So Proposition 6.7 yields a continuous homomorphism $\pi \colon (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ from $(\mathbb{F}_n, {\sim}\mathbb{F}_n)_{n \in \mathbb{N}}$ to $(\mathbb{F}_n, {\sim}R'_n)_{n \in \mathbb{N}}$, in which cases the mapping $\varphi \coloneqq \psi \circ \pi$ satisfies condition (2). ⊠

Following the usual abuse of language, we say that an equivalence relation is *finite* if each of its equivalence classes are finite. A Borel equivalence relation E on a Polish space is *hyperfinite* if there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations of E whose union is E. As the Lusin–Novikov uniformization theorem easily implies that every finite Borel equivalence relation on a Polish space is smooth, it follows that every hyperfinite Borel equivalence relation on a Polish space is hypersmooth.

We will need the following elementary result (see [CLM14, Proposition 5.1]):

Proposition 6.9. *Suppose that X is a Polish space, $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of smooth Borel equivalence relations on X, and* *there is a cover $(B_n)_{n \in \mathbb{N}}$ of X such that B_n is a Borel set on which $\bigcup_{m \in \mathbb{N}} E_m$ has countable index over E_n for all $n \in \mathbb{N}$. Then $\bigcup_{m \in \mathbb{N}} E_m$ is Borel reducible to a hyperfinite Borel equivalence relation on a Polish space.*

As a first application of our second dichotomy, we give a proof of the Kechris–Louveau dichotomy:

Theorem 6.10 (Kechris–Louveau)**.** *Suppose that E is a hypersmooth Borel equivalence relation on a Polish space. Then exactly one of the following holds:*

1. *There is a Borel reduction of E to a hyperfinite Borel equivalence relation on a Polish space.*
2. *There is a continuous embedding of $\mathbb{E}_1$ into E.*

Proof. The exclusivity of the two conditions comes from the fact, mentioned in the introduction, that $\mathbb{E}_1$ is not Borel reducible to a countable Borel equivalence relation on a Polish space; this can, for instance, be obtained as a consequence of [KL97, Theorem 4.1], [Kec92, §1.II.i], and Feldman–Moore's theorem (alternatively, a more elementary proof can be obtained from [CLM14, Proposition 2.4] and [CLM14, Proposition 2.5]). To see that at least one of the two conditions holds, write E as the increasing union of a sequence of smooth Borel subequivalence relations $(E_n)_{n \in \mathbb{N}}$. Refining the topology on X if necessary, we can assume that the E_n's are closed. Hence we can apply Theorem 6.8 and use Proposition 6.9 to complete the proof. ⊠

7. Primary results

The following fact is the main result of this paper:

Theorem 7.1. *Suppose that $\mathcal{F}$ is a class of strongly-idealistic potentially-F_σ equivalence relations on Polish spaces that is closed under countable disjoint unions and countable-index Borel superequivalence relations. If E is an equivalence relation on a Polish space that is a countable union of subequivalence relations that are Borel reducible to relations in $\mathcal{F}$, then at least one of the following holds:*

1. *There is a Borel reduction of E to a relation in $\mathcal{F}$.*
2. *There is a continuous embedding of $\mathbb{E}_1$ into E.*

Moreover, if every relation in $\mathcal{F}$ is also ccc idealistic, then exactly one of these conditions holds.

Proof. To see that the conditions are mutually exclusive when every relation in $\mathcal{F}$ is ccc idealistic, observe that if both hold, then there is a

Borel reduction of $\mathbb{E}_1$ to a ccc idealistic Borel equivalence relation on a Polish space, contradicting [KL97, Theorem 4.1].

To see that at least one of the conditions holds, note that, by Theorem 6.8, we can assume that there is a cover $(B_n)_{n \in \mathbb{N}}$ of X by Borel sets on which E has countable index over subequivalence relations that are Borel reducible to relations in $\mathcal{F}$. For each $n \in \mathbb{N}$, Proposition 3.8 yields a Borel reduction φ_n of $E \restriction B_n$ to some $F_n \in \mathcal{F}$, and Proposition 3.12 allows us to assume that B_n is E-invariant. Then the E-invariant Borel sets $B'_n \coloneqq B_n \setminus \bigcup_{m<n} B_m$ partition X, so the functions $\varphi_n \restriction B'_n$ can be combined to obtain a Borel reduction of E to $\coprod_{n \in \mathbb{N}} F_n$. ⊠

Theorem 1 follows from Theorem 7.1 and the previously mentioned fact that every countable Borel equivalence relation on a Polish space is strongly ccc idealistic and potentially F_σ.

Theorem 2 follows from Theorem 1, [KL97, Theorem 4.1], the Feldman–Moore theorem, and the fact that every countable Borel equivalence relation on a Polish space is ccc idealistic.

Theorem 3 follows from Proposition 3.17, Theorem 7.1, and [KL97, Theorem 4.1].

Theorem 4 follows from Theorem 5 and the fact that every countable Borel equivalence relation on a Polish space is strongly ccc idealistic and potentially F_σ.

Theorem 5 follows from Theorem 7.1 and [KL97, Theorem 4.1].

Theorem 6 follows from Theorem 5, Propositions 3.13, 3.15, and 3.17, and the observation that if F is a Borel equivalence relation on a Polish space, then the class $\mathcal{F}$ of equivalence relations on Polish spaces that are Borel isomorphic to countable-index Borel superequivalence relations of $F \times \Delta(\mathbb{N})$ is closed under countable disjoint unions and countable-index Borel superequivalence relations, and if F is strongly idealistic and potentially F_σ, then so too is $F \times \Delta(\mathbb{N})$.

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