Advances in Mathematics 265 (2014) 1–31

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Essential countability of treeable equivalence relations[☆]

John D. Clemens[a], Dominique Lecomte[b,c], Benjamin D. Miller[d,a,*]

[a] *Institut für Mathematische Logik und Grundlagenforschung, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstraße 62, 48149 Münster, Germany*
[b] *Université Paris 6, Institut de Mathématiques de Jussieu, Projet Analyse Fonctionnelle, Couloir 16-26, 4ème étage, Case 247, 4, place Jussieu, 75 252 Paris Cedex 05, France*
[c] *Université de Picardie, I.U.T de l'Oise, site de Creil, 13, allée de la faïencerie, 60 107 Creil, France*
[d] *Kurt Gödel Research Center for Mathematical Logic, Währinger Straße 25, 1090 Vienna, Austria*

ARTICLE INFO

Article history:
Received 23 April 2013
Accepted 28 July 2014
Available online xxxx
Communicated by H. Jerome Keisler

MSC:
primary 03E15
secondary 28A05

Keywords:
Dichotomy theorem
Essentially countable

ABSTRACT

We establish a dichotomy theorem characterizing the circumstances under which a treeable Borel equivalence relation E is essentially countable. Under additional topological assumptions on the treeing, we in fact show that E is essentially countable if and only if there is no continuous embedding of $\mathbb{E}_1$ into E. Our techniques also yield the first classical proof of the analogous result for hypersmooth equivalence relations, and allow us to show that up to continuous Kakutani embeddability, there is a minimum Borel function which is not essentially countable-to-one.

☆ The authors were supported in part by SFB Grant 878.

* Corresponding author.
E-mail addresses: jclemens@uni-muenster.de (J.D. Clemens), dominique.lecomte@upmc.fr (D. Lecomte), glimmeffros@gmail.com (B.D. Miller).
URLs: http://wwwmath.uni-muenster.de/u/john.clemens (J.D. Clemens), https://www.imj-prg.fr/~dominique.lecomte/ (D. Lecomte), http://wwwmath.uni-muenster.de/u/ben.miller (B.D. Miller).

http://dx.doi.org/10.1016/j.aim.2014.07.035
0001-8708/

Essentially hyperfinite
Hypersmooth

0. Introduction

Basic notions. A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is K_σ if it is a countable union of compact sets, F_σ if it is a countable union of closed sets, G_δ if it is a countable intersection of open sets, and *Borel* if it is in the σ-algebra generated by the underlying topology. A function between such spaces is *Borel* if pre-images of open sets are Borel. Every subset of a Polish space inherits the Borel structure consisting of its intersection with each Borel subset of the underlying space.

We endow $\mathbb{N}$ with the discrete topology. A subset of a Polish space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$. It is not hard to see that every non-empty analytic set is a continuous image of $\mathbb{N}^\mathbb{N}$ itself. A set is *co-analytic* if its complement is analytic. Following standard practice, we use Σ^1_1 and Π^1_1 to denote the classes of analytic and co-analytic subsets of Polish spaces.

Suppose that X and Y are Polish spaces and R and S are relations on X and Y. A *homomorphism* from R to S is a function $\phi: X \to Y$ sending R-related sequences to S-related sequences, a *cohomomorphism* from R to S is a function $\phi: X \to Y$ sending R-unrelated sequences to S-unrelated sequences, a *reduction* of R to S is a function satisfying both requirements, and an *embedding* of R into S is an injective reduction. Given sequences $(R_i)_{i\in I}$ and $(S_i)_{i\in I}$ of relations on X and Y, we use the analogous terminology to describe functions $\phi: X \to Y$ which have the desired property for all $i \in I$.

When E and F are equivalence relations on X and Y, the existence of a reduction of E to F is trivially equivalent to the existence of an injection of X/E into Y/F. By requiring that the former is Borel, we obtain a definable refinement of cardinality capable of distinguishing quotients whose classical cardinality is that of $\mathbb{R}$. Over the last few decades, this notion has been used to great effect in shedding new light on obstacles of definability inherent in classification problems throughout mathematics, particularly in the theories of countable groups and fields, probability measure-preserving group actions, separable C^* and von Neumann algebras, and separable Banach spaces. In order to better understand such results, it is essential to obtain the best possible understanding of the Borel reducibility hierarchy. The present paper is a contribution towards this goal.

An initial segment. It is easy to see that for each countable cardinal n, there is a Borel reduction of the equality relation on the n-point discrete space to any Borel equivalence relation with at least n classes. The first non-trivial theorem in the area appears in [9], implying that there is a Borel reduction of the equality relation on $\mathbb{R}$ to any Borel equivalence relation with uncountably many classes. That is, the continuum hypothesis holds in the definable context. Building on this, [2, Theorem 1] yields the analog of the continuum hypothesis at the next level, namely, that there is a Borel reduction of the

Vitalı equivalence relation on $\mathbb{R}$, i.e., the orbit equivalence relation induced by the action of $\mathbb{Q}$ on $\mathbb{R}$ under addition, into any Borel equivalence relation which is not Borel reducible to the equality relation on $\mathbb{R}$.

Going one step further, [5, Theorem 1] implies that under Borel reducibility, there is no Borel equivalence relation lying strictly between the Vitalı equivalence relation and the orbit equivalence relation induced by the action of $\mathbb{R}^{<\mathbb{N}}$ on $\mathbb{R}^{\mathbb{N}}$ under addition. It is well-known, however, that the full analog of [2, Theorem 1] cannot hold. This can be seen, for example, by noting that under Borel reducibility, the latter equivalence relation is incomparable with the orbit equivalence relation induced by the action of $\mathbb{Q}^{\mathbb{N}}$ on $\mathbb{R}^{\mathbb{N}}$. Nevertheless, in this paper we establish a generalization of [5, Theorem 1] of a substantially less local nature.

One should note that to facilitate both the proofs of these results as well as topological strengthenings in which Borel reducibility is replaced with continuous embeddability, one typically focuses on different equivalence relations. In [9], one uses the equality relation on $2^{\mathbb{N}}$. In [2], one uses the relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ given by $x \mathbin{\mathbb{E}_0} y \iff \exists n \in \mathbb{N}\ \forall m \geq n\ x(m) = y(m)$. And in [5], one uses the relation $\mathbb{E}_1$ on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by $x \mathbin{\mathbb{E}_1} y \iff \exists n \in \mathbb{N}\ \forall m \geq n\ x(m) = y(m)$.

Treeable equivalence relations. We identify graphs with their (ordered) edge sets, so that a *graph* on X is an irreflexive, symmetric binary relation G on X. A *cycle* through such a graph is a sequence $(x_i)_{i\leq n}$ such that $n \geq 3$, $(x_i)_{i<n}$ is injective, $x_i \mathbin{G} x_{i+1}$ for all $i < n$, and $x_0 = x_n$. We say that G is *acyclic* if it admits no such cycles. A *treeing* of an equivalence relation is an acyclic Borel graph whose connected components coincide with the classes of the relation. A Borel equivalence relation is *treeable* if it admits a Borel treeing. Examples include orbit equivalence relations associated with free Borel actions of countable discrete free groups. Such relations play a particularly significant role in the measure-theoretic context, due primarily to their susceptability to cocycle reduction techniques.

Beyond such applications, treeable equivalence relations play another important role as a proving ground for natural conjectures, where simpler arguments can often be used to obtain stronger results. One example appears in [3], in which a strengthening of [2, Theorem 1] is established for treeable Borel equivalence relations. Although the proof given there takes [2, Theorem 1] for granted, more direct arguments have since appeared (see, for example, [8, Theorem 22]). Moreover, in the presence of strong determinacy assumptions, the ideas behind this argument can be used to establish the natural generalizations of both [9] and [2, Theorem 1] to treeable equivalence relations of higher complexity.

Following the standard abuse of language, we say that an equivalence relation is *finite* if all of its classes are finite, and *countable* if all of its classes are countable. A Borel equivalence relation is *essentially* $\mathscr{E}$ if it is Borel reducible to a Borel equivalence relation in $\mathscr{E}$. In addition to the results just mentioned, [3] concludes with a question at the heart of our concerns here: is $\mathbb{E}_1$ the minimum treeable Borel equivalence relation which is not essentially countable?

Essential countability. In order to present our characterization of essential countability, we must first introduce some terminology. A *path* through a binary relation R on X is a sequence of the form $(x_i)_{i \le n}$, where $n \in \mathbb{N}$ and $x_i \; R \; x_{i+1}$ for all $i < n$. The n^{th} *iterate* of R is the binary relation $R^{(n)}$ consisting of all pairs (y, z) for which there is such a path with $x_0 = y$ and $x_n = z$. We use $R^{(\le n)}$ to denote $\bigcup_{m \le n} R^{(m)}$.

For all $n \in \mathbb{N}$, let $\mathbb{F}_n$ denote the equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by $x \; \mathbb{F}_n \; y \iff \forall m \ge n \; x(m) = y(m)$.

Theorem A. *Suppose that X is a Polish space, E is a treeable Borel equivalence relation on X, and G is a Borel treeing of E. Then exactly one of the following holds:*

(1) *The equivalence relation E is essentially countable.*
(2) *There exists a function $f: \mathbb{N} \to \mathbb{N}$ for which there is a continuous homomorphism $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{F}_{n+1} \setminus \mathbb{F}_n)_{n \in \mathbb{N}}$ to $(G^{(\le f(n+1))} \setminus G^{(\le f(n))})_{n \in \mathbb{N}}$.*

Although this stops somewhat short of yielding an answer to [3, Question 13], it does imply one of the main corollaries of a positive answer: among essentially treeable Borel equivalence relations, essential countability is *robust*, in the sense that it is equivalent to the existence of a universally measurable (or $\aleph_0$-universally Baire measurable) reduction of E to a countable equivalence relation.

Moreover, under appropriate topological assumptions, we do obtain a positive answer to the original question. We say that a Borel equivalence relation is *subtreeable-with-F_σ-iterates* if it has a Borel treeing contained in an acyclic graph with F_σ iterates.

Theorem B. *Suppose that X is a Polish space and E is a Borel equivalence relation on X which is essentially subtreeable-with-F_σ-iterates. Then exactly one of the following holds:*

(1) *The equivalence relation E is essentially countable.*
(2) *There is a continuous embedding $\pi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ of $\mathbb{E}_1$ into E.*

Although the restriction that E is subtreeable-with-F_σ-iterates might appear rather Machiavellian, it turns out that the family of such relations has unbounded potential complexity. While this fact is beyond the scope of the present paper, one should note that it has surprising consequences for the global structure of the Borel reducibility hierarchy. We say that a class $\mathscr{E}$ of Borel equivalence relations is *dichotomous* if there is a Borel equivalence relation $E_{\mathscr{E}}$ such that for every Borel equivalence relation E, either $E \in \mathscr{E}$ or there is a Borel reduction of $E_{\mathscr{E}}$ to E. Using the unbounded potential complexity of the family of Borel equivalence relations which are subtreeable-with-F_σ iterates, one can show that if a Borel equivalence relation is not Borel reducible to $\mathbb{E}_0$, then it is incomparable with Borel equivalence relations of unbounded potential complexity, strengthening [5, Theorem 2]. It follows that if $\mathscr{E}$ is a dichotomous class of equivalence relations of bounded potential complexity, then $\mathscr{E}$ consists solely of smooth equivalence relations. Consequently, the only non-trivial such families are those associated with the main results of [9] and [2], the classes of potentially open and potentially closed equivalence relations. These developments will be explored in a future paper.

Essentially countable-to-one functions. In the case of treeings induced by Borel functions, we obtain even stronger results. To describe these, we must again introduce some terminology. A *Kakutani embedding* of a function $T: X \to X$ into a function $U: Y \to Y$ is a Borel injection $\pi: X \to Y$ with the property that $(\pi \circ T)(x) = (U^n \circ \pi)(x)$, where $n > 0$ is least with $(U^n \circ \pi)(x) \in \pi(X)$, for all $x \in X$.

We say that a set $Y \subseteq X$ is T*-complete* if $X = \bigcup_{n \in \mathbb{N}} T^{-n}(Y)$, we say that a set $Y \subseteq X$ is T*-stable* if $T(Y) \subseteq Y$, and we say that a Borel function $T: X \to X$ is *essentially countable-to-one* if there is a T-complete, T-stable Borel set $B \subseteq X$ on which T is countable-to-one.

The *product* of functions $f: X \to X$ and $g: Y \to Y$ is the function $f \times g: X \times Y \to X \times Y$ given by $(f \times g)(x, y) = (f(x), g(y))$. The *successor function* on $\mathbb{N}$ is given by $S(n) = n+1$. The *unilateral shift* on $(2^{\mathbb{N}})^{\mathbb{N}}$ is given by $s((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}$.

Theorem C. *Suppose that X is a Polish space and $T: X \to X$ is a Borel function. Then exactly one of the following holds:*

(1) *The function T is essentially countable-to-one.*
(2) *There is a continuous Kakutani embedding $\pi: \mathbb{N} \times (2^{\mathbb{N}})^{\mathbb{N}} \to X$ of $S \times s$ into T.*

Organization. In Section 1, we review the basic descriptive set theory utilized throughout. In Section 2, we establish several Baire category results. In Section 3, we prove a parametrized version of an unpublished generalization of the main theorem of [9], due originally to Conley–Lecomte–Miller. In Section 4, we establish our main technical theorems, from which the results stated thus far are relatively straightforward corollaries. These technical results are essentially generalizations of Theorems A and B to Borel equivalence relations equipped with suitably definable assignments of quasi-metrics to their classes, although we state them in a somewhat different form so as to facilitate the exposition. In Section 5, we give the promised classical proof of [5, Theorem 1]. In Section 6, we establish Theorems A and B. In Section 7, we establish Theorem C.

1. Preliminaries

In this section, we review the basic descriptive set theory utilized throughout the paper.

Suppose that X and Y are topological spaces. The *compact-open topology* on the set of all continuous functions $f: X \to Y$ is that generated by the sets $\{f: X \to Y \mid f(K) \subseteq U\}$, where $K \subseteq X$ is compact and $U \subseteq Y$ is open. We use $C(X, Y)$ to denote the corresponding topological space. The following observations will aid complexity calculations involving this space.

Proposition 1.1. *Suppose that X is a compact Polish space and Y is a Polish space. Then the function $e: C(X, Y) \times X \to Y$ given by $e(f, x) = f(x)$ is continuous.*

Proof. See, for example, [7, §IV.44.II]. □

Proposition 1.2. *Suppose that X is a locally compact Polish space and Y is a Polish space. Then $C(X, Y)$ is a Polish space.*

Proof. See, for example, [7, §IV.44.VII]. □

Although Borel functions constitute a much broader class than continuous ones, the following fact often allows one to treat Borel functions as if they are continuous.

Proposition 1.3. *Suppose that X and Y are Polish spaces and $\mathscr{F}$ is a countable family of Borel functions $T: X \to Y$. Then there are finer Polish topologies on X and Y, whose Borel sets are the same as those of the original topologies, with respect to which every $T \in \mathscr{F}$ is continuous. Moreover, if $X = Y$ then the topologies on X and Y can be taken to be the same.*

Proof. See, for example, [4, §13]. □

When proving facts about Polish spaces, it is often notationally convenient (and perhaps conceptually clearer) to first focus on the special case of $\mathbb{N}^{\mathbb{N}}$. The desired general result is then typically obtained from a representation theorem such as the following.

Proposition 1.4. *Every Polish space is analytic.*

Proof. See, for example, [4, Theorem 7.9]. □

A *Baire space* is a topological space in which every countable intersection of dense open sets is dense. The following fact ensures that Baire category techniques are applicable in arbitrary Polish spaces.

Theorem 1.5 *(Baire). Every complete metric space is a Baire space.*

Proof. See, for example, [4, Theorem 8.4]. □

A set is *nowhere dense* if its closure does not contain a non-empty open set, a set is *meager* if it is contained in a countable union of nowhere dense sets, a set is *comeager* if its complement is meager, and a set has the *Baire property* if its symmetric difference with some open set is meager. One can view the latter three notions as topological analogs of μ-null sets, μ-conull sets, and μ-measurable sets, although the topological and measure-theoretic notions behave quite differently.

The following fact, known in some circles as *localization*, can be viewed as the Baire category analog of the Lebesgue density theorem.

Proposition 1.6. *Suppose that X is a Polish space and $B \subseteq X$ has the Baire property. Then B is non-meager if and only if there is a non-empty open set $U \subseteq X$ such that B is comeager in U.*

Proof. This easily follows from the definitions of a Baire space and the Baire property (see, for example, [4, Proposition 8.26]). □

A function $\phi: X \to Y$ is *Baire measurable* if for all open sets $V \subseteq Y$, the set $\phi^{-1}(V)$ has the Baire property. Again, this can be viewed as a topological analog of μ-measurability. The following observation is a very strong analog of the measure-theoretic fact that μ-measurable functions can be approximated by continuous ones on sets of arbitrarily large μ-measure.

Proposition 1.7. *Suppose that X and Y are Polish spaces and $\phi: X \to Y$ is Baire measurable. Then there is a dense G_δ set $C \subseteq X$ such that $\phi \upharpoonright C$ is continuous.*

Proof. See, for example, [4, Proposition 8.38]. □

Although our primary focus is on Borel sets, we will often consider analytic sets, in which case the following fact ensures that Baire category arguments remain applicable.

Proposition 1.8 *(Lusin–Sierpiński). Suppose that X is a Polish space and $A \subseteq X$ is analytic. Then A has the Baire property.*

Proof. See, for example, [4, Theorem 21.6]. □

A topological space X is T_0 if for all distinct $x, y \in X$, there is an open set $U \subseteq X$ containing exactly one of x and y. A set $Y \subseteq X$ is *invariant* with respect to an equivalence relation E on X if it is a union of E-classes. An equivalence relation E on X is *generically ergodic* if every invariant set $B \subseteq X$ with the Baire property is meager or comeager. The following consequence of generic ergodicity is often useful when dealing with parametrized dichotomy theorems.

Proposition 1.9. *Suppose that X is a Baire space, Y is a second countable T_0 space, E is a generically ergodic equivalence relation on X, and $\phi: X \to Y$ is a Baire measurable homomorphism from E to the equality relation on Y. Then there exists $y \in Y$ for which $\phi^{-1}(y)$ is comeager.*

Proof. Fix a basis $\{V_n \mid n \in \mathbb{N}\}$ for the topology of Y, let N denote the set of $n \in \mathbb{N}$ for which $\phi^{-1}(V_n)$ is comeager, and let y be the unique element of $\bigcap_{n\in N} V_n \setminus \bigcup_{n\in\sim N} V_n$. $\square$

The x^{th} *vertical section* and y^{th} *horizontal section* of a set $R \subseteq X \times Y$ are the sets R_x and R^y given by $R_x = \{y \in Y \mid x \mathrel{R} y\}$ and $R^y = \{x \in X \mid x \mathrel{R} y\}$. Given a property P, we write $\forall^* x\ P(x)$ to indicate that the set $\{x \in X \mid P(x)\}$ is comeager. The following fact can be viewed as the Baire category analog of Fubini's theorem.

Theorem 1.10 *(Kuratowski–Ulam). Suppose that X and Y are Baire spaces, Y is second countable, and $R \subseteq X \times Y$ has the Baire property.*

(1) *$\forall^* x \in X\ R_x$ has the Baire property.*
(2) *R is comeager $\Longleftrightarrow \forall^* x \in X\ R_x$ is comeager.*

Proof. See, for example, [4, Theorem 8.41]. $\square$

The following fact can often be used to reduce problems of finding perfect sets with desirable properties to questions of Baire category.

Theorem 1.11 *(Mycielski). Suppose that X is a non-empty Polish space and $R \subseteq X \times X$ is meager. Then there is a continuous cohomomorphism $\phi: 2^{\mathbb{N}} \to X$ from the equality relation on $2^{\mathbb{N}}$ to R.*

Proof. See, for example, [4, Theorem 19.1]. $\square$

We use $\sigma(\Sigma^1_1)$ to denote the class of subsets of Polish spaces which lie in the smallest σ-algebra containing the analytic sets, and we say that a function $f: X \to Y$ is *$\sigma(\Sigma^1_1)$-measurable* if for all open sets $U \subseteq Y$, the set $f^{-1}(U)$ is in $\sigma(\Sigma^1_1)$. We use proj_X to denote the *projection function* given by $\operatorname{proj}_X(x, y) = x$. A *uniformization* of a set $R \subseteq X \times Y$ is a function $f: \operatorname{proj}_X(R) \to Y$ whose graph is contained in R.

Theorem 1.12 *(Jankov–von Neumann). Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is an analytic set. Then there is a $\sigma(\Sigma^1_1)$-measurable uniformization of R.*

Proof. See, for example, [4, Theorem 18.1]. $\square$

Theorem 1.13 *(Lusin–Novikov). Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set all of whose vertical sections are countable. Then there are countably many Borel uniformizations of R whose graphs cover R.*

Proof. See, for example, [4, Theorem 18.10]. $\square$

The following facts will be useful in ensuring that various constructions yield Borel sets.

Theorem 1.14 *(Lusin). Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then $\{x \in X \mid \exists! y \in Y \ x \, R \, y\}$ is co-analytic.*

Proof. See, for example, [4, Theorem 18.11]. □

Theorem 1.15 *(Lusin). Suppose that X and Y are Polish spaces and $f: X \to Y$ is a countable-to-one Borel function. Then $f(X)$ is Borel.*

Proof. See, for example, [4, Lemma 18.12]. □

Although the class of analytic sets is clearly closed under projections, one must often consider analogs of projections in which the non-emptiness of the sections is replaced with stronger conditions. The following two facts ensure that the class of analytic sets is also closed under certain generalized projections of this form.

Theorem 1.16 *(Mazurkiewicz–Sierpiński). Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is analytic. Then so too is $\{x \in X \mid R_x$ is uncountable$\}$.*

Proof. See, for example, [4, Theorem 29.20]. □

Theorem 1.17 *(Novikov). Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is analytic. Then so too is $\{x \in X \mid R_x$ is comeager$\}$.*

Proof. See, for example, [4, Theorem 29.22]. □

Suppose that Γ and Γ' are classes of subsets of Polish spaces. A property P is *Γ-on-Γ'* if $\{x \in X \mid P(R_x)\} \in \Gamma$ whenever X and Y are Polish spaces and $R \subseteq X \times Y$ in Γ'. The following *reflection theorem* will help us to ensure that our constructions yield Borel sets.

Theorem 1.18 *(Harrington–Kechris–Moschovakis). Suppose that P is a Π^1_1-on-Σ^1_1 property. Then every analytic subset of a Polish space satisfying P is contained in a Borel set satisfying P.*

Proof. See, for example, [4, Theorem 35.10]. □

Rather than apply reflection directly, we will often use the following *separation theorem.*

Theorem 1.19 *(Lusin). Suppose that X is a Polish space and $A, A' \subseteq X$ are disjoint analytic sets. Then there is a Borel set $B \subseteq X$ such that $A \subseteq B$ and $A' \cap B = \emptyset$.*

Proof. This easily follows from Theorem 1.18 (a direct proof can be found, for example, in [4, Theorem 14.7]). □

This yields the following connection between analytic and Borel sets.

Theorem 1.20 *(Souslin). A subset of a Polish space is Borel if and only if it is both analytic and co-analytic.*

Proof. The fact that sets which are both analytic and co-analytic are Borel is a direct consequence of Theorem 1.19, and the converse follows from Proposition 1.4 and a straightforward induction (see, for example, [4, Theorem 14.11], although the latter part is proven there in a somewhat different fashion). $\square$

We will also use the following *generalized separation theorem*.

Theorem 1.21 *(Novikov). Suppose that X is a Polish space and $A_n \subseteq X$ are analytic sets for which $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Then there are Borel sets $B_n \subseteq X$ containing A_n for which $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$.*

Proof. This also follows easily from Theorem 1.18 (a direct proof can be found, for example, in [4, Theorem 28.5]). $\square$

Given $m, n \in \mathbb{N} \cup \{\mathbb{N}\}$, we say that a sequence $s \in 2^m$ is *extended by* a sequence $t \in 2^n$, or $s \sqsubseteq t$, if $s(i) = t(i)$ for all $i < m$. We use $s \frown t$ to denote the *concatenation* of s and t.

Fix sequences $s_n \in 2^n$ for which the set $\{s_n \mid n \in \mathbb{N}\}$ is *dense*, in the sense that $\forall s \in 2^{<\mathbb{N}}\ \exists n \in \mathbb{N}\ s \sqsubseteq s_n$. Let $\mathbb{G}_0$ denote the graph on $2^{\mathbb{N}}$ consisting of all pairs of the form $(s_n \frown (i) \frown x, s_n \frown (1-i) \frown x)$, where $i < 2$, $n \in \mathbb{N}$, and $x \in 2^{\mathbb{N}}$.

The *restriction* of a graph G on X to a set $Y \subseteq X$ is the graph $G \restriction Y$ on Y given by $G \restriction Y = G \cap (Y \times Y)$. Given a graph G on X, we say that a set $Y \subseteq X$ is *G-independent* if $G \restriction Y = \emptyset$.

Proposition 1.22 *(Kechris–Solecki–Todorcevic). Suppose that $B \subseteq 2^{\mathbb{N}}$ is a $\mathbb{G}_0$-independent set with the Baire property. Then B is meager.*

Proof. This is a direct consequence of the definition of $\mathbb{G}_0$ and Proposition 1.6 (see, for example, [6, Proposition 6.2]). $\square$

An *I-coloring* of G is a function $c: X \to I$ such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$. We say that G has *countable Borel chromatic number* if there is a Borel $\mathbb{N}$-coloring of G.

Theorem 1.23 *(Kechris–Solecki–Todorcevic). Suppose that X is a Polish space and G is an analytic graph on X. Then exactly one of the following holds:*

(1) *The graph G has countable Borel chromatic number.*
(2) *There is a continuous homomorphism from $\mathbb{G}_0$ to G.*

Proof. See, for example, [6, Theorem 6.4]. □

We say that a Borel equivalence relation is *smooth* if it is Borel reducible to the equality relation on a Polish space.

Theorem 1.24 *(Harrington–Kechris–Louveau). Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *The equivalence relation E is smooth.*
(2) *There is a continuous embedding $\pi: 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into E.*

Proof. See, for example, [2, Theorem 1.1]. □

We say that an equivalence relation is *hyper* $\mathscr{E}$ if it is the union of an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of relations in $\mathscr{E}$.

Theorem 1.25 *(Dougherty–Jackson–Kechris). Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. If E is hypersmooth, then E is hyperfinite.*

Proof. See, for example, [1, Theorem 5.1]. □

We say that a set $B \subseteq X$ is *E-complete* if it intersects every E-class. While not strictly necessary for our purposes here, the following fact is also useful in establishing closure properties of essential countability.

Theorem 1.26 *(Hjorth). Suppose that X is a Polish space and E is a treeable Borel equivalence relation on X. Then the following are equivalent:*

(1) *There is an E-complete Borel set on which E is countable.*
(2) *The equivalence relation E is essentially countable.*

Proof. See, for example, [3, Theorem 6]. □

Finally, we note that while the original proofs of Theorems 1.23, 1.24, and 1.26 utilized the effective theory, classical proofs have since appeared (see [8]). In particular, our reliance on these results does not prevent our arguments from being classical in nature.

2. Baire category results

In this section, we establish several Baire category results which will be useful throughout the paper.

A function is *meager-to-one* if pre-images of singletons are meager.

Proposition 2.1. *Suppose that X is a Polish space, $A \subseteq X$, G is a graph on X, and there is a meager-to-one Baire measurable function $\phi: 2^{\mathbb{N}} \to X$ for which the set $A' = \phi^{-1}(A)$ is comeager and the set $G' = (\phi \times \phi)^{-1}(G)$ is meager. Then there is a continuous injection $\pi: 2^{\mathbb{N}} \to A$ of $2^{\mathbb{N}}$ into a G-independent set.*

Proof. By Proposition 1.7, there is a dense G_δ set $B' \subseteq A'$ on which ϕ is continuous. Let E' denote the pullback of the equality relation on X through ϕ. The fact that ϕ is Baire measurable ensures that E' has the Baire property, and the fact that ϕ is meager-to-one implies that every vertical section of E' is meager, so E' is meager by Theorem 1.10. In particular, it follows that $(B' \times B') \setminus (E' \cup G')$ is a comeager subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, so Theorem 1.11 yields a continuous injection $\psi: 2^{\mathbb{N}} \to B'$ of $2^{\mathbb{N}}$ into a G'-independent set which is also a *partial transversal* of E', meaning that it intersects every equivalence class in at most one point. It follows that the function $\pi = \phi \circ \psi$ is as desired. $\square$

Throughout the paper, we will work with spaces of the form $(2^m)^n$, where $m, n \in \mathbb{N} \cup \{\mathbb{N}\}$. We use $\frown$ to denote horizontal concatenation, and $\oplus$ to denote vertical concatenation. We will abuse language by saying that a sequence $s \in (2^m)^n$ is *extended* by a sequence $s' \in (2^{m'})^{n'}$, or $s \sqsubseteq s'$, if $\forall i < m \ \forall j < n \ s(i)(j) = s'(i)(j)$.

Proposition 2.2. *Suppose that $k \in \mathbb{N}$ and $B \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is a set with the Baire property on which $\mathbb{F}_{k+1}$ has countable index over $\mathbb{F}_k$. Then B is meager.*

Proof. Suppose, towards a contradiction, that B is non-meager. Then Theorem 1.10 yields a non-meager set of $(x, z) \in (2^{\mathbb{N}})^k \times (2^{\mathbb{N}})^{\mathbb{N}}$ such that $\{y \in 2^{\mathbb{N}} \mid x \frown (y) \frown z \in B\}$ is non-meager, and therefore uncountable. As $(x \frown (y) \frown z, x \frown (y') \frown z) \in \mathbb{F}_{k+1} \setminus \mathbb{F}_k$ for distinct $y, y' \in 2^{\mathbb{N}}$, this contradicts the fact that $\mathbb{F}_{k+1}$ has countable index over $\mathbb{F}_k$ on B. $\square$

Remark 2.3. Suppose that μ is a Borel probability measure on $(2^{\mathbb{N}})^{\mathbb{N}}$ for which μ-almost every measure in the disintegration of μ with respect to the function deleting the k^{th} column is continuous (this holds, for example, if $\mu(U) = 1/2^n$ for every basic open set $U \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ specifying values on n coordinates). Then an essentially identical argument (using this assumption in place of Theorem 1.10) yields the analogous result in which B is μ-measurable instead of Baire measurable.

Proposition 2.4. *Suppose that $A \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ is an analytic set on which $\mathbb{E}_1$ has countable index over $\mathbb{F}_k$, for some $k \in \mathbb{N}$. Then $[A]_{\mathbb{E}_1}$ is meager.*

Proof. Note that $\mathbb{F}_{\ell+1}$ has countable index over $\mathbb{F}_\ell$ on $[A]_{\mathbb{F}_\ell}$, for all $\ell \geq k$. As each $[A]_{\mathbb{F}_\ell}$ is analytic, Proposition 1.8 ensures that it has the Baire property, so Proposition 2.2 implies that it is meager, thus so too is the set $[A]_{\mathbb{E}_1} = \bigcup_{\ell \geq k} [A]_{\mathbb{F}_\ell}$. $\square$

Suppose that X is a Polish space and E is a Borel equivalence relation on X. Theorem 1.13 immediately implies that if $B \subseteq X$ is an E-complete Borel set on which E is countable, then there is a Borel reduction of E to $E \upharpoonright B$, thus E is essentially countable. Together with Proposition 2.4, the following weak converse yields a simple proof of [5, Proposition 1.4], ruling out the existence of a Baire measurable reduction of $\mathbb{E}_1$ to a countable equivalence relation on a Polish space.

Proposition 2.5. *Suppose that X and Y are Polish spaces, E is an analytic equivalence relation on X, F is a countable equivalence relation on Y, and there is a Baire measurable reduction $\phi: X \to Y$ of E to F. Then there is a Borel set $B \subseteq X$ such that $E \upharpoonright B$ is countable and $[B]_E$ is comeager.*

Proof. By Proposition 1.7, there is a dense G_δ set $C \subseteq X$ on which ϕ is continuous. By Theorem 1.12, there is a $\sigma(\varSigma^1_1)$-measurable function $\phi': \phi(C) \to C$ such that $\phi \circ \phi'$ is the identity function. As pre-images of analytic sets under continuous functions are analytic, it follows that $\phi' \circ \phi$ is also $\sigma(\varSigma^1_1)$-measurable, so one more application of Proposition 1.7 yields a dense G_δ set $D \subseteq C$ on which it is continuous. Then the set $A = (\phi' \circ \phi)(D)$ is analytic. As E is countable on A and Theorem 1.16 ensures that this property is $\varPi^1_1$-on-$\varSigma^1_1$, Theorem 1.18 yields a Borel set $B \supseteq A$ on which E is countable. As $D \subseteq [B]_E$, it follows that the latter set is comeager. □

For each $k \in \mathbb{N}$, let $\mathbb{F}_k((2^m)^n)$ denote the equivalence relation on $(2^m)^n$ given by $x \ \mathbb{F}_k((2^m)^n) \ y \iff \forall i \geq k \ x(i) = y(i)$. We say that $\phi: (2^m)^n \to (2^{m'})^{n'}$ is *extended by* $\psi: (2^{m''})^{n''} \to (2^{m'''})^{n'''}$, or $\phi \sqsubseteq \psi$, if $s \sqsubseteq t \implies \phi(s) \sqsubseteq \psi(t)$ for all $s \in (2^m)^n$ and $t \in (2^{m''})^{n''}$.

Proposition 2.6. *Suppose that $m, m', n \in \mathbb{N}$, $\phi: (2^m)^n \to (2^{m'})^n$ is an embedding of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m'})^n))_{k<n}$, and $\mathcal{U}$ is a family of open subsets of $(2^{\mathbb{N}})^n$ whose union is dense. Then there exists $m'' \in \mathbb{N}$ for which there is an embedding $\psi: (2^m)^n \to (2^{m''})^n$ of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m''})^n))_{k<n}$ extending ϕ with $\forall s \in (2^m)^n \ \exists U \in \mathcal{U} \ \mathcal{N}_{\psi(s)} \subseteq U$.*

Proof. Fix an injective enumeration $(s_i)_{i<I}$ of $(2^m)^n$. Set $m_0 = m'$ and $\phi_0 = \phi$, and recursively find $m_{i+1} \in \mathbb{N}$ and $\phi_{i+1}: (2^m)^n \to (2^{m_{i+1}})^n$ of the form $\phi_{i+1}(s) = \phi_i(s) \oplus t$, where $t \in (2^{m_{i+1}-m_i})^n$ has the property that $\mathcal{N}_{\phi_i(s_i) \oplus t}$ is a subset of some $U \in \mathcal{U}$. Set $m'' = m_I$ and $\psi = \phi_I$. □

Proposition 2.7. *Suppose that $m, n \in \mathbb{N}$ and $\pi: (2^{\mathbb{N}})^n \to \mathbb{N}$ is Baire measurable. Then there exist $i: (2^m)^n \to \mathbb{N}$, $m' \in \mathbb{N}$, and an embedding $\phi: (2^m)^n \to (2^{m'})^n$ of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m'})^n))_{k<n}$, extending the identity function on $(2^m)^n$, with the property that $\forall s \in (2^m)^n \ \forall^* x \in (2^{\mathbb{N}})^n \ i(s) = \pi(\phi(s) \oplus x)$.*

Proof. Proposition 1.6 ensures that the family $\mathcal{U}$ of open sets $U \subseteq (2^{\mathbb{N}})^n$ with the property that $\exists i \in \mathbb{N}\ \forall^* x \in U\ i = \pi(x)$ has dense union. The desired result therefore follows from an application of Proposition 2.6 to the identity function on $(2^m)^n$. $\square$

Note that $\mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n)$ is homeomorphic to the product of $((2^k)^n \times (2^k)^n) \times (2^{\mathbb{N}})^{n-(k+1)}$ with the complement of the equality relation on $2^{\mathbb{N}}$. In particular, it is a locally compact Polish space, so Theorem 1.5 ensures that it is a Baire space.

Proposition 2.8. *Suppose that $\ell, m, m', n \in \mathbb{N}$, $\phi\colon (2^m)^n \to (2^{m'})^n$ is an embedding of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m'})^n))_{k<n}$, and $\mathcal{U}$ is a family of open subsets of $\mathbb{F}_{\ell+1}((2^{\mathbb{N}})^n)$ whose union is dense. Then there exists $m'' \in \mathbb{N}$ for which there is an embedding $\psi\colon (2^m)^n \to (2^{m''})^n$ of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m''})^n))_{k<n}$ extending ϕ with the property that $\forall (s,t) \in \mathbb{F}_{\ell+1}((2^m)^n) \setminus \mathbb{F}_\ell((2^m)^n)\ \exists U \in \mathcal{U}\ \mathcal{N}_{\psi(s)} \times \mathcal{N}_{\psi(t)} \subseteq U$.*

Proof. Fix an injective enumeration $(s_i, t_i)_{i<I}$ of $\mathbb{F}_{\ell+1}((2^m)^n) \setminus \mathbb{F}_\ell((2^m)^n)$. Define $m_0 = m'$ and $\phi_0 = \phi$, and recursively find $m_{i+1} \in \mathbb{N}$ and $\phi_{i+1}\colon (2^m)^n \to (2^{m_{i+1}})^n$ of the form $\phi_{i+1}(s) = \phi_i(s) \oplus \sigma(s)$, where $\sigma\colon (2^m)^n \to (2^{m_{i+1}-m_i})^n$ is itself of the form

$$\sigma(s) = \begin{cases} t & \text{if } s\ \mathbb{F}_\ell((2^m)^n)\ s_i \text{ and} \\ u & \text{otherwise,} \end{cases}$$

and $\mathbb{F}_{\ell+1}((2^{\mathbb{N}})^n) \cap (\mathcal{N}_{\phi_i(s_i)\oplus t} \times \mathcal{N}_{\phi_i(t_i)\oplus u})$ is a non-empty subset of some $U \in \mathcal{U}$. Set $m'' = m_I$ and $\psi = \phi_I$. $\square$

Proposition 2.9. *Suppose that $\ell, m, n \in \mathbb{N}$ and $\pi\colon \mathbb{F}_{\ell+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_\ell((2^{\mathbb{N}})^n) \to \mathbb{N}$ is Baire measurable. Then there exist $i\colon \mathbb{F}_{\ell+1}((2^m)^n) \setminus \mathbb{F}_\ell((2^m)^n) \to \mathbb{N}$, $m' \in \mathbb{N}$, and an embedding $\phi\colon (2^m)^n \to (2^{m'})^n$ of $(\mathbb{F}_k((2^m)^n))_{k<n}$ into $(\mathbb{F}_k((2^{m'})^n))_{k<n}$, extending the identity function on $(2^m)^n$, with the property that $\forall (s,t) \in \mathbb{F}_{\ell+1}((2^m)^n) \setminus \mathbb{F}_\ell((2^m)^n)\ \forall^* (x,y) \in \mathbb{F}_{\ell+1}((2^{\mathbb{N}})^n)\ i(s,t) = \pi(\phi(s) \oplus x, \phi(t) \oplus y)$.*

Proof. By Proposition 1.6, the family $\mathcal{U}$ of open sets $U \subseteq \mathbb{F}_{\ell+1}((2^{\mathbb{N}})^n)$ with the property that $\exists i \in \mathbb{N}\ \forall^* (x,y) \in U\ i = \pi(x,y)$ has dense union. The desired result therefore follows from an application of Proposition 2.8 to the identity function on $(2^m)^n$. $\square$

We next establish an analog of Theorem 1.11 for $(2^{\mathbb{N}})^n$.

Proposition 2.10. *Suppose that $m, m', n \in \mathbb{N}$, $\phi\colon (2^m)^n \to (2^{m'})^n$, $C \subseteq (2^{\mathbb{N}})^n$ is comeager, and $(R_k)_{k<n}$ is a sequence of subsets of $(2^{\mathbb{N}})^n \times (2^{\mathbb{N}})^n$ with the property that R_k is comeager in $\mathbb{F}_{k+1}((2^{\mathbb{N}})^n)$, for all $k < n$. Then ϕ extends to a continuous homomorphism $\psi\colon (2^{\mathbb{N}})^n \to C$ from $(\mathbb{F}_k((2^{\mathbb{N}})^n), \mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \setminus \mathbb{F}_k((2^{\mathbb{N}})^n))_{k<n}$ to $(\mathbb{F}_k((2^{\mathbb{N}})^n), R_k)_{k<n}$.*

Proof. Fix a sequence $(U_i)_{i\in\mathbb{N}}$ of dense open subsets of $(2^{\mathbb{N}})^n$ whose intersection is contained in C. For all $k < n$, fix a decreasing sequence $(U_{i,k})_{i\in\mathbb{N}}$ of dense open subsets of

$\mathbb{F}_{k+1}((2^{\mathbb{N}})^n)$ whose intersection is contained in R_k. We will recursively construct a strictly increasing sequence of natural numbers $\ell_i \in \mathbb{N}$ and embeddings $\phi_i: (2^i)^n \to (2^{\ell_i})^n$ of $(\mathbb{F}_k((2^i)^n))_{k<n}$ into $(\mathbb{F}_k((2^{\ell_i})^n))_{k<n}$ with the following properties:

(1) $\forall s \in (2^i)^n \ \forall t \in (2^{i+1})^n \ (s \sqsubseteq t \Longrightarrow \phi_i(s) \sqsubseteq \phi_{i+1}(t))$.
(2) $\forall s \in (2^{i+1})^n \ \mathcal{N}_{\phi_{i+1}(s)} \subseteq U_i$.
(3) $\forall k < n \ \forall (s,t) \in \mathbb{F}_{k+1}((2^{i+1})^n) \setminus \mathbb{F}_k((2^{i+1})^n) \ \mathbb{F}_{k+1}((2^{\mathbb{N}})^n) \cap (\mathcal{N}_{\phi_{i+1}(s)} \times \mathcal{N}_{\phi_{i+1}(t)}) \subseteq U_{i,k}$.

We begin by setting $\ell_m = m'$ and $\phi_m = \phi$. Given $\phi_i: (2^i)^n \to (2^{m_i})^n$, define $\phi_i': (2^{i+1})^n \to (2^{m_i+1})^n$ by $\phi_i'(s \oplus t) = \phi_i(s) \oplus t$. We then obtain $m_{i+1} \in \mathbb{N}$ and $\phi_{i+1}: (2^{i+1})^n \to (2^{m_{i+1}})^n$ by one application of Proposition 2.6 and n applications of Proposition 2.8. This completes the recursive construction, and the corresponding function $\psi: (2^{\mathbb{N}})^n \to (2^{\mathbb{N}})^n$, given by $\psi(x) = \bigcup_{i \geq m} \phi_i \circ \mathrm{proj}_{(2^i)^n}(x)$, is as desired. □

We next consider analogous results with $(2^{\mathbb{N}})^{\mathbb{N}}$ in place of $(2^{\mathbb{N}})^n$.

Proposition 2.11. *Suppose that $m, m', n, n' \in \mathbb{N}$, $\phi: (2^m)^n \to (2^{m'})^{n'}$ is an embedding of $(\mathbb{F}_k((2^m)^n))_{k \leq n}$ into $(\mathbb{F}_k((2^{m'})^{n'}))_{k \leq n}$, and $\mathcal{U}$ is a family of open subsets of $(2^{\mathbb{N}})^{\mathbb{N}}$ whose union is dense. Then there exist $m'', n'' \in \mathbb{N}$ for which there is an embedding $\psi: (2^m)^n \to (2^{m''})^{n''}$ of $(\mathbb{F}_k((2^m)^n))_{k \leq n}$ into $(\mathbb{F}_k((2^{m''})^{n''}))_{k \leq n}$ extending ϕ with the property that $\forall s \in (2^m)^n \ \exists U \in \mathcal{U} \ \mathcal{N}_{\psi(s)} \subseteq U$.*

Proof. Fix an injective enumeration $(s_i)_{i<I}$ of $(2^m)^n$. Set $m_0 = m'$, $n_0 = n'$, and $\phi_0 = \phi$, and recursively find $m_{i+1}, n_{i+1} \in \mathbb{N}$ and $\phi_{i+1}: (2^m)^n \to (2^{m_{i+1}})^{n_{i+1}}$ of the form $\phi_{i+1}(s) = (\phi_i(s) \frown u) \oplus v$, where $u \in (2^{m_i})^{n_{i+1}-n_i}$ and $v \in (2^{m_{i+1}-m_i})^{n_{i+1}}$ have the property that $\mathcal{N}_{(\phi_i(s_i) \frown u) \oplus v}$ is a subset of some $U \in \mathcal{U}$. Set $m'' = m_I$, $n'' = n_I$, and $\psi = \phi_I$. □

We say that an open set $U \subseteq (2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$ is *k-dense* if for all $m, n \in \mathbb{N}$ and $(s,t) \in \sim\mathbb{F}_k((2^m)^n)$, there exist $m', n' \in \mathbb{N}$ and extensions $s', t' \in (2^{m'})^{n'}$ of s, t such that $\mathcal{N}_{s'} \times \mathcal{N}_{t'} \subseteq U$ and

$$\forall i < m' \ \forall k < j < n' \ \bigl(s'(j)(i) \neq t'(j)(i) \implies (i < m \text{ and } j < n)\bigr).$$

Proposition 2.12. *Suppose that $\ell, m, m', n, n' \in \mathbb{N}$, $\phi: (2^m)^n \to (2^{m'})^{n'}$ is an embedding of $(\mathbb{F}_k((2^m)^n))_{k \leq n}$ into $(\mathbb{F}_k((2^{m'})^{n'}))_{k \leq n}$, and $\mathcal{U}$ is a family of open subsets of $(2^{\mathbb{N}})^{\mathbb{N}} \times (2^{\mathbb{N}})^{\mathbb{N}}$ whose union is ℓ-dense. Then there exist $m'', n'' \in \mathbb{N}$ for which there is an embedding $\psi: (2^m)^n \to (2^{m''})^{n''}$ of $(\mathbb{F}_k((2^m)^n))_{k \leq n}$ into $(\mathbb{F}_k((2^{m''})^{n''}))_{k \leq n}$ extending ϕ with the property that $\forall (s,t) \in \sim\mathbb{F}_\ell((2^m)^n) \ \exists U \in \mathcal{U} \ \mathcal{N}_{\psi(s)} \times \mathcal{N}_{\psi(t)} \subseteq U$.*

Proof. Fix an injective enumeration $(s_i, t_i)_{i<I}$ of $\sim\mathbb{F}_\ell((2^m)^n)$. Define $m_0 = m'$, $n_0 = n'$, and $\phi_0 = \phi$, and recursively find $m_{i+1}, n_{i+1} \in \mathbb{N}$ and $\phi_{i+1}: (2^m)^n \to (2^{m_{i+1}})^{n_{i+1}}$ of the form $\phi_{i+1}(s) = (\phi_i(s) \frown \sigma(s)) \oplus \tau(s)$, where $\sigma: (2^m)^n \to (2^{m_i})^{n_{i+1}-n_i}$ is of the form

$$\sigma(s) = \begin{cases} t & \text{if } s \, \mathbb{F}_\ell((2^m)^n) \, s_i \text{ and} \\ u & \text{otherwise,} \end{cases}$$

$\tau\colon (2^m)^n \to (2^{m_{i+1}-m_i})^{n_{i+1}}$ is of the form

$$\tau(s) = \begin{cases} v & \text{if } s \, \mathbb{F}_\ell((2^m)^n) \, s_i \text{ and} \\ w & \text{otherwise,} \end{cases}$$

and $(t,u) \in \mathbb{F}_{\max(0,\ell+1-n_i)}((2^{m_i})^{n_{i+1}-n_i})$ and $(v,w) \in \mathbb{F}_{\ell+1}((2^{m_{i+1}-m_i})^{n_{i+1}})$ are such that $\mathcal{N}_{(\phi_i(s_i)\frown t)\oplus v} \times \mathcal{N}_{(\phi_i(t_i)\frown u)\oplus w}$ is contained in some $U \in \mathcal{U}$. Set $m'' = m_I$ and $\psi = \phi_I$. $\square$

We say that a set $M \subseteq (2^\mathbb{N})^\mathbb{N}$ is *k-meager* if it is disjoint from the intersection of a countable family of k-dense open sets.

Proposition 2.13. *Suppose that $C \subseteq (2^\mathbb{N})^\mathbb{N}$ is comeager and $(R_k)_{k\in\mathbb{N}}$ is a sequence of subsets of $(2^\mathbb{N})^\mathbb{N} \times (2^\mathbb{N})^\mathbb{N}$ with the property that R_k is k-meager, for all $k \in \mathbb{N}$. Then there is a continuous homomorphism $\phi\colon (2^\mathbb{N})^\mathbb{N} \to C$ from $(\mathbb{F}_k, \sim\mathbb{F}_k)_{k\in\mathbb{N}}$ to $(\mathbb{F}_k, \sim R_k)_{k\in\mathbb{N}}$.*

Proof. Fix a sequence $(U_i)_{i\in\mathbb{N}}$ of dense open subsets of $(2^\mathbb{N})^\mathbb{N}$ whose intersection is contained in C. For all $k \in \mathbb{N}$, fix a decreasing sequence $(U_{i,k})_{i\in\mathbb{N}}$ of k-dense open subsets of $(2^\mathbb{N})^\mathbb{N} \times (2^\mathbb{N})^\mathbb{N}$ whose intersection is disjoint from R_k. We will recursively construct strictly increasing sequences of natural numbers $m_i, n_i \in \mathbb{N}$ and embeddings $\phi_i\colon (2^i)^i \to (2^{m_i})^{n_i}$ of $(\mathbb{F}_k((2^i)^i))_{k\le i}$ into $(\mathbb{F}_k((2^{m_i})^{n_i}))_{k\le i}$ such that:

(1) $\forall s \in (2^i)^i \ \forall t \in (2^{i+1})^{i+1} \ (s \sqsubseteq t \Longrightarrow \phi_i(s) \sqsubseteq \phi_{i+1}(t))$.
(2) $\forall s \in (2^{i+1})^{i+1} \ \mathcal{N}_{\phi_{i+1}(s)} \subseteq U_i$.
(3) $\forall k \le i \ \forall (s,t) \in \sim\mathbb{F}_k((2^{i+1})^{i+1}) \ \mathcal{N}_{\phi_{i+1}(s)} \times \mathcal{N}_{\phi_{i+1}(t)} \subseteq U_{i,k}$.

We begin by setting $m_0 = n_0 = 0$ and fixing $\phi_0\colon (2^0)^0 \to (2^0)^0$. Given $\phi_i\colon (2^i)^i \to (2^{m_i})^{n_i}$, define $\psi_i\colon (2^{i+1})^{i+1} \to (2^{m_i+1})^{n_i+1}$ by $\psi_i((s \frown t) \oplus u) = (\phi_i(s) \frown t) \oplus u$. We then obtain $m_{i+1}, n_{i+1} \in \mathbb{N}$ and $\phi_{i+1}\colon (2^{i+1})^{i+1} \to (2^{m_{i+1}})^{n_{i+1}}$ by one application of Proposition 2.11 and $i+1$ applications of Proposition 2.12. This completes the recursive construction. Define $\phi\colon (2^\mathbb{N})^\mathbb{N} \to (2^\mathbb{N})^\mathbb{N}$ by $\phi(x) = \bigcup_{i\in\mathbb{N}} \phi_i \circ \mathrm{proj}_{(2^i)^i}(x)$. $\square$

We next give a condition sufficient for ensuring k-meagerness.

Proposition 2.14. *Suppose that $k \in \mathbb{N}$ and $R \subseteq (2^\mathbb{N})^\mathbb{N} \times (2^\mathbb{N})^\mathbb{N}$ is an F_σ set disjoint from $\mathbb{E}_1 \setminus \mathbb{F}_k$. Then R is k-meager.*

Proof. It is sufficient to show that every open set $U \subseteq 2^\mathbb{N} \times 2^\mathbb{N}$ containing $\mathbb{E}_1 \setminus \mathbb{F}_k$ is k-dense. Towards this end, suppose that $m, n \in \mathbb{N}$ and $(s,t) \in \sim\mathbb{F}_k((2^m)^n)$. Let $x, y \in (2^\mathbb{N})^\mathbb{N}$ denote the extensions of s, t with constant value 0 off of the domains of s, t.

Then $(x, y) \in \mathbb{E}_1 \setminus \mathbb{F}_k$, so $(x, y) \in U$, thus there exist $m', n' \in \mathbb{N}$ and $s', t' \in (2^{m'})^{n'}$ such that $s \sqsubseteq s' \sqsubseteq x$, $t \sqsubseteq t' \sqsubseteq y$, and $\mathcal{N}_{s'} \times \mathcal{N}_{t'} \subseteq U$. □

We close this section with a closure property of the family of equivalence relations into which $\mathbb{E}_1$ is reducible.

Proposition 2.15. *Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, $A \subseteq X$ is analytic, and $\phi: A \to Y$ is a Borel reduction of E to F for which there is a Baire measurable reduction $\psi: (2^{\mathbb{N}})^{\mathbb{N}} \to \phi(A)$ of $\mathbb{E}_1$ to $F \restriction \phi(A)$. Then there is a continuous embedding of $\mathbb{E}_1$ into $E \restriction A$.*

Proof. By Proposition 1.7, there is a dense G_δ set $C \subseteq (2^{\mathbb{N}})^{\mathbb{N}}$ on which ψ is continuous. By Theorem 1.12, there is a $\sigma(\varSigma^1_1)$-measurable function $\phi': \phi(A) \to X$ for which $\phi \circ \phi'$ is the identity function. Then $\phi' \circ (\psi \restriction C)$ is a $\sigma(\varSigma^1_1)$-measurable reduction of $\mathbb{E}_1 \restriction C$ to E. One more appeal to Proposition 1.7 therefore yields a dense G_δ set $D \subseteq C$ for which it is a continuous reduction of $\mathbb{E}_1 \restriction D$ to E. As Propositions 2.13 and 2.14 ensure that there is a continuous embedding of $\mathbb{E}_1$ into $\mathbb{E}_1 \restriction D$, the proposition follows. □

3. Independent perfect sets

We say that a set $B \subseteq Y$ is $\aleph_0$-*universally Baire* if $f^{-1}(B)$ has the Baire property whenever X is a Polish space and $f: X \to Y$ is continuous. In this section, we establish a local version of the following generalization of the perfect set theorem for co-analytic equivalence relations (see [9]).

Proposition 3.1 (Conley–Lecomte–Miller). *Suppose that X is a Polish space, $A \subseteq X$ is analytic, G is an $\aleph_0$-universally Baire graph on X, R is a reflexive symmetric co-analytic binary relation on X, and $G^{(2)} \subseteq R$. Then at least one of the following holds:*

(1) *There is a Borel set $B \supseteq A$ on which $\sim R$ has countable Borel chromatic number.*
(2) *There is a continuous injection $\pi: 2^{\mathbb{N}} \to A$ of $2^{\mathbb{N}}$ into a G-independent set.*

Proof. As the property of being independent with respect to an analytic graph is $\varPi^1_1$-on-$\varSigma^1_1$, Theorem 1.18 ensures that every $(\sim R)$-independent analytic set is contained in a $(\sim R)$-independent Borel set. It follows that if $\chi_B(\sim R \restriction A) \leq \aleph_0$, then there is a Borel set $B \supseteq A$ for which $\chi_B(\sim R \restriction B) \leq \aleph_0$. Otherwise, Theorem 1.23 yields a continuous homomorphism $\phi: 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $(A \times A) \cap \sim R$. As $\mathbb{G}_0$ has full projection, it follows that $\phi(2^{\mathbb{N}}) \subseteq A$, and Proposition 1.22 ensures that ϕ is meager-to-one. So by Proposition 2.1, it only remains to verify that the graph $G' = (\phi \times \phi)^{-1}(G)$ is meager. Suppose, towards a contradiction, that this is not the case. By Theorem 1.10, there exists $x \in X$ for which G'_x is non-meager and has the Baire property. Proposition 1.22 then yields a pair $(y, z) \in \mathbb{G}_0 \restriction G'_x$, in which case the fact that $G^{(2)} \subseteq R$ implies that $\phi(y) \; R \; \phi(z)$, contradicting the fact that ϕ is a homomorphism from $\mathbb{G}_0$ to $\sim R$. □

We will need the following complexity calculation.

Proposition 3.2. *Suppose that X is a Polish space and G is a co-analytic graph on X. Then the property $P(A)$ that there is no continuous injection of $2^{\mathbb{N}}$ into a G-independent subset of A is Π^1_1-on-Σ^1_1.*

Proof. Let E denote the equality relation on X, and suppose that Y is a Polish space and $R \subseteq X \times Y$ is analytic. Then Proposition 2.1 ensures that the inexistence of a continuous injection of $2^{\mathbb{N}}$ into a G-independent subset of R^y is equivalent to the inexistence of a continuous function $\phi: 2^{\mathbb{N}} \to X$ for which $(\phi \times \phi)^{-1}((R^y \times R^y) \setminus (E \cup G))$ is comeager. Propositions 1.1 and 1.2 along with Theorem 1.17 imply that this latter property is Π^1_1-on-Σ^1_1. □

Given an equivalence relation E on X, we say that a graph G on X has *countable E-local Borel chromatic number* if its restriction to each equivalence class of E has countable Borel chromatic number.

Proposition 3.3. *Suppose that X is a Polish space, $A \subseteq X$ is analytic, E is an analytic equivalence relation on X, G is a co-analytic graph on X, R is a reflexive symmetric co-analytic binary relation on X, and $G^{(2)} \subseteq R$. Then at least one of the following holds:*

(1) *There is a Borel set $B \supseteq A$ on which $\sim R$ has countable E-local Borel chromatic number.*
(2) *There exists $x \in X$ for which there is a continuous injection $\pi: 2^{\mathbb{N}} \to A$ of $2^{\mathbb{N}}$ into a G-independent subset of $[x]_E$.*

Proof. By Proposition 3.2, the property $Q(A)$ that there is no $x \in X$ for which there is a continuous injection $\pi: 2^{\mathbb{N}} \to A$ of $2^{\mathbb{N}}$ into a G-independent subset of $[x]_E$ is Π^1_1-on-Σ^1_1. So if condition (2) fails, then Theorem 1.18 yields a Borel set $B \subseteq X$ containing A such that there is no $x \in X$ for which there is a continuous injection $\pi: 2^{\mathbb{N}} \to B$ of $2^{\mathbb{N}}$ into a G-independent subset of $[x]_E$. As Proposition 1.8 ensures that G is $\aleph_0$-universally Baire, Proposition 3.1 implies that $\sim R$ has countable E-local Borel chromatic number on B. □

4. Two dichotomy theorems

In this section, we establish the main technical results of the paper. We say that a sequence $(G_n)_{n \in \mathbb{N}}$ *eventually* has a property P if G_n has property P for all but finitely many $n \in \mathbb{N}$.

Theorem 4.1. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, and $(R_n)_{n \in \mathbb{N}}$ is an increasing sequence of reflexive symmetric co-analytic binary relations on X such that $E \subseteq \bigcup_{n \in \mathbb{N}} R_n$ and $R_n^{(2)} \subseteq R_{n+1}$ for all $n \in \mathbb{N}$. Then exactly one of the following holds:*

(1) *The set X is a countable union of Borel sets on which $(\sim R_n)_{n\in\mathbb{N}}$ eventually has countable E-local Borel chromatic number.*
(2) *There exists $f:\mathbb{N}\to\mathbb{N}$ for which there is a continuous homomorphism $\phi:(2^{\mathbb{N}})^{\mathbb{N}}\to X$ from $(\mathbb{F}_{n+1}\setminus\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E\cap R_{f(n+1)}\setminus R_{f(n)})_{n\in\mathbb{N}}$.*

Proof. Observe that if $f:\mathbb{N}\to\mathbb{N}$ and $\phi:(2^{\mathbb{N}})^{\mathbb{N}}\to X$ is a homomorphism from $(\mathbb{F}_{n+1}\setminus\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E\cap R_{f(n+1)}\setminus R_{f(n)})_{n\in\mathbb{N}}$, then ϕ is necessarily a homomorphism from $\mathbb{E}_1$ to E. Moreover, as each of the sets $\mathbb{F}_{n+1}\setminus\mathbb{F}_n$ is non-empty and $(R_n)_{n\in\mathbb{N}}$ is increasing, it follows that $f(n+1)>f(n)$ for all $n\in\mathbb{N}$, so $f(n)\geq n$ for all $n\in\mathbb{N}$, thus ϕ is in fact a homomorphism from $(\mathbb{E}_1\setminus\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E\setminus R_n)_{n\in\mathbb{N}}$.

To see that conditions (1) and (2) are mutually exclusive, observe that if both hold, then there is a non-meager analytic set $A\subseteq(2^{\mathbb{N}})^{\mathbb{N}}$ such that for all $x\in A$, there is an $\aleph_0$-coloring c of $\sim R_n\restriction[\phi(x)]_{E\restriction\phi(A)}$. Then for any such x and c, the function $c\circ\phi$ is a coloring of $\sim\mathbb{F}_n\restriction[x]_{\mathbb{E}_1\restriction A}$, so $\mathbb{E}_1$ has countable index over $\mathbb{F}_n$ on A, contradicting Proposition 2.4.

In order to show that at least one of conditions (1) and (2) does indeed hold, it will be convenient to assume that $X=\mathbb{N}^{\mathbb{N}}$. To see that this special case is sufficient to establish the theorem, note that we can assume X is non-empty, in which case Proposition 1.4 yields a continuous surjection $\pi:\mathbb{N}^{\mathbb{N}}\to X$. Set $E'=(\pi\times\pi)^{-1}(E)$ and $R'_n=(\pi\times\pi)^{-1}(R_n)$. If $f:\mathbb{N}\to\mathbb{N}$ and $\phi':(2^{\mathbb{N}})^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ is a continuous homomorphism from $(\mathbb{F}_{n+1}\setminus\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E'\cap R'_{f(n+1)}\setminus R'_{f(n)})_{n\in\mathbb{N}}$, then the map $\phi=\pi\circ\phi'$ is a homomorphism from $(\mathbb{F}_{n+1}\setminus\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E\cap R_{f(n+1)}\setminus R_{f(n)})_{n\in\mathbb{N}}$. On the other hand, suppose there are Borel sets $B'_n\subseteq\mathbb{N}^{\mathbb{N}}$ and natural numbers $k_n\in\mathbb{N}$ such that $\mathbb{N}^{\mathbb{N}}=\bigcup_{n\in\mathbb{N}}B'_n$ and $\sim R'_{k_n}\restriction B'_n$ has countable E'-local Borel chromatic number for all $n\in\mathbb{N}$. Then X is the union of the analytic sets $A_n=\pi(B'_n)$. If $x\in A_n$, then there exists $x'\in B'_n$ such that $\pi(x')=x$, and if $c':B'_n\to\mathbb{N}$ is a coloring of $\sim R'_{k_n}\restriction[x']_{E'\restriction B'_n}$, then the function $c(y)=\min\{c(y')\mid y=\pi(y')\}$ is a coloring of $\sim R_{k_n}\restriction[x]_{E\restriction A_n}$, so Proposition 3.3 yields Borel sets $B_n\supseteq A_n$ such that $\sim R_{k_n+1}\restriction B_n$ has countable E-local Borel chromatic number for all $n\in\mathbb{N}$.

We now proceed to the main argument. We will recursively define a decreasing sequence $(X^\alpha)_{\alpha<\omega_1}$ of Borel subsets of X, beginning with $X^0=X$, and taking intersections at limit ordinals. In order to describe the construction of $X^{\alpha+1}$ from X^α, we need several preliminaries.

Lemma 4.2. *There is an increasing sequence $(S_n)_{n\in\mathbb{N}}$ of reflexive symmetric analytic binary relations on X such that $\forall n\in\mathbb{N}\ S_n\subseteq R_n$, $E=\bigcup_{n\in\mathbb{N}}S_n$, and $\forall n\in\mathbb{N}\ S_n^{(2)}\subseteq S_{n+1}$.*

Proof. As $(E\setminus R_n)_{n\in\mathbb{N}}$ is a sequence of analytic sets with empty intersection, Theorem 1.21 yields a sequence $(R'_n)_{n\in\mathbb{N}}$ of Borel sets with empty intersection such that $\forall n\in\mathbb{N}\ E\setminus R_n\subseteq R'_n$. By replacing R'_n with $\bigcap_{i\leq n}R'_i$, we can ensure that $(R'_n)_{n\in\mathbb{N}}$ is decreasing. By replacing R'_n with $\{(x,y)\in R'_n\mid x\neq y\text{ and }(y,x)\in R'_n\}$, we can assume that each of these sets is irreflexive and symmetric. Set $R''_n = E \setminus R'_n$. Then $(R''_n)_{n\in\mathbb{N}}$ is an increasing sequence of reflexive symmetric analytic binary relations on X such that $\forall n \in \mathbb{N}\ R''_n \subseteq R_n$ and $E = \bigcup_{n\in\mathbb{N}} R''_n$. Set $S_0 = R''_0$, and recursively define $S_{n+1} = R''_{n+1} \cup S_n^{(2)}$. A straightforward induction shows that $\forall n \in \mathbb{N}\ S_n \subseteq R_n \cap S_{n+1}$, and it is clear that $E = \bigcup_{n\in\mathbb{N}} S_n$ and $\forall n \in \mathbb{N}\ S_n^{(2)} \subseteq S_{n+1}$. □

Fix trees $T_{m,n}$ on $(\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$ for which $p[T_{m,n}] = S_n \setminus R_m$. An *approximation* is a quadruple of the form $a = (n, f, \phi, (\psi_k)_{k<n})$, with the property that $n \in \mathbb{N}$, $f: \{0, \dots, n\} \to \mathbb{N}$, $\phi: (2^n)^n \to \mathbb{N}^n$, and $\psi_k: \mathbb{F}_{k+1}((2^n)^n) \setminus \mathbb{F}_k((2^n)^n) \to \mathbb{N}^n$ for all $k < n$.

We say that a is *extended* by another approximation b if $n^a \le n^b$, $f^a \sqsubseteq f^b$, $\phi^a \sqsubseteq \phi^b$, and $\psi^a \sqsubseteq \psi^b$ for all $k < n^a$. When $n^b = n^a + 1$, we say that b is a *one-step extension* of a.

A *configuration* is a quadruple of the form $\gamma = (n, f, \phi, (\psi_k)_{k<n})$, with the property that $n \in \mathbb{N}$, $f: \{0, \dots, n\} \to \mathbb{N}$, $\phi: (2^\mathbb{N})^n \to \mathbb{N}^\mathbb{N}$, and $\psi_k: \mathbb{F}_{k+1}((2^\mathbb{N})^n) \setminus \mathbb{F}_k((2^\mathbb{N})^n) \to \mathbb{N}^\mathbb{N}$ for all $k < n$.

For reasons of definability, it will be important to focus our attention on configurations which are *continuous*, in the sense that the functions ϕ and ψ_k are continuous. In the course of the argument, it will also be useful to consider configurations which are merely *Baire measurable*, in the sense that the functions ϕ and ψ_k are Baire measurable.

We say that γ is *compatible* with a set $Y \subseteq X$ if $\phi^\gamma(x) \in Y$ for all $x \in \mathrm{dom}(\phi^\gamma)$. We say that γ is *compatible* with the sequence $(T_{m,n})_{m,n\in\mathbb{N}}$ if $((\phi^\gamma(x), \phi^\gamma(y)), \psi_k^\gamma(x, y)) \in [T_{f(k),f(k+1)}]$ for all $k < n^\gamma$ and $(x, y) \in \mathrm{dom}(\psi_k^\gamma)$. We say that γ is *compatible* with an approximation a if $n^a = n^\gamma$, $f^a = f^\gamma$, $\phi^a \sqsubseteq \phi^\gamma$, and $\psi_k^a \sqsubseteq \psi_k^\gamma$ for all $k < n^a$.

Again for reasons of definability, it will be important to focus on the corresponding notions of *generic compatibility*, in which one only asks for the desired properties on a comeager set. Although it is possible to proceed with only this latter notion, the arguments provide a strong connection between the two, and only a modicum of further effort is required to elucidate the connection between them.

Given an embedding $\pi: (2^\mathbb{N})^{n^\gamma} \to (2^\mathbb{N})^{n^\gamma}$ of $(\mathbb{F}_k((2^\mathbb{N})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^\mathbb{N})^{n^\gamma}))_{k<n^\gamma}$, let $\pi_*\gamma$ denote the configuration δ given by $n^\delta = n^\gamma$, $f^\delta = f^\gamma$, $\phi^\delta = \phi^\gamma \circ \pi$, and $\psi_k^\delta = \psi_k^\gamma \circ (\pi \times \pi)$.

Lemma 4.3. *Suppose that a is an approximation, $B \subseteq X$ is a Borel set, γ is a Baire measurable configuration which is generically compatible with a, B, and $(T_{m,n})_{m,n\in\mathbb{N}}$, $m, m' \in \mathbb{N}$, and $\pi: (2^m)^{n^\gamma} \to (2^{m'})^{n^\gamma}$ is an embedding of $(\mathbb{F}_k((2^m)^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^{m'})^{n^\gamma}))_{k<n^\gamma}$. Then π extends to a continuous embedding $\pi': (2^\mathbb{N})^{n^\gamma} \to (2^\mathbb{N})^{n^\gamma}$ of $(\mathbb{F}_k((2^\mathbb{N})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^\mathbb{N})^{n^\gamma}))_{k<n^\gamma}$ for which $\pi'_*\gamma$ is continuous and compatible with a, B, and $(T_{m,n})_{m,n\in\mathbb{N}}$.*

Proof. By Proposition 1.7, there are comeager sets $C \subseteq \mathrm{dom}(\phi^\gamma)$ and $C_k \subseteq \mathrm{dom}(\psi_k^\gamma)$ for which $\phi^\gamma \restriction C$ and $\psi_k^\gamma \restriction C_k$ are continuous. Then the set $D = (\phi^\gamma)^{-1}(B) \cap C$ is comeager, as are the sets $D_k \subseteq \mathrm{dom}(\psi_k^\gamma)$ of $(x, y) \in C_k$ with $((\phi^\gamma(x), \phi^\gamma(y)), \psi_k^\gamma(x, y)) \in$

$[T_{f^\gamma(k),f^\gamma(k+1)}]$ and $(s,t) \in \mathrm{dom}(\psi^a_k) \implies \psi^a_k(s,t) \sqsubseteq \psi^\gamma_k(x,y)$, where s and t are the projections of x and y onto $\mathrm{dom}(\phi^a)$. But Proposition 2.10 ensures that the function π extends to a continuous homomorphism $\pi' \colon (2^{\mathbb{N}})^{n^\gamma} \to D$ from $(\mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}), \mathbb{F}_{k+1}((2^{\mathbb{N}})^{n^\gamma}) \setminus \mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}))_{k<n^\gamma}$ to $(\mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}), D_k)_{k<n^\gamma}$, and any such function is as desired. □

Given a natural number $n \in \mathbb{N}$ and an embedding $\pi \colon (2^{n^\gamma})^{n^\gamma} \to (2^n)^{n^\gamma}$ of $(\mathbb{F}_k((2^{n^\gamma})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^n)^{n^\gamma}))_{k<n^\gamma}$, let $\pi_*\gamma$ denote the configuration δ given by $n^\delta = n^\gamma$, $f^\delta = f^\gamma$, $\phi^\delta(s \oplus x) = \phi^\gamma(\pi(s) \oplus x)$, and $\psi^\delta_k(s \oplus x, t \oplus y) = \psi^\gamma_k(\pi(s) \oplus x, \pi(t) \oplus y)$.

Lemma 4.4. *Suppose that γ is a Baire measurable configuration. Then there exists $n \in \mathbb{N}$ for which there is an embedding $\pi \colon (2^{n^\gamma})^{n^\gamma} \to (2^n)^{n^\gamma}$ of $(\mathbb{F}_k((2^{n^\gamma})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^n)^{n^\gamma}))_{k<n^\gamma}$ with the property that $\pi_*\gamma$ is generically compatible with an approximation.*

Proof. This follows from one application of Proposition 2.7 and n^γ applications of Proposition 2.9. □

Let $\Gamma^\alpha(a)$ denote the set of all continuous configurations which are generically compatible with X^α, $(T_{m,n})_{m,n\in\mathbb{N}}$, and a. Theorem 1.17 ensures that $\Gamma^\alpha(a)$ is analytic (and even Borel).

Associate with each configuration γ the set $D^\gamma \subseteq (2^{\mathbb{N}})^{n^\gamma}$ given by

$$D^\gamma = \left\{x \in \left(2^{\mathbb{N}}\right)^{n^\gamma} \;\middle|\; \forall^* y \in \left(2^{\mathbb{N}}\right)^{n^\gamma}\ \phi^\gamma(x)\ S_{f^\gamma(n^\gamma)}\ \phi^\gamma(y)\right\}.$$

If γ is generically compatible with $(T_{m,n})_{m,n\in\mathbb{N}}$, then D^γ is comeager. As $S^{(2)}_{f^\gamma(n^\gamma)} \subseteq S_{f^\gamma(n^\gamma)+1}$, it follows that $\phi^\gamma(D^\gamma)$ is an $S_{f^\gamma(n^\gamma)+1}$-clique.

We say that a is *α-terminal* if $\Gamma^\alpha(b) = \emptyset$ for all one-step extensions b of a. Define $A^\alpha(a) = \bigcup_{\gamma\in\Gamma^\alpha(a)} \phi^\gamma(D^\gamma)$.

Lemma 4.5. *Suppose that a is an approximation for which there is a continuous injection $\pi \colon 2^{\mathbb{N}} \to A^\alpha(a)$ into an $(E \setminus R_{f^a(n^a)+2})$-clique. Then a is not α-terminal.*

Proof. We first note that E can be replaced with an appropriate S_n.

Sublemma 4.6. *There exists $n > f^a(n^a) + 2$ for which there is a continuous injection $\pi' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $(\pi \circ \pi')(2^{\mathbb{N}})$ is an S_n-clique.*

Proof. Fix $x \in X$ with $\pi(2^{\mathbb{N}}) \subseteq [x]_E$, and set $S'_m = \pi^{-1}(S^x_m)$ for all $m \in \mathbb{N}$. Then $2^{\mathbb{N}} = \bigcup_{m\in\mathbb{N}} S'_m$, so there exists $m \geq f^a(n^a) + 2$ for which S'_m is non-meager. As Proposition 1.8 ensures that S'_m has the Baire property, the one-dimensional analog of Theorem 1.11 (whose proof is even simpler than in the two-dimensional case) yields a continuous injection $\pi' \colon 2^{\mathbb{N}} \to S'_m$. Set $n = m + 1$. As $(\pi \circ \pi')(2^{\mathbb{N}}) \subseteq S^x_m$ and $S^{(2)}_m \subseteq S_n$, it follows that $(\pi \circ \pi')(2^{\mathbb{N}})$ is an S_n-clique. □

Replacing π with $\pi \circ \pi'$, we can assume that $\pi(2^{\mathbb{N}})$ is an S_n-clique.

Sublemma 4.7. *There is a continuous injection $\pi_\Gamma: 2^{\mathbb{N}} \to \Gamma^\alpha(a)$ for which there is a continuous injection $\pi': 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with the property that $(\pi \circ \pi')(x) \in \phi^{\pi_\Gamma(x)}(D^{\pi_\Gamma(x)})$ for all $x \in 2^{\mathbb{N}}$.*

Proof. Note that the set of pairs $(x, \gamma) \in 2^{\mathbb{N}} \times \Gamma^\alpha(a)$ with $\pi(x) \in \phi^\gamma(D^\gamma)$ is analytic (and even Borel). By Theorem 1.12, there is a $\sigma(\Sigma^1_1)$-measurable function $\pi'_\Gamma: 2^{\mathbb{N}} \to \Gamma^\alpha(a)$ such that $\pi(x) \in \phi^{\pi'_\Gamma(x)}(D^{\pi'_\Gamma(x)})$ for all $x \in 2^{\mathbb{N}}$. As π is injective and no two distinct points of $\pi(2^{\mathbb{N}})$ are $R_{f^a(n^a)+2}$-related, it follows that π'_Γ is injective. By Proposition 1.7, there is a comeager set $C \subseteq 2^{\mathbb{N}}$ on which π'_Γ is continuous. The one-dimensional analog of Theorem 1.11 therefore yields a continuous injection $\pi': 2^{\mathbb{N}} \to C$. Set $\pi_\Gamma = \pi'_\Gamma \circ \pi'$. □

Replacing π with $\pi \circ \pi'$, we can assume that $\pi(x) \in \phi^{\pi_\Gamma(x)}(D^{\pi_\Gamma(x)})$ for all $x \in 2^{\mathbb{N}}$. Note that $\phi^{\pi_\Gamma(x)}(x')$ $(S_{n+2} \setminus R_{f^a(n^a)})$ $\phi^{\pi_\Gamma(y)}(y')$ whenever $x, y \in 2^{\mathbb{N}}$ are distinct, $x' \in D^{\pi_\Gamma(x)}$, and $y' \in D^{\pi_\Gamma(y)}$. Observe further that by Proposition 1.1, the set of pairs $((x, x', y, y'), z) \in (2^{\mathbb{N}} \times (2^{\mathbb{N}})^{n^a} \times 2^{\mathbb{N}} \times (2^{\mathbb{N}})^{n^a}) \times \mathbb{N}^{\mathbb{N}}$ with the property that $((\phi^{\pi_\Gamma(x)}(x'), \phi^{\pi_\Gamma(y)}(y')), z) \in [T_{f^a(n^a),n+2}]$ is closed, so by Theorem 1.12, there is a $\sigma(\Sigma^1_1)$-measurable function $\psi: \mathbb{F}_{n^a+1}((2^{\mathbb{N}})^{n^a+1}) \setminus \mathbb{F}_{n^a}((2^{\mathbb{N}})^{n^a+1}) \to \mathbb{N}^{\mathbb{N}}$ such that

$$\big(\big(\phi^{\pi_\Gamma(x)}(x'), \phi^{\pi_\Gamma(y)}(y')\big), \psi\big(x' \frown (x), y' \frown (y)\big)\big) \in [T_{f^a(n^a),n+2}]$$

for all distinct $x, y \in 2^{\mathbb{N}}$, $x' \in D^{\pi_\Gamma(x)}$, and $y' \in D^{\pi_\Gamma(y)}$.

Let γ denote the Baire measurable configuration given by $n^\gamma = n^a + 1$, $f^\gamma \restriction \{0, \dots, n^a\} = f^a$, $f^\gamma(n^\gamma) = n + 2$, $\phi^\gamma(x \frown (z)) = \phi^{\pi_\Gamma(z)}(x)$, $\psi^\gamma_k(x \frown (z), y \frown (z)) = \psi^{\pi_\Gamma(z)}_k(x, y)$ for $k < n^\gamma$, and $\psi^\gamma_{n^a} = \psi$. Lemma 4.4 then yields an approximation b, a natural number n', and an embedding $\pi': (2^{n^\gamma})^{n^\gamma} \to (2^{n'})^{n^\gamma}$ of $(\mathbb{F}_k((2^{n^\gamma})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^{n'})^{n^\gamma}))_{k<n^\gamma}$, extending the identity function on $(2^{n^\gamma})^{n^\gamma}$, for which $\pi'_*\gamma$ is generically compatible with b, $(T_{m,n})_{m,n\in\mathbb{N}}$, and X^α. As $\pi'_*\gamma$ is made up of perfectly many configurations generically compatible with a, it follows that b is a one-step extension of a. As Lemma 4.3 yields a continuous embedding $\pi'': (2^{\mathbb{N}})^{n^\gamma} \to (2^{\mathbb{N}})^{n^\gamma}$ of $(\mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}))_{k<n^\gamma}$ into $(\mathbb{F}_k((2^{\mathbb{N}})^{n^\gamma}))_{k<n^\gamma}$, extending π', with the property that $\pi''_*\gamma$ is continuous and compatible with b, $(T_{m,n})_{m,n\in\mathbb{N}}$, and X^α, it follows that a is not α-terminal. □

Lemma 4.8. *Suppose that a is an α-terminal approximation. Then there is a Borel set $B \subseteq X$ containing $A^\alpha(a)$ on which $\sim R_{f^a(n^a)+3}$ has countable E-local Borel chromatic number.*

Proof. As $A^\alpha(a)$ is analytic and Lemma 4.5 ensures that there is no continuous injection $\pi: 2^{\mathbb{N}} \to A^\alpha(a)$ into an $(E \setminus R_{f^a(n^a)+2})$-clique, the desired result follows from Proposition 3.3. □

Let T^α denote the set of all α-terminal approximations. For every such approximation a, appeal to Lemma 4.8 to obtain a Borel set $B^\alpha(a) \subseteq X$ containing $A^\alpha(a)$ on which $\sim R_{f^a(n^a)+3}$ has countable E-local Borel chromatic number. Define $X^{\alpha+1} = X^\alpha \setminus \bigcup_{a \in T^\alpha} B^\alpha(a)$. This completes the recursive construction.

Lemma 4.9. *Suppose that a is an approximation whose one-step extensions are all α-terminal. Then a is $(\alpha+1)$-terminal.*

Proof. Suppose that b is a one-step extension of a. If γ is a continuous configuration generically compatible with b, then the α-terminality of b ensures that $\phi^\gamma(D^\gamma) \cap X^{\alpha+1} = \emptyset$. It follows that if γ is also generically compatible with $(T_{m,n})_{m,n\in\mathbb{N}}$, then it is not generically compatible with $X^{\alpha+1}$, thus $\Gamma^{\alpha+1}(b) = \emptyset$. $\square$

Note that the family of α-terminal approximations is increasing. As there are only countably many approximations, there exists $\alpha < \omega_1$ such that every $(\alpha+1)$-terminal approximation is α-terminal. If the unique approximation a for which $n^a = f^a(0) = 0$ is α-terminal, then $X^{\alpha+1} = \emptyset$, and Lemma 4.8 ensures that there are Borel sets $B_n \subseteq X$ and natural numbers $k_n \in \mathbb{N}$ such that $X = \bigcup_{n\in\mathbb{N}} B_n$ and $\sim R_{k_n}$ has countable E-local Borel chromatic number on B_n, for all $n \in \mathbb{N}$.

Otherwise, Lemma 4.9 allows us to recursively construct non-α-terminal approximations a_n with the property that a_{n+1} is a one-step extension of a_n. Define $f: \mathbb{N} \to \mathbb{N}$ by $f(n) = f^{a_n}(n)$; define $\phi: (2^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi(x) \restriction n = \phi^{a_n}(s)$, where s is the projection of x onto $(2^n)^n$; and define $\psi_k: \mathbb{F}_{k+1} \setminus \mathbb{F}_k \to \mathbb{N}^\mathbb{N}$ by $\psi_k(x,y) \restriction n = \psi_k^{a_n}(s,t)$, where $k < n$ and s and t are the projections of x and y onto $(2^n)^n$.

It remains to show that for all $k \in \mathbb{N}$, the function ϕ is a homomorphism from $\mathbb{F}_{k+1} \setminus \mathbb{F}_k$ to $S_{f(k+1)} \setminus R_{f(k)}$. Towards this end, suppose that $x \ \mathbb{F}_{k+1} \setminus \mathbb{F}_k \ y$, and fix $n > k$ sufficiently large that $s_n \ (\mathbb{F}_{k+1}((2^n)^n) \setminus \mathbb{F}_k((2^n)^n)) \ t_n$, where s_n and t_n are the projections of x and y onto $(2^n)^n$. Then there is a continuous configuration γ_n generically compatible with a_n and $(T_{m,n})_{m,n\in\mathbb{N}}$. Fix $(x_n, y_n) \in \operatorname{dom}(\psi_k^{\gamma_n})$ with the property that the projections of x_n and y_n onto $(2^n)^n$ are s and t; $\phi^{\gamma_n}(x_n)$, $\phi^{\gamma_n}(y_n)$, $\psi_k^{\gamma_n}(x_n, y_n)$ are extensions of $\phi^{a_n}(s_n)$, $\phi^{a_n}(t_n)$, and $\psi_k^{a_n}(s_n, t_n)$; and $((\phi^{\gamma_n}(x_n), \phi^{\gamma_n}(y_n)), \psi_k^{\gamma_n}(x_n, y_n)) \in [T_{f(k),f(k+1)}]$. In particular, it follows that $((\phi^{a_n}(s_n), \phi^{a_n}(t_n)), \psi_k^{a_n}(s_n, t_n)) \in T_{f(k),f(k+1)}$, so $((\phi(x), \phi(y)), \psi_k(x,y)) \in [T_{f(k),f(k+1)}]$, from which it follows that $\phi(x) \ (S_{f(k+1)} \setminus R_{f(k)}) \ \phi(y)$. $\square$

As a corollary, we obtain the following.

Theorem 4.10. *Suppose that X is a Polish space, E is an analytic equivalence relation on X, and $(R_n)_{n\in\mathbb{N}}$ is an increasing sequence of reflexive symmetric F_σ binary relations on X such that $E \subseteq \bigcup_{n\in\mathbb{N}} R_n$ and $R_n^{(2)} \subseteq R_{n+1}$ for all $n \in \mathbb{N}$. Then exactly one of the following holds:*

(1) *The set X is a countable union of Borel sets on which $(\sim R_n)n \in \mathbb{N}$ eventually has countable E-local Borel chromatic number.*
(2) *There exists $f: \mathbb{N} \to \mathbb{N}$ for which there is a continuous homomorphism $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{F}_n, \sim\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E \cap R_{f(n)}, \sim R_{f(n)})_{n\in\mathbb{N}}$.*

Proof. In light of Theorem 4.1, it is sufficient to show that if there is a continuous homomorphism $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{F}_{n+1} \setminus \mathbb{F}_n)_{n\in\mathbb{N}}$ into $(R_{n+1} \setminus R_n)_{n\in\mathbb{N}}$, then there is a continuous homomorphism from $(\mathbb{F}_n, \sim\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E \cap R_n, \sim R_n)_{n\in\mathbb{N}}$. Towards this end, define $E' = (\phi \times \phi)^{-1}(E)$ and $R'_n = (\phi \times \phi)^{-1}(R_n)$. As Proposition 2.14 ensures that R'_n is n-meager, Proposition 2.13 yields a continuous homomorphism $\psi: (2^{\mathbb{N}})^{\mathbb{N}} \to (2^{\mathbb{N}})^{\mathbb{N}}$ from $(\mathbb{F}_n, \sim\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(\mathbb{F}_n, \sim R'_n)_{n\in\mathbb{N}}$, in which case the function $\pi = \phi \circ \psi$ is a continuous homomorphism from $(\mathbb{F}_n, \sim\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(E \cap R_{f(n)}, \sim R_{f(n)})_{n\in\mathbb{N}}$. □

5. Hypersmooth equivalence relations

In this section, we give a classical proof of Theorem [5, Theorem 1]. We first note that for witnesses to hypersmoothness, the σ-ideal appearing in Theorems 4.1 and 4.10 has a much nicer characterization.

Proposition 5.1. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of smooth Borel equivalence relations on X whose union is E, and there are Borel sets $B_n \subseteq X$, on which $\sim E_n$ has countable E-local chromatic number, with $X = \bigcup_{n\in\mathbb{N}} B_n$. Then E is essentially hyperfinite.*

Proof. Set $C_n = \bigcup_{m<n} B_m$ and $D_n = B_n \setminus C_n$, and let F_n denote the equivalence relation on X given by

$$x \ F_n \ y \quad \Longleftrightarrow \quad (x, y \in C_n \text{ and } x \ E_n \ y) \quad \text{or} \quad \exists m \geq n \ (x, y \in D_m \text{ and } x \ E_m \ y).$$

Then $(F_n)_{n\in\mathbb{N}}$ is again an increasing sequence of smooth Borel equivalence relations whose union is E. In addition, E has countable index over F_0. Fix Borel reductions $\phi_m: X \to 2^{\mathbb{N}}$ of F_m to the equality relation on $2^{\mathbb{N}}$, and observe that the product $\phi: X \to (2^{\mathbb{N}})^{\mathbb{N}}$, given by $\phi(x)(n) = \phi_n(x)$, is a Borel reduction of E to $\mathbb{E}_1$. Then $A = \phi(X)$ is an analytic set on which $\mathbb{E}_1$ is countable, so Theorem 1.18 yields a Borel set $B \supseteq A$ on which $\mathbb{E}_1$ is countable. Theorem 1.25 then ensures that $\mathbb{E}_1 \restriction B$ is hyperfinite, thus E is essentially hyperfinite. □

As a corollary, we obtain a classical proof of [5, Theorem 1].

Theorem 5.2 *(Kechris–Louveau). Suppose that X is a Polish space and E is a hypersmooth Borel equivalence relation on X. Then exactly one of the following holds:*

(1) *The equivalence relation E is essentially hyperfinite.*
(2) *There is a continuous embedding $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ of $\mathbb{E}_1$ into E.*

Proof. Propositions 2.4 and 2.5 ensure that the two conditions are mutually exclusive.

To see that at least one of them holds, fix an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of smooth Borel equivalence relations on X whose union is E. By Proposition 1.3, we can assume that each E_n is closed, in which case Theorem 4.10 and Proposition 5.1 therefore yield the desired result. □

6. Treeable equivalence relations

In this section, we establish our dichotomy theorems for treeable Borel equivalence relations. Given a binary relation R on a set $Y \subseteq X$, we say that a set $Z \subseteq X$ is *R-complete* if $\forall y \in Y\ \exists z \in Z\ y\ R\ z$.

Proposition 6.1. *Suppose that X is a Polish space, $A \subseteq X$ is analytic, E is a Borel equivalence relation on X, G is a Borel treeing of E, and n is a natural number such that for all $x \in A$, there is a countable set $C \subseteq [x]_E$ which is complete with respect to $G^{(\leq n)} \restriction [x]_{E\restriction A}$. Then there is a $(G^{(\leq n)} \restriction A)$-complete Borel set $B \subseteq X$ on which E is countable.*

Proof. We proceed via induction on n. The base case $n = 0$ is trivial, so suppose that we have already established the proposition at some $n \in \mathbb{N}$, and for all $x \in A$, there is a countable set $C \subseteq [x]_E$ which is complete with respect to $G^{(\leq n+1)} \restriction [x]_{E\restriction A}$. Let A' denote the set of $x \in X$ for which there are uncountably many $y \in G_x$ such that for some $m > n$ there is an injective G-path $(z_i)_{i\leq m}$ with $x = z_0$, $y = z_1$, and $z_m \in A$. As Theorem 1.16 ensures that the property of being countable is Π^1_1-on-Σ^1_1, the set A' is analytic. Moreover, the acyclicity of G ensures that if $x \in A$ and $C \subseteq [x]_E$ is a countable set which is complete with respect to $G^{(\leq n+1)} \restriction [x]_{E\restriction A}$, then $A' \cap [x]_E \subseteq C$. In particular, it follows that E is countable on A'. As this latter property is again Π^1_1-on-Σ^1_1, Theorem 1.18 yields a Borel set $B' \supseteq A'$ on which E is countable. As Theorems 1.14 and 1.20 ensure that $G^{(\leq n+1)}$ is Borel, Theorem 1.15 implies that the set B'' of points $G^{(\leq n+1)}$-related to points in B' is Borel.

Define $A'' = A \setminus B''$, and observe that if $x \in A''$ and $C \subseteq [x]_E$ is a countable set which is complete with respect to $G^{(\leq n+1)} \restriction [x]_{E\restriction A}$, then there exists $y \in C \setminus B'$ such that x is $G^{(\leq n)}$-related to either y or one of its countably many neighbors z for which (y, z) extends to a G-path $(w_i)_{i\leq n+1}$ from y to A. In particular, the induction hypothesis yields a $(G^{(\leq n)} \restriction A'')$-complete Borel set $B''' \subseteq X$ on which E is countable, in which case the set $B = B' \cup B'''$ is as desired. □

As corollaries, we obtain the following dichotomy theorems.

Theorem 6.2. *Suppose that X is a Polish space, E is a treeable Borel equivalence relation on X, and G is a Borel treeing of E. Then exactly one of the following holds:*

(1) *There is an E-complete Borel set on which E is countable.*
(2) *There exists a function $f: \mathbb{N} \to \mathbb{N}$ for which there is a continuous homomorphism $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ from $(\mathbb{F}_{n+1} \setminus \mathbb{F}_n)_{n\in\mathbb{N}}$ to $(G^{(\leq f(n+1))} \setminus G^{(\leq f(n))})_{n\in\mathbb{N}}$.*

Proof. As condition (1) ensures that X is of the form $\bigcup_{n\in\mathbb{N}} B_n$, where each $B_n \subseteq X$ is a Borel set on which $\sim G^{(\leq n)}$ has countable E-local Borel chromatic number, Proposition 2.4 ensures that the two conditions are mutually exclusive. Theorem 4.1 and Proposition 6.1 imply that at least one of them holds. □

Theorem 6.3. *Suppose that X is a Polish space and E is a Borel equivalence relation on X which is subtreeable-with-F_σ-iterates. Then for every analytic set $A \subseteq X$, exactly one of the following holds:*

(1) *There is an $(E \restriction A)$-complete Borel set $B \subseteq X$ on which E is countable.*
(2) *There is a continuous embedding $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ of $\mathbb{E}_1$ into $E \restriction A$.*

Proof. Proposition 2.4 ensures that the two conditions are mutually exclusive, and Theorem 4.10 and Proposition 6.1 imply that at least one of them holds. □

We say that *embeddability of $\mathbb{E}_1$ is determined below E by $\mathscr{E}$* if for every analytic set $A \subseteq X$, either $E \restriction A \in \mathscr{E}$ or there is a continuous embedding of $\mathbb{E}_1$ into E. Theorem 6.3 implies Borel equivalence relations which are subtreeable-with-F_σ-iterates have this property, where $\mathscr{E}$ is the class of essentially countable Borel equivalence relations on Polish spaces. The following fact implies that this holds under the weaker assumption of being essentially subtreeable-with-F_σ-iterates.

Proposition 6.4. *Suppose that $\mathscr{E}$ is a class of Borel equivalence relations on Polish spaces. Then the class of Borel equivalence relations below which embeddability of $\mathbb{E}_1$ is determined by essentially $\mathscr{E}$ is closed under Borel reducibility.*

Proof. Suppose that X and Y are Polish spaces, E and F are Borel equivalence relations on X and Y, $\pi: X \to Y$ is a Borel reduction of E to F, and embeddability of $\mathbb{E}_1$ is determined below F by essentially $\mathscr{E}$. Given an analytic set $A \subseteq X$, either there is a Borel reduction ψ of $F \restriction \pi(A)$, and therefore of $E \restriction A$, to a Borel equivalence relation in $\mathscr{E}$, or there is a continuous embedding of $\mathbb{E}_1$ into $F \restriction \pi(A)$, in which case Proposition 2.15 yields a continuous embedding of $\mathbb{E}_1$ into $E \restriction A$. □

7. Borel functions

In this section, we establish a natural strengthening of Theorem 6.3 for graphs induced by functions. Although not strictly necessary to achieve this goal, we will first establish several preliminary results so as to further clarify the nature of essential countability in this context.

Proposition 7.1. *Suppose that X and Y are Polish spaces, E is a Borel equivalence relation on X, F is a countable equivalence relation on a subset of Y, and $\pi: X \to Y$ is a Borel reduction of E to F. Then there is a countable Borel equivalence relation F' on Y such that π is also a reduction of E to F'.*

Proof. Set $R = (\pi \times \pi)(E)$. The fact that π is a homomorphism from E to F ensures that $R \subseteq F$. As F is countable and π is a cohomomorphism from E to F, it follows that R is subset of $Y \times Y$, with countable horizontal and vertical sections, for which π is a cohomomorphism from E to the smallest equivalence relation on Y containing R. As R is analytic and this latter property is Π^1_1-on-Σ^1_1, Theorem 1.18 yields a Borel set $R' \supseteq R$, with countable horizontal and vertical sections, for which π is a cohomomorphism from E to the smallest equivalence relation on Y containing R'. Let F' denote the latter equivalence relation. As $R \subseteq F'$, it follows that π is also a homomorphism from E to F', and therefore π is a reduction of E to F'. As the horizontal and vertical sections of R' are countable, it follows that F' is countable, so Theorem 1.15 ensures that F' is Borel. □

Proposition 7.2. *Suppose that X is a Polish space, E and F are Borel equivalence relations on X, and $E \cap F$ has countable index in E and F. Then E is essentially countable $\Longleftrightarrow$ F is essentially countable.*

Proof. It is sufficient to handle the special case that $E \subseteq F$.

To see ($\Longrightarrow$), suppose that X' is a Polish space, E' is a countable equivalence relation on X', and $\pi: X \to X'$ is a Borel reduction of E to E'. Then π is a reduction of F to the countable equivalence relation $(\pi \times \pi)(F)$ on $\pi(X)$, so F is essentially countable by Proposition 7.1.

To see ($\Longleftarrow$), suppose that X' is a Polish space, F' is a countable equivalence relation on X', and $\phi: X \to X'$ is a Borel reduction of F to F'. Let D' denote the equality relation on X', and observe that the relation $D = (\phi \times \phi)^{-1}(D')$ has countable index in F. By ($\Longrightarrow$), it is enough to show that $D \cap E$ is smooth, thus essentially countable. Suppose, towards a contradiction, that this is not the case. Then Theorem 1.24 yields a continuous embedding $\psi: 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into $D \cap E$, in which case $\phi \circ \psi$ is a countable-to-one Borel homomorphism from $\mathbb{E}_0$ to D', contradicting Proposition 1.9. □

With these preliminaries out of the way, we now turn our attention to functions $T: X \to X$. Let $\mathbb{E}_t(T)$ denote the equivalence relation on X given by $x \mathbin{\mathbb{E}_t(T)} y \Longleftrightarrow \exists m, n \in \mathbb{N}\ T^m(x) = T^n(y)$.

The *eventually periodic part* of T is the set of $x \in X$ for which there are natural numbers $m < n$ with $T^m(x) = T^n(x)$, and T is *aperiodic* if its eventually periodic part is empty. The following observation will allow us to focus our attention on aperiodic functions.

Proposition 7.3. *Suppose that X is a Polish space and $T: X \to X$ is Borel. Then there is a Borel transversal of the restriction of $\mathbb{E}_t(T)$ to the eventually periodic part of T.*

Proof. The *periodic part* of T is the set of $x \in X$ for which there is a positive natural number n with $x = T^n(x)$. As the periodic part of T intersects every equivalence class of $\mathbb{E}_t(T)$ in a finite set, the desired result follows from the fact that every finite Borel equivalence relation on a Polish space has a Borel transversal, which itself is a consequence of Theorem 1.13. □

The following observation will allow us to apply our earlier results.

Proposition 7.4. *Suppose that X is a Polish space and $T: X \to X$ is Borel. Then $\mathbb{E}_t(T)$ is treeable.*

Proof. As Borel equivalence relations with Borel transversals are trivially treeable, Proposition 7.3 allows us to assume that T is aperiodic. Then the graph G_T on X given by $x \; G_T \; y \iff (T(x) = y$ or $T(y) = x)$ is a Borel treeing of E. □

This yields another characterization of essential countability of $\mathbb{E}_t(T)$.

Proposition 7.5. *Suppose that X is a Polish space, $T: X \to X$ is Borel, and $\mathbb{E}_t(T)$ is essentially countable. Then there is an $\mathbb{E}_t(T)$-complete Borel set on which $\mathbb{E}_t(T)$ is countable.*

Proof. By Proposition 7.4, the equivalence relation $\mathbb{E}_t(T)$ is treeable. The desired result is therefore a consequence of Theorem 1.26. Although this latter result has a classical proof (see [8]), we will give a simpler argument using the structure of T.

By Proposition 7.3, we can assume that T is aperiodic. By Proposition 1.3, we can assume that X carries a Polish topology with respect to which T is continuous. Then the iterates of G_T are closed. Theorem 6.3 therefore yields the desired $\mathbb{E}_t(T)$-complete Borel set on which $\mathbb{E}_t(T)$ is countable. □

Define $\mathbb{E}_0(T)$ on X by $x \; \mathbb{E}_0(T) \; y \iff \exists n \in \mathbb{N} \; T^n(x) = T^n(y)$. Note that $\mathbb{E}_0(T)$ is a countable index subequivalence relation of $\mathbb{E}_t(T)$.

Proposition 7.6. *Suppose that X is a Polish space and $T: X \to X$ is Borel. Then $\mathbb{E}_0(T)$ is essentially countable if and only if $\mathbb{E}_t(T)$ is essentially countable.*

Proof. This is a direct consequence of Proposition 7.2. □

Together with Proposition 7.5, the following fact ensures that $\mathbb{E}_t(T)$ is essentially countable if and only if T is essentially countable-to-one.

Proposition 7.7. *Suppose that X is a Polish space, $T: X \to X$ is Borel, and $B \subseteq X$ is a Borel set on which $\mathbb{E}_t(T)$ is countable. Then there is a T-stable Borel set $A \supseteq B$ on which $\mathbb{E}_t(T)$ is countable.*

Proof. Set $A = \bigcup_{n \in \mathbb{N}} T^n(B)$. Then A is T-stable, and Theorem 1.15 ensures that it is Borel. $\square$

Define $\mathbb{F}_n(T)$ on X by $x \; \mathbb{F}_n(T) \; y \iff T^n(x) = T^n(y)$.

Proposition 7.8. *Suppose that X is a Polish space, $T: X \to X$ is Borel, and there is a sequence $(B_n)_{n \in \mathbb{N}}$ of Borel sets for which $X = \bigcup_{n \in \mathbb{N}} B_n$ and $(\sim\mathbb{F}_k(T))_{k \in \mathbb{N}}$ eventually has countable $\mathbb{E}_0(T)$-local chromatic number for all $n \in \mathbb{N}$. Then T is essentially countable-to-one.*

Proof. Fix natural numbers $k_n \in \mathbb{N}$ such that $\sim\mathbb{F}_{k_n}(T)$ has countable $\mathbb{E}_0(T)$-local chromatic number on B_n for all $n \in \mathbb{N}$. Then $\mathbb{E}_0(T)$ is countable on the analytic set $A = \bigcup_{n \in \mathbb{N}} T^{k_n}(B_n)$. As Theorem 1.16 ensures that the property of being countable is Π^1_1-on-Σ^1_1, Theorem 1.18 yields a Borel set $B \supseteq A$ on which $\mathbb{E}_0(T)$ is countable. As $\mathbb{E}_t(T)$ must also be countable on this set, Proposition 7.7 ensures that T is essentially countable-to-one. $\square$

Define $\mathbb{R}_n(T)$ on X by $x \; \mathbb{R}_n(T) \; y \iff \exists i, j \leq n \; T^i(x) = T^j(y)$.

Proposition 7.9. *Suppose that X is a Polish space, $T: X \to X$ is aperiodic, $f: \mathbb{N} \to \mathbb{N}$, and $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ is a homomorphism from $(\mathbb{F}_n, \sim\mathbb{F}_n)_{n \in \mathbb{N}}$ into $(\mathbb{F}_{f(n)}(T), \sim\mathbb{R}_{f(n)}(T))_{n \in \mathbb{N}}$. Then the function $\pi: \mathbb{N} \times (2^{\mathbb{N}})^{\mathbb{N}} \to X$ given by $\pi(n, s^n(x)) = T^{f(n)} \circ \phi(x)$ defines a Kakutani embedding of $S \times s$ into T.*

Proof. To see that π is well-defined, note that if $s^n(x) = s^n(y)$, then $x \; \mathbb{F}_n \; y$, so $\phi(x) \; \mathbb{F}_{f(n)}(T) \; \phi(y)$, thus $T^{f(n)} \circ \phi(x) = T^{f(n)} \circ \phi(y)$. Note also that the set $B = \pi(\mathbb{N} \times (2^{\mathbb{N}})^{\mathbb{N}})$ is trivially T-*recurrent*, in the sense that $B \subseteq \bigcup_{n>0} T^{-n}(B)$.

To see that π is injective, suppose that $m, n \in \mathbb{N}$ and $x, y \in (2^{\mathbb{N}})^{\mathbb{N}}$ are such that $\pi(m, x) = \pi(n, y)$. By reversing the roles of x and y if necessary, we can assume that $m \leq n$. Fix $x', y' \in (2^{\mathbb{N}})^{\mathbb{N}}$ such that $x = s^m(x')$ and $y = s^n(y')$, and observe that $T^{f(m)} \circ \phi(x') = \pi(m, x) = \pi(n, y) = T^{f(n)} \circ \phi(y')$, so the fact that $f(m) \leq f(n)$ ensures that $\phi(x') \; \mathbb{R}_{f(n)}(T) \; \phi(y')$. As ϕ is a homomorphism from $\sim\mathbb{F}_n$ to $\sim\mathbb{R}_{f(n)}(T)$, it follows that $x' \; \mathbb{F}_n \; y'$. As ϕ is also a homomorphism from $\mathbb{F}_n$ to $\mathbb{F}_{f(n)}(T)$, it follows that $T^{f(n)} \circ \phi(x') = T^{f(n)} \circ \phi(y')$. Then $T^{f(m)} \circ \phi(x') = T^{f(n)} \circ \phi(x')$, so the injectivity of f and the aperiodicity of T ensure that $m = n$, thus $x = s^n(x') = s^n(y') = y$.

Suppose now that $n \in \mathbb{N}$ and $x \in (2^{\mathbb{N}})^{\mathbb{N}}$, and fix $x' \in (2^{\mathbb{N}})^{\mathbb{N}}$ for which $x = s^n(x')$. As ϕ is a homomorphism from $(\mathbb{E}_0(s), \sim\mathbb{E}_0(s))$ to $(\mathbb{E}_0(T), \sim\mathbb{E}_t(T))$, it follows that $\phi([x']_{\mathbb{E}_0(s)}) = \phi((2^{\mathbb{N}})^{\mathbb{N}}) \cap [\phi(x')]_{\mathbb{E}_0(T)} = \phi((2^{\mathbb{N}})^{\mathbb{N}}) \cap [\phi(x')]_{\mathbb{E}_t(T)}$, thus

$$\begin{aligned} T_{\phi(\mathbb{N}\times(2^{\mathbb{N}})^{\mathbb{N}})} \circ \pi(n,x) &= T_{\phi(\mathbb{N}\times(2^{\mathbb{N}})^{\mathbb{N}})} \circ \pi\big(n, s^n\big(x'\big)\big) \\ &= T_{\phi(\mathbb{N}\times(2^{\mathbb{N}})^{\mathbb{N}})} \circ T^{f(n)} \circ \phi\big(x'\big) \\ &= T^{f(n+1)} \circ \phi\big(x'\big) \\ &= \pi\big(n+1, s^{n+1}\big(x'\big)\big) \\ &= \pi\big((S\times s)(n,x)\big), \end{aligned}$$

thus π is a Kakutani embedding of $S \times s$ into T. □

We are now ready to establish our final result.

Theorem 7.10. *Suppose that X is a Polish space and $T: X \to X$ is Borel. Then exactly one of the following holds:*

(1) *The function T is essentially countable-to-one.*
(2) *There is a continuous Kakutani embedding $\phi: (2^{\mathbb{N}})^{\mathbb{N}} \to X$ of $S \times s$ into T.*

Proof. To see that the two conditions are mutually exclusive, suppose that B is a T-complete, T-stable Borel set on which T is countable-to-one, and $\pi: \mathbb{N} \times (2^{\mathbb{N}})^{\mathbb{N}} \to X$ is a Borel Kakutani embedding of $S \times s$ into T. Then $\pi^{-1}(B)$ is an $(S \times s)$-complete, $(S \times s)$-stable Borel set on which $S \times s$ is countable-to-one, so $\mathrm{proj}_{(2^{\mathbb{N}})^{\mathbb{N}}}(\pi^{-1}(B))$ is an $\mathbb{E}_1$-complete Borel set on which $\mathbb{E}_1$ is countable, contradicting Proposition 2.4.

It remains to check that at least one of the two conditions holds. By Proposition 7.3, we can assume that T is aperiodic. By Proposition 1.3, we can assume that T is continuous, in which case each of the relations $\mathbb{R}_n(T)$ is closed. Theorem 4.10 and Proposition 7.8 ensure that if T is not essentially countable-to-one, then there is a function $f: \mathbb{N} \to \mathbb{N}$ for which there is a continuous homomorphism from $(\mathbb{F}_n, {\sim}\mathbb{F}_n)_{n\in\mathbb{N}}$ to $(\mathbb{E}_0(T) \cap \mathbb{R}_{f(n)}(T), {\sim}\mathbb{R}_{f(n)}(T))_{n\in\mathbb{N}}$. As the aperiodicity of T implies that $\mathbb{F}_n(T) = \mathbb{E}_0(T) \cap \mathbb{R}_n(T)$ for all $n \in \mathbb{N}$, Proposition 7.9 yields a continuous Kakutani embedding of $S \times s$ into T. □

Acknowledgments

We would like to thank Clinton Conley for allowing us to include Proposition 3.1 and its proof, as well as Clinton Conley, Alexander Kechris, and the anonymous referee for their comments on earlier versions of the paper.

References

[1] R. Dougherty, S. Jackson, A.S. Kechris, The structure of hyperfinite Borel equivalence relations, Trans. Amer. Math. Soc. 341 (1) (1994) 193–225, MR 1149121 (94c:03066).
[2] L.A. Harrington, A.S. Kechris, A. Louveau, A Glimm–Effros dichotomy for Borel equivalence relations, J. Amer. Math. Soc. 3 (4) (1990) 903–928, MR 1057041 (91h:28023).

[3] G. Hjorth, Selection theorems and treeability, Proc. Amer. Math. Soc. 136 (10) (2008) 3647–3653, MR 2415050 (2009d:03108).

[4] A.S. Kechris, Classical Descriptive set Theory, Grad. Texts in Math., vol. 156, Springer-Verlag, New York, 1995, MR 1321597 (96e:03057).

[5] A.S. Kechris, A. Louveau, The classification of hypersmooth Borel equivalence relations, J. Amer. Math. Soc. 10 (1) (1997) 215–242, MR 1396895 (97e:03067).

[6] A.S. Kechris, S. Solecki, S. Todorcevic, Borel chromatic numbers, Adv. Math. 141 (1) (1999) 1–44, MR 1667145 (2000e:03132).

[7] K. Kuratowski, Topology, vol. II, new edition, revised and augmented, translated from the French by A. Kirkor, Academic Press, New York, 1968, MR 0259835 (41 #4467).

[8] B.D. Miller, The graph-theoretic approach to descriptive set theory, Bull. Symbolic Logic 18 (4) (2012) 554–575.

[9] J.H. Silver, Counting the number of equivalence classes of Borel and coanalytic equivalence relations, Ann. Math. Logic 18 (1) (1980) 1–28, MR 568914 (81d:03051).