THE GRAPH-THEORETIC APPROACH TO DESCRIPTIVE SET THEORY

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Abstract. We sketch the ideas behind the use of chromatic numbers in establishing descriptive set-theoretic dichotomy theorems.

1. Introduction

Cantor [Can74] initiated the systematic study of infinite sets with the realization that the cardinality of the natural numbers is strictly less than that of the real numbers. His subsequent conjecture that no cardinal lies strictly between, later known as the *Continuum Hypothesis*, was the first of the now famous problems posed by Hilbert [Hil02] at the second International Congress of Mathematicians in 1900.

Many decades passed before Gödel [Goe40] established the consistency of the Continuum Hypothesis with the standard axioms of set theory. Still later, Cohen [Coh63, Coh64] established the consistency of its negation. Nevertheless, Cantor made a significant first step through his perfect set theorem [Can84], showing that closed sets of real numbers cannot provide counterexamples to the Continuum Hypothesis. This was the first hint at the now well-known fact that while the standard axioms do not determine the answers to many questions concerning the structure of abstract sets, they are sufficient to provide a rich structure theory for sufficiently concrete sets. And so *descriptive set-theory*, the systematic study of definable subsets of the real numbers, was born.

Eventually Alexandrov [Ale16] and Hausdorff [Hau16] generalized the perfect set theorem to Borel sets. Soon thereafter, Souslin [Sou17] isolated the notion of analyticity, and further generalized the theorem to analytic sets.

These results were the first in a long line of descriptive set-theoretic dichotomy theorems, each showing that suitably definable mathematical structures fail to be simple exactly when they contain a copy of

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a canonical complex structure. For example, Cantor's perfect set theorem asserts that every closed set of real numbers either is countable or contains a perfect set, and therefore a copy of $2^{\mathbb{N}}$. Such theorems have become so prevalent that the search for new canonical objects and dichotomies has become a cornerstone of the subject.

The proofs of the results established in the first half-century after Souslin's theorem followed the same basic outline. Using the tree structure afforded by analyticity, one defines an appropriately chosen Cantor-Bendixson-style derivative, reducing the problem to a combinatorially and topologically simple special case with a straightforward solution.

This basic outline was cast aside, however, beginning with Silver's generalization [Sil80] of the perfect set theorem. His proof was a technical *tour de force*, relying on sophisticated techniques from mathematical logic in addition to a much larger fragment of the standard set-theoretic axioms than typical. Although Harrington later found a simpler proof [Har76], his argument still relied on a detailed recursion-theoretic analysis of the real numbers, as well as the method of forcing distilled from Cohen's proof of the independence of the Continuum Hypothesis.

Over the next thirty years, Harrington's techniques unearthed an astonishing number of structural properties of Borel sets. While the proofs of many results closely mirrored that of Silver's theorem, others used progressively more elaborate refinements of Harrington's ideas.

One very interesting example of a modern dichotomy theorem appeared in work of Kechris-Solecki-Todorcevic [KST99] concerning definable chromatic numbers of graphs. In particular, they constructed a combinatorially and topologically simple graph G_0 which is minimal among analytic graphs with uncountable Borel chromatic number.

Recent work has revealed a classical proof of this G_0 dichotomy closely resembling that of the original perfect set theorem. Building upon this, the G_0 dichotomy has been combined with simple Baire category arguments to obtain a great variety of descriptive set-theoretic dichotomy theorems. In addition to eliminating the need for sophisticated machinery from mathematical logic, this approach illuminates new connections between seemingly unrelated theorems, leads to a global view of dichotomy theorems from which new results emerge, and readily generalizes to broader classes of definable sets.

These notes include a large number of dichotomy theorems, both new and old, along with sketches of proofs using this new approach. Although some details are omitted, I hope that they will nevertheless serve as an accessible introduction to the subject, and a starting point for further investigation.

2. Preliminaries

Here we provide a brief summary of our notation, as well as a few basic descriptive set-theoretic facts which we take for granted. While there are many excellent introductions to descriptive set theory such as [Kec95] and [Sri98], we restrict our citations to the former. Although we occasionally appeal to the axiom of countable choice for the sake of notational convenience, it is never essential.

The *diagonal* on a set X is given by $\Delta(X) = \{(x, x) \mid x \in X\}$. Given a set $R \subseteq X \times X$, we define $\reflectbox{$R$} = \{(y, x) \mid x \; R \; y\}$. We say that R is *reflexive* if $x \; R \; x$ for all $x \in X$, we say that R is *symmetric* if $x \; R \; y \implies y \; R \; x$ for all $x, y \in X$, and we say that R is *transitive* if $x \; R \; y \; R \; z \implies x \; R \; z$ for all $x, y, z \in X$. We also say that R is *irreflexive* if $\neg x \; R \; x$ for all $x \in X$.

A *homomorphism* from $R \subseteq X^n$ to $S \subseteq Y^n$ is a function $\varphi \colon X \to Y$ sending R-related sequences to S-related sequences. More generally, a homomorphism from a sequence $(R_i)_{i \in I}$ of relations on X to a sequence $(S_i)_{i \in I}$ of relations on Y is a function $\varphi \colon X \to Y$ which is a homomorphism from R_i to S_i for all $i \in I$.

A *reduction* of R to S is a homomorphism from $(R, \sim R)$ to $(S, \sim S)$. An *embedding* is an injective reduction. These notions are extended to sequences of relations in the same fashion as for homomorphisms.

A topological space X is *second countable* if it has a countable basis, *separable* if it has a countable dense set, and *Polish* if it is separable and carries a compatible complete metric. Examples include countable discrete spaces and the space of real numbers, as well as their countable powers. When working with countably infinite products of discrete spaces, we employ the usual notation $\mathcal{N}_s$ to denote the basic open sets.

A set is *Borel* if it is in the closure of the open sets under complements, countable intersections, and countable unions. We say that a set is *analytic* if it is the continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and we say that a set is *co-analytic* if it is the complement of an analytic set. The basic fact relating these notions is the following (see [Kec95, Theorem 14.1]):

Theorem 1 (Lusin's first separation theorem)**.** *Suppose that X is a Polish space and $A_1, A_2 \subseteq X$ are disjoint analytic sets. Then there is a Borel set $B \subseteq X$ such that $A_1 \subseteq B$ and $A_2 \cap B = \emptyset$.*

In particular, this implies Souslin's theorem that a set is Borel if and only if it is both analytic and co-analytic (see [Kec95, Theorem 14.7]).

The x^{th} *vertical section* of a set $R \subseteq X \times Y$ is the set given by $R_x = \{y \in Y \mid x \; R \; y\}$. On a single occasion we will also need the following result (see [Kec95, Theorem 18.11]):

Theorem 2 (Lusin's theorem on the co-analyticity of sets of unicity)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is Borel. Then the set $\{x \in X \mid |R_x| = 1\}$ is co-analytic.*

A *Baire space* is a topological space in which countable intersections of dense open sets are dense. We say that a set is *comeager* if it contains a countable intersection of dense open sets, and we say that a set is *meager* if its complement is comeager. A set has the *Baire property* if it is the symmetric difference of a meager set with an open set.

The family of sets with the Baire property contains the open sets and forms a σ-algebra, and therefore contains every Borel set (see [Kec95, Proposition 8.22]). Lusin-Sierpiński have shown that every analytic subset of a Polish space has the Baire property (see [Kec95, Corollary 29.14] for a more general result due to Nikodým).

The following fact is the analog of the Lebesgue density theorem for Baire category (see [Kec95, Proposition 8.26]):

Proposition 3 (Localization)**.** *Suppose that X is a Baire space and $B \subseteq X$ has the Baire property. Then either B is meager or there is a non-empty open set $U \subseteq X$ in which $B \cap U$ is comeager.*

We write $\forall^* x \; \varphi(x)$ to indicate that the set $\{x \mid \varphi(x)\}$ is comeager. The following fact is the analog of Fubini's theorem for Baire category (see [Kec95, Theorem 8.41]):

Theorem 4 (Kuratowski-Ulam)**.** *Suppose that X and Y are Baire spaces, Y is second countable, and $R \subseteq X \times Y$ has the Baire property.*

- *The set $\{x \in X \mid R_x$ has the Baire property$\}$ is comeager.*
- *The set R is comeager if and only if $\forall^* x \; R_x$ is comeager.*

The arguments to come depend heavily on the interplay between the Kuratowski-Ulam theorem and variants of the following basic observation (see [Kec95, Theorem 19.1] for a stronger version):

Theorem 5 (Mycielski)**.** *Suppose that n is a positive integer, X is a non-empty Polish space, and $R \subseteq X^n$ is comeager. Then there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from the set of injective sequences of length n to R.*

3. Graphs

A *graph* on X is an irreflexive symmetric set $G \subseteq X \times X$. The *restriction* of G to a set $Y \subseteq X$ is the graph $G \upharpoonright Y$ on Y given by

$G \restriction Y = G \cap (Y \times Y)$. We say that Y is *G-independent* if $G \restriction Y = \emptyset$. A *$\kappa$-coloring* of G is a function $c \colon X \to I$, where I is a set of cardinality κ, such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$.

Given a set $S \subseteq 2^{<\mathbb{N}}$, let $G_S(2^{\mathbb{N}})$ denote the graph on $2^{\mathbb{N}}$ given by

$$G_S(2^{\mathbb{N}}) = \{(s^\frown(i)^\frown x, s^\frown(1-i)^\frown x) \mid i < 2,\, s \in S, \text{ and } x \in 2^{\mathbb{N}}\}.$$

The properties of such graphs are best understood via the finitary approximations $G_S(2^n)$ on 2^n given by

$$G_S(2^n) = \{(s^\frown(i)^\frown t, s^\frown(1-i)^\frown t) \mid i < 2,\, s \in S, \text{ and } t \in 2^{n-|s|-1}\}.$$

Note that $G_S(2^{n+1})$ can be alternatively characterized as the graph obtained by connecting two vertex disjoint copies of $G_S(2^n)$ via the edges indicated by $S \cap 2^n$. The graph $G_S(2^{\mathbb{N}})$ can then be written as

$$G_S(2^{\mathbb{N}}) = \{(s^\frown x, t^\frown x) \mid n \in \mathbb{N},\, s\ G_S(2^n)\ t, \text{ and } x \in 2^{\mathbb{N}}\}.$$

We say that a set $S \subseteq 2^{<\mathbb{N}}$ is *dense* if $\forall r \in 2^{<\mathbb{N}} \exists s \in S\ r \sqsubseteq s$.

Proposition 6. *Suppose that $S \subseteq 2^{<\mathbb{N}}$ is dense and $B \subseteq 2^{\mathbb{N}}$ is a non-meager set with the Baire property. Then B is not $G_S(2^{\mathbb{N}})$-independent.*

Proof. Fix $r \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_r$. Then there exists $s \in S$ with $r \sqsubseteq s$. As B is also comeager in $\mathcal{N}_s$ and the function which flips the $|s|^{\text{th}}$ coordinate of its input is a homeomorphism, there are comeagerly many $x \in 2^{\mathbb{N}}$ for which $s^\frown(0)^\frown x\ (G_S(2^{\mathbb{N}}) \restriction B)\ s^\frown(1)^\frown x$. □

Corollary 7. *Suppose that $S \subseteq 2^{<\mathbb{N}}$ is dense. Then there is no Baire measurable $\aleph_0$-coloring of $G_S(2^{\mathbb{N}})$.*

We say that a set $S \subseteq 2^{\mathbb{N}}$ is *sparse* if $|S \cap 2^n| \leq 1$ for all $n \in \mathbb{N}$, in which case each of the graphs $G_S(2^n)$ is acyclic, thus so too is $G_S(2^{\mathbb{N}})$.

Theorem 8 (Kechris-Solecki-Todorcevic)**.** *Suppose that X is a Polish space and G is an analytic graph on X. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

1. *There is a Borel $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism from $G_S(2^{\mathbb{N}})$ to G.*

Moreover, for each dense set $S \subseteq 2^{<\mathbb{N}}$, at most one of these holds.

Proof (Sketch). The fact that the two conditions are mutually exclusive when S is dense is a direct consequence of the inexistence of Borel $\aleph_0$-colorings of $G_S(2^{\mathbb{N}})$. Short of giving an outline of the proof that at least one of these conditions holds when S is sparse, we will simply try to convey some intuition as to how the inability to construct homomorphisms leads to Borel $\aleph_0$-colorings.

Let $\mathrm{Hom}(G, H)$ denote the set of homomorphisms from G to H. The basic problem arising when constructing a homomorphism from $G_S(2^{\mathbb{N}})$ to G is that given a set $\Phi \subseteq \mathrm{Hom}(G_S(2^n), G)$, one must find $\psi \in \mathrm{Hom}(G_S(2^{n+1}), G)$ of the form $\psi(s^\frown(i)) = \varphi_i(s)$, where $\varphi_0, \varphi_1 \in \Phi$. While such a task is trivial when S has no sequences of length n, if there is such a sequence $s \in S$, then this comes down to finding $\varphi_0, \varphi_1 \in \Phi$ for which $\varphi_0(s) \mathrel{G} \varphi_1(s)$. So the inexistence of such a ψ ensures that the set $A = \{\varphi(s) \mid \varphi \in \Phi\}$ is G-independent. Moreover, if Φ is analytic, then so too is A, in which case Lusin's first separation theorem can be used to obtain a G-independent Borel set $B \supseteq A$.

The union of all G-independent sets arising from Φ in this fashion is a Borel set on which G has a Borel $\aleph_0$-coloring. By restricting our attention to those $\varphi \in \Phi$ whose ranges are disjoint from this set and considering the corresponding family of G-independent sets, one obtains a potentially larger Borel set on which G has a Borel $\aleph_0$-coloring. Repeating this process transfinitely often to appropriate sets of homomorphisms, one obtains a classical proof of Theorem 8 resembling that of Cantor's perfect set theorem via the Cantor-Bendixson derivative.

Let $\mathcal{I}_G$ denote the σ-ideal generated by the family of G-independent Borel sets. Then for each $n \in \mathbb{N}$, there is a corresponding σ-ideal $\mathcal{I}_n$ on $\mathrm{Hom}(G_S(2^n), G)$ given by

$$\Phi \in \mathcal{I}_n \iff \exists B \in \mathcal{I}_G \forall \varphi \in \Phi \exists s \in 2^n \ \varphi(s) \in B.$$

Clearly $\mathcal{I}_0$ and $\mathcal{I}_G$ are identical. Moreover, if $\Phi \subseteq \mathrm{Hom}(G_S(2^n), G)$ is $\mathcal{I}_n$-positive, then the set $\Psi \subseteq \mathrm{Hom}(G_S(2^{n+1}), G)$ of homomorphisms obtained by combining pairs of homomorphisms in Φ is $\mathcal{I}_{n+1}$-positive. This observation can be used to give a proof of Theorem 8 without iteration into the transfinite, resembling that of Cantor's perfect set theorem via condensation points. □

An instructive first application of Theorem 8 yields Souslin's perfect set theorem. Recall that a non-empty closed set $P \subseteq X$ is *perfect* if, when endowed with the subspace topology, it has no isolated points.

Theorem 9 (Souslin). *Suppose that X is a Polish space and $A \subseteq X$ is analytic. Then exactly one of the following holds:*

1. *The set A is countable.*
2. *There is a perfect subset of A.*

Proof (Sketch). It is clear that the two conditions are mutually exclusive. To see that at least one holds, consider the graph G on X given by $G = (A \times A) \setminus \Delta(A)$. As every G-independent set includes at most one point of A, the existence of an $\aleph_0$-coloring of G ensures that A is countable. By Theorem 8, we can therefore assume there is a dense set $S \subseteq 2^{<\mathbb{N}}$ such that $\emptyset \in S$ and there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G.

As $\varphi[2^{\mathbb{N}}] \subseteq A$ and $\varphi \restriction U$ is non-constant for all non-empty open sets $U \subseteq 2^{\mathbb{N}}$, one can recursively construct a continuous map $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ with $(\varphi \circ \psi)[\mathcal{N}_s] \cap (\varphi \circ \psi)[\mathcal{N}_t] = \emptyset$ whenever $n \in \mathbb{N}$ and $s, t \in 2^n$ are distinct. Then the perfect set $P = (\varphi \circ \psi)[2^{\mathbb{N}}]$ is contained in A. □

More generally, the above argument yields a strengthening of the analytic special case of Todorcevic's Open Coloring Axiom (see [Fen93]). A set $Y \subseteq X$ is a *G-clique* if all distinct points of Y are G-neighbors.

Theorem 10 (Feng)**.** *Suppose that X is a Polish space, $A \subseteq X$ is analytic, and G is a relatively open graph on A. Then exactly one of the following holds:*

(1) There is an $\aleph_0$-coloring of G.
(2) There is a perfect G-clique.

A *partial transversal* of an equivalence relation is a set intersecting every class in at most one point.

Theorem 11 (Silver)**.** *Suppose that X is a Polish space and E is a co-analytic equivalence relation on X. Then exactly one of the following holds:*

(1) The equivalence relation E has only countably many classes.
(2) There is a perfect partial transversal of E.

Proof (Sketch). Clearly the two conditions are mutually exclusive. To see that at least one holds, consider the graph $G = \sim E$. As every G-independent set is contained in a single E-class, it follows that if there is an $\aleph_0$-coloring of G, then E has only countably many classes. By Theorem 8, we can assume there is a dense set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G.

As every equivalence class of the relation $E' = (\varphi \times \varphi)^{-1}(E)$ is meager, the Kuratowski-Ulam theorem implies that E' is meager, so Mycielski's theorem yields a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\Delta(2^{\mathbb{N}})$ into E'. Then the perfect set $P' = \psi[2^{\mathbb{N}}]$ is a partial transveral of E', so the perfect set $P = \varphi[P']$ is a partial transversal of E. □

A similar result concerns metric spaces (see [Mil95, Theorem 31.2]). A *pseudo-metric* is a function $d \colon X \times X \to \{r \in \mathbb{R} \mid r \geq 0\}$ such that $d(x, x) = 0$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. The *uniformity* associated with d is the sequence $(U_\epsilon)_{\epsilon > 0}$ given by $U_\epsilon = \{(x, y) \in X \times X \mid d(x, y) \leq \epsilon\}$. We say that a set $Y \subseteq X$ is *dense* if $\forall \epsilon > 0 \forall x \in X \exists y \in Y \ d(x, y) \leq \epsilon$, and we say that a set

$Y \subseteq X$ is *discrete* if $\exists \epsilon > 0 \forall y, z \in Y \ d(y,z) \geq \epsilon$. We say that (X, d) is *separable* if it admits a countable dense set.

Theorem 12 (Friedman, Harrington, Kechris)**.** *Suppose that X is a Polish space and d is a pseudo-metric on X such that the sets along its uniformity are co-analytic. Then exactly one of the following holds:*

1. *The pseudo-metric space (X, d) is separable.*
2. *There is a discrete perfect subset of X.*

Proof (Sketch). Clearly the two conditions are mutually exclusive. To see that at least one holds, consider the graphs G_ϵ on X given by

$$G_\epsilon = \sim U_\epsilon = \{(x, y) \in X \times X \mid d(x, y) > \epsilon\}.$$

If c is an $\aleph_0$-coloring of G_ϵ and $Y \subseteq X$ includes points of every color, then $\forall x \in X \exists y \in Y \ d(x, y) \leq \epsilon$. It follows that if there are $\aleph_0$-colorings of G_ϵ for all $\epsilon > 0$, then (X, d) is separable. By Theorem 8, we can assume that for some $\epsilon > 0$ there is a dense set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to $G_{2\epsilon}$.

The Kuratowski-Ulam theorem and the triangle inequality imply that the graph $G'_\epsilon = (\varphi \times \varphi)^{-1}(G_\epsilon)$ is comeager, so Mycielski's theorem yields a perfect G'_ϵ-clique $P' \subseteq 2^{\mathbb{N}}$, in which case the perfect set $P = \varphi[P']$ is discrete. □

Another related theorem concerns linear orders. Recall that a *quasi-order* on X is a reflexive transitive set $R \subseteq X \times X$. We use $\equiv_R$ to denote the equivalence relation on X given by $\equiv_R = R \cap \Re$, and we use $<_R$ to denote the relation on X given by $<_R = R \setminus \Re$. We say that R is *linear* if $x \ R \ y$ or $y \ R \ x$ for all $x, y \in X$. We say that a set $Y \subseteq X$ is *dense* if it intersects every non-empty open interval. We say that R is *separable* if it admits a countable dense set, and we say that R is *ccc* if every set of pairwise disjoint non-empty open intervals is countable.

The *Souslin Hypothesis* is the statement that every ccc linear order is separable. Much as the Continuum Hypothesis led to the perfect set theorem, the Souslin Hypothesis led to the following (see [She84]):

Theorem 13 (Friedman, Shelah)**.** *Suppose that X is a Polish space and R is a linear co-analytic quasi-order on X. Then exactly one of the following holds:*

1. *The quasi-order R is separable.*
2. *There is a perfect subset of $X \times X$ with the property that the corresponding closed intervals have non-empty interior and are pairwise disjoint.*

Proof (Sketch). Clearly the two conditions are mutually exclusive. To see that at least one holds, note first that the set $A \subseteq X \times X$ given by

$$A = \{(x, y) \in X \times X \mid (x, y)_R \neq \emptyset\}$$

is analytic, as is the graph G on $X \times X$ given by

$$G = \{((x_1, y_1), (x_2, y_2)) \in A \times A \mid [x_1, y_1]_R \cap [x_2, y_2]_R = \emptyset\}.$$

We say that a family of sets is *intersecting* if no two sets in the family are disjoint. If there is an $\aleph_0$-coloring of G, then the set of closed intervals with non-empty interiors can be written as the union of countably many intersecting families. From this one can define a countable dense set, thus R is separable. By Theorem 8, we can therefore assume there is a dense set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi\colon 2^{\mathbb{N}} \to X \times X$ from $G_S(2^{\mathbb{N}})$ to G.

Using the Kuratowski-Ulam theorem, one can show that the graph $G' = (\varphi \times \varphi)^{-1}(G)$ is comeager, so Mycielski's theorem yields a perfect G'-clique $P' \subseteq 2^{\mathbb{N}}$, in which case the closed intervals associated with the perfect set $P = \varphi[P']$ are pairwise disjoint. □

Whereas the Continuum Hypothesis and Souslin Hypothesis are results from abstract set theory which have had a tremendous impact on descriptive set theory, results from outside of logic have also had an important influence. One example is the Glimm-Effros dichotomy, which was originally uncovered in the study of operator algebras (see [Gli61, Eff65]), but whose generalizations play a fundamental role in the study of definable equivalence relations (see [HKL90]).

The equivalence relation $E_0(2^{\mathbb{N}})$ on $2^{\mathbb{N}}$ is given by

$$x \mathrel{E_0(2^{\mathbb{N}})} y \iff \exists n \in \mathbb{N} \forall m \geq n \ x(m) = y(m).$$

As $G_S(2^{\mathbb{N}}) \subseteq E_0(2^{\mathbb{N}})$ for every set $S \subseteq 2^{<\mathbb{N}}$, it follows that every partial transversal of $E_0(2^{\mathbb{N}})$ with the Baire property is meager. We say that an equivalence relation is *countable* if all of its equivalence classes are countable. We can already establish the Glimm-Effros dichotomy for such equivalence relations:

Theorem 14 (Glimm, Effros, Jackson-Kechris-Louveau, Weiss)**.** *Suppose that X is a Polish space and E is a countable analytic equivalence relation on X. Then exactly one of the following holds:*

1. *The set X is the union of countably many Borel partial transversals of E.*
2. *There is a continuous embedding of $E_0(2^{\mathbb{N}})$ into E.*

Proof (Sketch). The fact that every Borel partial transversal of $E_0(2^{\mathbb{N}})$ is meager ensures that the two conditions are mutually exclusive. To see that at least one holds, consider the graph $G = E \setminus \Delta(X)$. Clearly any Borel $\aleph_0$-coloring of G gives rise to countably many Borel partial transversals whose union is X.

We say that a set $S \subseteq 2^{\mathbb{N}}$ is *full* if $|S \cap 2^n| \geq 1$ for all $n \in \mathbb{N}$. If S is full, then each of the graphs $G_S(2^n)$ is connected, from which it follows that the connected components of $G_S(2^{\mathbb{N}})$ are exactly the equivalence classes of $E_0(2^{\mathbb{N}})$.

By Theorem 8, we can assume there is a dense full set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G. The Kuratowski-Ulam theorem then ensures that the equivalence relation $E' = (\varphi \times \varphi)^{-1}(E)$ is meager. As $G_S(2^{\mathbb{N}}) \subseteq E'$, it follows that $E_0(2^{\mathbb{N}}) \subseteq E'$, so one can recursively construct a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $E_0(2^{\mathbb{N}})$ into E', in which case $\varphi \circ \psi$ is a continuous reduction of $E_0(2^{\mathbb{N}})$ to E. In particular, it is countable-to-one, so the Lusin-Novikov uniformization theorem (see Theorem 32) ensures that there is a non-meager Borel set $B \subseteq 2^{\mathbb{N}}$ on which $\varphi \circ \psi$ is injective. One more recursive construction yields a continuous embedding $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $E_0(2^{\mathbb{N}})$ into $E_0(2^{\mathbb{N}}) \restriction B$, and it follows that $\varphi \circ \psi \circ \pi$ is a continuous embedding of $E_0(2^{\mathbb{N}})$ into E. □

We next mention just one of many analogs of the Glimm-Effros dichotomy for quotient spaces. Suppose that $F \subseteq E$ are equivalence relations on X. A set $Y \subseteq X$ is a *partial transversal* of E over F if $E \restriction Y = F \restriction Y$. We say that F is of *index two* below E if every E-class is the disjoint union of two F-classes. We use $F_0(2^{\mathbb{N}})$ to denote the equivalence relation of index two below $E_0(2^{\mathbb{N}})$ given by

$$x \; F_0(2^{\mathbb{N}}) \; y \iff \exists n \in \mathbb{N} \forall m > n \; \textstyle\sum_{k<m} x(k) \equiv \sum_{k<m} y(k) \pmod 2.$$

As $G_S(2^{\mathbb{N}}) \subseteq E_0(2^{\mathbb{N}}) \setminus F_0(2^{\mathbb{N}})$ for every $S \subseteq 2^{<\mathbb{N}}$, every partial transversal of $E_0(2^{\mathbb{N}})$ over $F_0(2^{\mathbb{N}})$ with the Baire property is meager.

Theorem 15 (Louveau)**.** *Suppose that X is a Polish space, E is an analytic equivalence relation on X, and F is a co-analytic equivalence relation of index two below E. Then exactly one of the following holds:*

1. *(1) The set X is the union of countably many Borel partial transversals of E over F.*
2. *(2) There is a continuous embedding of $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}}))$ into (E, F).*

Proof (Sketch). The fact that every Borel partial transversal of $E_0(2^{\mathbb{N}})$ over $F_0(2^{\mathbb{N}})$ is meager ensures that the two conditions are mutually exclusive. To see that at least one holds, consider the graph on X given by $G = E \setminus F$. Clearly every Borel $\aleph_0$-coloring of this graph gives rise to a countable family of Borel partial transversals of E over

F whose union is X. By Theorem 8, we can therefore assume there is a dense full set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to G.

The Kuratowski-Ulam theorem ensures that the equivalence relation $F' = (\varphi \times \varphi)^{-1}(F)$ is meager, thus so too is the equivalence relation $E' = (\varphi \times \varphi)^{-1}(E)$. As $G_S(2^{\mathbb{N}}) \subseteq E' \setminus F'$, it follows that $E_0(2^{\mathbb{N}}) \subseteq E'$ and $F_0(2^{\mathbb{N}}) \subseteq F'$, so a recursive construction yields a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}}))$ into (E', F'), in which case $\varphi \circ \psi$ is a continuous reduction of $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}}))$ to (E, F). In particular, it is countable-to-one, so the Lusin-Novikov uniformization theorem yields a non-meager Borel set $B \subseteq 2^{\mathbb{N}}$ on which $\varphi \circ \psi$ is injective. One more recursive construction yields a continuous embedding $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}}))$ into $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}})) \restriction B$, so $\varphi \circ \psi \circ \pi$ is a continuous embedding of $(E_0(2^{\mathbb{N}}), F_0(2^{\mathbb{N}}))$ into (E, F). $\square$

4. Digraphs

A *digraph* on a set X is an irreflexive set $G \subseteq X \times X$. The notions of *restriction*, *independence*, *coloring*, and *clique* are defined as for graphs. We say that a set $Y \subseteq X$ is a *G-half-clique* if $G \restriction Y$ is a strict linear ordering of Y.

Louveau has noted the natural generalization of Theorem 8 in which $G_S(2^{\mathbb{N}})$ is replaced with the digraph $G_S^{\rightarrow}(2^{\mathbb{N}})$ on $2^{\mathbb{N}}$ given by

$$G_S^{\rightarrow}(2^{\mathbb{N}}) = \{(s \frown (0) \frown x, s \frown (1) \frown x) \mid s \in S \text{ and } x \in 2^{\mathbb{N}}\}.$$

Indeed, every known proof of Theorem 8 can be easily modified to establish this result. Alternatively, a recursive construction can be used to obtain it as a direct consequence of Theorem 8.

Theorems 10, 11, and 12 have natural generalizations to asymmetric structures, where the existence of a perfect clique for an appropriate auxiliary graph is replaced with the existence of a perfect clique or a perfect half-clique. (A similar remark holds of Theorem 14.) While it is tempting to establish these generalizations by replacing the use of Theorem 8 with its generalization to digraphs in our proofs, one can obtain them directly from their symmetric counterparts and the fact that every perfect $(G \cup \reflectbox{$G$})$-clique contains a perfect G-clique or a perfect G-half-clique.

5. Hypergraphs

Given a natural number $d \geq 2$, a *d-dimensional hypergraph* on X is a set $G \subseteq X^d$ of non-constant sequences which is invariant under the action of the group of permutations of d by coordinate permutation.

The *restriction* of G to a set $Y \subseteq X$ is the d-dimensional hypergraph $G \restriction Y$ on Y given by $G \restriction Y = G \cap Y^d$. We say that Y is *G-independent* if $G \restriction Y = \emptyset$. A *$\kappa$-coloring* of G is a function $c\colon X \to I$, where I is a set of cardinality κ, such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$.

Given a set $S \subseteq d^{<\mathbb{N}}$, let $G_S(d^{\mathbb{N}})$ denote the d-dimensional hypergraph on $d^{\mathbb{N}}$ consisting of all sequences of the form $(s^\frown(\tau(i))^\frown x)_{i<d}$, where $s \in S$, τ is a permutation of d, and $x \in d^{\mathbb{N}}$. The notions of *dense* and *sparse* for subsets of $d^{<\mathbb{N}}$ are defined as for subsets of $2^{<\mathbb{N}}$. The following fact is the natural generalization of Theorem 8 to d-dimensional hypergraphs (see [Lec09, Theorem 1.4]):

Theorem 16. *Suppose that $d \geq 2$, X is a Polish space, and G is a d-dimensional analytic hypergraph on X. Then for each sparse set $S \subseteq d^{<\mathbb{N}}$, at least one of the following holds:*

1. *There is a Borel $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism from $G_S(d^{\mathbb{N}})$ to G.*

Moreover, for each dense set $S \subseteq d^{<\mathbb{N}}$, at most one of these holds.

Proof (Sketch). All known proofs of Theorem 8 trivially generalize to give Theorem 16. □

One application is the following simultaneous generalization of Silver's theorem and [vEKM89, Theorem 3]:

Theorem 17. *Suppose that X is a Polish space endowed with a vector space structure (or more generally, a pregeometry) whose corresponding linear dependence relation is co-analytic. Then for every analytic set $A \subseteq X$ and $d \in \mathbb{N}$, exactly one of the following holds:*

1. *The set A is contained in the union of countably many affine subspaces of dimension at most d.*
2. *There is a perfect subset of A whose subsets of cardinality $d+2$ are affinely independent.*

Proof (Sketch). It is clear that the two conditions are mutually exclusive. To see that at least one holds, let G_i be the i-dimensional hypergraph consisting of all affinely independent sequences $x \in A^i$. Then every $\aleph_0$-coloring of G_{d+2} yields countably many affine subspaces of dimension at most d whose union contains A. By Theorem 16, we can assume there exist a dense set $S \subseteq (d+2)^{<\mathbb{N}}$ with $\emptyset \in S$ and a continuous homomorphism $\varphi\colon (d+2)^{\mathbb{N}} \to X$ from $G_S((d+2)^{\mathbb{N}})$ to G.

An induction using the Kuratowski-Ulam theorem ensures that the hypergraphs $G_i' = (\varphi \times \varphi)^{-1}(G_i)$ are comeager for $2 \leq i \leq d+2$. Mycielski's theorem yields a perfect set $P' \subseteq (d+2)^{\mathbb{N}}$ all of whose injective

$(d+2)$-sequences are G'_{d+2}-related, thus every subset of the perfect set $P = \varphi[P']$ of cardinality $d+2$ is affinely independent. $\square$

An $\aleph_0$-*dimensional dihypergraph* on X is a set $G \subseteq X^{\mathbb{N}}$ of non-constant sequences. The notions of *restriction*, *independence*, and *coloring* are defined as for d-dimensional hypergraphs.

Given a set $S \subseteq \mathbb{N}^{<\mathbb{N}}$, let $G_S(\mathbb{N}^{\mathbb{N}})$ denote the $\aleph_0$-dimensional dihypergraph on $\mathbb{N}^{\mathbb{N}}$ given by

$$G_S(\mathbb{N}^{\mathbb{N}}) = \{(s^\frown(n)^\frown x)_{n\in\mathbb{N}} \mid s \in S \text{ and } x \in \mathbb{N}^{\mathbb{N}}\}.$$

The notions of *dense* and *sparse* for subsets of $\mathbb{N}^{<\mathbb{N}}$ are defined as for subsets of $2^{<\mathbb{N}}$. Let C_0 denote the comeager subset of $\mathbb{N}^{\mathbb{N}}$ given by

$$C_0 = \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n \in \mathbb{N} \exists m \geq n \forall i < m \; x(i) < m\},$$

Let $G_S(C_0)$ denote the restriction of $G_S(\mathbb{N}^{\mathbb{N}})$ to C_0. The following fact is a slight modification of Lecomte's generalization of Theorem 8 to $\aleph_0$-dimensional dihypergraphs (see [Lec09, Theorem 1.6]):

Theorem 18 (Lecomte)**.** *Suppose that X is a Polish space and G is an analytic $\aleph_0$-dimensional dihypergraph on X. Then for each sparse set $S \subseteq \mathbb{N}^{<\mathbb{N}}$, at least one of the following holds:*

1. *There is a Borel $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism from $G_S(C_0)$ to G.*

Moreover, for each dense set $S \subseteq \mathbb{N}^{<\mathbb{N}}$, at most one of these holds.

Proof (Sketch). While Lecomte has noted the failure of the analogous result in which $G_S(C_0)$ is replaced with $G_S(\mathbb{N}^{\mathbb{N}})$ (see [Lec09, Theorem 1.5]), all known proofs of Theorem 8 easily adapt to establish Theorem 18 (see [Mil11]). $\square$

Much as Theorem 8 can be used to establish the perfect set theorem, Theorem 18 can be used to establish the superperfect set theorem (see [Kec95, Theorem 21.23]). Recall that a set is K_σ if it is the union of countably many compact sets. We say that a non-empty closed set $P \subseteq X$ is *superperfect* if, when endowed with subspace topology, its compact subsets have empty interiors.

Theorem 19 (Kechris, Saint Raymond)**.** *Suppose that X is a Polish space and $A \subseteq X$ is analytic. Then exactly one of the following holds:*

1. *There is a K_σ set containing A.*
2. *There is a superperfect subset of A.*

Proof (Sketch). Much as the space $2^{\mathbb{N}}$ can be continuously injected into any perfect subset of X, the space $\mathbb{N}^{\mathbb{N}}$ can be continuously injected into any superperfect subset of X via a closed map. So if both conditions held, then $\mathbb{N}^\mathbb{N}$ would be K_σ, which is absurd.

To see that at least one of the conditions holds, fix a compatible Polish metric d on X, and let G denote the $\aleph_0$-dimensional dihypergraph on X consisting of all sequences $x \in A^\mathbb{N}$ for which there exists $\epsilon > 0$ with the property that $d(x(i), x(j)) > \epsilon$ for all distinct $i, j \in \mathbb{N}$. It is easy to see that if $Y \subseteq X$ is G-independent, then $A \cap Y$ is totally bounded, so its closure is compact. It follows that if there is an $\aleph_0$-coloring of G, then there is a K_σ set containing A. By Theorem 18, we can assume there is a dense set $S \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $\emptyset \in S$ and there is a continuous homomorphism $\varphi \colon C_0 \to X$ from $G_S(C_0)$ to G.

Note that for each non-empty open set $U \subseteq \mathbb{N}^\mathbb{N}$, there exists a sequence $x \in (C_0 \cap U)^\mathbb{N}$ such that $(\varphi \circ x(n))_{n\in\mathbb{N}} \in G$. As φ is continuous, a recursive construction yields a continuous function $\psi \colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with the property that for all $s \in \mathbb{N}^{<\mathbb{N}}$ there exists $\epsilon > 0$ such that for all distinct $i, j \in \mathbb{N}$, any two points $y \in (\varphi \circ \psi)[\mathcal{N}_{s^\frown(i)}]$ and $z \in (\varphi \circ \psi)[\mathcal{N}_{s^\frown(j)}]$ are of distance at least ϵ from one another. It follows that the set $P = (\varphi \circ \psi)[\mathbb{N}^\mathbb{N}]$ is superperfect. □

The same idea can again be used to establish a version of the analytic case of Todorcevic's Open Coloring Axiom. An $\aleph_0$-*dimensional hypergraph* is an $\aleph_0$-dimensional dihypergraph invariant under the action of the group of permutations of $\mathbb{N}$ by coordinate permutation. The *evenly-splitting* $\aleph_0$-*dimensional hypergraph* on $\mathbb{N}^\mathbb{N}$ is the set of sequences $(s^\frown(\tau(i))^\frown x)_{i\in\mathbb{N}}$, where $s \in \mathbb{N}^{<\mathbb{N}}$, τ is a permutation of $\mathbb{N}$, and $x \in \mathbb{N}^\mathbb{N}$.

Theorem 20. *Suppose that X is a Polish space, $A \subseteq X$ is analytic, and G is a relatively box-open $\aleph_0$-dimensional hypergraph on A. Then exactly one of the following holds:*

1. *There is an $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism from the evenly-splitting $\aleph_0$-dimensional hypergraph to G.*

6. Sequences

Suppose that $\mathbf{G} = (G_n)_{n\in N}$ is a countable sequence of graphs on X. A set $Y \subseteq X$ is $\mathbf{G}$-*independent* if it is G_n-independent for some $n \in N$. A κ-*coloring* of $\mathbf{G}$ is a function $c \colon X \to I$, where I is a set of cardinality κ, such that $c^{-1}(\{i\})$ is $\mathbf{G}$-independent for all $i \in I$.

Given a sequence $\mathbf{S} = (S_n)_{n\in N}$ of subsets of $2^{<\mathbb{N}}$, we use $\mathbf{G_S}(2^\mathbb{N})$ to denote the sequence $(G_{S_n}(2^\mathbb{N}))_{n\in N}$.

Theorem 21. *Suppose that X is a Polish space and $\mathbf{G} = (G_n)_{n\in N}$ is a countable sequence of analytic graphs on X. Then for each sequence*

$\mathbf{S} = (S_n)_{n \in N}$ of subsets of $2^{<\mathbb{N}}$ whose union is sparse, at least one of the following holds:

1. *There is a Borel $\aleph_0$-coloring of $\mathbf{G}$.*
2. *There is a continuous homomorphism from $\mathbf{G_S}(2^{\mathbb{N}})$ to $\mathbf{G}$.*

Moreover, for each sequence $\mathbf{S} = (S_n)_{n \in N}$ of dense subsets of $2^{<\mathbb{N}}$, at most one of these holds.

Proof (Sketch). Every known proof of Theorem 8 can easily be modified to give this result. Alternatively, a recursive construction yields a direct proof from Theorem 18 (see [CCCM11, Proposition 2]). □

As a first application, we establish another Glimm-Effros-style dichotomy theorem (see [Hjo08]). A set is a *transversal* of an equivalence relation if it intersects every equivalence class in exactly one point. We say that a Borel equivalence relation is *treeable* if there is an acyclic Borel graph whose connected components coincide with its classes.

Theorem 22 (Hjorth)**.** *Suppose that X is a Polish space and E is a treeable Borel equivalence relation on X. Then exactly one of the following holds:*

1. *There is a Borel transversal of E.*
2. *There is a continuous embedding of $E_0(2^{\mathbb{N}})$ into E.*

Proof (Sketch). The fact that every partial transversal of $E_0(2^{\mathbb{N}})$ with the Baire property is meager implies that the two conditions are mutually exclusive. To see that at least one holds, fix an acyclic Borel graph G whose connected components coincide with the equivalence classes of E, and let $\mathbf{G} = (G_n)_{n \in \mathbb{N}}$ denote the sequence of graphs in which G_n consists of all pairs of E-related points between which there is an injective G-path of length strictly greater than n. An inductive argument using both Lusin's first separation theorem and his result on the co-analyticity of sets of unicity can be used to show that if $B \subseteq X$ is a $\mathbf{G}$-independent Borel set, then there is a Borel transversal of $E \upharpoonright [B]_E$, and it follows that if there is a Borel $\aleph_0$-coloring of $\mathbf{G}$, then there is a Borel transversal of E.

By Theorem 21, we can assume there is a sequence $\mathbf{S} = (S_n)_{n \in \mathbb{N}}$ of dense subsets of $2^{<\mathbb{N}}$ with full union for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $\mathbf{G_S}(2^{\mathbb{N}})$ to $\mathbf{G}$. The Kuratowski-Ulam theorem then implies that the equivalence relation $E' = (\varphi \times \varphi)^{-1}(E)$ is meager. As $E_0(2^{\mathbb{N}}) \subseteq E'$, we can repeat the second half of the proof of Theorem 14 to obtain a continuous embedding of $E_0(2^{\mathbb{N}})$ into E. □

We next turn to a generalization of Silver's theorem. We say that a set is a *partial transversal* of a sequence of equivalence relations if it is a partial transversal of every equivalence relation along the sequence.

Theorem 23. *Suppose that X is a Polish space and $\mathbf{E} = (E_n)_{n \in \mathbb{N}}$ is a sequence of co-analytic equivalence relations on X. Then exactly one of the following holds:*

1. *The set X is the union of countably many equivalence classes.*
2. *There is a perfect partial transversal of $\mathbf{E}$.*

Proof (Sketch). It is clear that the conditions are mutually exclusive. To see that at least one holds, let $\mathbf{G} = (G_n)_{n \in \mathbb{N}}$ denote the sequence of graphs on X given by $G_n = \sim E_n$. As every G_n-independent set is contained in an equivalence class of E_n, it follows that if there is an $\aleph_0$-coloring of $\mathbf{G}$, then X is the union of countably many equivalence classes. By Theorem 21, we can therefore assume that there is a sequence $\mathbf{S} = (S_n)_{n \in \mathbb{N}}$ of dense sets for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $\mathbf{G}_{\mathbf{S}}(2^{\mathbb{N}})$ to $\mathbf{G}$.

Essentially repeating our earlier proof of Theorem 11, the Kuratowski-Ulam theorem then ensures that each of the equivalence relations $E'_n = (\varphi \times \varphi)^{-1}(E_n)$ is meager, so Mycielski's theorem yields a perfect partial transversal $P' \subseteq 2^{\mathbb{N}}$ of the sequence $\mathbf{E}' = (E'_n)_{n \in \mathbb{N}}$, and it follows that the perfect set $P = \varphi[P']$ is a partial transversal of $\mathbf{E}$. □

Variants of Theorem 21 obtained by mixing hypergraphs of different dimensions are also quite often useful:

Theorem 24. *Suppose that X is a Polish space endowed with a vector space structure (or more generally, a pregeometry) whose linear dependence relation is co-analytic. Then for every analytic set $A \subseteq X$, exactly one of the following holds:*

1. *The span of A is countable dimensional.*
2. *There is a linearly independent perfect subset of A.*

Proof (Sketch). It is clear that the two conditions are mutually exclusive. To see that at least one holds, consider the sequence $\mathbf{G} = (G_d)_{d \geq 2}$, where G_d is the d-dimensional hypergraph on X consisting of all linearly independent sequences $x \in A^d$. As every G_d-independent set intersects A in a set of dimension strictly less than d, it follows that if there is an $\aleph_0$-coloring of $\mathbf{G}$, then the span of A is countable dimensional.

Fix a sequence $d \in \{2, 3, \ldots\}^{\mathbb{N}}$ in which every possible value appears infinitely often. The analog of Theorem 21 in which $2^{\mathbb{N}}$ is replaced with $\prod_{n \in \mathbb{N}} d(n)$ ensures the existence of a sequence $\mathbf{S} = (S_n)_{n \geq 2}$ of dense sets $S_n \subseteq \bigcup_{k \in d^{-1}(\{n\})} \prod_{i<k} d(i)$ with $\emptyset \in \bigcup_{n \geq 2} S_n$ and a continuous homomorphism $\varphi \colon \prod_{n \in \mathbb{N}} d(n) \to X$ from $\mathbf{G_S}(\prod_{n \in \mathbb{N}} d(n))$ to $\mathbf{G}$.

An induction using the Kuratowski-Ulam theorem shows that each of the graphs $G'_i = (\varphi \times \varphi)^{-1}(G_i)$ is comeager. Mycielski's theorem then yields a perfect set $P' \subseteq \prod_{n \in \mathbb{N}} d(n)$ for which every injective sequence of elements of P' of length i is in G'_i, and it follows that the perfect set $P = \varphi[P']$ is linearly independent. □

We next consider more complicated graphs. Suppose that F is an equivalence relation on X and G is a graph on X. A set $Y \subseteq X$ is *F-locally G-independent* if it is $(F \cap G)$-independent. An *F-local κ-coloring* of G is a κ-coloring of $F \cap G$. A Borel equivalence relation on a Polish space is *smooth* if it is Borel reducible to $\Delta(2^{\mathbb{N}})$.

Given a set $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$, let $H_T(2^{\mathbb{N}})$ denote the graph on $2^{\mathbb{N}}$ consisting of all pairs of the form $(t(i) \frown (i) \frown x, t(1-i) \frown (1-i) \frown x)$, where $i < 2$, $t \in T$, and $x \in 2^{\mathbb{N}}$.

We say that a set $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ is *sparse* if $|T \cap (2^n \times 2^n)| \leq 1$ for all $n \in \mathbb{N}$. If T is sparse, then $H_T(2^{\mathbb{N}})$ is acyclic.

We say that a set $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ is *dense* if

$$\forall s \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \exists t \in T \forall i < 2 \; s(i) \sqsubseteq t(i).$$

If $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ are dense, then there is no smooth equivalence relation $F \supseteq H_T(2^{\mathbb{N}})$ for which there is a Baire measurable F-local $\aleph_0$-coloring of $G_S(2^{\mathbb{N}})$.

Theorem 25. *Suppose that X is a Polish space, G is an analytic graph on X, and E is an analytic equivalence relation on X. Then for all sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ with the property that $\Delta(S) \cup T$ is sparse, at least one of the following holds:*

1. *There is a smooth equivalence relation F on X such that $E \subseteq F$ and G admits a Borel F-local $\aleph_0$-coloring.*
2. *There is a continuous homomorphism from $(G_S(2^{\mathbb{N}}), H_T(2^{\mathbb{N}}))$ to (G, E).*

Moreover, for all dense sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$, at most one of these holds.

Proof (Sketch). The fact that the two conditions are mutually exclusive when S and T are dense is a direct consequence of the inexistence of a smooth equivalence relation $F \supseteq H_T(2^{\mathbb{N}})$ for which there is a Borel F-local $\aleph_0$-coloring of $G_S(2^{\mathbb{N}})$.

The proof that one of the conditions holds when $\Delta(S) \cup T$ is sparse is similar to that of Theorem 8, albeit somewhat more elaborate. Instead of using the inability to construct a homomorphism to gradually construct G-independent Borel sets which cover the space, we build up increasingly finer smooth equivalence relations containing E and locally G-independent Borel sets which cover the space. □

As a consequence, we can establish the full generalization of the Glimm-Effros dichotomy to Borel equivalence relations (see [HKL90]):

Theorem 26 (Harrington-Kechris-Louveau)**.** *Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

1. *The equivalence relation E is smooth.*
2. *There is a continuous embedding of $E_0(2^{\mathbb{N}})$ into E.*

Proof (Sketch). Set $G = {\sim}E$. Using Lusin's first separation theorem, one can show that if F is a smooth equivalence relation containing E and there is an F-local Borel $\aleph_0$-coloring of G, then there is such a coloring which is F-invariant. The existence of the latter implies that E is smooth. By Theorem 25, we can assume there are dense sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $(G_S(2^{\mathbb{N}}), H_T(2^{\mathbb{N}}))$ to (G, E).

The Kuratowski-Ulam theorem ensures that the equivalence relation $E' = (\varphi \times \varphi)^{-1}(E)$ is meager, and a recursive construction using the fact that $H_T(2^{\mathbb{N}}) \subseteq E'$ yields a continuous embedding $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $E_0(2^{\mathbb{N}})$ into E'. Then $\varphi \circ \psi$ is a continuous reduction of $E_0(2^{\mathbb{N}})$ to E, and the second half of the proof of Theorem 14 can be repeated so as to produce a continuous embedding of $E_0(2^{\mathbb{N}})$ into E. □

Let $\mathbf{E_0}(2^{\mathbb{N}})$ denote the $\mathbb{N}$-sequence with constant value $E_0(2^{\mathbb{N}})$. More generally, we have the following:

Theorem 27. *Suppose that X is a Polish space and $\mathbf{E} = (E_n)_{n \in \mathbb{N}}$ is a sequence of Borel equivalence relations on X. Then exactly one of the following holds:*

1. *There is a smooth equivalence relation E on X with the property that $\bigcap_{n \in \mathbb{N}} E_n \subseteq E$ and $\forall x \in X \exists n \in \mathbb{N}\ [x]_E \subseteq [x]_{E_n}$.*
2. *There is a continuous embedding of $\mathbf{E_0}(2^{\mathbb{N}})$ into $\mathbf{E}$.*

Proof (Sketch). Just as the proof of Theorem 23 is essentially the proof of Theorem 11 in which the use of Theorem 8 is replaced with an application of Theorem 21, this is proven in essentially the same fashion as Theorem 26, replacing the use of Theorem 25 with its generalization to countable sequences of pairs of the form $(G_S(2^{\mathbb{N}}), H_T(2^{\mathbb{N}}))$. □

There are also asymmetric results along these lines. Given a set $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$, let $H_T^{\rightarrow}(2^{\mathbb{N}})$ denote the graph on $2^{\mathbb{N}}$ given by

$$H_T^{\rightarrow}(2^{\mathbb{N}}) = \{(t(0)\frown(0)\frown x, t(1)\frown(1)\frown x) \mid t \in T \text{ and } x \in 2^{\mathbb{N}}\}.$$

We say that a quasi-order is *lexicographically reducible* if it is Borel reducible to the lexicographical ordering of 2^α, for some $\alpha < \omega_1$.

Theorem 28. *Suppose that X is a Polish space, G is an analytic digraph on X, and R is an analytic quasi-order on X. Then for all sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ with the property that $\Delta(S) \cup T$ is sparse, at least one of the following holds:*

1. *There is a lexicographically reducible quasi-order R' on X such that $R \subseteq R'$ and G admits a Borel $\equiv_{R'}$-local $\aleph_0$-coloring.*
2. *There is a continuous homomorphism from $(G_S^{\rightarrow}(2^{\mathbb{N}}), H_T^{\rightarrow}(2^{\mathbb{N}}))$ to (G, E).*

Moreover, for all dense sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$, at most one of these holds.

Proof (Sketch). This is proven in the same fashion as Theorem 25. □

We say that a quasi-order R on X is *lexicographically linearizable* if there is a lexicographically reducible quasi-order $S \supseteq R$ with the property that $\equiv_R = \equiv_S$. The following sufficient condition for lexicographical linearizability ensures that Borel linear orderings of Polish spaces are lexicographically reducible (see [HMS88, Theorem 3.1]):

Theorem 29 (Harrington-Marker-Shelah)**.** *Suppose that X is a Polish space and R is a Borel quasi-order on X with no perfect set of pairwise R-incomparable points. Then R is lexicographically linearizable.*

Proof (Sketch). Set $G = \sim R$. Using Lusin's first separation theorem, one can show that if R' is a lexicographically reducible quasi-order containing R and there is an $\equiv_{R'}$-local Borel $\aleph_0$-coloring of G, then there is such a coloring which is suitably invariant, thus R is lexicographically linearizable. By Theorem 28, we can assume there are dense sets $S \subseteq 2^{<\mathbb{N}}$ and $T \subseteq \bigcup_{n \in \mathbb{N}} 2^n \times 2^n$ and a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $(G_S^{\rightarrow}(2^{\mathbb{N}}), H_T^{\rightarrow}(2^{\mathbb{N}}))$ to (G, E).

Set $R' = (\varphi \times \varphi)^{-1}(R)$. The fact that $G_S^{\rightarrow}(2^{\mathbb{N}})$ is disjoint from R' ensures that $\equiv_{R'}$ is meager, and since $H_T^{\rightarrow}(2^{\mathbb{N}})$ is contained in R', so too is R'. Mycielski's theorem yields a perfect set $P' \subseteq 2^{\mathbb{N}}$ of pairwise R'-incomparable points, in which case the perfect set $P = \varphi[P']$ consists of pairwise R-incomparable points, a contradiction. □

Let $R_0(2^{\mathbb{N}})$ denote the sub-quasi-order of $E_0(2^{\mathbb{N}})$ in which $x <_{R_0(2^{\mathbb{N}})} y$ if $x(n) < y(n)$, where n is the last coordinate on which x and y differ.

The following fact is the natural generalization of the Glimm-Effros dichotomy to quasi-orders (see [Kan98, Theorem 3]):

Theorem 30 (Kanovei, Louveau)**.** *Suppose that X is a Polish space and R is a Borel quasi-order on X. Then exactly one of the following holds:*

1. *The quasi-order R is lexicographically linearizable.*
2. *There is a continuous embedding of $E_0(2^{\mathbb{N}})$ or $R_0(2^{\mathbb{N}})$ into R.*

Proof (Sketch). This follows essentially from combining the proofs of Theorems 26 and 29. □

7. Parametrizations

We say that a graph G on $X \times Y$ is *vertically invariant* if $x_1 = x_2$ whenever $(x_1, y_1)\ G\ (x_2, y_2)$. We use $G(x)$ to denote the graph on Y given by $y\ G(x)\ z \iff (x, y)\ G\ (x, z)$.

Theorem 31. *Suppose that X and Y are Polish spaces and G is a vertically invariant analytic graph on $X \times Y$. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

1. *There is a Borel $\aleph_0$-coloring of G.*
2. *There is a continuous homomorphism from $G_S(2^{\mathbb{N}})$ to $G(x)$ for some $x \in X$.*

Moreover, for each dense set $S \subseteq 2^{<\mathbb{N}}$, at most one of these holds.

Proof (Sketch). By Theorem 8, it is enough to observe that if $S \subseteq 2^{\mathbb{N}}$ is full and $\varphi\colon 2^{\mathbb{N}} \to X \times Y$ is a continuous homomorphism from $G_S(2^{\mathbb{N}})$ to G, then $\mathrm{proj}_X \circ \varphi$ is constant. This is a direct consequence of the inexistence of non-trivial $E_0(2^{\mathbb{N}})$-invariant open sets. □

As a corollary, we obtain the parametrized perfect set theorem (see [Kec95, Theorem 18.10]):

Theorem 32 (Lusin-Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is an analytic set whose vertical sections do not have perfect subsets. Then there are partial functions f_n with relatively Borel graphs such that $R = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n)$.*

Proof (Sketch). Let G denote the graph on $X \times Y$ given by

$$G = \{((x, y), (x, z)) \in R \times R \mid x \in X \text{ and } y \neq z\}.$$

Clearly every G-independent set intersects R in the graph of a partial function, so if there is a Borel $\aleph_0$-coloring of G, then there are partial functions f_n with relatively Borel graphs such that $R = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n)$.

Suppose, towards a contradiction, that there is no Borel $\aleph_0$-coloring of G. By Theorem 31, there is a dense set $S \subseteq 2^{<\mathbb{N}}$ such that $\emptyset \in S$ and there is a continuous homomorphism $\varphi\colon 2^{\mathbb{N}} \to Y$ from $G_S(2^{\mathbb{N}})$ to $G(x)$ for some $x \in X$. One can then repeat the second half of our proof of Theorem 9 to obtain a perfect set $P \subseteq R_x$, a contradiction. □

Along similar lines, we also obtain the parametrized superperfect set theorem (see [Kec95, Theorem 35.46]):

Theorem 33 (Saint Raymond)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is an analytic set whose vertical sections do not have superperfect subsets. Then there are relatively Borel sets $R_n \subseteq R$ with compact vertical sections such that $R = \bigcup_{n \in \mathbb{N}} R_n$.*

Proof (Sketch). Much as Theorem 32 is proven by replacing the use of Theorem 8 in the proof of Theorem 9 with an application of Theorem 31 , this result is proven by replacing the use of Theorem 18 in the proof of Theorem 19 with an application of its parametrized analog. □

8. Generalizations

There are a number of directions in which consequences of the G_0 dichotomies generalize. The first observation to make along these lines is that Lusin's first separation theorem goes through in Hausdorff spaces, and as a result, one can abstractly obtain the graph-theoretic results on Hausdorff spaces as a consequence of their special cases for Polish spaces. As the other dichotomy theorems are then obtained by using the graph-theoretic results and Baire category arguments on $2^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$, this allows us to generalize all of these theorems to Hausdorff spaces.

Alternatively, one can forget about Polish spaces altogether and work with Hausdorff spaces from the beginning. This simplifies the statements and proofs of many results, as one can then work directly with analytic graphs on analytic subsets of a space, rather than having to consider such objects as living on the underlying space itself. Indeed, we have only focused upon Polish spaces here so as to appeal to the reader's intuition, although the special structure of Polish spaces is occasionally necessary to obtain the cleanest possible statements.

Our arguments also generalize to more complicated sets. One example is the graph-theoretic proof of the Glimm-Effros dichotomy for treeable equivalence relations. Unlike Hjorth's original argument, ours generalizes to yield an analogous result for analytically treeable equivalence relations.

Moving beyond the first level of the projective hierarchy, we say that a set is *κ-Souslin* if it is the continuous image of a closed subset of

$\kappa^{\mathbb{N}}$. The following fact (see [Kan97]) facilitates the application of the graph-theoretic approach to such sets:

Theorem 34 (Kanovei)**.** *Suppose that X is a Polish space and G is a κ-Souslin graph on X. Then for each sparse set $S \subseteq 2^{<\mathbb{N}}$, at least one of the following holds:*

1. *There is a κ^+-Borel κ-coloring of G.*
2. *There is a continuous homomorphism from $G_S(2^{\mathbb{N}})$ to G.*

Moreover, if the union of κ-many meager subsets of $2^{\mathbb{N}}$ is meager and $S \subseteq 2^{<\mathbb{N}}$ is dense, then at most one of these holds.

While it is unknown whether there is a classical proof of Kanovei's theorem, the weakening in which the definability of the coloring is removed follows from our proof of the original G_0 dichotomy after removing all applications of Lusin's first separation theorem. As we have already seen, this weakening is sufficient for many applications.

There is a further difficulty stemming from the fact that the κ-Souslin generalizations of many dichotomy theorems are consistently false. Fortunately, the use of Baire category in the graph-theoretic proofs of such results necessitates the further restriction that the relevant structures are $\aleph_0$*-universally Baire,* meaning that their pre-images under continuous functions from Polish spaces have the Baire property.

In the special case that $\kappa = \aleph_1$, this allows us to study $\aleph_0$-universally Baire structures at the second level of the projective hierarchy. For instance, it yields the generalization of Burgess's analog of Silver's theorem (see [Bur79]) to $\aleph_0$-universally Baire $\mathbf{\Pi^1_2}$ equivalence relations.

It also yields a generalization of Burgess's theorem in another direction. Let $G^{(2)}$ denote the graph consisting of all pairs of distinct points connected by a G-path of length two.

Theorem 35 (Conley-Lecomte-Miller)**.** *Suppose that X is a Polish space and G is an analytic graph on X. Then at least one of the following holds:*

1. *The set X is the union of $\aleph_1$-many $\aleph_2$-Borel $G^{(2)}$-cliques.*
2. *There is a perfect G-independent set.*

Proof (Sketch). Let H denote the graph consisting of all pairs of distinct points which are not connected by a G-path of length two. Then H is co-analytic, and therefore $\aleph_1$-Souslin. By Theorem 34, we can assume there is a dense set $S \subseteq 2^{<\mathbb{N}}$ for which there is a continuous homomorphism $\varphi \colon 2^{\mathbb{N}} \to X$ from $G_S(2^{\mathbb{N}})$ to H.

By the Kuratowski-Ulam theorem, the graph $G' = (\varphi \times \varphi)^{-1}(G)$ is meager, so Mycielski's theorem yields a perfect G'-independent set $P' \subseteq 2^{\mathbb{N}}$, in which case the perfect set $P = \varphi[P']$ is G-independent. $\square$

By using the generalizations of our results to κ-Souslin structures and appealing to the known structure theory for projective sets under AD, one obtains the natural generalizations to the odd levels of the projective hierarchy. By employing $\mathsf{AD}_{\mathbb{R}}$, one obtains the natural generalizations to arbitrary structures. Similar results along these lines can be found in [CK11].

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