ORTHOGONAL MEASURES AND ERGODICITY

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ABSTRACT. Burgess-Mauldin have proven the Ramsey-theoretic result that continuous sequences $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of pairwise orthogonal Borel probability measures admit continuous orthogonal subsequences. We establish an analogous result for sequences indexed by $2^{\mathbb{N}} / \mathbb{E}_0$, the next Borel cardinal. As a corollary, we obtain a strengthening of the Harrington-Kechris-Louveau $\mathbb{E}_0$ dichotomy for restrictions of measure equivalence. We then use this to characterize the family of countable Borel equivalence relations which are non-hyperfinite with respect to an ergodic Borel probability measure which is not strongly ergodic.

INTRODUCTION

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is *Borel* if it is in the σ-algebra generated by the underlying topology.

A *standard Borel space* is a set X equipped with the family of Borel sets associated with a Polish topology on X. Every subset of a standard Borel space inherits the σ-algebra consisting of its intersection with each Borel subset of the original space; this restriction is again standard Borel exactly when the subset in question is Borel (see, for example, [Kec95, Corollary 13.4 and Theorem 15.1]). The *product* of standard Borel spaces X and Y is the set $X \times Y$, equipped with the σ-algebra generated by the family of all sets of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are Borel. A function between standard Borel spaces is *Borel* if pre-images of Borel sets are Borel.

Suppose that E and F are equivalence relations on X and Y. A *homomorphism* from E to F is a function $\phi \colon X \to Y$ sending E-equivalent points to F-equivalent points, a *reduction* of E to F is a homomorphism sending E-inequivalent points to F-inequivalent points, and an *embedding* of E into F is an injective reduction.

Following the standard abuse of language, we say that an equivalence relation is *finite* if all of its classes are finite, and *countable* if all of its classes are countable. We say that a Borel equivalence relation is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations. We say that a Borel equivalence relation is *smooth* if it is Borel reducible to equality on a standard Borel space. By the Lusin-Novikov uniformization theorem for Borel subsets of the plane with countable vertical sections (see, for example, [Kec95, Theorem 18.10]), examples of smooth Borel equivalence relations include all finite Borel equivalence relations on standard Borel spaces. However, the relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$, given by

$$c \mathbb{E}_0 d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m),$$

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is an example of a hyperfinite Borel equivalence relation which is not smooth. In fact, the Harrington-Kechris-Louveau $\mathbb{E}_0$ dichotomy (see [HKL90, Theorem 1.1]) ensures that, under Borel reducibility, it is the minimal non-smooth Borel equivalence relation.

The *inner saturation* of a set $Y \subseteq X$ under an equivalence relation E on X is given by $(Y)_E = \{y \in Y \mid [y]_E \subseteq Y\}$, whereas the *outer saturation* of Y under E is given by $[Y]_E = \bigcup_{y \in Y} [y]_E$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that when E is a countable Borel equivalence relation and Y is Borel, then both types of saturations are also Borel.

Much recent work on countable Borel equivalence relations has utilized measure-theoretic techniques. A *Borel measure* on a standard Borel space X is a function μ associating an element of $[0, \infty]$ with each Borel subset of X in such a fashion that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$, whenever $(B_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of X. We say that μ is a *Borel probability measure* if $\mu(X) = 1$. We say that a Borel set $B \subseteq X$ is *μ-null* if $\mu(B) = 0$. Otherwise, we say that B is *μ-positive*. We say that a Borel set is *μ-conull* if its complement is μ-null. Two Borel measures are *equivalent* if they have the same μ-null Borel sets. We say that a countable Borel equivalence relation E on X is *μ-hyperfinite* if there is a μ-conull Borel set on which E is hyperfinite. We say that μ is *E-quasi-invariant* if the family of μ-null Borel sets is closed under outer E-saturation. We say that μ is *E-ergodic* if every E-invariant Borel set is either μ-null or μ-conull, or equivalently, if every Borel homomorphism from E to equality on a standard Borel space is constant on a μ-conull Borel set. More generally, we say that μ is *(E, F)-ergodic* if for every Borel homomorphism from E to F, the induced homomorphism from E to equality on Y/F is constant on a μ-conull Borel set. In the special case that $F = \mathbb{E}_0$, this notion is also referred to as *strong E-ergodicity*, and has played a pivotal role in descriptive set theory and ergodic theory over the last few years.

Given a Polish space X, we use $P(X)$ to denote the set of all Borel probability measures on X, equipped with the smallest (necessarily Polish) topology making the functions of the form $\Lambda_f(\mu) = \int f(x)\ d\mu(x)$ continuous, where $f \colon X \to \mathbb{R}$ varies over all bounded continuous functions (see, for example, [Kec95, Theorem 17.19]). It is not difficult to see that measure equivalence is Borel with respect to this topology (see, for example, [Kec95, Exercise 17.39]).

Much recent work in descriptive set theory has focused upon Borel reducibility of countable Borel equivalence relations, which refines the classical notion of cardinality and has shed new light on the nature of classification problems throughout mathematics (see, for example, [Hjo99, AK00, Hjo01, Tho03, FW04, FRW06, Tho06, FLR09, FRW11, Ros11, FTT13a, FTT13b, Sab13]). In [CM14], a somewhat weaker notion of *measure reducibility* was considered, in which one merely requires that for every Borel probability measure μ on X, there is a μ-conull Borel set on which E is Borel reducible to F. A basic idea there was to consider the connection between E and measure equivalence on $\mathcal{EQ}_E \setminus \mathcal{H}_E$, where $\mathcal{EQ}_E$ denotes the Borel set of E-ergodic E-quasi-invariant Borel probability measures on X (see, for example, [Dit92, Theorem 2 of Chapter 2]), and $\mathcal{H}_E$ denotes the Borel set of Borel probability measures μ on X with respect to which E is μ-hyperfinite (see [Seg97], or [CM14, Theorem J.10] for a simpler exposition in English). In particular, it was noted that if the latter relation has only countably many equivalence classes, then E is a countable disjoint union of successors of $\mathbb{E}_0$ (under measure reducibility), and every measure in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is strongly E-ergodic. As the latter property has proven quite useful, it is natural to ask under what circumstances such a conclusion can be obtained. Our primary goal here is to establish the following characterization.

Theorem A. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *The restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth.*
2. *Every measure in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is strongly E-ergodic.*

The proof of (2) $\implies$ (1) goes through variants of a pair of well-known results, worth mentioning in their own right.

We say that Borel probability measures μ and ν on X are *orthogonal* if there is a μ-conull ν-null Borel subset of X. It is clear that if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of pairwise orthogonal Borel probability measures on X, then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint Borel subsets of X such that B_n is μ_n-conull, for all $n \in \mathbb{N}$. On the other hand, it is not difficult to produce sequences $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of pairwise orthogonal Borel probability measures on a standard Borel space for which there is no sequence $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint Borel sets such that B_c is μ_c-conull, for all $c \in 2^{\mathbb{N}}$. Nevertheless, Burgess-Mauldin have discovered a Ramsey-theoretic salvage: If X is a Polish space and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ is a continuous sequence of pairwise orthogonal Borel probability measures on X, then there is a continuous injection $\pi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ for which there is a K_σ sequence $(K_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint sets such that K_c is $\mu_{\pi(c)}$-conull, for all $c \in 2^{\mathbb{N}}$ (see [BM81, Theorem 1]). Here, we say that a sequence $(x_i)_{i \in I}$ of points of X is *continuous* if it is continuous when viewed as a function from I to X, and we say that a sequence $(K_c)_{c \in 2^{\mathbb{N}}}$ of subsets of X is K_σ if $\{(c, x) \in 2^{\mathbb{N}} \times X \mid x \in K_c\}$ is a countable union of compact sets.

The primary result underlying our proof of (2) $\implies$ (1) is the analogous result for sequences indexed by $2^{\mathbb{N}} / \mathbb{E}_0$, the Borel-cardinal successor of $2^{\mathbb{N}}$. We say that measure-equivalence classes $\mathcal{C}$ and $\mathcal{D}$ of Borel probability measures on X are *orthogonal* if there are orthogonal measures $\mu \in \mathcal{C}$ and $\nu \in \mathcal{D}$. We say that a sequence $(\mathcal{C}_C)_{C \in I/E}$ of measure-equivalence classes of Borel probability measures on X is *continuous* if there is a continuous sequence $(\mu_i)_{i \in I}$ of Borel probability measures on X with the property that $\mu_i \in \mathcal{C}_{[i]_E}$, for all $i \in I$. We say that a sequence $(K_C)_{C \in I/E}$ of subsets of X is K_σ if the sequence $(K_i)_{i \in I}$ given by $K_i = K_{[i]_E}$ is K_σ. We say that a function $\pi \colon X/E \to Y/F$ is *continuous* if there is a continuous function $\phi \colon X \to Y$ such that $\phi(x) \in \pi([x]_E)$, for all $x \in X$.

Theorem B. *Suppose that X is a Polish space and $(\mathcal{C}_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ is a continuous sequence of pairwise orthogonal measure-equivalence classes of Borel probability measures on X. Then there is a continuous injection $\pi \colon 2^{\mathbb{N}}/\mathbb{E}_0 \to 2^{\mathbb{N}}/\mathbb{E}_0$ for which there is a K_σ sequence $(K_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ of pairwise disjoint sets such that K_C is μ-conull, for all $C \in 2^{\mathbb{N}}/\mathbb{E}_0$ and $\mu \in \mathcal{C}_{\pi(C)}$.*

Our use of this result in the proof of (2) $\implies$ (1) is via a corollary, strengthening the $\mathbb{E}_0$ dichotomy for restrictions of measure equivalence. We say that a subset of a standard Borel space is *analytic* if it is the image of a standard Borel space under a Borel function.

Theorem C. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathcal{A} \subseteq \mathcal{EQ}_E$ is analytic. Then exactly one of the following holds:*

(1) *The restriction of measure equivalence to $\mathcal{A}$ is smooth.*
(2) *There is a continuous embedding $\pi\colon 2^{\mathbb{N}} \to \mathcal{A}$ of $\mathbb{E}_0$ into measure equivalence for which there is a K_σ sequence $(K_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ of pairwise disjoint subsets of X such that K_C is $\pi(c)$-conull, for all $C \in 2^{\mathbb{N}}/\mathbb{E}_0$ and $c \in C$.*

In §1, we prove Theorems B and C. In §2, we establish Theorem A.

1. Orthogonal sequences of measures

We begin this section by establishing the analog of [BM81, Theorem 1] for sequences indexed by $2^{\mathbb{N}}/\mathbb{E}_0$.

Theorem 1. *Suppose that X is a Polish space and $(\mathcal{C}_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ is a continuous sequence of pairwise orthogonal measure-equivalence classes of Borel probability measures on X. Then there is a continuous injection $\pi\colon 2^{\mathbb{N}}/\mathbb{E}_0 \to 2^{\mathbb{N}}/\mathbb{E}_0$ for which there is a K_σ sequence $(K_C)_{C \in 2^{\mathbb{N}}/\mathbb{E}_0}$ of pairwise disjoint sets such that K_C is μ-conull, for all $C \in 2^{\mathbb{N}}/\mathbb{E}_0$ and $\mu \in \mathcal{C}_{\pi(C)}$.*

Proof. Fix a continuous sequence $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X such that $\mu_c \in \mathcal{C}_{[c]_{\mathbb{E}_0}}$ for all $c \in 2^{\mathbb{N}}$, as well as positive real numbers δ_n and ϵ_n with $\lim_{n \to \infty} \delta_n = 0$ and $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. We will recursively construct, for $i < 2$ and $n \in \mathbb{N}$, positive natural numbers k_n and sets $U_{n,i} \subseteq X$, expressible as unions of finitely many open sets of diameter at most δ_n, such that $\overline{U_{n,0}} \cap \overline{U_{n,1}} = \emptyset$ and

$$\forall n \in \mathbb{N} \forall s \in 2^{n+1} \forall c \in 2^{\mathbb{N}}\ \mu_{\phi_{n+1}(s) \frown c}(U_{n,s(n)}) > 1 - \epsilon_n,$$

where $\phi_{n+1}(s) = (s(0))^{k_0} \frown \cdots \frown (s(n))^{k_n}$.

Suppose that $n \in \mathbb{N}$ and we have already found k_m and $U_{m,i}$ for $i < 2$ and $m < n$. To handle the case $n = 0$ at the same time as the others, set $\phi_0(\emptyset) = \emptyset$. For all $s_0, s_1 \in 2^n$, the $\mathbb{E}_0$-inequivalence of $\phi_n(s_0) \frown (0)^\infty$ and $\phi_n(s_1) \frown (1)^\infty$ yields disjoint Borel sets $B_{n,s_0,s_1,0}, B_{n,s_0,s_1,1} \subseteq X$ with $\mu_{\phi_n(s_i) \frown (i)^\infty}(B_{n,s_0,s_1,i}) = 1$ for all $i < 2$. The tightness of Borel probability measures on Polish spaces (see, for example, [Kec95, Theorem 17.11]) then gives rise to compact sets $K_{n,s_0,s_1,i} \subseteq B_{n,s_0,s_1,i}$ such that $\mu_{\phi_n(s_i) \frown (i)^\infty}(K_{n,s_0,s_1,i}) > 1 - \epsilon_n/2^n$ for all $i < 2$. The disjointness of $K_{n,s_0,s_1,0}$ and $K_{n,s_0,s_1,1}$ ensures that they are of positive distance apart, in which case they are contained in open sets $U_{n,s_0,s_1,0}$ and $U_{n,s_0,s_1,1}$ with disjoint closures. As each $K_{n,s_0,s_1,i}$ is compact, we can assume that $U_{n,s_0,s_1,0}$ and $U_{n,s_0,s_1,1}$ are expressible as finite unions of open sets of diameter at most δ_n.

Now define $U_{n,s_i,i} = \bigcap_{s_{1-i} \in 2^n} U_{n,s_0,s_1,i}$ and $U_{n,i} = \bigcup_{s_i \in 2^n} U_{n,s_i,i}$. The latter sets are unions of finitely many open sets of diameter at most δ_n, since the property of being such a union is itself closed under finite intersections and finite unions. Moreover, an elementary calculation reveals that they have disjoint closures and $\mu_{\phi_n(s) \frown (i)^\infty}(U_{n,i}) > 1 - \epsilon_n$ for all $i < 2$ and $s \in 2^n$. As each of the functions $\mu \mapsto \mu(U_{n,i})$ is upper semi-continuous (see, for example, [Kec95, Corollary 17.21]), there is a natural number $k_n > 0$ such that

$$\forall i < 2 \forall s \in 2^n \forall c \in 2^{\mathbb{N}}\ \mu_{\phi_n(s) \frown (i)^{k_n} \frown c}(U_{n,i}) > 1 - \epsilon_n.$$

This completes the construction. Note that the sequences $(K_{n,c})_{c \in 2^{\mathbb{N}}}$ given by $K_{n,c} = \bigcap_{m \geq n} \overline{U_{m,c(m)}}$ are closed and totally bounded, and therefore compact (see, for example, [Kec95, Proposition 4.2]). It follows that the sequence $(K_c)_{c \in 2^{\mathbb{N}}}$ given by $K_c = \bigcup_{n \in \mathbb{N}} K_{n,c}$ is K_σ. Clearly $K_c = K_d$ whenever $c \mathbin{\mathbb{E}_0} d$, and $K_c \cap K_d = \emptyset$

whenever $\neg c \mathbb{E}_0 d$. Define a continuous embedding $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ into itself by setting $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \upharpoonright n)$, and observe that

$$\forall c \in 2^{\mathbb{N}} \ \mu_{\phi(c)}(K_c) = \sup_{n \in \mathbb{N}} \mu_{\phi(c)}(K_{n,c}) \geq 1 - \inf_{n \in \mathbb{N}} \sum_{m \geq n} \epsilon_m = 1,$$

thus the function $\pi \colon 2^{\mathbb{N}} / \mathbb{E}_0 \to 2^{\mathbb{N}} / \mathbb{E}_0$ induced by ϕ is as desired. ⊠

Much as the previously mentioned result of Burgess-Mauldin can be combined with Souslin's perfect set theorem (see, for example, [Kec95, Exercise 14.13]) to yield a strengthening of the special case of the latter for analytic families of pairwise orthogonal Borel probability measures, Theorem 1 can be combined with the $\mathbb{E}_0$ dichotomy to yield a strengthening of the special case of the latter for measure equivalence.

Theorem 2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathcal{A} \subseteq \mathcal{EQ}_E$ is analytic. Then exactly one of the following holds:*

1. *The restriction of measure equivalence to $\mathcal{A}$ is smooth.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \to \mathcal{A}$ of $\mathbb{E}_0$ into measure equivalence for which there is a K_σ sequence $(K_C)_{C \in 2^{\mathbb{N}} / \mathbb{E}_0}$ of pairwise disjoint subsets of X such that K_C is $\pi(c)$-conull, for all $C \in 2^{\mathbb{N}} / \mathbb{E}_0$ and $c \in C$.*

Proof. The easy direction of the $\mathbb{E}_0$ dichotomy yields (1) $\implies$ $\neg$(2). To see $\neg$(1) $\implies$ (2), appeal to the difficult direction of the $\mathbb{E}_0$ dichotomy to obtain a continuous embedding $\psi \colon 2^{\mathbb{N}} \to \mathcal{A}$ of $\mathbb{E}_0$ into measure equivalence. Theorem 1 then yields a continuous embedding $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $\mathbb{E}_0$ into $\mathbb{E}_0$ for which there is a K_σ sequence $(K_C)_{C \in 2^{\mathbb{N}} / \mathbb{E}_0}$ such that $K_{[c]_{\mathbb{E}_0}}$ is $(\psi \circ \phi)(c)$-conull, for all $c \in 2^{\mathbb{N}}$. Then the function $\pi = \psi \circ \phi$ is as desired. ⊠

2. Smoothness and strong ergodicity

As before, we say that a sequence $(x_i)_{i \in I}$ of points of X is *Borel* if it is Borel when viewed as a function from I to X, we say that a sequence $(B_i)_{i \in I}$ of subsets of X is *Borel* if $\{(i, x) \in I \times X \mid x \in B_i\}$ is Borel, and we say that a sequence $(B_C)_{C \in I/E}$ of subsets of X is *Borel* if the sequence $(B_i)_{i \in I}$ given by $B_i = B_{[i]_{\mathbb{E}_0}}$ is Borel.

Suppose that $(\mu_c)_{c \in 2^{\mathbb{N}}}$ is a Borel sequence of Borel probability measures on X for which there is a Borel sequence $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint sets such that B_c is μ_c-conull, for all $c \in C$. Then for any Borel probability measure $\mu_{2^{\mathbb{N}}}$ on $2^{\mathbb{N}}$ which is *continuous* in the sense that every singleton is μ-null, the corresponding Borel probability measure μ on X given by $\mu = \int \mu_c \ d\mu_{2^{\mathbb{N}}}(c)$ is orthogonal to every μ_c. However, Fubini's Theorem (see, for example, [Kec95, §17.A]) ensures that every μ-positive Borel set is also μ_c-positive, for $\mu_{2^{\mathbb{N}}}$-positively many $c \in 2^{\mathbb{N}}$. In particular, it follows that there is no μ-conull Borel subset of the complement of $\bigcup_{c \in 2^{\mathbb{N}}} B_c$. Nevertheless, it is not difficult to see that if $\mu_c \in \mathcal{EQ}_E$ for all $c \in 2^{\mathbb{N}}$, then there is no $\mu \in \mathcal{EQ}_E$ with this property. The following similar observation provides a useful sufficient condition for orthogonality of measures arising from integration.

Proposition 3. *Suppose that I, J, and X are Polish spaces, E is a countable Borel equivalence relation on X, μ and ν are Borel probability measures on I and J, $(\mu_i)_{i \in I}$ and $(\nu_j)_{j \in J}$ are Borel sequences of E-ergodic E-quasi-invariant Borel probability measures on X such that μ_i and ν_j are orthogonal for all $i \in I$ and $j \in J$, and $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ are Borel sequences of pairwise disjoint sets such* *that A_i is μ_i-conull and B_j is ν_j-conull for all $i \in I$ and $j \in J$. Then the measures $\mu' = \int \mu_i \ d\mu(i)$ and $\nu' = \int \nu_j \ d\nu(j)$ are orthogonal.*

Proof. If λ is an E-quasi-invariant Borel probability measure on X and $C \subseteq X$ is a λ-conull Borel set, then $(C)_E$ is also λ-conull. We can therefore assume that the A_i and B_j are E-invariant.

As a result of Lusin-Souslin ensures that images of Borel sets under Borel injections are Borel (see, for example, [Kec95, Theorem 15.1]), it follows that for all Borel sets $I' \subseteq I$ and $J' \subseteq J$, the sets $\bigcup_{i \in I'} A_i$ and $\bigcup_{j \in J'} B_j$ are also Borel. Define $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$.

We say that a sequence $(J_n)_{n \in \mathbb{N}}$ of subsets of J *separates points* if for all distinct $j, k \in J$, there exists $n \in \mathbb{N}$ such that $j \in J_n$ and $k \notin J_n$. Given such a sequence $(J_n)_{n \in \mathbb{N}}$ of Borel sets, note that $\bigcup_{j \in J_n} B_j$ is either μ_i-conull or μ_i-null, for all $n \in \mathbb{N}$. In particular, it follows that if $i \in I$ and B is μ_i-conull, then there is a unique $j \in J$ for which B_j is μ_i-conull.

Set $I_k = \{i \in I \mid \mu_i(B) = k\}$, for $k < 2$. For each $i \in I_1$, let $\phi(i)$ denote the unique $j \in J$ for which $\mu_i(B_j) = 1$. As the graph of ϕ is Borel (see, for example, [Kec95, Theorem 17.25]), so too is ϕ (see, for example, [Kec95, Theorem 14.12]).

Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a basis for X. For each sequence $t \in \mathbb{N}^{\mathbb{N}}$, define $U_t = \bigcup_{n \in \mathbb{N}} U_{t(n)}$. For each sequence $t \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, define $G_t = \bigcap_{n \in \mathbb{N}} U_{t(n)}$.

The regularity of Borel probability measures on Polish spaces (see, for example, [Kec95, Theorem 17.10]) ensures that for all $i \in I_1$, there exists $t \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that G_t is μ_i-conull and $\nu_{\phi(i)}$-null. Moreover, the set of all such pairs (i, t) is Borel (see, for example, [Kec95, Theorem 17.25]).

We use $\sigma(\mathbf{\Sigma}^1_1)$ to denote the σ-algebra generated by the family of analytic sets. By the Jankov-von Neumann uniformization theorem for analytic subsets of the plane (see, for example, [Kec95, Theorem 18.1]), there is a $\sigma(\mathbf{\Sigma}^1_1)$-measurable function $\tau \colon I_1 \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that $G_{\tau(i)}$ is μ_i-conull and $\nu_{\phi(i)}$-null, for all $i \in I_1$. As Lusin's theorem on the measurability of analytic sets (see, for example, [Kec95, Theorem 29.7]) ensures that τ is $(\mu \restriction I_1)$-measurable, Lusin's theorem on the approximation of measurable functions by continuous ones (see, for example, [Kec95, Theorem 17.12]) yields a $(\mu \restriction I_1)$-conull Borel set $C_1 \subseteq I_1$ on which τ is Borel.

Set $A'_i = A_i \setminus B$ for all $i \in I_0$, and $A'_i = A_i \cap B_{\phi(i)} \cap (G_{\tau(i)})_E$ for all $i \in C_1$. Then A'_i is μ_i-conull for all i in the set $C = I_0 \cup C_1$, so the set $A' = \bigcup_{i \in C} A'_i$ is μ'-conull. As above, if $j \in J$ and A' is ν_j-conull, then there is a unique $i \in C$ for which A'_i is ν_j-conull. So $A'_i \cap B_j \neq \emptyset$, thus $\phi(i) = j$, in which case the definition of A'_i ensures that it is ν_j-null, a contradiction. It follows that A' is ν_j-null for all $j \in J$, and therefore ν'-null, thus μ' and ν' are orthogonal. ⊠

The *push-forward* of a Borel measure μ on X through a Borel function $\phi \colon X \to Y$ is the Borel measure $\phi_* \mu$ on Y given by $(\phi_* \mu)(B) = \mu(\phi^{-1}(B))$.

Proposition 4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $B \subseteq X$ is Borel. Then there is a Borel reduction of measure equivalence on $\mathcal{EQ}_{E \restriction B} \setminus \mathcal{H}_{E \restriction B}$ to measure equivalence on $\mathcal{EQ}_E \setminus \mathcal{H}_E$.*

Proof. Fix $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n = 1$, appeal to the uniformization theorem for Borel subsets of the plane with countable vertical sections to obtain Borel functions $\phi_n \colon B \to X$ with $E \cap (B \times X) = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$, and define $\pi \colon P(B) \to P(X)$ by $\pi(\mu) = \sum_{n \in \mathbb{N}} \epsilon_n (\phi_n)_* \mu$. Clearly $\pi[\mathcal{EQ}_{E \restriction B}] \subseteq \mathcal{EQ}_E$, and since the family of

Borel sets on which E is hyperfinite is closed under E-saturation (see, for example, [DJK94, Proposition 5.2]), it follows that $\pi[\mathcal{EQ}_{E \upharpoonright B} \setminus \mathcal{H}_{E \upharpoonright B}] \subseteq \mathcal{EQ}_E \setminus \mathcal{H}_E$, so π is the desired reduction. ⊠

We say that E is *μ-nowhere hyperfinite* if there is no μ-positive Borel set on which E is hyperfinite.

Theorem 5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X for which the restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth, and μ is a Borel probability measure on X for which E is μ-nowhere hyperfinite. If there is a μ-null-to-one Borel homomorphism from E to $\mathbb{E}_0$, then there is a μ-null-to-one Borel homomorphism from E to equality on $2^{\mathbb{N}}$. In particular, it follows that μ is not E-ergodic.*

Proof. We will establish that every E-invariant μ-positive Borel set has an E-invariant μ-positive Borel subset of strictly smaller μ-measure. By a straightforward measure exhaustion argument, this implies the apparently stronger statement that every E-invariant μ-positive Borel set has an E-invariant μ-positive Borel subset of exactly half its μ-measure. And a straightforward recursive application of this latter statement then yields the desired conclusion.

As Proposition 4 ensures that our hypotheses are preserved under passage to μ-positive Borel subsets, it is sufficient to show that μ is not E-ergodic.

Suppose that $\phi \colon X \to 2^{\mathbb{N}}$ is a μ-null-to-one Borel homomorphism from E to $\mathbb{E}_0$. A *disintegration* of μ through ϕ is a sequence $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X such that (1) $\mu_c(\phi^{-1}(c)) = 1$ for all $c \in \phi[X]$, and (2) $\mu = \int \mu_c \, d(\phi_* \mu)(c)$. Fix a Borel disintegration $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of μ through ϕ (the existence of which follows, for example, from [Kec95, Exercise 17.35]).

By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $\phi_n \colon X \to X$ with the property that $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\phi_n)$. Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers for which $\sum_{n \in \mathbb{N}} \epsilon_n = 1$. Then for each $c \in 2^{\mathbb{N}}$, the Borel probability measure ν_c on X given by $\nu_c = \sum_{n \in \mathbb{N}} \epsilon_n (\phi_n)_* \mu_c$ is E-quasi-invariant.

Lemma 6. *The relation E is ν_c-nowhere hyperfinite, for $(\phi_* \mu)$-almost every $c \in 2^{\mathbb{N}}$.*

Proof. By a result from [Seg97] (see [CM14, Theorem J.8] for a simpler proof in English)), there is a Borel set $F \subseteq (\mathbb{N} \times (X \times X)) \times P(X)$ such that for all $\nu \in P(X)$, the following conditions hold:

1. The sets $(F^\nu)_n$ form an increasing sequence of finite Borel subequivalence relations of E.
2. The set $B^\nu = \{x \in X \mid [x]_E \neq \bigcup_{n \in \mathbb{N}} [x]_{(F^\nu)_n}\}$ does not contain a ν-positive Borel subset on which E is hyperfinite.

In particular, it follows that E is ν-hyperfinite if and only if $\nu(B^\nu) = 1$, for all Borel probability measures ν on X. And the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that the set $B = \{(x, \nu) \in X \times P(X) \mid x \in B^\nu\}$ is Borel. It follows that $\{\nu \in P(X) \mid E \text{ is } \nu\text{-nowhere hyperfinite}\}$ is Borel (see, for example, [Kec95, Theorem 17.25]), thus so too is the set of $c \in 2^{\mathbb{N}}$ for which E is μ_c-nowhere hyperfinite. As ϕ is a homomorphism from E to $\mathbb{E}_0$ and the latter is hyperfinite, it easily follows from [Seg97] ([CM14, Propositon J.13] for a simpler proof in English) that this set is $(\phi_* \mu)$-conull. But the nowhere hyperfiniteness of E with respect to μ_c and ν_c are equivalent (see, for example, [DJK94, Proposition 5.2]). ⊠

We say that a sequence $(\phi_i)_{i \in I}$ of functions from X to Y is *Borel* if the corresponding function $\phi \colon I \times X \to Y$ given by $\phi(i, x) = \phi_i(x)$ is Borel.

A function $\rho \colon E \to \mathbb{R}^+$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \mathrel{E} y \mathrel{E} z$. For each $c \in 2^{\mathbb{N}}$, fix a Borel cocycle $\rho_c \colon E \to \mathbb{R}^+$ with respect to which ν_c is *invariant*, in the sense that

$$\nu_c(T^{-1}(B)) = \int_B \rho_c(T^{-1}(x), x) \; d\nu_c(x),$$

whenever $B \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in E. The existence of such cocycles follows, for example, from [KM04, Proposition 8.3], and a rudimentary inspection of the proof of the latter reveals that it is sufficiently uniform so as to ensure the existence of such cocycles for which the corresponding sequence $(\rho_c)_{c \in 2^{\mathbb{N}}}$ is Borel.

For each $c \in 2^{\mathbb{N}}$, fix a sequence $(\nu_{c,x})_{x \in X}$ of E-ergodic ρ_c-invariant Borel probability measures on X which forms an *ergodic decomposition* of ρ_c, in the sense that (1) $\nu(\{x \in X \mid \nu_{c,x} = \nu\}) = 1$ for all E-ergodic ρ_c-invariant Borel probability measures ν on X, and (2) $\nu = \int \nu_{c,x} \; d\nu(x)$ for all ρ_c-invariant Borel probability measures ν on X. Ditzen has established the existence of Borel ergodic decompositions (see [Dit92, Theorem 6 of Chapter 2]), and a rudimentary inspection of the proof again reveals that it is sufficiently uniform so as to ensure the existence of such decompositions for which the corresponding sequence $(\nu_{c,x})_{c \in 2^{\mathbb{N}}, x \in X}$ is Borel.

Lemma 7. *Suppose that $c \in 2^{\mathbb{N}}$. Then $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull, for ν_c-almost every $x \in X$.*

Proof. If $n \in \mathbb{N}$ and $x \in \phi^{-1}(c)$, then the fact that ϕ is a homomorphism from E to $\mathbb{E}_0$ ensures that $\phi(x) \mathrel{\mathbb{E}_0} (\phi \circ \phi_n)(x)$, or equivalently, that $x \in (\phi \circ \phi_n)^{-1}([c]_{\mathbb{E}_0})$, so $\phi^{-1}(c) \subseteq (\phi \circ \phi_n)^{-1}([c]_{\mathbb{E}_0})$. As $\phi^{-1}(c)$ is μ_c-conull, so too is $(\phi \circ \phi_n)^{-1}([c]_{\mathbb{E}_0})$, from which it follows that $\nu_c(\phi^{-1}([c]_{\mathbb{E}_0})) = \sum_{n \in \mathbb{N}} \epsilon_n (\phi_n)_* \mu_c(\phi^{-1}([c]_{\mathbb{E}_0})) = \sum_{n \in \mathbb{N}} \epsilon_n = 1$. As $\nu_c(\phi^{-1}([c]_{\mathbb{E}_0})) = \int \nu_{c,x}(\phi^{-1}([c]_{\mathbb{E}_0})) \; d\nu_c(x)$, it also follows that $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull, for ν_c-almost every $x \in X$. ⊠

Let $\mathcal{B}$ denote the set of Borel probability measures of the form $\nu_{c,x}$, where $c \in 2^{\mathbb{N}}$, $x \in X$, and $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull.

Lemma 8. *The set $\mathcal{B}$ is Borel.*

Proof. As ϕ is Borel, so too is $\{(c, x) \in 2^{\mathbb{N}} \times X \mid x \in \phi^{-1}([c]_{\mathbb{E}_0})\}$ (see, for example, [Kec95, Proposition 12.4]), thus the set

$$R = \{(\nu, c) \in (\mathcal{EQ}_E \setminus \mathcal{H}_E) \times 2^{\mathbb{N}} \mid \nu(\phi^{-1}([c]_{\mathbb{E}_0})) = 1\}$$

is also Borel (see, for example, [Kec95, Theorem 17.25]).

As $(\nu_{c,x})_{c \in 2^{\mathbb{N}}, x \in X}$ is Borel, another appeal to [Kec95, Proposition 12.4] ensures that so too is $\{(\nu, (c, x)) \in P(X) \times (2^{\mathbb{N}} \times X) \mid \nu = \nu_{c,x}\}$. And one more appeal to [Kec95, Theorem 17.25] yields that the set

$$S = \{(\nu, c) \in P(X) \times 2^{\mathbb{N}} \mid \nu(\{x \in X \mid \nu = \nu_{c,x}\}) = 1\}$$

is also Borel.

As $\nu \in \mathcal{B}$ if and only if there exists $c \in 2^{\mathbb{N}}$ such that $(\nu, c) \in R \cap S$, the lemma now follows from the uniformization theorem for Borel subsets of the plane with countable vertical sections. ⊠

Lemma 9. *The restriction of measure equivalence to $\mathcal{B}$ is countable.*

Proof. The definition of ergodic decomposition ensures that if $c \in 2^{\mathbb{N}}$, $x, y \in X$, and $\nu_{c,x} \neq \nu_{c,y}$, then $\nu_{c,x}$ and $\nu_{c,y}$ are orthogonal. Similarly, if $c, d \in 2^{\mathbb{N}}$ are not $\mathbb{E}_0$-related, $x, y \in X$, $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull, and $\phi^{-1}([d]_{\mathbb{E}_0})$ is $\nu_{d,y}$-conull, then $\nu_{c,x}$ and $\nu_{d,y}$ are orthogonal. The countability of $\mathbb{E}_0$ therefore yields that of the restriction of measure equivalence to $\mathcal{B}$. ⊠

A *transversal* of an equivalence relation is a set intersecting every equivalence class in exactly one point. As the uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that smooth countable Borel equivalence relations have Borel transversals, it follows that the restriction of measure equivalence to $\mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E)$ has a Borel transversal. Together with another application of the uniformization theorem for Borel subsets of the plane with countable vertical sections, this yields the existence of a measure-equivalence-invariant Borel function $\psi \colon \mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E) \to 2^{\mathbb{N}}$ such that $\phi^{-1}([\psi(\nu)]_{\mathbb{E}_0})$ is ν-conull, for all $\nu \in \mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E)$.

For each $c \in 2^{\mathbb{N}}$, let Λ_c denote the Borel probability measure on $P(X)$ obtained by pushing forward ν_c through the function $x \mapsto \nu_{c,x}$.

Lemma 10. *Suppose that $c \in 2^{\mathbb{N}}$ and E is ν_c-nowhere hyperfinite. Then the intersection $\mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E)$ is Λ_c-conull.*

Proof. The fact that $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull for ν_c-almost every $x \in X$ ensures that $\Lambda_c(\mathcal{B}) \geq \nu_c(\{x \in X \mid \phi^{-1}([c]_{\mathbb{E}_0}) \text{ is } \nu_{c,x}\text{-conull}\}) = 1$. As every $\nu_{c,x}$ is E-ergodic and ρ_c-invariant, it follows that $\Lambda_c(\mathcal{EQ}_E) = \nu_c(\{x \in X \mid \nu_{c,x} \in \mathcal{EQ}_E\}) = 1$. As E is ν_c-nowhere hyperfinite, one more appeal to [Seg97] (or [CM14, Theorem J.8]) ensures that $\Lambda_c(\mathcal{H}_E) = \nu_c(\{x \in X \mid E \text{ is } \nu_{c,x}\text{-hyperfinite}\}) = 0$. ⊠

Let R denote the set of pairs $(c, d) \in \mathbb{E}_0$ for which $\psi^{-1}(d)$ is Λ_c-positive. As the graph of ψ is Borel (see, for example, [Kec95, Proposition 12.4]), it follows that R is Borel (see, for example, [Kec95, Theorem 17.25]).

Lemma 11. *Suppose that $c \in 2^{\mathbb{N}}$ and E is ν_c-nowhere hyperfinite. Then there exists $d \in 2^{\mathbb{N}}$ such that $c \mathrel{R} d$.*

Proof. As $\mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E)$ is Λ_c-conull, it follows that $\nu_{c,x} \in \mathcal{B} \cap (\mathcal{EQ}_E \setminus \mathcal{H}_E)$ for ν_c-almost every $x \in X$. As $\phi^{-1}([c]_{\mathbb{E}_0})$ is $\nu_{c,x}$-conull for ν_c-almost every $x \in X$, it follows that $\nu_{c,x} \in \psi^{-1}([c]_{\mathbb{E}_0})$ for ν_c-almost every $x \in X$. Fix $d \in [c]_{\mathbb{E}_0}$ with the property that $\nu_{c,x} \in \psi^{-1}(d)$ for ν_c-positively many $x \in X$, and observe that $\psi^{-1}(d)$ is Λ_c-positive, thus $c \mathrel{R} d$. ⊠

By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel sets $B_n \subseteq 2^{\mathbb{N}}$ and Borel injections $\psi_n \colon B_n \to 2^{\mathbb{N}}$ such that $R = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\psi_n)$. Fix $n \in \mathbb{N}$ such that B_n is $(\phi_* \mu)$-positive.

For each $b \in B_n$, let λ_b be the E-quasi-invariant Borel measure on X given by

$$\lambda_b = \int_{\psi^{-1}(\psi_n(b))} \nu \, d\Lambda_b(\nu).$$

Proposition 3 ensures that λ_a and λ_b are orthogonal, for all distinct $a, b \in B_n$.

Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a basis for X. For each sequence $t \in \mathbb{N}^{\mathbb{N}}$, define $U_t = \bigcup_{n \in \mathbb{N}} U_{t(n)}$. For each sequence $t \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, define $G_t = \bigcap_{n \in \mathbb{N}} U_{t(n)}$.

Another application of [Kec95, Theorem 17.25] ensures that we obtain a Borel set by considering the family of pairs $(b, t) \in B_n \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that G_t is λ_b-conull and λ_a-null, for all $a \in [b]_{\mathbb{E}_0} \cap B_n$ other than b itself. The uniformization theorem for analytic subsets of the plane then yields a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable function associating with every $b \in B_n$ a sequence $t_b \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ for which (b, t_b) is in this set. As every such function is necessarily $((\phi_* \mu) \upharpoonright B_n)$-measurable (see, for example, [Kec95, Theorem 29.7]), by deleting a $(\phi_* \mu)$-null Borel set from B_n, we can assume that this function is Borel (see, for example, [Kec95, Theorem 17.12]).

As ϕ is μ-null-to-one, it follows that $\phi_* \mu$ is continuous, so there is a Borel set $A_n \subseteq B_n$ with the property that $0 < (\phi_* \mu)(A_n) < (\phi_* \mu)(B_n)$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, the E-invariant set $A \subseteq X$ given by

$$A = \bigcup_{a \in A_n, b \in [a]_{\mathbb{E}_0}} (G_{t_a})_E \cap \phi^{-1}(b)$$

is Borel (recall that $(G_{t_a})_E$ in the above expression denotes the inner saturation). Note that if $a \in A_n$, then the fact that G_{t_a} is λ_a-conull ensures that A is λ_a-conull, so the set of $\nu \in \psi^{-1}(\psi_n(a))$ for which A is ν-conull is Λ_a-positive. This means that the set of $x \in X$ for which A is $\nu_{a,x}$-conull is ν_a-positive, so the definition of ergodic decomposition ensures that A is ν_a-positive. As A is E-invariant, it is necessarily μ_a-positive. And since A_n is itself $(\phi_* \mu)$-positive, the definition of measure disintegration ensures that A is μ-positive. However, if $b \in B_n \setminus A_n$, then the fact that G_{t_b} is λ_a-null for all $b \in [a]_{\mathbb{E}_0}$ other than b itself ensures that A is λ_b-null, so the set of $\nu \in \psi^{-1}(\psi_n(b))$ for which A is ν-null is Λ_b-conull. This means that the set of $x \in X$ for which A is $\nu_{b,x}$-null is ν_b-conull, so the definition of ergodic decomposition ensures that A is ν_b-null. As A is E-invariant, it is necessarily μ_b-null. And since $B_n \setminus A_n$ is itself $(\phi_* \mu)$-positive, the definition of measure disintegration ensures that the complement of A is μ-positive, thus μ is not E-ergodic. ⊠

We are now prepared to establish our primary result.

Theorem 12. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *The restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth.*
2. *Every measure in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is strongly E-ergodic.*

Proof. To see (1) $\implies$ (2), note that if the restriction of measure equivalence to $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is smooth and $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, then Theorem 5 ensures that there is no μ-null-to-one Borel homomorphism from E to $\mathbb{E}_0$, thus μ is strongly E-ergodic.

To see $\neg$(1) $\implies$ $\neg$(2), appeal to Theorem 2 to obtain a Borel embedding $\pi \colon 2^{\mathbb{N}} \to \mathcal{EQ}_E \setminus \mathcal{H}_E$ of $\mathbb{E}_0$ into measure equivalence and a Borel sequence $(B_C)_{C \in 2^{\mathbb{N}} / \mathbb{E}_0}$ of pairwise disjoint sets such that $B_{[c]_{\mathbb{E}_0}}$ is $\pi(c)$-conull, for all $c \in 2^{\mathbb{N}}$. By replacing the sets along this sequence with their inner E-saturations, we can assume that they are E-invariant. Define $B_c = B_{[c]_{\mathbb{E}_0}}$ for $c \in 2^{\mathbb{N}}$, fix a continuous $\mathbb{E}_0$-ergodic Borel probability measure $\mu_{2^{\mathbb{N}}}$ on $2^{\mathbb{N}}$, and let μ denote the Borel probability measure on X given by $\mu = \int \pi(c) \; d\mu_{2^{\mathbb{N}}}(c)$.

To see that μ is E-ergodic, note that if $B \subseteq X$ is an E-invariant Borel set, then the $\mathbb{E}_0$-invariant set $A = \{c \in 2^{\mathbb{N}} \mid B \text{ is } \pi(c)\text{-conull}\}$ is also Borel, in which case

$\mu_{2^{\mathbb{N}}}(A) = 0$ or $\mu_{2^{\mathbb{N}}}(A) = 1$, so the definition of μ ensures that $\mu_{2^{\mathbb{N}}}(A) = \mu(B)$, thus $\mu(B) = 0$ or $\mu(B) = 1$. To see that μ is E-quasi-invariant, note that if $B \subseteq X$ is Borel, then

$$\begin{aligned} \mu(B) > 0 &\iff \mu_{2^{\mathbb{N}}}(\{c \in 2^{\mathbb{N}} \mid \pi(c)(B) > 0\}) > 0 \\ &\iff \mu_{2^{\mathbb{N}}}(\{c \in 2^{\mathbb{N}} \mid \pi(c)([B]_E) > 0\}) > 0 \\ &\iff \mu([B]_E) > 0. \end{aligned}$$

To see that E is not μ-hyperfinite, note that otherwise E is $\pi(c)$-hyperfinite for a $\mu_{2^{\mathbb{N}}}$-conull set of $c \in 2^{\mathbb{N}}$, contradicting our choice of π. To see that μ is not strongly E-ergodic, apply the uniformization theorem for Borel subsets of the plane with countable vertical sections to the set $B = \{(x, c) \in X \times 2^{\mathbb{N}} \mid x \in B_c\}$ to obtain a Borel homomorphism $\phi\colon \mathrm{proj}_X[B] \to 2^{\mathbb{N}}$ from $E \restriction \mathrm{proj}_X[B]$ to $\mathbb{E}_0$ such that $c \; \mathbb{E}_0 \; \phi(x)$ for all $c \in 2^{\mathbb{N}}$ and all $x \in B_c$. As $\mathrm{proj}_X[B]$ is E-invariant, we can extend ϕ to a homomorphism from E to $\mathbb{E}_0$ by fixing any $b \in 2^{\mathbb{N}}$ and asking that $\phi(x) = b$ for all $x \in X \setminus \mathrm{proj}_X[B]$. It only remains to observe that if $c, d \in 2^{\mathbb{N}}$ are $\mathbb{E}_0$-inequivalent, then $\phi^{-1}(c)$ is $\pi(d)$-null, and since $\mu_{2^{\mathbb{N}}}$ is continuous, it follows that $\phi^{-1}(c)$ is μ-null for all $c \in 2^{\mathbb{N}}$. ⊠

Remark 13. As there is a homeomorphism $\phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ sending $\mathbb{E}_0$ to $\mathbb{E}_0 \times \mathbb{E}_0$, it follows that if there exists $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$ which is not strongly E-ergodic, then measure equivalence is non-smooth on the set of all such μ.

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References

[AK00] S. Adams and A.S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943 (electronic). MR 1775739 (2001g:03086)

[BM81] J.P. Burgess and R.D. Mauldin, *Conditional distributions and orthogonal measures*, Ann. Probab. **9** (1981), no. 5, 902–906.

[CM14] C.T. Conley and B.D. Miller, *Definable cardinals just beyond* $\mathbb{R}/\mathbb{Q}$, preprint available at https://dl.dropboxusercontent.com/u/47430894/Papers/rigidity/rigidity.pdf, 2014.

[Dit92] A. Ditzen, *Definable equivalence relations on Polish spaces*, Ph.D. thesis, California Institute of Technology, Pasadena, California, 1992.

[DJK94] R.L. Dougherty, S.C. Jackson, and A.S. Kechris, *The structure of hyperfinite Borel equivalence relations*, Trans. Amer. Math. Soc. **341** (1994), no. 1, 193–225. MR 1149121 (94c:03066)

[FLR09] V. Ferenczi, A. Louveau, and C. Rosendal, *The complexity of classifying separable Banach spaces up to isomorphism*, J. Lond. Math. Soc. (2) **79** (2009), no. 2, 323–345. MR 2496517 (2010c:46017)

[FRW06] M.D. Foreman, D.J. Rudolph, and B. Weiss, *On the conjugacy relation in ergodic theory*, C. R. Math. Acad. Sci. Paris **343** (2006), no. 10, 653–656. MR 2271741 (2007h:37004)

[FRW11] M. Foreman, D.J. Rudolph, and B. Weiss, *The conjugacy problem in ergodic theory*, Ann. of Math. (2) **173** (2011), no. 3, 1529–1586. MR 2800720 (2012k:37006)

[FTT13a] I. Farah, A.S. Toms, and A. Tornquist, *The descriptive set theory of* C^**-algebra invariants*, To appear in Int. Math. Res. Notices, 2013.

[FTT13b] ______, *Turbulence, orbit equivalence, and the classification of nuclear* C^**-algebras*, To appear in J. für die reine und angwandte Math., 2013.

[FW04] M. Foreman and B. Weiss, *An anti-classification theorem for ergodic measure preserving transformations*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 3, 277–292. MR 2060477 (2005c:37006)

[Hjo99] G. Hjorth, *Around nonclassifiability for countable torsion free abelian groups*, Abelian groups and modules (Dublin, 1998), Trends Math., Birkhäuser, Basel, 1999, pp. 269–292. MR 1735575 (2001c:20113)

[Hjo01] ______, *On invariants for measure preserving transformations*, Fund. Math. **169** (2001), no. 1, 51–84. MR 1852353 (2002j:03048)

[HKL90] L.A. Harrington, A.S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928.

[Kec95] A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[KM04] A.S. Kechris and B.D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics, vol. 1852, Springer-Verlag, Berlin, 2004. MR 2095154 (2005f:37010)

[Ros11] C. Rosendal, *Descriptive classification theory and separable Banach spaces*, Notices Amer. Math. Soc. **58** (2011), no. 9, 1251–1262. MR 2868100

[Sab13] M. Sabok, *Completeness of the isomorphism problem for separable C^*-algebras*, preprint, 2013.

[Seg97] M. Segal, *Hyperfiniteness*, Ph.D. thesis, Hebrew University, Jerusalem, Israel, 1997.

[Tho03] S. Thomas, *The classification problem for torsion-free abelian groups of finite rank*, J. Amer. Math. Soc. **16** (2003), no. 1, 233–258. MR 1937205 (2004b:20081)

[Tho06] ______, *Borel superrigidity and the classification problem for the torsion-free abelian groups of finite rank*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 93–116. MR 2275590 (2007m:20085)

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