SIGMA-CONTINUITY WITH CLOSED WITNESSES

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Abstract. We use variants of the $\mathbb{G}_0$ dichotomy to establish a refinement of Solecki's basis theorem for the family of Baire-class one functions which are not σ-continuous with closed witnesses.

Introduction

A subset of a topological space is F_σ if it is a union of countably-many closed sets, *Borel* if it is in the σ-algebra generated by the closed sets, and *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$.

A function between topological spaces is *σ-continuous with closed witnesses* if its domain is a union of countably-many closed sets on which it is continuous, *Baire class one* if preimages of open sets are F_σ, *strongly σ-closed-to-one* if its domain is a union of countably-many analytic sets intersecting the preimage of each singleton in a closed set, *F_σ-to-one* if the preimage of each singleton is F_σ, and *Borel* if preimages of open sets are Borel.

A *topological embedding* of a topological space X into a topological space Y is a function $\pi \colon X \to Y$ which is a homeomorphism onto its image, where the latter is endowed with the subspace topology. A *topological embedding* of a set $A \subseteq X$ into a set $B \subseteq Y$ is a topological embedding π of X into Y such that $A = \pi^{-1}(B)$. A *topological embedding* of a function $f \colon X \to Y$ into a function $f' \colon X' \to Y'$ is a pair (π_X, π_Y), consisting of topological embeddings π_X of X into X' and π_Y of $f(X)$ into $f'(X')$, with $f' \circ \pi_X = \pi_Y \circ f$.

Some time ago, Jayne-Rogers showed that a function between Polish spaces is σ-continuous with closed witnesses if and only if preimages of closed sets are F_σ (see [JR82, Theorem 1]). Solecki later refined this result by providing a two-element basis, under topological embeddability, for the family of Baire-class one functions which do not have this property (see [Sol98, Theorem 3.1]). Here we use variants of the $\mathbb{G}_0$ dichotomy (see [KST99]) to establish a pair of dichotomies which together refine Solecki's theorem.

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In §1, we provide a simple characterization of Baire-class one functions that is used throughout the remainder of the paper. As a first application, we use the Lecomte-Zeleny $\mathbf{\Delta}^0_2$-measurable analog of the $\mathbb{G}_0$ dichotomy theorem (see [LZ14, Corollary 4.5]) to establish that the property of being Baire class one is determined by behaviour on countable sets.

In §2, we use the Hurewicz dichotomy theorem to provide a one-element basis, under topological embeddability, for the family of Baire-class one functions which are not F_σ-to-one. To be precise, let $\mathbb{N}^{\leq\mathbb{N}}_*$ denote the set $\mathbb{N}^{\leq\mathbb{N}}$ equipped with the smallest topology making the sets $\mathcal{N}^*_s = \{t \in \mathbb{N}^{\leq\mathbb{N}} \mid s \sqsubseteq t\}$ clopen for all $s \in \mathbb{N}^{<\mathbb{N}}$, and fix a function $f_0 \colon \mathbb{N}^{\leq\mathbb{N}}_* \to \mathbb{R}$ such that $f_0 \restriction \mathbb{N}^\mathbb{N}$ has constant value zero and $f_0 \restriction \mathbb{N}^{<\mathbb{N}}$ is an injection into $\{1/n \mid n \in \mathbb{N}\}$.

Theorem 1. *Suppose that X and Y are Polish spaces and $f \colon X \to Y$ is a Baire-class one function. Then exactly one of the following holds:*

1. *The function f is F_σ-to-one.*
2. *There is a topological embedding of f_0 into f.*

In §3, we use the sequential $\aleph_0$-dimensional analog of the $\mathbb{G}_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) to provide a one-element basis, under topological embeddability, for the family of F_σ-to-one Baire-class one functions which are not σ-continuous with closed witnesses. To be precise, let $\mathbb{N}^{\leq\mathbb{N}}_{**}$ denote the set $\mathbb{N}^{\leq\mathbb{N}}$ equipped with the smallest topology making the sets $\mathcal{N}^*_s$ and $\{s\}$ clopen for all $s \in \mathbb{N}^{<\mathbb{N}}$, and define $f_1 \colon \mathbb{N}^{\leq\mathbb{N}}_* \to \mathbb{N}^{\leq\mathbb{N}}_{**}$ by $f_1(s) = s$.

Theorem 2. *Suppose that X and Y are Polish spaces and $f \colon X \to Y$ is an F_σ-to-one Baire-class one function. Then exactly one of the following holds:*

1. *The function f is σ-continuous with closed witnesses.*
2. *There is a topological embedding of f_1 into f.*

Theorem 2 trivially yields the following.

Theorem 3 (Jayne-Rogers)**.** *Suppose that X and Y are Polish spaces, and $f \colon X \to Y$ is a function with the property that $f^{-1}(C)$ is F_σ, for all closed subsets C of Y. Then f is σ-continuous with closed witnesses.*

And Theorems 1 and 2 trivially yield the following.

Theorem 4 (Solecki)**.** *Suppose that X and Y are Polish spaces and f is a Baire-class one function. Then exactly one of the following holds:*

(1) *The function f is σ-continuous with closed witnesses.*
(2) *There is a topological embedding of f_0 or f_1 into f.*

We close the paper with an appendix in which we prove several auxiliary facts needed for the proof of the main results above. Along the way, we use Lecomte's $\aleph_0$-dimensional analog of the $\mathbb{G}_0$ dichotomy theorem (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) to give a new proof of a special case of Hurewicz's dichotomy theorem (see, for example, [Kec95, Theorem 21.18]), yielding the existence of a one-element basis, under topological embeddability, for the family of Borel sets which are not F_σ. To be precise, we show that if X is a Polish space and $B \subseteq X$ is Borel, then either B is F_σ, or there is a topological embedding $\pi \colon \mathbb{N}^{\leq \mathbb{N}}_* \to X$ of $\mathbb{N}^\mathbb{N}$ into B. We note that the same argument, using the parametrized $\aleph_0$-dimensional analog of the $\mathbb{G}_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) in lieu of its non-parametrized counterpart, yields a slight weakening of Saint Raymond's parametrized analog of Hurewicz's result (see, for example, [Kec95, Theorem 35.45]). As a corollary, we show that F_σ-to-one Borel functions between Polish spaces are strongly σ-closed-to-one.

1. Baire-class one functions

Throughout the rest of the paper, we will rely on the following characterization of Baire-class one functions.

Proposition 1.1. *Suppose that X is a topological space, Y is a second countable metric space, and $f \colon X \to Y$ is a function. Then the following are equivalent:*

(1) *The function f is Baire class one.*
(2) *For all $\epsilon > 0$, there is a cover of X by countably-many closed subsets whose f-images have d_Y-diameter strictly less than ϵ.*

Proof. To see (1) $\implies$ (2), it is sufficient to show that for all real numbers $\epsilon > 0$ and open sets $V \subseteq Y$ of d_Y-diameter strictly less than ϵ, the set $f^{-1}(V)$ is a union of countably-many closed subsets of X. But this follows from the fact that $f^{-1}(V)$ is F_σ.

To see (2) $\implies$ (1), it is sufficient to show that for all real numbers $\epsilon > 0$ and open sets $V \subseteq Y$, there is an F_σ set $F \subseteq X$ such that $f^{-1}(V_\epsilon) \subseteq F \subseteq f^{-1}(V)$, where $V_\epsilon = \{y \in Y \mid \mathcal{B}_{d_Y}(y, \epsilon) \subseteq V\}$. Towards this end, fix a cover $(C_n)_{n \in \mathbb{N}}$ of X by closed sets whose f-images have d_Y-diameter strictly less than ϵ, define $N = \{n \in \mathbb{N} \mid f(C_n) \cap V_\epsilon \neq \emptyset\}$, and observe that the set $F = \bigcup_{n \in N} C_n$ is as desired. $\square$

As a corollary, we obtain the following.

Theorem 1.2. *Suppose that X and Y are Polish spaces, d_Y is a compatible metric on Y, and $f \colon X \to Y$ is Borel. Suppose further that for all countable sets $C \subseteq X$ and real numbers $\epsilon > 0$, there is a Baire-class one function $g \colon X \to Y$ with $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$. Then f is Baire class one.*

Proof. Suppose, towards a contradiction, that f is not Baire class one, fix a compatible metric d_Y on Y, and appeal to Proposition 1.1 to find $\delta > 0$ for which there is no cover of X by countably-many closed subsets whose f-images have d_Y-diameter at most δ.

A *digraph* on a set X is an irreflexive subset of $X \times X$. A *homomorphism* from a digraph G on X to a digraph H on Y is a function $\phi \colon X \to Y$ sending G-related points to H-related points.

Let $G_{\delta,f}$ denote the digraph on X consisting of all $(w, x) \in X \times X$ for which $d_Y(f(w), f(x)) > \delta$. We say that a set $W \subseteq X$ is *$G_{\delta,f}$-independent* if $G_{\delta,f} \restriction W = \emptyset$. Our choice of δ ensures that X is not the union of countably-many closed $G_{\delta,f}$-independent sets.

Fix $s_n^{\mathbf{\Delta}_2^0} \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq s_n^{\mathbf{\Delta}_2^0}$, as well as $z_n \in 2^{\mathbb{N}}$ for all $n \in \mathbb{N}$. Now define a digraph on $2^{\mathbb{N}}$ by setting

$$\mathbb{G}_0^{\mathbf{\Delta}_2^0} = \{(s_n^{\mathbf{\Delta}_2^0} \frown (0) \frown z_n, s_n^{\mathbf{\Delta}_2^0} \frown (1) \frown z_n) \mid n \in \mathbb{N}\}.$$

The Lecomte-Zeleny dichotomy theorem characterizing analytic graphs of uncountable $\mathbf{\Delta}_2^0$-measurable chromatic number (see [LZ14, Corollary 4.5]) yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from this digraph to $G_{\delta,f}$. Set

$$C = \phi(\{s_n^{\mathbf{\Delta}_2^0} \frown (i) \frown z_n \mid i < 2 \text{ and } n \in \mathbb{N}\})$$

and $\epsilon = \delta/3$.

It only remains to check that no function $g \colon X \to Y$ with the property that $\sup_{x \in C} d_Y(f(x), g(x)) \leq \epsilon$ is Baire class one. As ϕ is necessarily a homomorphism from the above digraph to the digraph $G_{\epsilon,g}$ associated with such a function, there can be no cover of X by countably-many closed subsets whose g-images have d_Y-diameter at most ϵ, so one more appeal to Proposition 1.1 ensures that g is not Baire class one. $\square$

2. F_σ-to-one functions

The proof of Theorem 1 is based on a technical but useful sufficient condition for the topological embeddability of f_0.

Proposition 2.1. *Suppose that Y is a Polish space and $f\colon \mathbb{N}^{\leq\mathbb{N}}_* \to Y$ is a Baire-class one function for which there exists $y \in Y$ such that $\mathbb{N}^\mathbb{N} = f^{-1}(y)$. Then there is a topological embedding of f_0 into f.*

Proof. Fix a compatible metric d_Y on Y.

Lemma 2.2. *Suppose that $\epsilon > 0$. Then there is a dense open subset U of $\mathbb{N}^{\leq\mathbb{N}}_*$ such that $f(U) \subseteq \mathcal{B}_{d_Y}(y, \epsilon)$.*

Proof. By Proposition 1.1, there is a partition $(C_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^{\leq\mathbb{N}}_*$ into closed sets whose f-images have d_Y-diameter strictly less than ϵ. Then for each non-empty open set $V \subseteq \mathbb{N}^{\leq\mathbb{N}}_*$, there exists $n \in \mathbb{N}$ for which C_n is non-meager in V, so there is a non-empty open set $W \subseteq V$ such that C_n is comeager in W. As C_n is closed, it follows that $W \subseteq C_n$, thus the diameter of $f(W)$ is strictly less than ϵ. As W necessarily contains a point of $\mathbb{N}^\mathbb{N}$, it follows that $f(W) \subseteq \mathcal{B}_{d_Y}(y, \epsilon)$. The union of the non-empty open sets $W \subseteq \mathbb{N}^{\leq\mathbb{N}}_*$ obtained in this way from non-empty open sets $V \subseteq \mathbb{N}^{\leq\mathbb{N}}_*$ is therefore as desired. □

Fix an injective enumeration $(s_n)_{n\in\mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ with the property that $s_m \sqsubseteq s_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, fix a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of strictly positive real numbers such that $0 = \lim_{n\to\infty} \epsilon_n$, and for each $n \in \mathbb{N}$, set $m(n+1) = \max\{m \leq n \mid s_m \sqsubseteq s_{n+1}\}$. Define $u_\emptyset = \emptyset$, and recursively appeal to Lemma 2.2 to obtain sequences $u_{s_{n+1}} \in \mathbb{N}^{<\mathbb{N}}$, for all $n \in \mathbb{N}$, with the property that $u_{s_{m(n+1)}} \frown s_{n+1}(|s_{m(n+1)}|) \sqsubseteq u_{s_{n+1}}$ and $f(\mathcal{N}^*_{u_{s_{n+1}}}) \subseteq \mathcal{B}_{d_Y}(y, \min\{\epsilon_n, d_Y(y, f(u_{s_n}))\})$.

Define $\pi_X\colon \mathbb{N}^{\leq\mathbb{N}}_* \to \mathbb{N}^{\leq\mathbb{N}}_*$ by

$$\pi_X(s) = \begin{cases} u_s & \text{if } s \in \mathbb{N}^{<\mathbb{N}}, \text{ and} \\ \bigcup_{n\in\mathbb{N}} u_{s\upharpoonright n} & \text{otherwise.} \end{cases}$$

Define $\pi_Y\colon f_0(\mathbb{N}^{\leq\mathbb{N}}_*) \to f(X)$ by $\pi_Y(0) = y$ and $\pi_Y(f_0(s)) = f(u_s)$, for all $s \in \mathbb{N}^{<\mathbb{N}}$. As both of these functions are continuous injections with compact domains, they are necessarily topological embeddings, thus (π_X, π_Y) is a topological embedding of f_0 into f. □

Proof of Theorem 1. Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see $\neg(1) \implies (2)$, suppose that there exists $y \in Y$ such that $f^{-1}(y)$ is not F_σ, and appeal to Theorem A.4 to obtain a topological embedding $\pi\colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ of $\mathbb{N}^\mathbb{N}$ into $f^{-1}(y)$. Proposition 2.1 then yields a topological embedding (π_X, π_Y) of f_0 into $f \circ \pi$, and it follows that $(\pi \circ \pi_X, \pi_Y)$ is a topological embedding of f_0 into f. □

3. Sigma-continuous functions

We begin with a technical but useful sufficient condition for the topological embeddability of f_1.

Proposition 3.1. *Suppose that X and Y are metric spaces, $f\colon X \to Y$, and there are a dense G_δ set $C \subseteq \mathbb{N}^\mathbb{N}$, a set $W \subseteq X$ intersecting the f-preimage of every singleton in a closed set, and a function $\phi\colon C \to W$, such that both ϕ and $f \circ \phi$ are continuous, which is a homomorphism from $\mathbb{G}^\mathbb{N}_{0,m} \restriction C$ to the $\aleph_0$-dimensional dihypergraph G_m consisting of all convergent sequences $(x_n)_{n\in\mathbb{N}} \in X^\mathbb{N}$ with $f(\lim_{n\to\infty} x_n) \neq \lim_{n\to\infty} f(x_n)$ but $\{f(x_n) \mid n \in \mathbb{N}\} \subseteq \mathcal{B}_{d_Y}(f(\lim_{n\to\infty} x_n), 1/m)$, for all $m \in \mathbb{N}$. Then there is a topological embedding of f_1 into f.*

Proof. Fix dense open sets $U_n \subseteq \mathbb{N}^\mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} U_n \subseteq C$. We will recursively construct sequences $(u_s)_{s\in\mathbb{N}^n}$ of elements of $\mathbb{N}^{<\mathbb{N}}$, sequences $(V_s)_{s\in\mathbb{N}^n}$ of open subsets of Y, and sequences $(x_s)_{s\in\mathbb{N}^n}$ of elements of X, for all $n \in \mathbb{N}$, such that:

1. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ u_s \sqsubset u_{s\frown(i)}$.
2. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \mathcal{N}_{u_{s\frown(i)}} \subseteq U_{|s|}$.
3. $\forall s \in \mathbb{N}^{<\mathbb{N}}\ (f \circ \phi)(\mathcal{N}_{u_s}) \cup \{f(x_s)\} \subseteq V_s$.
4. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ V_{s\frown(i)} \subseteq V_s$.
5. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \mathrm{diam}_{d_X}(\phi(\mathcal{N}_{u_{s\frown(i)}})) < 1/|s|$.
6. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \mathrm{diam}_{d_Y}(V_{s\frown(i)}) < 1/|s|$.
7. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \overline{\phi(\mathcal{N}_{u_{s\frown(i)}})} \subseteq \mathcal{B}_{d_X}(x_s, 1/i)$.
8. $\forall s \in \mathbb{N}^{<\mathbb{N}}\ f(x_s) \notin \overline{\bigcup_{i\in\mathbb{N}} V_{s\frown(i)}}$.
9. $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ V_{s\frown(i)} \cap \overline{\bigcup_{j\in\mathbb{N}\setminus\{i\}} V_{s\frown(j)}} = \emptyset$.

We begin by setting $u_\emptyset = \emptyset$ and $V_\emptyset = Y$. Suppose now that $n \in \mathbb{N}$ and we have already found $(u_s)_{s\in\mathbb{N}^{\le n}}$, $(V_s)_{s\in\mathbb{N}^{\le n}}$, and $(x_s)_{s\in\mathbb{N}^{<n}}$. For each $s \in \mathbb{N}^n$, fix $\delta_s > 0$ as well as $u'_s \in \mathbb{N}^{<\mathbb{N}}$ such that $u_s \sqsubseteq u'_s$, $\mathcal{N}_{u'_s} \subseteq U_n$, $\mathrm{diam}_{d_X}(\phi(\mathcal{N}_{u'_s})) < 1/n$, $\mathrm{diam}_{d_Y}((f \circ \phi)(\mathcal{N}_{u'_s})) < 3/2n$, and $\mathcal{B}_{d_Y}((f \circ \phi)(\mathcal{N}_{u'_s}), \delta_s) \subseteq V_s$. Fix a natural number $n_s \ge 1/\delta_s$ such that $u'_s \sqsubseteq s^\mathbb{N}_{n_s}$, appeal to the Baire category theorem to find $z_s \in \mathbb{N}^\mathbb{N}$ with the property that $s^\mathbb{N}_{n_s} \frown (i) \frown z_s \in C$ for all $i \in \mathbb{N}$, and define $x_{i,s} = \phi(s^\mathbb{N}_{n_s} \frown (i) \frown z_s)$ and $y_{i,s} = f(x_{i,s})$ for all $i \in \mathbb{N}$, as well as $x_s = \lim_{i\to\infty} x_{i,s}$. The fact that $f(x_s) \neq \lim_{i\to\infty} y_{i,s}$ ensures the existence of an infinite set $I_s \subseteq \mathbb{N}$ for which $f(x_s) \notin \overline{\{y_{i,s} \mid i \in I_s\}}$. Note that there can be no infinite set $J \subseteq I_s$ such that $(y_{j,s})_{j\in J}$ is constant, since otherwise the fact that $\phi(C) \subseteq W$ would imply that $f(x_s) = y_{j,s}$, for all $j \in J$. So by passing to an infinite subset of I_s, we can assume that $(y_{i,s})_{i\in I_s}$ is injective. By passing to a further infinite subset of I_s, we can ensure that $(y_{i,s})_{i\in I_s}$ has at most one limit point. By eliminating this limit point from the sequence if necessary, we can therefore ensure that $y_{i,s} \notin \overline{\{y_{j,s} \mid j \in I_s \setminus \{i\}\}}$, for all $i \in I_s$. Similarly, we can assume that $x_s \notin \{x_{i,s} \mid i \in I_s\}$. By passing one last time to an infinite subset of I_s, we can assume that $d_X(x_s, x_{i_{k,s},s}) < 1/k$ for all $k \in \mathbb{N}$, where $(i_{k,s})_{k\in\mathbb{N}}$ is the strictly increasing enumeration of I_s. For each $k \in \mathbb{N}$, fix $\epsilon^X_{k,s} > 0$ strictly less than $1/k - d_X(x_s, x_{i_{k,s},s})$, and fix $\epsilon^Y_{k,s} > 0$ strictly less than $d_Y(f(x_s), y_{i_{k,s},s})/2$ and $d_Y(y_{i,s}, y_{i_{k,s},s})/3$, for all $i \in I_s \setminus \{i_{k,s}\}$. Set $V_{s\frown(k)} = \mathcal{B}_{d_Y}(y_{i_{k,s},s}, \epsilon^Y_{k,s}) \cap V_s$, and fix an initial segment $u_{s\frown(k)}$ of $s^{\mathbb{N}}_{n_s} \frown (i_{k,s}) \frown z_s$ of length at least $n_s + 1$ with the property that $\phi(\mathcal{N}_{u_{s\frown(k)}}) \subseteq \mathcal{B}_{d_X}(x_{i_{k,s},s}, \epsilon^X_{k,s})$ and $(f \circ \phi)(\mathcal{N}_{u_{s\frown(k)}}) \subseteq V_{s\frown(k)}$. Our choice of u'_s ensures that conditions (1), (2), and (5) hold, and along with the fact that ϕ is a homomorphism, that condition (3) holds as well. Condition (4) holds trivially, and the remaining conditions follow from our upper bounds on $\epsilon^X_{k,s}$ and $\epsilon^Y_{k,s}$. This completes the recursive construction.

By condition (1), we obtain a continuous function $\psi \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by setting $\psi(s) = \bigcup_{n\in\mathbb{N}} u_{s\upharpoonright n}$. Condition (2) ensures that $\psi(\mathbb{N}^{\mathbb{N}}) \subseteq C$. Set $x_s = (\phi \circ \psi)(s)$ for $s \in \mathbb{N}^{\mathbb{N}}$, and define $\pi_X \colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ and $\pi_Y \colon \mathbb{N}^{\leq\mathbb{N}}_{**} \to Y$ by $\pi_X(s) = x_s$ and $\pi_Y = f \circ \pi_X$. We will show that (π_X, π_Y) is a topological embedding of f_1 into f.

Lemma 3.2. *Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then $\pi_X(\mathcal{N}^*_s) \subseteq \overline{\phi(\mathcal{N}_{u_s})}$.*

Proof. Simply observe that

$$\begin{aligned}\pi_X(\mathcal{N}^*_s) &= (\phi \circ \psi)(\mathcal{N}_s) \cup \{x_t \mid t \in \mathcal{N}^*_s \setminus \mathcal{N}_s\} \\ &\subseteq \phi(\mathcal{N}_{u_s}) \cup \textstyle\bigcup_{t\in\mathcal{N}^*_s\setminus\mathcal{N}_s} \overline{\phi(\mathcal{N}_{u_t})} \\ &\subseteq \overline{\phi(\mathcal{N}_{u_s})},\end{aligned}$$

by conditions (1) and (7). ☐

Lemma 3.3. *Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then $\pi_Y(\mathcal{N}^*_s) \subseteq V_s$.*

Proof. Simply observe that

$$\begin{aligned}\pi_Y(\mathcal{N}^*_s) &= (f \circ \phi \circ \psi)(\mathcal{N}_s) \cup \{f(x_t) \mid t \in \mathcal{N}^*_s \setminus \mathcal{N}_s\} \\ &\subseteq (f \circ \phi)(\mathcal{N}_{u_s}) \cup \{f(x_t) \mid t \in \mathcal{N}^*_s \setminus \mathcal{N}_s\} \\ &\subseteq V_s \cup \textstyle\bigcup_{t\in\mathcal{N}^*_s\setminus\mathcal{N}_s} V_t \\ &\subseteq V_s,\end{aligned}$$

by conditions (3) and (4). ☐

To see that π_X and π_Y are injective, it is enough to check that the latter is injective. Towards this end, suppose that $s, t \in \mathbb{N}^{\leq\mathbb{N}}$ are distinct. If there is a least $n \leq \min\{|s|,|t|\}$ with $s \upharpoonright n \neq t \upharpoonright n$, then condition (9) ensures that $V_{s \upharpoonright n}$ and $V_{t \upharpoonright n}$ are disjoint, and since Lemma 3.3 implies that $\pi_Y(s)$ is in the former and $\pi_Y(t)$ is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of s and t if necessary, we can assume that there exists $n < |t|$ for which $s = t \upharpoonright n$. But then condition (8) ensures that $\pi_Y(s) \notin V_{t \upharpoonright (n+1)}$, while Lemma 3.3 implies that $\pi_Y(t) \in V_{t \upharpoonright (n+1)}$, thus $\pi_Y(s) \neq \pi_Y(t)$.

To see that π_X is a topological embedding, it only remains to show that it is continuous (since $\mathbb{N}^{\leq \mathbb{N}}_*$ is compact). And for this, it is enough to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}^{\leq \mathbb{N}}_*$, there is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$. Towards this end, observe that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma 3.2 ensures that $\pi_X(\mathcal{N}^*_{s \upharpoonright (n+1)}) \subseteq \overline{\phi(\mathcal{N}_{u_{s \upharpoonright (n+1)}})}$, so condition (5) implies that $\mathcal{N}^*_{s \upharpoonright (n+1)}$ is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$. And if $s \in \mathbb{N}^{<\mathbb{N}}$, then Lemma 3.2 ensures that

$$\begin{aligned}\pi_X(\mathcal{N}^*_s \setminus \textstyle\bigcup_{i<n} \mathcal{N}^*_{s \frown (i)}) &= \pi_X(\{s\} \cup \textstyle\bigcup_{i \geq n} \mathcal{N}^*_{s \frown (i)}) \\ &\subseteq \{\pi_X(s)\} \cup \textstyle\bigcup_{i \geq n} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})},\end{aligned}$$

so condition (7) implies that $\mathcal{N}^*_s \setminus \bigcup_{i<n} \mathcal{N}^*_{s \frown (i)}$ is an open neighborhood of s whose image under π_X is a subset of $\mathcal{B}_{d_X}(\pi_X(s), 1/n)$.

To see that π_Y is continuous, it is sufficient to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}^{\leq \mathbb{N}}_{**}$, there is an open neighborhood of s whose image under π_Y is contained in $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$. Towards this end, observe that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma 3.3 ensures that $\pi_Y(\mathcal{N}^*_{s \upharpoonright (n+1)}) \subseteq V_{s \upharpoonright (n+1)}$, so condition (6) implies that $\mathcal{N}^*_{s \upharpoonright (n+1)}$ is an open neighborhood of s whose image under π_Y is contained in $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$. And if $s \in \mathbb{N}^{<\mathbb{N}}$, then $\{s\}$ is an open neighborhood of s whose image under π_Y is a subset of $\mathcal{B}_{d_Y}(\pi_Y(s), 1/n)$.

Before showing that π_Y is a topological embedding, we first establish several lemmas.

Lemma 3.4. *Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then $\pi_Y(\mathcal{N}^*_s) = \overline{V_s} \cap \pi_Y(\mathbb{N}^{\leq \mathbb{N}}_{**})$.*

Proof. Lemma 3.3 ensures that $\pi_Y(\mathcal{N}^*_s) \subseteq \overline{V_s} \cap \pi_Y(\mathbb{N}^{\leq \mathbb{N}}_{**})$, so it is enough to show that $\pi_Y(\mathbb{N}^{\leq \mathbb{N}}_{**} \setminus \mathcal{N}^*_s) \cap \overline{V_s} = \emptyset$. Towards this end, note that if $t \in \mathbb{N}^{\leq \mathbb{N}}_{**} \setminus \mathcal{N}^*_s$, then either there exists a least $n \leq \min\{|s|,|t|\}$ for which $s \upharpoonright n$ and $t \upharpoonright n$ are incompatible, or $t \sqsubset s$. In the former case, condition (9) implies that $\overline{V_{s \upharpoonright n}}$ and $V_{t \upharpoonright n}$ are disjoint, and since Lemma 3.3 implies that $\pi_Y(t)$ is in the latter, it is not in the former. But then it is also not in $\overline{V_s}$, by condition (4). In the latter case, set $n = |t|$, and appeal to condition (8) to see that $\pi_Y(t)$ is not in $\overline{V_{s \upharpoonright (n+1)}}$. But then it is also not in $\overline{V_s}$, by condition (4). ☐

Lemma 3.5. *Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then*

$$\pi_Y(\mathcal{N}_s^*) = \pi_Y(\mathbb{N}_{**}^{\leq\mathbb{N}}) \setminus (\overline{\bigcup_{t\perp s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\}).$$

Proof. To see that $\pi_Y(\mathcal{N}_s^*) \cap (\overline{\bigcup_{t\perp s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\}) = \emptyset$, note that if $t \perp s$, then there is a maximal $n < \min\{|s|, |t|\}$ with the property that $s \upharpoonright n = t \upharpoonright n$, in which case t is an extension of $(s \upharpoonright n) \frown (j)$, for some $j \in \mathbb{N} \setminus \{s(n)\}$. Condition (4) therefore ensures that

$$\overline{\bigcup_{t\perp s} V_t} = \bigcup_{n<|s|} \overline{\bigcup_{j\in\mathbb{N}\setminus\{s(n)\}} V_{(s\upharpoonright n)\frown(j)}}.$$

As Lemma 3.3 implies that $\pi_Y(\mathcal{N}_s^*) \subseteq V_s$, and condition (4) ensures that $V_s \subseteq V_{s\upharpoonright(n+1)}$ for all $n < |s|$, it follows from condition (9) that $\pi_Y(\mathcal{N}_s^*) \cap \overline{\bigcup_{t\perp s} V_t} = \emptyset$.

To see that $\pi_Y(\mathbb{N}_{**}^{\leq\mathbb{N}} \setminus \mathcal{N}_s^*) \subseteq \overline{\bigcup_{t\perp s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\}$, note that if $t \in \mathbb{N}_{**}^{\leq\mathbb{N}} \setminus \mathcal{N}_s^*$ and $t \not\sqsubset s$, then there exists $n \leq \min\{|s|, |t|\}$ such that $s \upharpoonright n$ and $t \upharpoonright n$ are incompatible, so Lemma 3.3 ensures that $\pi_Y(t) \in \overline{\bigcup_{t\perp s} V_t}$. □

Lemma 3.6. *Suppose that $s \in \mathbb{N}^{<\mathbb{N}}$. Then $\pi_Y(s)$ is the unique element of $\pi_Y(\mathbb{N}_{**}^{\leq\mathbb{N}}) \setminus (\overline{\bigcup_{t\not\sqsubseteq s} V_t} \cup \{\pi_Y(t) \mid t \sqsubset s\})$.*

Proof. As $\overline{\bigcup_{t\not\sqsubseteq s} V_t} = \overline{\bigcup_{t\perp s} V_t} \cup \overline{\bigcup_{i\in\mathbb{N}} V_{s\frown(i)}}$ by condition (4), Lemma 3.5 ensures that we need only show that $\pi_Y(s)$ is the unique element of $\pi_Y(\mathcal{N}_s^*) \setminus \overline{\bigcup_{i\in\mathbb{N}} V_{s\frown(i)}}$. Condition (8) ensures that $\pi_Y(s)$ is in this set, while Lemma 3.3 implies that the other points of $\pi_Y(\mathcal{N}_s^*)$ are not. □

It remains to show that $\pi_Y(\mathcal{N}_s^*)$ and $\{\pi_Y(s)\}$ are clopen in $\pi_Y(\mathbb{N}_{**}^{\leq\mathbb{N}})$, for all $s \in \mathbb{N}^{\mathbb{N}}$. The former is a consequence of Lemmas 3.4 and 3.5, while the latter follows from Lemma 3.6. □

Proposition 3.7. *Suppose that X and Y are Polish spaces, $f \colon X \to Y$, G is the $\aleph_0$-dimensional dihypergraph on X consisting of all convergent sequences $(x_n)_{n\in\mathbb{N}}$ such that $f(\lim_{n\to\infty} x_n) \neq \lim_{n\to\infty} f(x_n)$, and $W \subseteq X$ is G-independent. Then $\overline{W}$ is G-independent.*

Proof. Fix compatible metrics d_X and d_Y on X and Y, respectively. We must show that if $x = \lim_{n\to\infty} \overline{w}_n$ and each $\overline{w}_n$ is in $\overline{W}$, then $f(x) = \lim_{n\to\infty} f(\overline{w}_n)$. For each $n \in \mathbb{N}$, write $\overline{w}_n = \lim_{m\to\infty} w_{m,n}$, where each $w_{m,n}$ is in W. The fact that W is G-independent then ensures that $f(\overline{w}_n) = \lim_{m\to\infty} f(w_{m,n})$. Fix $m_n \in \mathbb{N}$ with the property that both $d_X(w_{m_n,n}, \overline{w}_n)$ and $d_Y(f(w_{m_n,n}), f(\overline{w}_n))$ are at most $1/n$. It follows that $x = \lim_{n\to\infty} w_{m_n,n}$, so one more appeal to the fact that W is G-independent yields that $f(x) = \lim_{n\to\infty} f(w_{m_n,n}) = \lim_{n\to\infty} f(\overline{w}_n)$. □

Proof of Theorem 2. Clearly, if (1) holds then $f^{-1}(C)$ is F_σ for every closed set $C \subseteq Y$. Hence conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let G denote the $\aleph_0$-dimensional dihypergraph on X consisting of all convergent sequences $(x_n)_{n\in\mathbb{N}}$ such that $f(\lim_{n\to\infty} x_n) \neq \lim_{n\to\infty} f(x_n)$.

As f is continuous on a closed set if and only if the set in question is G-independent, Proposition 3.7 ensures that if X is a union of countably-many G-independent sets, then f is σ-continuous with closed witnesses. We can therefore focus on the case that X is not a union of countably-many G-independent sets. While it is not difficult to see that condition (1) fails in this case, simply applying the dichotomy for $\aleph_0$-dimensional analytic dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) will not yield the sort of homomorphism we require. So instead, we will use our further assumptions to obtain a homomorphism with stronger properties.

Fix a compatible metric d_Y on Y, and for each $\epsilon > 0$, let G_ϵ denote the $\aleph_0$-dimensional dihypergraph on X consisting of all sequences $(x_n)_{n\in\mathbb{N}} \in G$ with $\{f(x_n) \mid n \in \mathbb{N}\} \subseteq \mathcal{B}_{d_Y}(f(\lim_{n\to\infty} x_n), \epsilon)$. Note that if $B \subseteq X$ is a G_ϵ-independent set and $C \subseteq X$ is a closed set whose f-image has d_Y-diameter strictly less than ϵ, then $B \cap C$ is G-independent. As Proposition 1.1 ensures that X is a union of countably-many closed sets whose f-images have d_Y-diameter strictly less than ϵ, it follows that every G_ϵ-independent set is a union of countably-many G-independent sets.

We say that a set $W \subseteq X$ is *eventually* $(G_\epsilon)_{\epsilon>0}$*-independent* if there exists $\epsilon > 0$ for which it is G_ϵ-independent. As X is not a union of countably-many G-independent sets, it follows that it is not a union of countably-many eventually $(G_\epsilon)_{\epsilon>0}$-independent sets. Again, however, simply applying the sequential $\aleph_0$-dimensional analog of the $\mathbb{G}_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) will not yield the sort of homomorphism we require, and we must once more appeal to our further assumptions.

Theorem A.6 ensures that X is a union of countably-many analytic sets whose intersection with the f-preimage of each singleton is closed. As X is not a union of countably-many eventually $(G_\epsilon)_{\epsilon>0}$-independent sets, it follows that there is an analytic set $A \subseteq X$, whose intersection with the f-preimage of each singleton is closed, that is not a union of countably-many eventually $(G_\epsilon)_{\epsilon>0}$-independent sets.

At long last, we now appeal to the sequential $\aleph_0$-dimensional analog of the $\mathbb{G}_0$ dichotomy theorem (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 21]) to obtain a dense G_δ set $C \subseteq \mathbb{N}^\mathbb{N}$ and a continuous function $\phi\colon C \to A$ which is a homomorphism from $\mathbb{G}_{0,n}^\mathbb{N} \upharpoonright C$ to $G_{1/n}$, for all $n \in \mathbb{N}$. In fact, by first replacing the given topology of X with a finer Polish topology consisting only of Borel sets but with respect to which f is continuous (see, for example, [Kec95, Theorem 13.11]), we can ensure that $f \circ \phi$ is continuous as well. An application of Proposition 3.1 therefore yields the desired topological embedding of f_1 into f. □

Appendix: F_σ sets

We begin the appendix with a straightforward observation.

Proposition A.1. (a) *The set* $\mathbb{N}^\mathbb{N}$ *is not an* F_σ *subspace of* $\mathbb{N}^{\leq\mathbb{N}}_*$.
(b) *The set* $\mathbb{N}^\mathbb{N}$ *is a closed subspace of* $\mathbb{N}^{\leq\mathbb{N}}_{**}$.

Proof. To see (a), note that a subset of a topological space is G_δ if it is an intersection of countably-many open sets. As $\mathbb{N}^{<\mathbb{N}}$ is countable and $\mathbb{N}^\mathbb{N}$ is dense in $\mathbb{N}^{\leq\mathbb{N}}_*$, it follows that $\mathbb{N}^\mathbb{N}$ is a dense G_δ subspace of $\mathbb{N}^{\leq\mathbb{N}}_*$. As $\mathbb{N}^{<\mathbb{N}}$ is also dense in $\mathbb{N}^{\leq\mathbb{N}}_*$, the Baire category theorem (see, for example, [Kec95, Theorem 8.4]) ensures that it is not a G_δ subspace of $\mathbb{N}^{\leq\mathbb{N}}_*$, thus $\mathbb{N}^\mathbb{N}$ is not an F_σ subspace of $\mathbb{N}^{\leq\mathbb{N}}_*$.

To see (b), note that $\{s\}$ is clopen in $\mathbb{N}^{\leq\mathbb{N}}_{**}$ for all $s \in \mathbb{N}^{<\mathbb{N}}$, so $\mathbb{N}^{<\mathbb{N}}$ is open in $\mathbb{N}^{\leq\mathbb{N}}_{**}$, thus $\mathbb{N}^\mathbb{N}$ is closed in $\mathbb{N}^{\leq\mathbb{N}}_{**}$. □

An $\aleph_0$-*dimensional dihypergraph* on a set X is a set of non-constant elements of $X^\mathbb{N}$. A *homomorphism* from an $\aleph_0$-dimensional dihypergraph G on X to an $\aleph_0$-dimensional dihypergraph H on Y is a function $\phi\colon X \to Y$ sending elements of G to elements of H.

Fix sequences $s_n^\mathbb{N} \in \mathbb{N}^n$ such that $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq s_n^\mathbb{N}$, and define $\aleph_0$-dimensional dihypergraphs on $\mathbb{N}^\mathbb{N}$ by setting

$$\mathbb{G}_{0,n}^\mathbb{N} = \{(s_n^\mathbb{N} \frown (i) \frown z)_{i\in\mathbb{N}} \mid z \in \mathbb{N}^\mathbb{N}\},$$

for all $n \in \mathbb{N}$, and $\mathbb{G}_0^\mathbb{N} = \bigcup_{n\in\mathbb{N}} \mathbb{G}_{0,n}^\mathbb{N}$.

We now establish a technical but useful sufficient condition for the topological embeddability of $\mathbb{N}^\mathbb{N}$.

Proposition A.2. *Suppose that* X *is a metric space,* $Y \subseteq X$ *is a set, and there are a dense* G_δ *set* $C \subseteq \mathbb{N}^\mathbb{N}$ *and a continuous homomorphism* $\phi\colon C \to Y$ *from* $\mathbb{G}_0^\mathbb{N} \upharpoonright C$ *to the* $\aleph_0$*-dimensional dihypergraph*

$$G = \{(y_n)_{n\in\mathbb{N}} \in Y^\mathbb{N} \mid \exists x \in X \setminus Y\ x = \lim_{n\to\infty} y_n\}.$$

Then there is a topological embedding $\pi\colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ *of* $\mathbb{N}^\mathbb{N}$ *into* Y.

Proof. Fix dense open sets $U_n \subseteq \mathbb{N}^\mathbb{N}$ such that $\bigcap_{n\in\mathbb{N}} U_n \subseteq C$. We will recursively construct sequences $(u_s)_{s\in\mathbb{N}^n}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ and sequences $(x_s)_{s\in\mathbb{N}^n}$ of elements of X, for all $n \in \mathbb{N}$, such that:

(1) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ u_s \sqsubset u_{s\frown(i)}$.
(2) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \mathcal{N}_{u_{s\frown(i)}} \subseteq U_{|s|}$.
(3) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \mathrm{diam}_{d_X}(\phi(\mathcal{N}_{u_{s\frown(i)}})) < 1/|s|$.
(4) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ \overline{\phi(\mathcal{N}_{u_{s\frown(i)}})} \subseteq \mathcal{B}_{d_X}(x_s, 1/i)$.
(5) $\forall i \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ x_s \notin \overline{\phi(\mathcal{N}_{u_{s\frown(i)}})}$.
(6) $\forall i, j \in \mathbb{N} \forall s \in \mathbb{N}^{<\mathbb{N}}\ (i \neq j \implies \overline{\phi(\mathcal{N}_{u_{s\frown(i)}})} \cap \overline{\phi(\mathcal{N}_{u_{s\frown(j)}})} = \emptyset)$.

We begin by setting $u_\emptyset = \emptyset$. Suppose now that $n \in \mathbb{N}$ and we have already found $(u_s)_{s\in\mathbb{N}^{\leq n}}$ and $(x_s)_{s\in\mathbb{N}^{<n}}$. For each $s \in \mathbb{N}^n$, fix $u'_s \in \mathbb{N}^{<\mathbb{N}}$ such that $u_s \sqsubseteq u'_s$, $\mathcal{N}_{u'_s} \subseteq U_n$, and $\mathrm{diam}_{d_X}(\phi(\mathcal{N}_{u'_s})) < 1/n$, fix $n_s \in \mathbb{N}$ for which $u'_s \sqsubseteq s^\mathbb{N}_{n_s}$, and appeal to the Baire category theorem to find $z_s \in \mathbb{N}^\mathbb{N}$ with the property that $s^\mathbb{N}_{n_s} \frown (i) \frown z_s \in C$, for all $i \in \mathbb{N}$. Set $y_{i,s} = \phi(s^\mathbb{N}_{n_s} \frown (i) \frown z_s)$ for all $i \in \mathbb{N}$, as well as $x_s = \lim_{n\to\infty} y_{i,s}$. As $x_s \notin \{y_{i,s} \mid i \in \mathbb{N}\}$, there is an infinite set $I_s \subseteq \mathbb{N}$ for which $(y_{i,s})_{i\in I_s}$ is injective. By passing to an infinite subset of I_s, we can assume that $d_X(x_s, y_{i_{k,s},s}) < 1/k$ for all $k \in \mathbb{N}$, where $(i_{k,s})_{k\in\mathbb{N}}$ is the strictly increasing enumeration of I_s. For each $k \in \mathbb{N}$, fix $\epsilon_{k,s} > 0$ strictly less than $1/k - d_X(x_s, y_{i_{k,s},s})$, $d_X(x_s, y_{i_{k,s},s})$, and $d_X(y_{i,s}, y_{i_{k,s},s})/2$ for all $i \in I_s \setminus \{i_{k,s}\}$, and fix an initial segment $u_{s\frown(k)}$ of $s^\mathbb{N}_{n_s} \frown (i_{k,s}) \frown z_s$ of length at least $n_s + 1$ with the property that $\phi(\mathcal{N}_{u_{s\frown(k)}}) \subseteq \mathcal{B}_{d_X}(y_{i_{k,s},s}, \epsilon_{k,s})$. Our choice of u'_s ensures that conditions (1) – (3) hold, and our strict upper bounds on $\epsilon_{k,s}$ yield the remaining conditions. This completes the recursive construction.

Condition (1) ensures that we obtain a function $\psi\colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by setting $\psi(s) = \bigcup_{n\in\mathbb{N}} u_{s\restriction n}$, and condition (2) implies that $\psi(\mathbb{N}^\mathbb{N}) \subseteq C$. Set $x_s = (\phi \circ \psi)(s)$ for $s \in \mathbb{N}^\mathbb{N}$, and define $\pi\colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ by $\pi(s) = x_s$. We will show that π is a topological embedding of $\mathbb{N}^\mathbb{N}$ into Y.

Lemma A.3. *Suppose that* $s \in \mathbb{N}^{<\mathbb{N}}$. *Then* $\pi(\mathcal{N}^*_s) \subseteq \overline{\phi(\mathcal{N}_{u_s})}$.

Proof. Simply observe that

$$\begin{aligned}\pi(\mathcal{N}^*_s) &= (\phi \circ \psi)(\mathcal{N}_s) \cup \{x_t \mid t \in \mathcal{N}^*_s \setminus \mathcal{N}_s\} \\ &\subseteq \phi(\mathcal{N}_{u_s}) \cup \textstyle\bigcup_{t\in\mathcal{N}^*_s\setminus\mathcal{N}_s} \overline{\phi(\mathcal{N}_{u_t})} \\ &\subseteq \overline{\phi(\mathcal{N}_{u_s})},\end{aligned}$$

by conditions (1) and (4). □

To see that π is injective, suppose that $s, t \in \mathbb{N}^{\leq \mathbb{N}}$ are distinct. If there is a least $n \leq \min\{|s|, |t|\}$ with $s \upharpoonright n \neq t \upharpoonright n$, then condition (6) ensures that $\overline{\phi(\mathcal{N}_{u_{s \upharpoonright n}})}$ and $\overline{\phi(\mathcal{N}_{u_{t \upharpoonright n}})}$ are disjoint, and since Lemma A.3 implies that $\pi(s)$ is in the former and $\pi(t)$ is in the latter, it follows that they are distinct. Otherwise, after reversing the roles of s and t if necessary, we can assume that there exists $n < |t|$ for which $s = t \upharpoonright n$. But then condition (5) ensures that $\pi(s) \notin \overline{\phi(\mathcal{N}_{u_{t \upharpoonright (n+1)}})}$, while Lemma A.3 implies that $\pi(t) \in \overline{\phi(\mathcal{N}_{u_{t \upharpoonright (n+1)}})}$, thus $\pi(s) \neq \pi(t)$.

As $\mathbb{N}^{\leq \mathbb{N}}_*$ is compact, it only remains to check that π is continuous. And for this, it is enough to check that for all $n \in \mathbb{N}$ and $s \in \mathbb{N}^{\leq \mathbb{N}}_*$, there is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. Towards this end, note first that if $s \in \mathbb{N}^{\mathbb{N}}$, then Lemma A.3 ensures that $\pi(\mathcal{N}^*_{s \upharpoonright (n+1)}) \subseteq \overline{\phi(\mathcal{N}_{u_{s \upharpoonright (n+1)}})}$, so condition (3) implies that $\mathcal{N}^*_{s \upharpoonright (n+1)}$ is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. On the other hand, if $s \in \mathbb{N}^{<\mathbb{N}}$, then Lemma A.3 ensures that

$$\begin{aligned}\pi(\mathcal{N}^*_s \setminus \textstyle\bigcup_{i<n} \mathcal{N}^*_{s \frown (i)}) &= \pi(\{s\} \cup \textstyle\bigcup_{i \geq n} \mathcal{N}^*_{s \frown (i)}) \\ &\subseteq \{\pi(s)\} \cup \textstyle\bigcup_{i \geq n} \overline{\phi(\mathcal{N}_{u_{s \frown (i)}})},\end{aligned}$$

so condition (4) implies that $\mathcal{N}^*_s \setminus \bigcup_{i<n} \mathcal{N}^*_{s \frown (i)}$ is an open neighborhood of s whose image under π is a subset of $\mathcal{B}_{d_X}(\pi(s), 1/n)$. □

As a corollary, we obtain the following dichotomy theorem characterizing the family of Borel sets which are F_σ.

Theorem A.4 (Hurewicz)**.** *Suppose that X is a Polish space and $B \subseteq X$ is Borel. Then exactly one of the following holds:*

(1) *The set B is F_σ.*
(2) *There is a topological embedding $\pi \colon \mathbb{N}^{\leq \mathbb{N}}_* \to X$ of $\mathbb{N}^{\mathbb{N}}$ into B.*

Proof. Proposition A.1 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, let G denote the $\aleph_0$-dimensional dihypergraph consisting of all sequences $(y_n)_{n \in \mathbb{N}}$ of points of B converging to a point of $X \setminus B$. We say that a set $W \subseteq X$ is *G-independent* if $G \upharpoonright W = \emptyset$. Note that the closure of every such subset of B is contained in B. In particular, it follows that if B is a union of countably-many G-independent sets, then it is F_σ. Otherwise, Lecomte's dichotomy theorem for $\aleph_0$-dimensional dihypergraphs of uncountable chromatic number (see [Lec09, Theorem 1.6] or [Mil11, Theorem 18]) yields a dense G_δ set $C \subseteq \mathbb{N}^{\mathbb{N}}$ for which there is a continuous homomorphism $\phi \colon C \to B$ from $\mathbb{G}^{\mathbb{N}}_0 \upharpoonright C$ to G, in which case Proposition A.2 yields a topological embedding $\pi\colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ of $\mathbb{N}^\mathbb{N}$ into B. □

There is also a parametrized form of this theorem.

Theorem A.5 (Saint Raymond)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set with F_σ horizontal sections. Then R is a union of countably-many analytic subsets with closed horizontal sections.*

Proof. The parametrized form of our earlier dihypergraph is given by

$$G = \{((x_n)_{n\in\mathbb{N}}, y) \in (R^y)^\mathbb{N} \times Y \mid \exists x \in X \setminus R^y\ x = \lim_{n\to\infty} x_n\}.$$

We say that a set $S \subseteq X \times Y$ is *G-independent* if S^y is G^y-independent, for all $y \in Y$. Note that the closure of every horizontal section of every such subset of R is contained in the corresponding horizontal section of R. Moreover, if $S \subseteq R$ is analytic, then so too is the set $\{(x, y) \in X \times Y \mid x \in \overline{S^y}\}$. In particular, it follows that if R is a union of countably-many G-independent analytic subsets, then it is a union of countably-many analytic subsets with closed horizontal sections. Otherwise, the parametrized form of the dichotomy theorem for $\aleph_0$-dimensional dihypergraphs of uncountable Borel chromatic number (i.e., the straightforward common generalization of [Mil12, Theorems 18 and 31]) yields a dense G_δ set $C \subseteq \mathbb{N}^\mathbb{N}$ and $y \in Y$ for which there is a continuous homomorphism $\phi\colon C \to R^y$ from $\mathbb{G}_0^\mathbb{N} \restriction C$ to G^y, in which case Proposition A.2 yields a continuous embedding $\pi\colon \mathbb{N}^{\leq\mathbb{N}}_* \to X$ of $\mathbb{N}^\mathbb{N}$ into R^y. As the latter is F_σ, this contradicts Proposition A.1. □

Our use of this result is via the following corollary.

Theorem A.6. *Suppose that X and Y are Polish spaces and $f\colon X \to Y$ is an F_σ-to-one Borel function. Then f is strongly σ-closed-to-one.*

Proof. As the set $R = \text{graph}(f)$ is Borel (see, for example, [Kec95, Proposition 12.4]) and has F_σ horizontal sections, an application of Theorem A.5 ensures that it is a union of countably-many analytic sets with closed horizontal sections. As the projections of these sets onto X intersect the preimage of each singleton in a closed set, it follows that f is strongly σ-closed-to-one. □

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