SCRAMBLED CANTOR SETS

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ABSTRACT. We show that Li–Yorke chaos ensures the existence of a scrambled Cantor set.

INTRODUCTION

A *dynamical system* is a pair (X, f), where X is a metric space and $f: X \to X$ is a continuous function. Given such a system, we say that points $x, y \in X$ are *proximal* if $\liminf_{n\to\infty} d_X(f^n(x), f^n(y)) = 0$, and *asymptotic* if $\limsup_{n\to\infty} d_X(f^n(x), f^n(y)) = 0$. The pair (x, y) is *Li–Yorke* if x and y are proximal but not asymptotic, a set $Y \subseteq X$ is *scrambled* if (x, y) is Li–Yorke for all distinct $x, y \in Y$, and the system (X, f) is *Li–Yorke chaotic* if there is an uncountable scrambled set $Y \subseteq X$. In [LY75], Li and Yorke showed that every dynamical system on the unit interval with a point of period three is Li–Yorke chaotic.

The scrambled set constructed in [LY75] is indexed by an interval on the real line, and therefore has cardinality $2^{\aleph_0}$. Moreover, subsequent constructions of uncountable scrambled sets in the literature typically gave rise to sets of cardinality $2^{\aleph_0}$, or even *Cantor sets*, i.e., homeomorphic copies of the Cantor space $2^{\mathbb{N}}$. One example is the construction, in [BGKM02], of uncountable scrambled sets in dynamical systems of positive topological entropy.

A metric space is *Polish* if it is complete and separable, and a dynamical system (X, f) is *Polish* if X is Polish. At the end of [BHS08, §3], Blanchard, Huang, and Snoha asked whether every Li–Yorke chaotic Polish dynamical system admits a scrambled Cantor set. Here we utilize the descriptive set theory of definable graphs to obtain the following answer to their question:

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Theorem 1. *Suppose that (X, f) is a Li–Yorke chaotic Polish dynamical system. Then there is a scrambled Cantor set $C \subseteq X$.*

In §1, we establish an analog of the Kechris–Solecki–Todorcevic characterization of the existence of $\aleph_0$-colorings (see [KST99, Theorem 6.3]) within cliques. In §2, we use this to establish a similar analog of Silver's perfect set theorem for co-analytic equivalence relations (see [Sil80]). In §3, we use the latter to establish Theorem 1. And in §4, we discuss several generalizations.

1. Colorings in cliques

Endow $\mathbb{N}$ with the discrete topology, and $\mathbb{N}^\mathbb{N}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$, and *Polish* if it is second countable and admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets, and *co-analytic* if its complement is analytic. Every non-empty Polish space is a continuous image of $\mathbb{N}^\mathbb{N}$ (see, for example, [Kec95, Theorem 7.9]), thus so too is every non-empty analytic space, and a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, the proof of [Kec95, Theorem 14.11]).

A *digraph* on a set X is an irreflexive binary relation G on X. A set $Y \subseteq X$ is *G-independent* if $G \restriction Y = \emptyset$. An *$I$-coloring* of G is a function $c \colon X \to I$ such that $c(x) \neq c(y)$ for all $(x, y) \in G$, or equivalently, such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$. A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\phi \colon X \to Y$ for which $(\phi \times \phi)(R) \subseteq S$. We say that a set $Y \subseteq X$ is an *R-clique* if $x \mathrel{R} y$ for all distinct $x, y \in Y$.

We use $X^{<\mathbb{N}}$ to denote $\bigcup_{n \in \mathbb{N}} X^n$, (i) to denote the singleton sequence with value i, and $\sqsubseteq$ to denote extension. Following standard practice, we also use $\mathcal{N}_s$ to denote $\{b \in \mathbb{N}^\mathbb{N} \mid s \sqsubseteq b\}$ or $\{c \in 2^\mathbb{N} \mid s \sqsubseteq c\}$ (with the context determining which of the two we have in mind). Fix sequences $\mathbb{s}_n \in 2^n$ such that $\forall s \in 2^{\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq \mathbb{s}_n$, and let $\mathbb{G}_0$ denote the digraph on $2^\mathbb{N}$ given by $\mathbb{G}_0 = \{(\mathbb{s}_n \frown (i) \frown c)_{i<2} \mid c \in 2^\mathbb{N} \text{ and } n \in \mathbb{N}\}$. A subset of a topological space is G_δ if it is a countable intersection of open sets.

Theorem 1.1. *Suppose that X is a Hausdorff space, G is an analytic digraph on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

1. *For every R-clique $Y \subseteq X$, there is an $\mathbb{N}$-coloring of $G \restriction Y$.*
2. *There is a continuous homomorphism $\phi \colon 2^\mathbb{N} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^\mathbb{N})$ is an R-clique.*

Proof. Suppose that there is an R-clique $Y \subseteq X$ for which there is no $\mathbb{N}$-coloring of $G \restriction Y$. Then $G \neq \emptyset$, so there are continuous surjections $\phi_G \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow G$ and $\phi_X \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow \bigcup_{i<2} \mathrm{proj}_i(G)$. Fix a decreasing sequence $(R_n)_{n\in\mathbb{N}}$ of open subsets of $X \times X$ whose intersection is R.

We will define a decreasing sequence $(Y^\alpha)_{\alpha<\omega_1}$ of subsets of Y, off of which there are $\mathbb{N}$-colorings of $G \restriction Y$. We begin by setting $Y^0 = Y$. For all limit ordinals $\lambda < \omega_1$, we set $Y^\lambda = \bigcap_{\alpha<\lambda} Y^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{<\mathbb{N}}$, $\psi^a_n \colon 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for all $n < n^a$, and $\phi_X(\mathcal{N}_{\phi^a(s)}) \times \phi_X(\mathcal{N}_{\phi^a(t)}) \subseteq R_{n^a}$ for all distinct $s, t \in 2^{n^a}$. A *one-step extension* of an approximation a is an approximation b such that:

(a) $n^b = n^a + 1$.
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}\ (s \sqsubset t \implies \phi^a(s) \sqsubset \phi^b(t))$.
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}\ (s \sqsubset t \implies \psi^a_n(s) \sqsubset \psi^b_n(t))$.

Similarly, a *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^\mathbb{N}$, $\psi^\gamma_n \colon 2^{n^\gamma-(n+1)} \to \mathbb{N}^\mathbb{N}$ for all $n < n^\gamma$, and $(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (i) \frown t))_{i<2}$ for all $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. We say that γ is *compatible* with a set $Y' \subseteq Y$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq Y'$, and *compatible* with a if:

(i) $n^a = n^\gamma$.
(ii) $\forall t \in 2^{n^a}\ \phi^a(t) \sqsubseteq \phi^\gamma(t)$.
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}\ \psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

An approximation a is *Y'-terminal* if no configuration is compatible with both Y' and a one-step extension of a. Let $A(a, Y')$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(\mathbb{s}_{n^a})$, where γ varies over all configurations compatible with a and Y'.

Lemma 1.2. *Suppose that $Y' \subseteq Y$ and a is a Y'-terminal approximation. Then $A(a, Y')$ is G-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and Y', with the property that $((\phi_X \circ \phi^{\gamma_i})(\mathbb{s}_{n^a}))_{i<2} \in G$. Fix a sequence $d \in \mathbb{N}^\mathbb{N}$ with the property that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_i})(\mathbb{s}_{n^a}))_{i<2}$, and let γ be the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi^\gamma_n(t \frown (i)) = \psi^{\gamma_i}_n(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi^\gamma_{n^a}(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is Y'-terminal. ⊠

Let $Y^{\alpha+1}$ be the difference of Y^α and the union of the sets of the form $A(a, Y^\alpha)$, where a varies over all Y^α-terminal approximations.

Lemma 1.3. *Suppose that $\alpha < \omega_1$ and a is a non-$Y^{\alpha+1}$-terminal approximation. Then a has a non-Y^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $Y^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(\mathbb{s}_{n^b}) \in Y^{\alpha+1}$, so b is not Y^α-terminal. ⊠

Fix $\alpha < \omega_1$ such that the families of Y^α- and $Y^{\alpha+1}$-terminal approximations coincide, and note that the triple $a_0 = (n^{a_0}, \phi^{a_0}, (\psi_n^{a_0})_{n<n^{a_0}})$, where $n^{a_0} = 0$ and $\phi^{a_0} \colon 2^0 \to \mathbb{N}^0$, is an approximation (as the fact that 2^0 has only one element ensures that a_0 vacuously satisfies the final clause in the definition of approximation). As $A(a_0, Y') = Y' \cap \bigcup_{i<2} \mathrm{proj}_i(G)$ for all $Y' \subseteq Y$, we can assume that a_0 is not Y^α-terminal, since otherwise there is an $\mathbb{N}$-coloring of $G \restriction Y$.

By recursively applying Lemma 1.3, we obtain non-Y^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi', \psi_n \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi'(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \restriction n)$ and $\psi_n(c) = \bigcup_{m>n} \psi_n^{a_m}(c \restriction (m-(n+1)))$ for all $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$. Clearly these functions are continuous.

To see that the function $\phi = \phi_X \circ \phi'$ is a homomorphism from $\mathbb{G}_0$ to G, we will show the stronger fact that if $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, then $(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi')(\mathbb{s}_n \frown (i) \frown c))_{i<2}$. It is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi')(\mathbb{s}_n \frown (i) \frown c))_{i<2}$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\prod_{i<2} \phi_X(\mathcal{N}_{\phi^{a_m}(\mathbb{s}_n \frown (i) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$, where $s = c \restriction (m-(n+1))$. As a_m is not Y^α-terminal, there is a configuration γ compatible with a_m, so $((\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (i) \frown s))_{i<2} \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, thus $U \cap V \neq \emptyset$.

To see that $\phi(2^\mathbb{N})$ is an R-clique, observe that if $c, d \in 2^\mathbb{N}$ are distinct and $n \in \mathbb{N}$ is sufficiently large that $c \restriction n \neq d \restriction n$, then $\phi(c) \in \phi_X(\mathcal{N}_{\phi^{a_n}(c \restriction n)})$ and $\phi(d) \in \phi_X(\mathcal{N}_{\phi^{a_n}(d \restriction n)})$, so $\phi(c) \mathrel{R_n} \phi(d)$. ⊠

2. Separability in cliques

The following well-known fact rules out the existence of a Baire-measurable $\mathbb{N}$-coloring of $\mathbb{G}_0$:

Proposition 2.1. *Suppose that $B \subseteq 2^\mathbb{N}$ is a non-meager set with the Baire property. Then B is not $\mathbb{G}_0$-independent.*

Proof. Fix a sequence $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s$ (see, for example, [Kec95, Proposition 8.26]). Then there exists $n \in \mathbb{N}$ for which $s \sqsubseteq \mathbb{s}_n$. Let ι be the involution of $\mathcal{N}_{\mathbb{s}_n}$ sending $\mathbb{s}_n \frown (0) \frown c$ to $\mathbb{s}_n \frown (1) \frown c$ for all $c \in 2^\mathbb{N}$. As ι is a homeomorphism, it follows that $B \cap \iota(B)$ is comeager in $\mathcal{N}_{\mathbb{s}_n}$ (see, for example, [Kec95, Exercise 8.45]), so

$B \cap \iota(B) \cap \mathcal{N}_{\mathfrak{s}_n \frown (0)} \neq \emptyset$. As $(c, \iota(c)) \in \mathbb{G}_0 \upharpoonright B$ for all $c \in B \cap \iota(B) \cap \mathcal{N}_{\mathfrak{s}_n \frown (0)}$, it follows that B is not $\mathbb{G}_0$-independent. ⊠

The following corollary is also well known:

Proposition 2.2. *Suppose that E is a non-meager equivalence relation on $2^{\mathbb{N}}$ with the Baire property. Then E is not disjoint from $\mathbb{G}_0$.*

Proof. By the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]), there is a sequence $c \in 2^{\mathbb{N}}$ for which $[c]_E$ has the Baire property and is not meager, so Proposition 2.1 ensures that $[c]_E$ is not $\mathbb{G}_0$-independent. ⊠

A *partial transversal* of an equivalence relation E on a set X is a set $Y \subseteq X$ that does not contain distinct E-related points.

Theorem 2.3. *Suppose that X is a Hausdorff space, E is a co-analytic equivalence relation on X, and R is a reflexive G_δ binary relation on X for which there is an R-clique $Y \subseteq X$ intersecting uncountably-many E-classes. Then there is a Cantor set $C \subseteq X$ that is both a partial transversal of E and an R-clique.*

Proof. Set $G = {\sim}E$, appeal to Theorem 1.1 to obtain a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique, and let F be the pullback of E through ϕ. As $\mathbb{G}_0 \cap F = \emptyset$, Proposition 2.2 implies that F is meager, thus Mycielski's Theorem (see, for example, [Kec95, Theorem 19.1]) yields a continuous injection $\psi \colon 2^{\mathbb{N}} \hookrightarrow 2^{\mathbb{N}}$ sending distinct elements of $2^{\mathbb{N}}$ to F-inequivalent elements of $2^{\mathbb{N}}$, in which case the set $C = (\phi \circ \psi)(2^{\mathbb{N}})$ is as desired. ⊠

3. Li-Yorke chaos

We say that a dynamical system (X, f) is *analytic* if X is analytic. As every Polish dynamical system is analytic, Theorem 1 is a consequence of the following result:

Theorem 3.1. *Suppose that (X, f) is a Li–Yorke chaotic analytic dynamical system. Then there is a scrambled Cantor set $C \subseteq X$.*

Proof. Note first that the set

$$\begin{aligned} R &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are proximal}\} \\ &= \textstyle\bigcap_{\epsilon > 0} \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{(x, y) \in X \times X \mid d_X(f^m(x), f^m(y)) < \epsilon\} \end{aligned}$$

is G_δ, and the equivalence relation

$$\begin{aligned} E &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are asymptotic}\} \\ &= \textstyle\bigcap_{\epsilon > 0} \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \{(x, y) \in X \times X \mid d_X(f^m(x), f^m(y)) \leq \epsilon\} \end{aligned}$$

is Borel. As the fact that (X, f) is Li–Yorke chaotic yields an R-clique intersecting uncountably-many E-classes, Theorem 2.3 yields a scrambled Cantor set. ⊠

4. Further generalizations

If $\phi \colon S \to \mathbb{R}$ is a function on a metric space, then $\liminf_{d_S(s,s_0)\to\infty} \phi(s)$ and $\limsup_{d_S(s,s_0)\to\infty} \phi(s)$ do not depend on the choice of $s_0 \in S$, so we can safely denote them by $\liminf_{||s||\to\infty} \phi(s)$ and $\limsup_{||s||\to\infty} \phi(s)$. A *metric semigroup* is a semigroup S equipped with a metric d_S (we do not require that multiplication is continuous). Given an action $S \curvearrowright X$ of a metric semigroup on a metric space, we say that points $x, y \in X$ are *proximal* if $\liminf_{||s||\to\infty} d_X(s \cdot x, s \cdot y) = 0$, and *asymptotic* if $\limsup_{||s||\to\infty} d_X(s \cdot x, s \cdot y) = 0$. The pair (x, y) is *Li–Yorke* if x and y are proximal but not asymptotic, a set $Y \subseteq X$ is *scrambled* if (x, y) is Li–Yorke for all distinct points $x, y \in Y$, and the action $S \curvearrowright X$ is *Li–Yorke chaotic* if there is an uncountable scrambled set $Y \subseteq X$.

Generalizing Theorem 3.1, we have the following:

Theorem 4.1. *Suppose that $S \curvearrowright X$ is a Li–Yorke-chaotic action of a metric semigroup by continuous functions on an analytic metric space. Then there is a scrambled Cantor set $C \subseteq X$.*

Proof. Fix $s_0 \in S$, and note that the set

$$\begin{aligned} R &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are proximal}\} \\ &= \textstyle\bigcap_{\epsilon>0} \bigcap_{n\in\mathbb{N}} \bigcup_{d_S(s,s_0)\geq n} \{(x, y) \in X \times X \mid d_X(s \cdot x, s \cdot y) < \epsilon\} \end{aligned}$$

is G_δ, and the equivalence relation

$$\begin{aligned} E &= \{(x, y) \in X \times X \mid x \text{ and } y \text{ are asymptotic}\} \\ &= \textstyle\bigcap_{\epsilon>0} \bigcup_{n\in\mathbb{N}} \bigcap_{d_S(s,s_0)\geq n} \{(x, y) \in X \times X \mid d_X(s \cdot x, s \cdot y) \leq \epsilon\} \end{aligned}$$

is Borel. As the fact that $S \curvearrowright X$ is Li–Yorke chaotic yields an R-clique intersecting uncountably-many E-classes, Theorem 2.3 yields a scrambled Cantor set. ⊠

Given any real number $\delta \geq 0$, we say that points $x, y \in X$ are $(\leq\delta)$-*proximal* if $\liminf_{n\to\infty} d_X(f^n(x), f^n(y)) \leq \delta$. The proof of Theorem 3.1 easily generalizes to yield the analogous result for the version of Li–Yorke chaos where proximality is replaced with $(\leq\delta)$-proximality.

Along similar lines, we say that points $x, y \in X$ are $(\leq\delta)$-*asymptotic* if $\limsup_{n\to\infty} d_X(f^n(x), f^n(y)) \leq \delta$. We next describe the additional results necessary to apply our arguments to the analog of Li–Yorke chaos where asymptoticity is replaced with $(\leq\delta)$-asymptoticity.

Given a set $R \subseteq X \times Y$, define $R^{-1} = \{(y,x) \in Y \times X \mid x \mathrel{R} y\}$. Given a set $S \subseteq Y \times Z$, define $RS = \{(x,z) \in X \times Z \mid \exists y \in Y\ x \mathrel{R} y \mathrel{S} z\}$. Slightly generalizing Proposition 2.2, we have the following:

Proposition 4.2. *Suppose that R is a binary relation on $2^{\mathbb{N}}$ with the Baire property and $\mathbb{G}_0 \cap R^{-1}R = \emptyset$. Then R is meager.*

Proof. Suppose, towards a contradiction, that R is not meager. The Kuratowski-Ulam theorem then yields a sequence $c \in 2^{\mathbb{N}}$ for which the set $R_c = \{d \in 2^{\mathbb{N}} \mid c \mathrel{R} d\}$ has the Baire property and is not meager. As $R_c \times R_c \subseteq R^{-1}R$, it follows that R_c is $\mathbb{G}_0$-independent, contradicting Proposition 2.1. ⊠

An *extended-valued quasi-metric* on X is a map $d \colon X \times X \to [0,\infty]$ such that $d(x,x) = 0$ for all $x \in X$, $d(x,y) = d(y,x)$ for all $x, y \in X$, and $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$. Given $\delta > 0$, we say that (X,d) is *δ-discrete* if $d(x,y) > \delta$ for all distinct $x, y \in X$. We say that a set $Y \subseteq X$ is *$\aleph_0$-universally Baire* if its pre-image under every continuous function $\phi \colon 2^{\mathbb{N}} \to X$ has the Baire property.

Generalizing Theorem 2.3, we have the following:

Theorem 4.3. *Suppose that $\delta \geq 0$, $\epsilon \geq 2\delta$, X is a Hausdorff space, d is an extended-valued quasi-metric on X for which $d^{-1}([0,\delta])$ is an $\aleph_0$-universally-Baire binary relation on X and $d^{-1}([0,\epsilon])$ is a co-analytic binary relation on X, and R is a reflexive G_δ binary relation on X. Then at least one of the following holds:*

1. *Every R-clique $Y \subseteq X$ is a countable union of sets of d-diameter at most ϵ.*
2. *There is a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ for which $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), d \restriction \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. Suppose that condition (1) fails, fix an R-clique $Y \subseteq X$ for which there is no cover of Y by countably-many sets of d-diameter at most ϵ, set $G = d^{-1}((\epsilon,\infty])$, note that Theorem 1.1 yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G for which $\phi(2^{\mathbb{N}})$ is an R-clique, and define $G' = (d \circ (\phi \times \phi))^{-1}([0,\delta])$. As $\epsilon \geq 2\delta$, the triangle inequality ensures that $\mathbb{G}_0 \cap (G')^{-1}G' = \emptyset$, so Proposition 4.2 implies that G' is meager, thus Mycielski's Theorem yields a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from the *complete graph* $\mathbb{K}_{2^{\mathbb{N}}} = \{c \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \mid c(0) \neq c(1)\}$ on $2^{\mathbb{N}}$ to $\sim G'$. Then the function $\pi = \phi \circ \psi$ is as desired. ⊠

While not necessary for our results on Li–Yorke chaos, the following additional generalization of Theorem 2.3 is perhaps worth noting:

Theorem 4.4. *Suppose that X is a Hausdorff space, d is an extended-valued quasi-metric on X for which there are arbitrarily small $\epsilon > 0$ such that $d^{-1}([0, \epsilon])$ is a co-analytic binary relation on X, and R is a reflexive G_δ binary relation on X. Then exactly one of the following holds:*

1. *For every R-clique $Y \subseteq X$, the space $(Y, d \restriction Y)$ is second countable.*
2. *There exists $\delta > 0$ for which there is a continuous injection $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ with the property that $\pi(2^{\mathbb{N}})$ is an R-clique and $(\pi(2^{\mathbb{N}}), d \restriction \pi(2^{\mathbb{N}}))$ is δ-discrete.*

Proof. It is sufficient to show that if condition (2) fails, then condition (1) holds. Towards this end, suppose that $Y \subseteq X$ is an R-clique, and fix real numbers $\epsilon_n > 0$ such that $d^{-1}([0, \epsilon_n])$ is co-analytic and $\epsilon_{n+1} \leq \epsilon_n/2$ for all $n \in \mathbb{N}$. As pre-images of analytic sets under continuous functions with Polish domains are analytic (see, for example, the proof of [Kec95, Proposition 14.4]) and analytic subsets of Polish spaces have the Baire property (see, for example, [Kec95, Theorem 21.6]), it follows that for all $n \in \mathbb{N}$, the set $d^{-1}([0, \epsilon_n])$ is $\aleph_0$-universally Baire, so Theorem 4.3 yields a countable cover $\mathcal{Y}_n$ of Y by sets of d-diameter at most ϵ_n. Setting $\mathcal{U}_n = \{\mathcal{B}_d(Y', \epsilon_n) \cap Y \mid Y' \in \mathcal{Y}_n\}$ for all $n \in \mathbb{N}$, it only remains to observe that the set $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ is a basis for $(Y, d \restriction Y)$. ⊠

We say that a set $Y \subseteq X$ is $(\leq\delta)$-*scrambled* if x and y are proximal but not $(\leq\delta)$-asymptotic for all distinct $x, y \in Y$. By replacing the use of Theorem 2.3 with that of Theorem 4.3 in the proof of Theorem 3.1, one can show that the existence of an uncountable $(\leq\delta)$-scrambled set ensures the existence of a $(\leq\delta/2)$-scrambled Cantor set.

However, when $\delta > 0$, a substantially stronger result can be obtained through substantially simpler means. We say that points $x, y \in X$ are $(<\delta)$-*asymptotic* if $\limsup_{n \to \infty} d_X(f^n(x), f^n(y)) < \delta$, or equivalently, if there exist $\epsilon < \delta$ and $n \in \mathbb{N}$ such that $\forall m \geq n \; d_X(f^m(x), f^m(y)) \leq \epsilon$. Blanchard, Huang, and Snoha have established the analog of Theorem 1 where asymptoticity is replaced with $(<\delta)$-asymptoticity (see [BHS08, Theorem 16]). As the negation of $(<\delta)$-asymptoticity is a G_δ condition, so too is the corresponding analog of Li–Yorke pair, thus their result follows from the special case of Theorem 2.3 where E is equality, which is far simpler to establish (see [She99, Remark 1.14]).

We say that a dynamical system (X, f) is *complete* if X is complete. At the end of [Aki04, §6], Akin noted that the analog of Theorem 3.1 for complete dynamical systems is open in the case that X is perfect.

Recall that the *density* of a topological space X is the minimal cardinal κ for which there is a dense set $Y \subseteq X$ of cardinality κ. While our approach does not fully resolve Akin's problem beyond the separable case, it does yield a weaker generalization of Theorem 1:

Theorem 4.5. *Suppose that (X, f) is a complete dynamical system that admits a scrambled set of cardinality strictly greater than the density of X. Then there is a scrambled Cantor set $C \subseteq X$.*

Much as we obtained Theorem 1 from Theorem 3.1, we will obtain Theorem 4.5 from a generalization to a broader class of topological spaces. Endow each infinite cardinal κ with the discrete topology, and $\kappa^{\mathbb{N}}$ with the corresponding product topology. A topological space is *κ-Souslin* if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$. The idea behind the proof that every Polish space is analytic works just as well to show that every complete metric space of density κ is κ-Souslin. Other examples of κ-Souslin spaces include all definable subsets of analytic Hausdorff spaces, under definable determinacy. For instance, every co-analytic subset of an analytic Hausdorff space is $\aleph_1$-Souslin (see, for example, [Kec95, Theorem 36.12]). More generally, $\boldsymbol{\Delta}^1_{2n}$-determinacy ensures that every $\boldsymbol{\Pi}^1_{2n+1}$ subset of an analytic Hausdorff space is $\boldsymbol{\delta}^1_{2n+1}$-Souslin (see, for example, [Kec95, Corollary 39.9]).

We say that a dynamical system (X, f) is *κ-Souslin* if X is κ-Souslin. Theorem 4.5 is a consequence of the following generalization of Theorem 3.1:

Theorem 4.6. *Suppose that κ is an infinite cardinal and (X, f) is a κ-Souslin dynamical system that admits a scrambled set of cardinality strictly greater than κ. Then there is a scrambled Cantor set $C \subseteq X$.*

Proof. The proof of Theorem 1.1 easily adapts to show the analogous result in which the digraph G is merely κ-Souslin and the $\mathbb{N}$-coloring in condition (1) is replaced with a κ-coloring. The proof of Theorem 2.3 therefore adapts to show the analogous result in which E is $\aleph_0$-universally Baire and co-κ-Souslin, and Y intersects strictly more than κ classes. But this can be plugged into the proof of Theorem 3.1 to obtain the desired result. ⊠

It is easy to see that the generalizations mentioned in this section can all be combined with one another.

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