

# RECURRENCE AND THE EXISTENCE OF INVARIANT MEASURES

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**Abstract.** We show that recurrence conditions do not yield invariant Borel probability measures in the descriptive set-theoretic milieu, in the strong sense that if a Borel action of a locally compact Polish group on a standard Borel space satisfies such a condition but does not have an orbit supporting an invariant Borel probability measure, then there is an invariant Borel set on which the action satisfies the condition but does not have an invariant Borel probability measure.

Suppose that  $X$  is a Borel space and  $T: X \rightarrow X$  is a Borel automorphism. Given a set  $S \subseteq \mathbb{Z}$ , a set  $Y \subseteq X$  is *S-wandering* if  $T^m(Y) \cap T^n(Y) = \emptyset$  for all distinct  $m, n \in S$ , and *weakly wandering* if there is an infinite set  $S \subseteq \mathbb{Z}$  for which it is *S-wandering*. We say that a set  $Y \subseteq X$  is *T-complete* if it intersects each orbit of  $T$ , and a Borel probability measure  $\mu$  on  $X$  is *T-invariant* if  $\mu(B) = \mu(T(B))$  for all Borel sets  $B \subseteq X$ .

It is well-known that if  $\nu$  is a Borel probability measure on  $X$ , then the inexistence of a weakly-wandering  $\nu$ -positive set yields a  $T$ -invariant Borel probability measure  $\mu \gg \nu$  (see [Zak93] for the generalization to groups of Borel automorphisms). In light of this and his own characterization of the existence of invariant Borel probability measures (see [Nad90]), Nadkarni asked whether the inexistence of a weakly-wandering  $T$ -complete Borel set yields a  $T$ -invariant Borel probability measure in the special case that  $X$  is a standard Borel space.

We say that a set  $Y \subseteq X$  is *T-invariant* if  $Y = T(Y)$ , and a Borel probability measure  $\mu$  on  $X$  is *T-ergodic* if every  $T$ -invariant Borel set is  $\mu$ -conull or  $\mu$ -null. When  $X$  is a standard Borel space, a Borel equivalence relation  $F$  on  $X$  is *smooth* if there is a standard Borel space  $Z$  for which there is a Borel function  $\pi: X \rightarrow Z$  such that  $x F y \iff \pi(x) = \pi(y)$  for all  $x, y \in X$ .

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Eigen-Hajian-Nadkarni negatively answered Nadkarni's question by providing a standard Borel space  $X$ , a Borel automorphism  $T: X \rightarrow X$ , and a smooth Borel superequivalence relation  $F$  of the orbit equivalence relation  $E_T^X$  such that there is no weakly-wandering  $T$ -complete Borel set, but for each  $F$ -class  $C$  there is a weakly-wandering  $(T \upharpoonright C)$ -complete Borel set (see [EHN93]). To see that the latter condition rules out the existence of a  $T$ -invariant Borel probability measure, observe that the ergodic decomposition theorem ensures that the existence of a  $T$ -invariant Borel probability measure yields the existence of a  $T$ -ergodic  $T$ -invariant Borel probability measure (see [LM97, Theorem 3.2] for the generalization to analytic equivalence relations), the smoothness of  $F$  implies that every  $T$ -ergodic Borel measure concentrates on some  $F$ -class  $C$ , and the existence of a weakly-wandering  $(T \upharpoonright C)$ -complete Borel set rules out the existence of a  $(T \upharpoonright C)$ -invariant Borel probability measure.

However, Eigen-Hajian-Nadkarni also noted that this leaves open the question as to whether the inexistence of such an equivalence relation yields a  $T$ -invariant Borel probability measure.

The  $T$ -saturation of a set  $Y \subseteq X$  is given by  $[Y]_T = \bigcup_{n \in \mathbb{Z}} T^n(Y)$ . When  $X$  is a topological space, a homeomorphism  $T: X \rightarrow X$  is *topologically transitive* if for all non-empty open sets  $U, V \subseteq X$  there exists  $n \in \mathbb{Z}$  such that  $T^n(U) \cap V \neq \emptyset$ , and *minimal* if every orbit is dense. The *odometer* is the isometry of  $2^{\mathbb{N}}$  given by  $\sigma((1)^n \frown (0) \frown c) = (0)^n \frown (1) \frown c$  for all  $c \in 2^{\mathbb{N}}$  and  $n \in \mathbb{N}$ . Let  $m$  denote the Borel measure on  $2^{\mathbb{N}}$  given by  $m(\mathcal{N}_s) = 1/2^{|s|}$  for all  $s \in 2^{<\mathbb{N}}$ . It is easy to see that the odometer is minimal, and therefore topologically transitive, and that  $m$  is the unique  $\sigma$ -invariant Borel probability measure.

As noted in [Mil04, Example 3.13], if  $C \subseteq 2^{\mathbb{N}}$  is a non-meager  $m$ -null Borel set and  $X = [C]_\sigma$ , then  $\sigma \upharpoonright X$  also yields a negative answer to Nadkarni's question. To see that there is no weakly-wandering  $(\sigma \upharpoonright X)$ -complete Borel set, observe that if  $B \subseteq 2^{\mathbb{N}}$  is a non-meager set with the Baire property, then there is a non-empty open set  $U \subseteq 2^{\mathbb{N}}$  in which  $B$  is comeager (see, for example, [Kec95, Proposition 8.26]), and the fact that  $\sigma$  is a homeomorphism ensures that the set  $M = [U \setminus B]_\sigma$  is meager (see, for example, [Kec95, Exercise 8.45]). Note also that the compactness of  $2^{\mathbb{N}}$  and minimality of  $\sigma$  yield a finite set  $Z \subseteq \mathbb{Z}$  such that  $2^{\mathbb{N}} = \bigcup_{n \in Z} \sigma^n(U)$ , in which case  $2^{\mathbb{N}} = \bigcup_{n \in Z} \sigma^n(\sigma^k(U))$  for all  $k \in \mathbb{Z}$ . In particular, it follows that if  $x \in X \setminus M$ , then for all  $k \in \mathbb{Z}$  there exists  $n \in Z$  such that

$\sigma^{-n}(x) \in \sigma^k(U)$ . Letting  $\mathcal{O}$  denote the orbit of  $x$  under  $\sigma$ , the pigeon-hole principle therefore ensures that there is no set  $S \subseteq \mathbb{Z}$  of cardinality strictly greater than that of  $Z$  for which  $U \cap \mathcal{O}$  is  $S$ -wandering, thus the same holds of  $B \cap \mathcal{O}$ .

The transformation  $\sigma \upharpoonright X$  also yields a negative answer to the subsequent question of Eigen-Hajian-Nadkarni, as the topological transitivity of  $\sigma$  ensures that every  $\sigma$ -invariant Borel set is comeager or meager, thus every smooth Borel superequivalence relation of  $E_\sigma^X$  has a comeager equivalence class.

Following [Tse15], we say that a set  $Y \subseteq X$  is *locally weakly-wandering* if its intersection with each orbit of  $T$  is weakly wandering. As noted by Tserunyan, the transformation  $\sigma \upharpoonright X$  also yields a negative answer to the question as to whether the inexistence of a locally-weakly-wandering  $T$ -complete Borel set yields a  $T$ -invariant Borel probability measure. This is a direct consequence of the above proof that there is no weakly-wandering  $(\sigma \upharpoonright X)$ -complete Borel set.

We say that a set  $Y \subseteq X$  is *very weakly wandering* if there are arbitrarily large finite sets  $S \subseteq \mathbb{Z}$  for which it is  $S$ -wandering. The above arguments also yield negative answers to the analogous questions in which weak wandering is replaced with very weak wandering.

Here we generalize these observations from Borel actions of  $\mathbb{Z}$  to Borel actions of locally compact Polish groups, and from Borel probability measures that are invariant with respect to a Borel action to Borel probability measures that are invariant with respect to a Borel cocycle. We also show that not only are there examples of Borel actions yielding negative answers to the generalizations of the questions considered by Nadkarni, Eigen-Hajian-Nadkarni, and Tserunyan, but that it was wholly unnecessary to search for them in the first place, as they lie within every Borel action that could possibly contain them. Moreover, rather than just establishing this for variants of weak wandering, we show that no recurrence condition whatsoever yields the existence of an invariant Borel probability measure.

In §1, we establish the basic properties of the *recurrence spectrum* of a Borel action of a Polish group on a standard Borel space, which codifies the suitably robust forms of recurrence that it satisfies.

In §2, we show that locally-compact non-compact Polish groups have free Borel actions on Polish spaces with maximal recurrence spectra.

In §3, we show that the existence of weakly-wandering and very-weakly-wandering suitably-complete Borel sets, as well as suitably-complete Borel sets satisfying the minimal non-trivial notion of transience corresponding to the failure of the strongest notion of recurrence, in addition to the Eigen-Hajian-Nadkarni-style refinements thereof, can be characterized in terms of the recurrence spectrum. Our arguments also yield complexity bounds leading to implications between many of these notions. For instance, it follows that if  $X$  is a standard Borel space,  $T: X \rightarrow X$  is a Borel automorphism, and there is no smooth Borel superequivalence relation  $F$  of  $E_T^X$  with the property that there is a weakly-wandering  $(T \upharpoonright C)$ -complete Borel set for every  $F$ -class  $C$ , then there is no locally-weakly-wandering  $T$ -complete Borel set.

In §4, we generalize the generic compressibility theorem of Kechris-Miller (see [KM04, Theorem 13.1]) to Borel actions of locally compact Polish groups on standard Borel spaces. We simultaneously replace comeagerness with a stronger notion under which the recurrence spectrum is invariant, thereby insuring that no condition on the latter yields an invariant Borel probability measure.

**§1. The recurrence spectrum.** Suppose that  $G \curvearrowright X$  is a group action. We say that a set  $Y \subseteq X$  is *complete* if  $X = GY$ , and  *$\sigma$ -complete* if there is a countable set  $H \subseteq G$  for which  $X = HY$ . The following observation ensures that, under mild hypotheses, these notions coincide on open sets.

**PROPOSITION 1.1.** *Suppose that  $G$  is a topological group,  $H \subseteq G$  is dense,  $X$  is a topological space,  $G \curvearrowright X$  is continuous-in- $G$ , and  $U \subseteq X$  is open. Then  $GU = HU$ .*

**PROOF.** Suppose that  $g \in G$  and  $x \in U$ . As  $G \curvearrowright X$  is continuous-in- $G$ , there is an open neighborhood  $V \subseteq G$  of  $1_G$  for which  $Vx \subseteq U$ . Fix an open neighborhood  $W \subseteq G$  of  $g$  for which  $W^{-1}g \subseteq V$ . As  $H$  is dense, there exists  $h \in H \cap W$ , and it only remains to observe that  $h^{-1}g \in V$ , so  $h^{-1}g \cdot x \in U$ , thus  $g \cdot x \in hU$ .  $\square$

Let  $\mathbb{E}_0$  denote the equivalence relation on  $2^{\mathbb{N}}$  given by  $c \mathbb{E}_0 d \iff \exists n \in \mathbb{N} \forall m \geq n \ c(m) = d(m)$ . We will abuse language by saying that a subset of  $X$  is  $\aleph_0$ -*universally Baire* if its pre-image under every Borel function from a Polish space to  $X$  has the Baire property, and an  $\aleph_0$ -*universally-Baire* equivalence relation  $E$  on  $X$  is *smooth* if there is no Borel function  $\pi: 2^{\mathbb{N}} \rightarrow X$  such that  $c \mathbb{E}_0 d \iff \pi(c) E \pi(d)$

for all  $c, d \in 2^{\mathbb{N}}$ . The Harrington-Kechris-Louveau generalization of the Glimm-Effros dichotomy (see [HKL90, Theorem 1.1]) ensures that this is compatible with the usual notion of smoothness for Borel equivalence relations on standard Borel spaces.

When  $X$  is a topological space, a continuous-in- $X$  action  $G \curvearrowright X$  is *topologically transitive* if for all non-empty open sets  $U, V \subseteq X$  there exists  $g \in G$  such that  $gU \cap V \neq \emptyset$ , and *minimal* if every orbit is dense.

**PROPOSITION 1.2.** *Suppose that  $G$  is a group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous-in- $X$ ,  $B \subseteq X$  is  $E_G^X$ -invariant,  $C \subseteq X$  is an  $E_G^X$ -invariant  $G_\delta$  set for which  $G \curvearrowright C$  is topologically transitive and in which  $B$  is comeager, and  $F$  is a smooth  $\aleph_0$ -universally-Baire superequivalence relation of  $E_G^B$ . Then there is an  $F$ -class that is comeager in  $C$ .*

**PROOF.** Fix a dense  $G_\delta$  set  $C' \subseteq C$  contained in  $B$ , and note that  $F$  has the Baire property in  $C' \times C'$ , thus in  $C \times C$ . The straightforward generalization of the Becker-Kechris criterion for continuously embedding  $\mathbb{E}_0$  from orbit equivalence relations induced by groups of homeomorphisms (see [BK96, Theorem 3.4.5]) to superequivalence relations of such orbit equivalence relations (see, for example, [KMS14, Theorem 2.1]) ensures that the union of  $F$  and  $(C \setminus B) \times (C \setminus B)$  is non-meager in  $C \times C$ , so the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields an  $F$ -class that is non-meager and has the Baire property in  $C$ , thus comeager in  $C$  by topological transitivity.  $\square$

For each set  $Y \subseteq X$ , define  $\Delta(Y) = \{g \in G \mid Y \cap gY \neq \emptyset\}$ . The following fact is the obvious generalization of Pettis's Lemma (see, for example, [Kec95, Theorem 9.9]) to group actions.

**PROPOSITION 1.3.** *Suppose that  $G$  is a group,  $X$  is a Baire space,  $G \curvearrowright X$  is continuous-in- $X$ ,  $U \subseteq X$  is non-empty and open, and  $B \subseteq U$  is comeager. Then  $\Delta(U) \subseteq \Delta(B)$ .*

**PROOF.** Note that for all  $g \in G$ , the fact that  $G \curvearrowright X$  is continuous-in- $X$  ensures that  $gU$  is open and  $gB$  is comeager in  $gU$ . In particular, it follows that if  $U \cap gU$  is non-empty, then  $B \cap gB$  is comeager in  $U \cap gU$ , in which case the fact that  $X$  is a Baire space ensures that  $B \cap gB$  is also non-empty, thus  $\Delta(U) \subseteq \Delta(B)$ .  $\square$

Given an upward-closed family  $\mathcal{F} \subseteq \mathcal{P}(G)$  and any family  $\Gamma \subseteq \mathcal{P}(X)$ , we say that an action  $G \curvearrowright X$  is  $\mathcal{F}$ -recurrent on  $\Gamma$  if  $\Delta(B) \in \mathcal{F}$

for all  $B \in \Gamma$ . This generalizes the usual notion of recurrence in topological dynamics, where one says that a continuous-in- $X$  action of a group on a topological space  $X$  is  $\mathcal{F}$ -recurrent if it is  $\mathcal{F}$ -recurrent on non-empty open sets. We next note that, under mild hypotheses, this latter notion propagates to recurrence on non-meager sets with the Baire property.

**PROPOSITION 1.4.** *Suppose that  $G$  is a group,  $X$  is a Baire space,  $\mathcal{F} \subseteq \mathcal{P}(G)$  is upward closed, and  $G \curvearrowright X$  is continuous-in- $X$  and  $\mathcal{F}$ -recurrent. Then  $G \curvearrowright X$  is  $\mathcal{F}$ -recurrent on non-meager sets with the Baire property.*

**PROOF.** If  $B \subseteq X$  is a non-meager set with the Baire property, then there is a non-empty open set  $U \subseteq X$  in which  $B$  is comeager, and Proposition 1.3 ensures that  $\Delta(U) \subseteq \Delta(B)$ , thus  $\Delta(B) \in \mathcal{F}$ .  $\square$

When  $X$  is a Polish space, the *decomposition into minimal components* of a continuous action  $G \curvearrowright X$  is the equivalence relation on  $X$  given by  $x F_G^X y \iff \overline{Gx} = \overline{Gy}$ . It is easy to see that  $F_G^X$  is  $G_\delta$  and smooth (although the latter is also a direct consequence of the former and [HKL90, Corollary 1.2]), and that for each  $F_G^X$ -class  $C$ , the action  $G \curvearrowright C$  is minimal.

When  $G$  is Borel, the *recurrence spectrum* of a Borel action  $G \curvearrowright X$  is the collection of all upward-closed families  $\mathcal{F} \subseteq \mathcal{P}(G)$  such that every smooth Borel superequivalence relation  $F$  of  $E_G^X$  has an equivalence class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent on  $\sigma$ -complete Borel sets. As a theorem of Becker-Kechris ensures that every Borel action of a Polish group on a standard Borel space is Borel isomorphic to a continuous action on a Polish space (see [BK96, Theorem 5.2.1]), the following observation ensures that, under mild hypotheses, the notion of recurrence spectrum is robust, in the sense that it does not depend on the particular underlying notion of definability, and in the sense that it is invariant under passage to sufficiently large  $E_G^X$ -invariant subsets.

**PROPOSITION 1.5.** *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous,  $\mathcal{F} \subseteq \mathcal{P}(G)$  is upward closed, and  $B \subseteq X$  is  $E_G^X$ -invariant and comeager in every  $F_G^X$ -class. Then the following are equivalent:*

1. *Every smooth  $\aleph_0$ -universally-Baire superequivalence relation  $F$  of  $E_G^B$  has a class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent on  $\sigma$ -complete  $\aleph_0$ -universally-Baire sets.*
2. *There is an  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent.*

PROOF. To see (1)  $\implies$  (2), fix an  $F_G^X$ -class  $C$  for which  $G \curvearrowright B \cap C$  is  $\mathcal{F}$ -recurrent on  $\sigma$ -complete open sets. To see that  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent, suppose that  $U \subseteq C$  is a non-empty open set, and note that the minimality of  $G \curvearrowright C$  ensures that  $U$  is complete, and therefore  $\sigma$ -complete by Proposition 1.1, thus  $\Delta(U) \in \mathcal{F}$ .

To see (2)  $\implies$  (1), fix an  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent, and suppose that  $F$  is a smooth  $\aleph_0$ -universally-Baire superequivalence relation of  $E_G^B$ . Proposition 1.2 then yields an  $F$ -class  $D$  that is comeager in  $C$ . To see that  $G \curvearrowright D$  is  $\mathcal{F}$ -recurrent on  $\sigma$ -complete  $\aleph_0$ -universally-Baire sets, suppose that  $A \subseteq D$  is such a set, and note that  $\sigma$ -completeness ensures that  $A$  is non-meager in  $C$ . Fix a dense  $G_\delta$  set  $C' \subseteq C$  contained in  $D$ , and note that  $A \cap C'$  has the Baire property in  $C'$ , so  $A$  has the Baire property in  $C$ , thus Proposition 1.4 ensures that  $\Delta(A) \in \mathcal{F}$ .  $\square$

Let  $\check{\Gamma}$  denote the family of sets whose complements are in  $\Gamma$ , let  $\Gamma \setminus \Gamma$  denote the family of differences of sets in  $\Gamma$ , and let  $\Gamma_\sigma$  denote the family of countable unions of sets in  $\Gamma$ .

The *horizontal sections* of a set  $R \subseteq X \times Y$  are the sets of the form  $R^y = \{x \in X \mid x R y\}$  for  $y \in Y$ , whereas the *vertical sections* of a set  $R \subseteq X \times Y$  are the sets of the form  $R_x = \{y \in Y \mid x R y\}$  for  $x \in X$ . We say that  $\mathcal{F}$  is  $\Gamma$ -on-open if  $\{x \in X \mid U^x \in \mathcal{F}\} \in \Gamma$  for all open sets  $U \subseteq G \times X$ .

Given a superequivalence relation  $E$  of  $E_G^X$ , we say that an action  $G \curvearrowright X$  is  $E$ -locally  $\mathcal{F}$ -recurrent on  $\Gamma$  if for all  $B \in \Gamma$ , there is an  $E$ -class  $C$  such that  $\Delta(B \cap C) \in \mathcal{F}$ . We next note that, under mild hypotheses, the recurrence spectrum can also be characterized in terms of local recurrence of  $G \curvearrowright X$  itself.

PROPOSITION 1.6. *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous,  $\Gamma \subseteq \mathcal{P}(X)$  is a family of  $\aleph_0$ -universally-Baire sets containing the open sets and closed under finite intersections and finite unions, and  $\mathcal{F} \subseteq \mathcal{P}(G)$  is upward closed and  $\check{\Gamma}$ -on-open. Then the following are equivalent:*

1. *There is an  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent.*
2. *The action  $G \curvearrowright X$  is  $F_G^X$ -locally  $\mathcal{F}$ -recurrent on  $\sigma$ -complete  $(\Gamma \setminus \Gamma)_\sigma$  sets.*

PROOF. To see  $\neg(2) \implies \neg(1)$ , observe that if  $B \subseteq X$  is a  $\sigma$ -complete  $(\Gamma \setminus \Gamma)_\sigma$  set such that  $\Delta(B \cap C) \notin \mathcal{F}$  for every  $F_G^X$ -class  $C$ , then it is non-meager and has the Baire property in every such

class, so there is no  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent by Proposition 1.4.

To see  $\neg(1) \implies \neg(2)$ , fix a basis  $(U_n)_{n \in \mathbb{N}}$  for  $X$ . For all  $n \in \mathbb{N}$ , define  $V_n = \{(g, x) \in G \times X \mid U_n \cap gU_n \cap [x]_{F_G^X} \neq \emptyset\}$ . Observe that if  $(g, x) \in V_n$ , then the minimality of  $G \curvearrowright [x]_{F_G^X}$  yields  $h \in G$  for which  $h \cdot x \in U_n \cap gU_n$ , so the continuity of  $G \curvearrowright X$  yields open neighborhoods  $U_g \subseteq G$  of  $g$  and  $U_x \subseteq X$  of  $x$  such that  $hU_x \cup U_g^{-1}hU_x \subseteq U_n$ , thus  $U_g \times U_x \subseteq V_n$ , hence  $V_n$  is open. It follows that the  $F_G^X$ -invariant sets  $A_n = \{x \in GU_n \mid V_n^x \notin \mathcal{F}\}$  are in  $\Gamma$ , so the sets  $B_n = A_n \setminus \bigcup_{m < n} A_m$  are in  $\Gamma \setminus \Gamma$ , thus the set  $B = \bigcup_{n \in \mathbb{N}} B_n \cap U_n$  is in  $(\Gamma \setminus \Gamma)_\sigma$ . But if there is no  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent, then  $B$  is complete, and therefore  $\sigma$ -complete by Proposition 1.1, thus  $G \curvearrowright X$  is not  $F_G^X$ -locally  $\mathcal{F}$ -recurrent on  $\sigma$ -complete  $(\Gamma \setminus \Gamma)_\sigma$  sets.  $\square$

We next show that, under an additional mild hypothesis on  $G$ , Propositions 1.3, 1.4, and 1.5 can be strengthened so as to show that the recurrence spectrum is also robust in the sense that it does not depend on whether the underlying notion of recurrence is local.

The following fact is a somewhat more intricate generalization of the special case of Pettis's Lemma for second-countable groups to continuous actions of such groups.

**PROPOSITION 1.7.** *Suppose that  $G$  is a second-countable Baire group,  $X$  is a second-countable Baire space,  $G \curvearrowright X$  is continuous,  $U \subseteq X$  is non-empty and open, and  $B \subseteq U$  is comeager. Then  $\Delta(U \cap Gx) \subseteq \Delta(B \cap Gx)$  for comeagerly many  $x \in X$ .*

**PROOF.** We write  $\forall^* x \in X \phi(x)$  to indicate that  $\{x \in X \mid \phi(x)\}$  is comeager. As the fact that  $G \curvearrowright X$  is continuous-in- $X$  ensures that it is open, it follows that  $\{(g, x) \in G \times X \mid g \cdot x \notin U \setminus B\}$  is comeager, so the set  $C = \{x \in X \mid \forall^* g \in G \ g \cdot x \notin U \setminus B\}$  is comeager by the Kuratowski-Ulam theorem. Observe that if  $h \in G$  and  $x \in X$ , then  $\{g \in G \mid g \cdot x \notin h(U \setminus B)\} = h\{g \in G \mid g \cdot x \notin U \setminus B\}$ , so the fact that  $G \curvearrowright X$  is continuous-in- $X$  also ensures that if  $x \in C$ , then  $\forall^* g \in G \ g \cdot x \notin (U \setminus B) \cup h(U \setminus B)$ , in which case the fact that  $(U \cap hU) \setminus (B \cap hB) \subseteq (U \setminus B) \cup h(U \setminus B)$  therefore implies that  $\forall^* g \in G \ g \cdot x \notin (U \cap hU) \setminus (B \cap hB)$ . Note now that for all  $h \in G$ , the fact that  $G \curvearrowright X$  is continuous-in- $X$  ensures that  $U \cap hU$  is open, so the fact that  $G \curvearrowright X$  is continuous-in- $G$  implies that if  $x \in X$  and  $U \cap hU \cap Gx$  is non-empty, then there are non-meagerly many  $g \in G$  for which  $g \cdot x \in U \cap hU$ . In particular, it follows that if



$x \in C$  and  $U \cap hU \cap Gx$  is non-empty, then so too is  $B \cap hB \cap Gx$ , hence  $\Delta(U \cap Gx) \subseteq \Delta(B \cap Gx)$  for all  $x \in C$ .  $\square$

We next note that, under mild hypotheses,  $\mathcal{F}$ -recurrence of topologically transitive actions not only propagates to  $\mathcal{F}$ -recurrence on non-meager sets with the Baire property, but to its  $E_G^X$ -local strengthening.

**PROPOSITION 1.8.** *Suppose that  $G$  is a second-countable Baire group,  $X$  is a second-countable Baire space,  $\mathcal{F} \subseteq \mathcal{P}(G)$  is upward closed, and  $G \curvearrowright X$  is continuous,  $\mathcal{F}$ -recurrent, and topologically transitive. Then  $G \curvearrowright X$  is  $E_G^X$ -locally  $\mathcal{F}$ -recurrent on non-meager sets with the Baire property.*

**PROOF.** Suppose that  $B \subseteq X$  is a non-meager set with the Baire property, and fix a non-empty open set  $U \subseteq X$  in which  $B$  is comeager. The topological transitivity of  $G \curvearrowright X$  ensures that the set  $C = \{x \in X \mid Gx \text{ is dense}\}$  is comeager, and Proposition 1.7 implies that the set  $D = \{x \in X \mid \Delta(U \cap Gx) \subseteq \Delta(B \cap Gx)\}$  is comeager. So it only remains to observe that if  $x \in C \cap D$ , then  $\Delta(U) \subseteq \Delta(U \cap Gx) \subseteq \Delta(B \cap Gx)$ , thus  $\Delta(B \cap Gx) \in \mathcal{F}$ .  $\square$

We can now establish the promised robustness result.

**PROPOSITION 1.9.** *Suppose that  $G$  is a second-countable Baire group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous,  $\mathcal{F} \subseteq \mathcal{P}(G)$  is upward closed, and  $B \subseteq X$  is  $E_G^X$ -invariant and comeager in every  $F_G^X$ -class. Then the following are equivalent:*

1. *Every smooth  $\aleph_0$ -universally-Baire superequivalence relation  $F$  of  $E_G^B$  has a class  $C$  for which  $G \curvearrowright C$  is  $E_G^C$ -locally  $\mathcal{F}$ -recurrent on  $\sigma$ -complete  $\aleph_0$ -universally-Baire sets.*
2. *There is an  $F_G^X$ -class  $C$  for which  $G \curvearrowright C$  is  $\mathcal{F}$ -recurrent.*

**PROOF.** Exactly as in the proof of Proposition 1.5, but replacing the use of Proposition 1.4 with that of Proposition 1.8.  $\square$

**§2. The strongest notion of recurrence.** Given a set  $S \subseteq G$ , we say that a set  $Y \subseteq X$  is  $S$ -transient if  $Y \cap SY = \emptyset$ .

**PROPOSITION 2.1.** *Suppose that  $G$  is a topological group,  $X$  is a Hausdorff space,  $G \curvearrowright X$  is continuous,  $K \subseteq G$  is compact, and  $x \in X$  is not fixed by any element of  $K$ . Then there is a  $K$ -transient open neighborhood of  $x$ .*

**PROOF.** For each  $g \in K$ , the fact that  $X$  is Hausdorff yields disjoint open neighborhoods  $V_g \subseteq X$  of  $x$  and  $W_g \subseteq X$  of  $g \cdot x$ , and the

continuity of  $G \curvearrowright X$  yields open neighborhoods  $U_g \subseteq G$  of  $g$  and  $V'_g \subseteq V_g$  of  $x$  for which  $U_g V'_g \subseteq W_g$ , thus  $U_g V'_g \cap V'_g = \emptyset$ . The compactness of  $K$  then yields a finite set  $F \subseteq K$  for which  $K \subseteq \bigcup_{g \in F} U_g$ , in which case  $\bigcap_{g \in F} V'_g$  is a  $K$ -transient open neighborhood of  $x$ .  $\square$

It follows that upward-closed families whose corresponding notions of recurrence are realizable by suitable free actions necessarily contain all co-compact neighborhoods of the identity.

**PROPOSITION 2.2.** *Suppose that  $G$  is a topological group and  $\mathcal{F} \subseteq \mathcal{P}(G)$  is an upward closed family for which there is an  $\mathcal{F}$ -recurrent continuous free action of  $G$  on a Hausdorff space. Then  $\mathcal{F}$  contains every co-compact neighborhood of  $1_G$ .*

**PROOF.** This is a direct consequence of Proposition 2.1.  $\square$

Given sets  $Y, Z \subseteq X$ , define  $\Delta(Y, Z) = \{g \in G \mid gY \cap Z \neq \emptyset\}$ . When  $X$  is a topological space, we say that a continuous-in- $X$  action  $G \curvearrowright X$  is *topologically mixing* if  $\Delta(U, V)$  is co-compact for all non-empty open sets  $U, V \subseteq X$ .

**PROPOSITION 2.3.** *Suppose that  $G$  is a topological group,  $X$  is a topological space,  $\mathcal{F} \subseteq \mathcal{P}(G)$  is the family of co-pre-compact subsets of  $G$  containing  $1_G$ , and  $G \curvearrowright X$  is continuous-in- $X$ ,  $\mathcal{F}$ -recurrent, and topologically transitive. Then  $G \curvearrowright X$  is topologically mixing.*

**PROOF.** Given non-empty open sets  $U, V \subseteq X$ , the topological transitivity of  $G \curvearrowright X$  yields  $g \in G$  for which  $gU \cap V \neq \emptyset$ , so the fact that  $G \curvearrowright X$  is  $\mathcal{F}$ -recurrent ensures that  $\Delta(gU \cap V)$  is co-compact. As  $h \in \Delta(gU \cap V) \implies hgU \cap V \neq \emptyset \implies hg \in \Delta(U, V)$ , it follows that  $\Delta(gU \cap V)g \subseteq \Delta(U, V)$ , so  $\Delta(U, V)$  is co-compact.  $\square$

It follows that the existence of a suitable free Borel action of  $G$  whose recurrence spectrum contains the family  $\mathcal{F}$  of co-pre-compact subsets of  $G$  containing  $1_G$  is equivalent to the existence of a suitable continuous topologically-mixing free action of  $G$ .

**PROPOSITION 2.4.** *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{F} \subseteq \mathcal{P}(G)$  is the family of co-pre-compact subsets of  $G$  containing  $1_G$ . Then  $\mathcal{F}$  is in the recurrence spectrum of  $G \curvearrowright X$  if and only if there is an equivalence class  $C$  of  $F_G^X$  for which  $G \curvearrowright C$  is topologically mixing.*

**PROOF.** This is a direct consequence of Propositions 1.5 and 2.3.  $\square$

To our surprise, we were unable to find a proof in the literature of the fact that locally-compact non-compact Polish groups have free topologically-mixing continuous actions on Polish spaces. In a pair of private emails, Glasner-Weiss suggested that the strengthening in which the underlying space is compact should be a consequence of generalizations of the results of [Wei12] to locally compact groups, and that a substantially simpler construction should yield the aforementioned result. However, we give an elementary proof by checking that the action of  $G$  by left multiplication on the space  $F(G)$  of closed subsets of  $G$  is topologically mixing, where  $F(G)$  is equipped with the *Fell topology* generated by the sets  $V_K = \{F \in F(G) \mid F \cap K = \emptyset\}$  and  $W_U = \{F \in F(G) \mid F \cap U \neq \emptyset\}$ , where  $K \subseteq G$  compact and  $U \subseteq G$  open. It is well-known that  $F(G)$  is a compact Polish space (see, for example, [Kec95, Exercise 12.7]).

**PROPOSITION 2.5.** *Suppose that  $G$  is a locally-compact non-compact Polish group. Then there is a Polish space  $X$  for which there is a free topologically-mixing continuous action  $G \curvearrowright X$ .*

**PROOF.** While it is well-known that  $G \curvearrowright F(G)$  is continuous, we will provide a proof for the reader's convenience. Towards this end, it is sufficient to show that if  $g \in G$ ,  $F \in F(G)$ , and  $U_{gF} \subseteq F(G)$  is an open neighborhood of  $gF$ , then there are open neighborhoods  $U_g \subseteq G$  of  $g$  and  $U_F \subseteq F(G)$  of  $F$  for which  $U_g U_F \subseteq U_{gF}$ . Clearly we can assume that  $U_{gF} = V_K$  for some compact  $K \subseteq G$ , or  $U_{gF} = W_U$  for some open set  $U \subseteq G$ . In the former case, it follows that  $F \cap g^{-1}K = \emptyset$ , so the local compactness of  $G$  ensures that for all  $h \in K$  there are a pre-compact open neighborhood  $U_{g,h} \subseteq G$  of  $g$  and an open neighborhood  $V_{g,h} \subseteq G$  of  $h$  such that  $F \cap \overline{U_{g,h}}^{-1} V_{g,h} = \emptyset$ , and the compactness of  $K$  yields a finite set  $L \subseteq K$  such that  $K \subseteq \bigcup_{h \in L} V_{g,h}$ , in which case the sets  $U_g = \bigcap_{h \in L} U_{g,h}$  and  $U_F = V_{\overline{U_g}^{-1}K}$  are as desired. In the latter case, there exists  $h \in F$  for which  $gh \in U$ , so there are open neighborhoods  $U_g \subseteq G$  of  $g$  and  $U_h \subseteq G$  of  $h$  such that  $U_g U_h \subseteq U$ , thus the sets  $U_g$  and  $U_F = W_{U_h}$  are as desired.

To see that  $G \curvearrowright F(G)$  is topologically mixing, it is sufficient to show that if  $U = V_K \cap \bigcap_{i < m} W_{U_i}$  and  $U' = V_{K'} \cap \bigcap_{j < n} W_{U'_j}$  are non-empty, where  $K, K' \subseteq G$  are compact and  $U_i \subseteq \sim K$  and  $U'_j \subseteq \sim K'$  are open for all  $i < m$  and  $j < n$ , then  $\{g \in G \mid gU \cap U' = \emptyset\}$  is compact. Towards this end, note that for all  $i < m$  and  $j < n$ , the

sets

$$L_i = \{g \in G \mid gU_i \subseteq K'\} = \bigcap_{h \in U_i} \{g \in G \mid gh \in K'\} = \bigcap_{h \in U_i} K'h^{-1}$$

and

$$L'_j = \{g \in G \mid U'_j \subseteq gK\} = \bigcap_{h \in U'_j} \{g \in G \mid h \in gK\} = \bigcap_{h \in U'_j} hK^{-1}$$

are compact, set  $L = \bigcup_{i < m} L_i \cup \bigcup_{j < n} L'_j$ , and observe that if  $g \in \sim L$ , then there exist  $g_i \in gU_i \setminus K'$  for all  $i < m$  and  $g'_j \in U'_j \setminus gK$  for all  $j < n$ . But then the set  $F = \{g_i \mid i < m\} \cup \{g'_j \mid j < n\}$  is in  $gU \cap U'$ .

The *free part* of the action  $G \curvearrowright F(G)$  is the set  $X$  of  $F \in F(G)$  that are not fixed by any non-identity element of  $G$ . The local-compactness and separability of  $G$  ensure that  $X$  is the intersection of countably-many sets of the form  $X_K = \{F \in F(G) \mid \forall g \in K \ gF \neq F\}$ , where  $K \subseteq G \setminus \{1_G\}$  is compact. As Proposition 2.1 ensures that each  $X_K$  is open, it follows that  $X$  is  $G_\delta$ , and therefore Polish. To see that  $G \curvearrowright X$  is the desired action, it only remains to establish that  $X$  is comeager. And for this, it is sufficient to show that if  $K \subseteq G \setminus \{1_G\}$  is compact, then  $X_K$  is dense. Towards this end, suppose that  $U = V_L \cap \bigcap_{i < n} W_{U_i}$  is non-empty, where  $L \subseteq G$  is compact and  $U_i \subseteq \sim L$  is open for all  $i < n$ , and fix  $g_i \in U_i$  for all  $i < n$ . As  $G$  is locally compact, by passing to open neighborhoods of  $g_i$  contained in  $U_i$ , we can assume that each of the sets  $U_i$  is precompact. As  $G$  is not compact, there exists  $g \in \sim(L \cup \bigcup_{i < n} K^{-1}U_i)$ . Then the set  $F = \{g\} \cup \{g_i \mid i < n\}$  is in  $U$ , and the fact that  $F \cap Kg = \emptyset$  ensures that  $F \in X_K$ .  $\square$

**§3. Transience.** Following the usual convention, let  $\Sigma_1^0$  denote the pointclass of open sets, and recursively define  $\Pi_n^0$  to be the pointclass of complements of  $\Sigma_n^0$  sets, and  $\Sigma_{n+1}^0$  to be the pointclass of countable unions of  $\Pi_n^0$  sets. It is well known that  $(\Sigma_n^0 \setminus \Sigma_n^0)_\sigma$ ,  $(\Pi_n^0 \setminus \Pi_n^0)_\sigma$ , and  $\Sigma_{n+1}^0$  coincide on metric spaces (see, for example, [Kec95, §11.B]).

Given a set  $S \subseteq G$ , we use  $\mathcal{F}_S$  to denote the family of sets  $T \subseteq G$  for which  $S \cap T \neq \emptyset$ .

**PROPOSITION 3.1.** *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $S \subseteq G$ . Then the following are equivalent:*

1. *The family  $\mathcal{F}_S$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*

2. *There is a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $S$ -transient  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.*
3. *The action  $G \curvearrowright X$  has an  $S$ -transient  $\sigma$ -complete  $\Sigma_2^0$  set.*

PROOF. As a set  $Y \subseteq X$  is  $S$ -transient if and only if  $\Delta(Y) \notin \mathcal{F}_S$ , Proposition 1.5 yields (1)  $\iff$  (2). Note that if  $U \subseteq G \times X$  and  $x \in X$ , then  $U^x \in \mathcal{F}_S \iff x \in \bigcup_{g \in S} U_g$ , so  $\mathcal{F}_S$  is  $\Sigma_1^0$ -on-open. As a set  $Y \subseteq X$  is  $S$ -transient if and only if  $\Delta(C \cap Y) \notin \mathcal{F}_S$  for all equivalence classes  $C$  of  $F_G^X$ , Proposition 1.6 yields (1)  $\iff$  (3).  $\square$

Given a set  $\mathcal{S} \subseteq \mathcal{P}(G)$ , we say that a set  $Y \subseteq X$  is  $\mathcal{S}$ -transient if there is a set  $S \in \mathcal{S}$  for which  $Y$  is  $S$ -transient.

PROPOSITION 3.2. *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S} \subseteq \mathcal{P}(G)$ . Then the following are equivalent:*

1. *There exists  $S \in \mathcal{S}$  for which the family  $\mathcal{F}_S$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
2. *There exist  $S \in \mathcal{S}$  and a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $S$ -transient  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.*
3. *The action  $G \curvearrowright X$  has an  $\mathcal{S}$ -transient  $\sigma$ -complete  $\Sigma_2^0$  set.*

PROOF. This is a direct consequence of Proposition 3.1.  $\square$

Given a set  $S \subseteq G$ , we say that a set  $Y \subseteq X$  is  $S$ -wandering if  $gY \cap hY = \emptyset$  for all distinct  $g, h \in S$ , and *weakly wandering* if there exists an infinite set  $S \subseteq G$  for which  $Y$  is  $S$ -wandering.

PROPOSITION 3.3. *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S}$  is the family of sets of the form  $S^{-1}S \setminus \{1_G\}$ , where  $S \subseteq G$  is infinite. Then the following are equivalent:*

1. *There exists  $S \in \mathcal{S}$  for which the family  $\mathcal{F}_S$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
2. *There exist an infinite set  $S \subseteq G$  and a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $S$ -wandering  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.*
3. *The action  $G \curvearrowright X$  has a weakly-wandering  $\sigma$ -complete  $\Sigma_2^0$  set.*

PROOF. Observe that if  $S \subseteq G$ , then a set  $Y \subseteq X$  is  $S$ -wandering if and only if it is  $(S^{-1}S \setminus \{1_G\})$ -transient, and appeal to Proposition 3.2.  $\square$

A subset of a standard Borel space is *analytic* if it is the image of a standard Borel space under a Borel function, and *co-analytic* if its complement is analytic. A result of Lusin-Sierpiński (see, for example, [Kec95, Theorem 21.6]) ensures that analytic subsets of standard Borel spaces are  $\aleph_0$ -universally Baire. Let  $\Sigma_1^1$  denote the pointclass of analytic sets, and let  $\Pi_1^1$  denote the pointclass of co-analytic sets.

Given a superequivalence relation  $E$  of  $E_G^X$ , we say that a set  $Y \subseteq X$  is  *$E$ -locally weakly-wandering* if its intersection with each  $E$ -class is weakly wandering. Given a set  $\mathcal{S} \subseteq \mathcal{P}(G)$ , define  $\mathcal{F}_{\mathcal{S}} = \bigcap_{S \in \mathcal{S}} \mathcal{F}_S$ .

**PROPOSITION 3.4.** *Suppose that  $G$  is a Polish group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S}$  is the family of sets of the form  $S^{-1}S \setminus \{1_G\}$ , where  $S \subseteq G$  is infinite. Then the following are equivalent:*

1. *The family  $\mathcal{F}_{\mathcal{S}}$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
2. *There is a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $E_G^X$ -locally-weakly-wandering  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.*
3. *The action  $G \curvearrowright X$  has an  $F_G^X$ -locally-weakly-wandering  $\sigma$ -complete  $(\Sigma_1^1 \setminus \Sigma_1^1)_{\sigma}$  set.*

**PROOF.** As a set  $Y \subseteq X$  is  $E_G^X$ -locally weakly-wandering if and only if  $\Delta(C \cap Y) \notin \mathcal{F}_{\mathcal{S}}$  for all equivalence classes  $C$  of  $E_G^X$ , Proposition 1.9 yields (1)  $\iff$  (2). Note that if  $U \subseteq G \times X$  and  $x \in X$ , then  $U^x \in \mathcal{F}_{\mathcal{S}} \iff \forall (g_i)_{i \in \mathbb{N}} \in G^{\mathbb{N}} \exists i \neq j (g_i = g_j \text{ or } g_i^{-1}g_j \in U^x)$ , so  $\mathcal{F}_{\mathcal{S}}$  is  $\Pi_1^1$ -on-open. As a set  $Y \subseteq X$  is  $F_G^X$ -locally weakly-wandering if and only if  $\Delta(C \cap Y) \notin \mathcal{F}_{\mathcal{S}}$  for all equivalence classes  $C$  of  $F_G^X$ , Proposition 1.6 yields (1)  $\iff$  (3).  $\square$

We say that a set  $Y \subseteq X$  is *very weakly wandering* if there are arbitrarily large finite sets  $S \subseteq G$  for which  $Y$  is  $S$ -wandering.

**PROPOSITION 3.5.** *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S}$  is the family of sets of the form  $\bigcup_{n \in \mathbb{N}} S_n^{-1}S_n \setminus \{1_G\}$ , where  $S_n \subseteq G$  has cardinality  $n$  for all  $n \in \mathbb{N}$ . Then the following are equivalent:*

1. *There exists  $S \in \mathcal{S}$  for which the family  $\mathcal{F}_{\mathcal{S}}$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
2. *There exist sets  $S_n \subseteq G$  of cardinality  $n$  and a smooth  $\aleph_0$ -universally Baire superequivalence relation  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has a  $\sigma$ -complete  $\aleph_0$ -universally-Baire set that is  $S_n$ -wandering for all  $n \in \mathbb{N}$ .*

3. *The action  $G \curvearrowright X$  has a very-weakly-wandering  $\sigma$ -complete  $\Sigma_2^0$  set.*

PROOF. Observe that if  $S_n \subseteq G$  for all  $n \in \mathbb{N}$ , then a set  $Y \subseteq X$  is  $S_n$ -wandering for all  $n \in \mathbb{N}$  if and only if it is  $(\bigcup_{n \in \mathbb{N}} S_n^{-1} S_n \setminus \{1_G\})$ -transient, and appeal to Proposition 3.2.  $\square$

Although we are already in position to establish the analog of Proposition 3.4 for very weak wandering, the following observations will allow us to obtain a substantially stronger complexity bound.

PROPOSITION 3.6. *Suppose that  $G$  is a topological group,  $X$  is a topological space,  $G \curvearrowright X$  is continuous, and  $U \subseteq X$  is a non-empty open set. Then there exist a non-empty open set  $V \subseteq U$  and an open neighborhood  $W \subseteq G$  of  $1_G$  for which  $W\Delta(V)W^{-1} \subseteq \Delta(U)$ .*

PROOF. The continuity of  $G \curvearrowright X$  yields a non-empty open set  $V \subseteq U$  and an open neighborhood  $W \subseteq G$  of  $1_G$  for which  $WV \subseteq U$ . To see that  $W\Delta(V)W^{-1} \subseteq U$ , note that if  $g \in \Delta(V)$  and  $g_x, g_y \in W$ , then there exist  $x, y \in V$  for which  $g \cdot x = y$ , in which case the points  $x' = g_x \cdot x$  and  $y' = g_y \cdot y$  are in  $U$ , so the fact that  $g_y g g_x^{-1} \cdot x' = y'$  ensures that  $g_y g g_x^{-1} \in \Delta(U)$ .  $\square$

PROPOSITION 3.7. *Suppose that  $G$  is a topological group,  $X$  is a topological space,  $G \curvearrowright X$  is continuous, and  $S \subseteq G$ . Then every  $S$ -wandering non-empty open set  $U \subseteq X$  has a non-empty open subset  $V \subseteq U$  such that for all dense sets  $H \subseteq G$ , there is an injection  $\phi: S \rightarrow H$  with the property that  $V$  is  $\phi(S)$ -wandering.*

PROOF. By Proposition 3.6, there exist a non-empty open set  $V \subseteq U$  and an open neighborhood  $W \subseteq G$  of  $1_G$  for which  $W\Delta(V)W^{-1} \subseteq \Delta(U)$ . Note that if  $g, h \in S$  and  $(gW)^{-1}(hW) \cap \Delta(V) \neq \emptyset$ , then the fact that  $(gW)^{-1}(hW) = W^{-1}g^{-1}hW$  yields that  $g^{-1}h \in W\Delta(V)W^{-1} \subseteq \Delta(U)$ , thus  $g = h$ . But if  $H \subseteq G$  is dense, then there is a function  $\phi: S \rightarrow H$  with the property that  $\phi(g) \in gW$  for all  $g \in S$ , and it follows that  $\phi$  is injective and  $V$  is  $\phi(S)$ -wandering.  $\square$

Given a superequivalence relation  $E$  of  $E_G^X$ , we say that a set  $Y \subseteq X$  is  $E$ -locally very-weakly-wandering if its intersection with each  $E$ -class is very weakly wandering.

PROPOSITION 3.8. *Suppose that  $G$  is a Polish group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S}$  is the family of sets of the form  $\bigcup_{n \in \mathbb{N}} S_n^{-1} S_n \setminus \{1_G\}$ , where  $S_n \subseteq G$  has cardinality  $n$  for all  $n \in \mathbb{N}$ . Then the following are equivalent:*

1. The family  $\mathcal{F}_S$  is not in the recurrence spectrum of  $G \curvearrowright X$ .
2. There is a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $E_G^X$ -locally very-weakly-wandering  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.
3. The action  $G \curvearrowright X$  has an  $F_G^X$ -locally very-weakly-wandering  $\sigma$ -complete  $\Sigma_4^0$  set.

PROOF. As a set  $Y \subseteq X$  is  $E_G^X$ -locally very-weakly-wandering if and only if  $\Delta(C \cap Y) \notin \mathcal{F}_S$  for all equivalence classes  $C$  of  $E_G^X$ , Proposition 1.9 yields (1)  $\iff$  (2). The fact that every  $F_G^X$ -locally very-weakly-wandering set  $Y \subseteq X$  is  $E_G^X$ -locally very-weakly-wandering yields (3)  $\implies$  (2). To see (1)  $\implies$  (3), fix a countable dense set  $H \subseteq G$ , and let  $\mathcal{T}$  denote the family of sets of the form  $\bigcup_{n \in \mathbb{N}} T_n^{-1} T_n \setminus \{1_H\}$ , where  $T_n \subseteq H$  has cardinality  $n$  for all  $n \in \mathbb{N}$ . Now observe that if condition (1) holds, then Proposition 1.5 ensures that there is no equivalence class  $C$  of  $F_G^X$  for which  $G \curvearrowright C$  is  $\mathcal{F}_S$ -recurrent, so Proposition 3.7 implies that there is no equivalence class  $C$  of  $F_G^X$  for which  $G \curvearrowright C$  is  $\mathcal{F}_\mathcal{T}$ -recurrent, thus  $\mathcal{F}_\mathcal{T}$  is not in the recurrence spectrum of  $G \curvearrowright X$ . Note that if  $U \subseteq G \times X$  and  $x \in X$ , then  $U^x \in \mathcal{F}_\mathcal{T} \iff \exists n \in \mathbb{N} \forall (h_i)_{i < n} \in H^n \exists i \neq j (h_i = h_j \text{ or } h_i^{-1} h_j \in U^x)$ , so  $\mathcal{F}_\mathcal{T}$  is  $\Sigma_3^0$ -on-open. As every set  $Y \subseteq X$  with the property that  $\Delta(C \cap Y) \notin \mathcal{F}_\mathcal{T}$  for all equivalence classes  $C$  of  $F_G^X$  is  $F_G^X$ -locally very-weakly-wandering, Proposition 1.6 yields condition (3).  $\square$

We say that a set  $Y \subseteq X$  is *non-trivially transient* if there is a non-pre-compact set  $S \subseteq G$  for which  $Y$  is  $S$ -transient.

PROPOSITION 3.9. *Suppose that  $G$  is a separable group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous, and  $\mathcal{S}$  is the family of non-pre-compact sets  $S \subseteq G$ . Then the following are equivalent:*

1. There exists  $S \in \mathcal{S}$  for which the family  $\mathcal{F}_S$  is not in the recurrence spectrum of  $G \curvearrowright X$ .
2. There exist a non-pre-compact set  $S \subseteq G$  and a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $S$ -transient  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.
3. The action  $G \curvearrowright X$  has a non-trivially-transient  $\sigma$ -complete  $\Sigma_2^0$  set.

PROOF. Observe that a set  $Y \subseteq X$  is non-trivially transient if and only if it is  $\mathcal{S}$ -transient, and appeal to Proposition 3.2.  $\square$



Although we are already in position to establish the analog of Proposition 3.4 for non-trivial transience, we will again first establish an observation yielding a substantially stronger complexity bound.

**PROPOSITION 3.10.** *Suppose that  $G$  is a locally compact group,  $X$  is a topological space,  $G \curvearrowright X$  is continuous, and  $S \subseteq G$  is not pre-compact. Then every  $S$ -transient non-empty open set  $U \subseteq X$  has a non-empty open subset  $V \subseteq U$  such that for all dense sets  $H \subseteq G$ , there is a function  $\phi: S \rightarrow H$  with the property that  $\phi(S)$  is not pre-compact and  $V$  is  $\phi(S)$ -transient.*

**PROOF.** By Proposition 3.6, there exist a non-empty open set  $V \subseteq U$  and an open neighborhood  $W \subseteq G$  of  $1_G$  for which  $\Delta(V)W^{-1} \subseteq \Delta(U)$ . As  $G$  is locally compact, we can assume that  $W$  is pre-compact. As  $H \subseteq G$  is dense, there is a function  $\phi: S \rightarrow H$  with the property that  $\phi(g) \in gW$  for all  $g \in S$ . Then  $S \subseteq \phi(S)W^{-1}$ , so  $\phi(S)$  is not pre-compact. And  $\Delta(V)W^{-1} \cap S \subseteq \Delta(U) \cap S = \emptyset$ , so  $\Delta(V) \cap \phi(S) \subseteq \Delta(V) \cap SW = \emptyset$ , thus  $V$  is  $\phi(S)$ -transient.  $\square$

Given a superequivalence relation  $E$  of  $E_G^X$ , we say that a set  $Y \subseteq X$  is  $E$ -locally non-trivially-transient if its intersection with each  $E$ -class is non-trivially transient.

**PROPOSITION 3.11.** *Suppose that  $G$  is a locally compact Polish group,  $X$  is a Polish space,  $G \curvearrowright X$  is continuous,  $\mathcal{F}$  is the family of co-pre-compact sets  $S \subseteq G$  containing  $1_G$ , and  $\mathcal{S}$  is the family of non-pre-compact sets  $S \subseteq G$ . Then the following are equivalent:*

1. *The family  $\mathcal{F}$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
2. *The family  $\mathcal{F}_{\mathcal{S}}$  is not in the recurrence spectrum of  $G \curvearrowright X$ .*
3. *There is a smooth  $\aleph_0$ -universally Baire superequivalence  $F$  of  $E_G^X$  for which each action  $G \curvearrowright [x]_F$  has an  $E_G^X$ -locally non-trivially-transient  $\sigma$ -complete  $\aleph_0$ -universally-Baire set.*
4. *The action  $G \curvearrowright X$  has an  $F_G^X$ -locally non-trivially-transient  $\sigma$ -complete  $\Sigma_4^0$  set.*

**PROOF.** As  $\mathcal{F}_{\mathcal{S}}$  is the family of co-pre-compact sets  $S \subseteq G$ , the fact that  $1_G \in \Delta(Y)$  for all  $Y \subseteq X$  yields (1)  $\iff$  (2). As a set  $Y \subseteq X$  is  $E_G^X$ -locally non-trivially-transient if and only if  $\Delta(C \cap Y) \notin \mathcal{F}_{\mathcal{S}}$  for all equivalence classes  $C$  of  $E_G^X$ , Proposition 1.9 yields (2)  $\iff$  (3). The fact that every  $F_G^X$ -locally non-trivially-transient set  $Y \subseteq X$  is  $E_G^X$ -locally non-trivially-transient yields (4)  $\implies$  (2). To see (2)  $\implies$  (4), fix a countable dense set  $H \subseteq G$ , and let  $\mathcal{T}$  denote the family of non-pre-compact sets  $T \subseteq G$  that are moreover contained in  $H$ .

Now observe that if condition (2) holds, then Proposition 1.5 ensures that there is no equivalence class  $C$  of  $F_G^X$  for which  $G \curvearrowright C$  is  $\mathcal{F}_S$ -recurrent, so Proposition 3.10 implies that there is no equivalence class  $C$  of  $F_G^X$  for which  $G \curvearrowright C$  is  $\mathcal{F}_T$ -recurrent, thus  $\mathcal{F}_T$  is not in the recurrence spectrum of  $G \curvearrowright X$ . Fix an increasing sequence  $(K_n)_{n \in \mathbb{N}}$  of compact subsets of  $G$  that is *cofinal* in the sense that every compact set  $K \subseteq G$  is contained in some  $K_n$ , and note that if  $U \subseteq G \times X$  and  $x \in X$ , then  $U^x \in \mathcal{F}_T \iff \exists n \in \mathbb{N} H \subseteq K_n \cup U^x$ , so  $\mathcal{F}_T$  is  $\Sigma_3^0$ -on-open. As every set  $Y \subseteq X$  with the property that  $\Delta(C \cap Y) \notin \mathcal{F}_T$  for all equivalence classes  $C$  of  $F_G^X$  is  $F_G^X$ -locally non-trivially-transient, Proposition 1.6 yields condition (4).  $\square$

**§4. Generic compressibility.** Suppose that  $E$  is a Borel equivalence relation on  $X$  that is *countable*, in the sense that all of its equivalence classes are countable. We say that a function  $\rho: E \rightarrow (0, \infty)$  is a *cocycle* if  $\rho(x, z) = \rho(x, y)\rho(y, z)$  whenever  $x E y E z$ . When  $\rho: E \rightarrow (0, \infty)$  is a Borel cocycle, we say that a Borel measure  $\mu$  on  $X$  is  $\rho$ -invariant if  $\mu(T(B)) = \int_B \rho(T(x), x) d\mu(x)$  for all Borel sets  $B \subseteq X$  and Borel automorphisms  $T: X \rightarrow X$  such that  $\text{graph}(T) \subseteq E$ . We say that  $\rho$  is *aperiodic* if  $\sum_{y \in [x]_E} \rho(y, x) = \infty$  for all  $x \in X$ . Here we generalize the following fact to orbit equivalence relations induced by Borel actions of locally compact Polish groups, while simultaneously strengthening comeagerness to a notion under which the recurrence spectrum is invariant.

**THEOREM 4.1 (Kechris-Miller).** *Suppose that  $X$  is a standard Borel space,  $E$  is a countable Borel equivalence relation on  $X$ , and  $\rho: E \rightarrow (0, \infty)$  is an aperiodic Borel cocycle. Then there is an  $E$ -invariant comeager Borel set  $C \subseteq X$  that is null with respect to every  $\rho$ -invariant Borel probability measure.*

A function  $\phi: X \rightarrow Z$  is  $E$ -invariant if  $\phi(x) = \phi(y)$  whenever  $x E y$ . The  $E$ -saturation of a set  $Y \subseteq X$  is the set of  $x \in X$  for which there exists  $y \in Y$  such that  $x E y$ . We say that a Borel probability measure  $\mu$  on  $X$  is  $E$ -quasi-invariant if the  $E$ -saturation of every  $\mu$ -null set  $N \subseteq X$  is  $\mu$ -null. Let  $P(X)$  denote the standard Borel space of Borel probability measures on  $X$  (see, for example, [Kec95, §17.E]). The *push-forward* of a Borel measure  $\mu$  on  $X$  through a Borel function  $\phi: X \rightarrow Y$  is the Borel measure  $\phi_*\mu$  on  $Y$  given by  $(\phi_*\mu)(B) = \mu(\phi^{-1}(B))$  for all Borel sets  $B \subseteq Y$ .

PROPOSITION 4.2. *Suppose that  $X$  is a standard Borel space,  $E$  is a countable Borel equivalence relation on  $X$ , and  $\phi: X \rightarrow P(X)$  is an  $E$ -invariant Borel function such that  $\mu$  is  $E$ -quasi-invariant and  $\phi^{-1}(\mu)$  is  $\mu$ -conull for all  $\mu \in \phi(X)$ . Then there is a Borel cocycle  $\rho: E \rightarrow (0, \infty)$  such that  $\mu$  is  $\rho$ -invariant for all  $\mu \in \phi(X)$ .*

PROOF. By standard change of topology results (see, for example, [Kec95, §13]), we can assume that  $X$  is a zero-dimensional Polish space. Fix a compatible complete ultrametric on  $X$ . By [FM77, Theorem 1], there is a countable group  $\Gamma$  of Borel automorphisms of  $X$  whose induced orbit equivalence relation is  $E$ . For all  $\gamma \in \Gamma$ , define  $\rho_\gamma: X_\gamma \rightarrow (0, \infty)$  by  $\rho_\gamma(x) = \lim_{\epsilon \rightarrow 0} ((\gamma^{-1})_*\phi(x))(\mathcal{B}(x, \epsilon)) / \phi(x)(\mathcal{B}(x, \epsilon))$ , where  $X_\gamma$  is the set of  $x \in X$  for which this limit exists and lies in  $(0, \infty)$ .

Note that if  $\gamma \in \Gamma$ ,  $\mu \in \phi(X)$ , and  $\psi: X \rightarrow (0, \infty)$  is a Radon-Nikodým derivative of  $(\gamma^{-1})_*\mu$  with respect to  $\mu$  (see, for example, [Kec95, §17.A]), then the straightforward generalization of the Lebesgue density theorem for Polish ultrametric spaces (see, for example, [Mil08a, Proposition 2.10]) to integrable functions ensures that  $\psi(x) = \lim_{\epsilon \rightarrow 0} \int_{\mathcal{B}(x, \epsilon)} \psi d\mu / \mu(\mathcal{B}(x, \epsilon)) = \rho_\gamma(x)$  for  $\mu$ -almost all  $x \in X$ .

It immediately follows that for all  $\gamma \in \Gamma$ , the complement of  $X_\gamma$  is null with respect to every  $\mu \in \phi(X)$ . Moreover, if  $B \subseteq X$  is Borel,  $\gamma, \delta \in \Gamma$ , and  $\mu \in \phi(X)$ , then

$$\begin{aligned} (\gamma\delta)_*^{-1}\mu(B) &= \int_{\delta(B)} \rho_\gamma(x) d\mu(x) \\ &= \int_B \rho_\gamma(\delta \cdot x) d((\delta^{-1})_*\mu)(x) \\ &= \int_B \rho_\gamma(\delta \cdot x)\rho_\delta(x) d\mu(x), \end{aligned}$$

so the almost-everywhere uniqueness of Radon-Nikodým derivatives ensures that the set of  $x \in X$  for which there exist  $\gamma, \delta \in \Gamma$  such that  $\rho_{\gamma\delta}(x) \neq \rho_\gamma(\delta \cdot x)\rho_\delta(x)$  is null with respect to every  $\mu \in \phi(X)$ .

Let  $N$  denote the  $E$ -saturation of the union of these sets, and let  $\rho: E \rightarrow (0, \infty)$  be the extension of the constant cocycle on  $E \upharpoonright N$  given by  $\rho(\gamma \cdot x, x) = \rho_\gamma(x)$  for all  $\gamma \in \Gamma$  and  $x \in X$ .  $\square$

As a consequence, we obtain the following.

THEOREM 4.3. *Suppose that  $X$  is a Polish space,  $E$  is a Borel equivalence relation on  $X$  admitting a Borel complete set  $B \subseteq X$  on which  $E$  is countable,  $F$  is a superequivalence relation of  $E$  for which*

every  $F$ -class is  $G_\delta$  and the  $F$ -saturation of every open set is Borel, and  $\phi: X \rightarrow P(X)$  is an  $E$ -invariant Borel function for which every measure  $\mu \in \phi(X)$  has  $\mu$ -conull  $\phi$ -preimage and concentrates off of Borel sets on which  $E$  is smooth. Then there is an  $E$ -invariant Borel set  $C \subseteq X$  that is comeager in every  $F$ -class, but null with respect to every measure in  $\phi(X)$ .

PROOF. By the Lusin-Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]), there is a Borel extension  $\pi: X \rightarrow B$  of the identity function on  $B$  whose graph is contained in  $E$ . Fix a sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  of positive real numbers whose sum is 1, in addition to a countable group  $\{\gamma_n \mid n \in \mathbb{N}\}$  of Borel automorphisms of  $B$  whose induced orbit equivalence relation is  $E \upharpoonright B$ , and define  $\psi: B \rightarrow P(B)$  by  $\psi(x) = \sum_{n \in \mathbb{N}} (\gamma_n \circ \pi)_* \phi(x) / \epsilon_n$ . As each  $\nu \in \psi(B)$  is  $(E \upharpoonright B)$ -quasi-invariant, Proposition 4.2 yields a Borel cocycle  $\rho: E \upharpoonright B \rightarrow (0, \infty)$  such that every  $\nu \in \psi(B)$  is  $\rho$ -invariant.

Given  $\nu \in \psi(B)$ , fix  $x \in B$  for which  $\nu = \psi(x)$ , set  $\mu = \phi(x)$ , and observe that  $\nu(\psi^{-1}(\nu)) \geq \mu(\phi^{-1}(\mu)) = 1$ . Moreover, as  $E$  is smooth on the periodic part  $P = \{x \in B \mid \sum_{y \in [x]_{E \upharpoonright B}} \rho(y, x) < \infty\}$  of  $\rho$  (see, for example, [Mil08b, Proposition 2.1.1]), and therefore on its  $E$ -saturation, it follows that  $[P]_E$  is null with respect to every measure in  $\phi(X)$ , thus  $P$  is null with respect to every measure in  $\psi(B)$ .

By the proof of Theorem 4.1 (see [KM04, Theorem 13.1]), there is a Borel set  $R \subseteq \mathbb{N}^{\mathbb{N}} \times B$ , whose vertical sections are  $(E \upharpoonright B)$ -invariant and null with respect to every  $\rho$ -invariant Borel probability measure, such that every  $x \in B$  is contained in comeagerly-many vertical sections of  $R$ . It follows that the vertical sections of the set  $S = (\text{id} \times \pi)^{-1}(R)$  are  $E$ -invariant and null with respect to every measure in  $\phi(X)$ , and every  $x \in X$  is contained in comeagerly-many vertical sections of  $S$ . The Kuratowski-Ulam theorem therefore ensures that for all  $x \in X$ , comeagerly-many vertical sections of  $S$  are comeager in  $[x]_F$ .

By [Sri79], there is a Borel set  $D \subseteq X$  intersecting every  $F$ -class in a single point. As the  $F$ -saturation of every open set is Borel, the usual proof of the Montgomery-Novikov theorem that the pointclass of Borel sets is closed under category quantifiers (see, for example, [Kec95, Theorem 16.1]) shows that  $\{(b, x) \in \mathbb{N}^{\mathbb{N}} \times X \mid S_b \text{ is comeager in } [x]_F\}$  and  $\{(b, x) \in \mathbb{N}^{\mathbb{N}} \times X \mid S_b \text{ is non-meager in } [x]_F\}$  are Borel, so [Kec95, Theorem 18.6] yields a Borel function  $\beta: D \rightarrow$

$\mathbb{N}^{\mathbb{N}}$  such that  $S_{\beta(x)}$  is comeager in  $[x]_F$  for all  $x \in D$ . Then the set  $C = \bigcup_{x \in D} S_{\beta(x)} \cap [x]_F$  is as desired.  $\square$

We say that a function  $\rho: G \times X \rightarrow (0, \infty)$  is a *cocycle* if  $\rho(gh, x) = \rho(g, h \cdot x)\rho(h, x)$  for all  $g, h \in G$  and  $x \in X$ . When  $\rho: G \times X \rightarrow (0, \infty)$  is a Borel cocycle, we say that a Borel measure  $\mu$  on  $X$  is  $\rho$ -invariant if  $\mu(gB) = \int_B \rho(g, x) d\mu(x)$  for all Borel sets  $B \subseteq X$  and group elements  $g \in G$ . The following fact is the desired generalization of Theorem 4.1.

**THEOREM 4.4.** *Suppose that  $G$  is a locally compact Polish group,  $X$  is a Polish space,  $G \curvearrowright X$  is a continuous action,  $F$  is a superequivalence relation of  $E_G^X$  for which every  $F$ -class is  $G_\delta$  and the  $F$ -saturation of every open set is Borel, and  $\rho: G \times X \rightarrow (0, \infty)$  is a Borel cocycle with the property that every  $G$ -orbit is null with respect to every  $\rho$ -invariant Borel probability measure. Then there is a  $G$ -invariant Borel set  $C \subseteq X$  that is comeager in every  $F$ -class, but null with respect to every  $\rho$ -invariant Borel probability measure.*

**PROOF.** By [Kec92, Theorem 1.1], there is a complete Borel set  $B \subseteq X$  on which  $E_G^X$  is countable. Fix a  $\rho$ -invariant uniform ergodic decomposition  $\phi: X \rightarrow P(X)$  of  $G \curvearrowright X$  (see [GS00, Theorem 5.2]), and appeal to Theorem 4.3.  $\square$

We next check that the special case of Theorem 4.4 for  $F = F_G^X$  provides a proper strengthening of Theorem 4.1. While this can be seen as a consequence of the Kuratowski-Ulam theorem, we will show that the usual proof of the latter easily adapts to yield a generalization to a natural class of equivalence relations containing  $F_G^X$ .

**THEOREM 4.5.** *Suppose that  $X$  is a second-countable Baire space,  $E$  is an equivalence relation on  $X$  such that every  $E$ -class is a Baire space and the  $E$ -saturation of every open subset of  $X$  is open, and  $B \subseteq X$  has the Baire property. Then:*

1.  $\forall^* x \in X$   $B$  has the Baire property in  $[x]_E$ .
2.  $B$  is comeager  $\iff \forall^* x \in X$   $B$  is comeager in  $[x]_E$ .

**PROOF.** We begin with a simple observation.

**LEMMA 4.6.** *Suppose that  $U \subseteq X$  is a non-empty open set and  $V \subseteq U$  is a dense open set. Then  $[V]_E$  is dense in  $[U]_E$ .*

**PROOF.** If  $W \subseteq X$  is open and  $[V]_E \cap W = \emptyset$ , then  $V \cap [W]_E = \emptyset$ , so the openness of  $[W]_E$  ensures that  $\overline{V} \cap [W]_E = \emptyset$ , thus the density of  $V$  implies that  $U \cap [W]_E = \emptyset$ , hence  $[U]_E \cap W = \emptyset$ .  $\square$

To see the special case of  $(\implies)$  of (2) when  $B \subseteq X$  is open, note that if  $U \subseteq X$  is non-empty and open, then Lemma 4.6 yields that  $[B \cap U]_E$  is dense in  $[U]_E$ , and therefore  $\forall^* x \in X$  ( $x \in [U]_E \implies x \in [B \cap U]_E$ ), or equivalently,  $\forall^* x \in X$  ( $U \cap [x]_E \neq \emptyset \implies B \cap U \cap [x]_E \neq \emptyset$ ). As  $X$  is second countable, it follows that  $\forall^* x \in X$   $B$  is dense in  $[x]_E$ .

To see  $(\implies)$  of (2), suppose that  $B \subseteq X$  is comeager, fix dense open sets  $B_n \subseteq X$  for which  $\bigcap_{n \in \mathbb{N}} B_n \subseteq B$ , and appeal to the special case for open sets to obtain that  $\forall^* x \in X$   $\bigcap_{n \in \mathbb{N}} B_n$  is comeager in  $[x]_E$ .

To see (1), fix an open set  $U \subseteq X$  for which  $B \triangle U$  is meager, and note that  $\forall^* x \in X$   $B \triangle U$  is meager in  $[x]_E$ , by  $(\implies)$  of (2).

To see  $(\impliedby)$  of (2), suppose that  $B$  is not comeager, fix a non-empty open set  $V \subseteq X$  in which  $B$  is meager, note that  $\forall x \in V$   $V \cap [x]_E \neq \emptyset$ , and appeal to  $(\implies)$  of (2) to obtain that  $\forall^* x \in X$   $B \cap V$  is meager in  $[x]_E$ , thus  $\forall^* x \in V$   $B$  is not comeager in  $[x]_E$ .  $\square$

Finally, we check that no condition on the recurrence spectrum can yield the existence of an invariant Borel probability measure.

**THEOREM 4.7.** *Suppose that  $G$  is a locally compact Polish group,  $X$  is a standard Borel space,  $G \curvearrowright X$  is Borel, and  $\rho: G \times X \rightarrow (0, \infty)$  is a Borel cocycle for which every  $G$ -orbit is null with respect to every  $\rho$ -invariant Borel probability measure. Then there is a  $G$ -invariant Borel set  $B \subseteq X$  that is null with respect to every  $\rho$ -invariant Borel probability measure but for which the recurrence spectra of  $G \curvearrowright B$  and  $G \curvearrowright X$  coincide.*

**PROOF.** We can assume that  $X$  is a Polish space and  $G \curvearrowright X$  is continuous. By Theorem 4.4, there is an  $E_G^X$ -invariant Borel set  $B \subseteq X$  that is comeager in every  $F_G^X$ -class, but null with respect to every  $\rho$ -invariant Borel probability measure. Proposition 1.5 then ensures that  $G \curvearrowright B$  and  $G \curvearrowright X$  have the same recurrence spectra.  $\square$

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