MARKERS AND THE RATIO ERGODIC THEOREM

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Abstract. We establish a generalization and strengthening of the marker lemma for Borel automorphisms that can also be viewed as a measureless strengthening of Dowker's ratio ergodic theorem.

Introduction

Identify $[\mathbb{N}]^{<\mathbb{N}}$ with the set of strictly increasing sequences of natural numbers of finite length. A decreasing sequence $(X_i)_{i\in\mathbb{N}}$ is *vanishing* if $\bigcap_{i\in\mathbb{N}} X_i = \emptyset$. The *forward orbit* of $x \in X$ under $T\colon X \to X$ is given by $[x]_T^{\rightarrow} = \{T^k(x) \mid k \in \mathbb{N}\}$. A set $Y \subseteq X$ is *forward T-invariant* if $T(Y) \subseteq Y$, *T-bounded* if there exists $n \in \mathbb{N}$ for which $X = \bigcup_{m\leq n} T^{-m}(Y)$, and *$T$-complete* if $X = \bigcup_{n\in\mathbb{N}} T^{-n}(Y)$. In the latter case, define $n_Y^T(x) = \min\{n \in \mathbb{Z}^+ \mid T^n(x) \in Y\}$. Given $w\colon X \to (0,\infty)$, define $\rho_w^T\colon X \times \mathbb{N} \to (0,\infty)$ by $\rho_w^T(x,k) = \prod_{j<k}(w \circ T^j)(x)$. Given $f\colon X \to \mathbb{R}$, define $S_n(f,T,w)\colon X \to \mathbb{R}$ by $S_n(f,T,w)(x) = \sum_{k<n}(f \circ T^k)(x)\rho_w^T(x,k)$ for all $n \in \mathbb{N}$. Given $g\colon X \to (0,\infty)$, define $R_n(f,g,T,w)\colon X \to \mathbb{R}$ as well as $\overline{R}(f,g,T,w)\colon X \to [-\infty,\infty]$ by $R_n(f,g,T,w)(x) = S_n(f,T,w)(x)/S_n(g,T,w)(x)$ for all $n \in \mathbb{N}$ and $\overline{R}(f,g,T,w)(x) = \limsup_{n\to\infty} R_n(f,g,T,w)(x)$. A *Borel space* is a set equipped with a distinguished σ-algebra of Borel subsets. A function between Borel spaces is *Borel* if preimages of Borel sets are Borel. Here we establish the following measureless version of [Dow50, Theorem II]:

Theorem 1. *Suppose that X is a Borel space, $f\colon X \to \mathbb{R}$, $g\colon X \to (0,\infty)$, $h\colon X \to \mathbb{R}$, $T\colon X \to X$, and $w\colon X \to (0,\infty)$ are Borel, and $h(x) < \overline{R}(f,g,T,w)(x)$ for all $x \in X$. Then there exist a forward T-invariant T-complete Borel set $C \subseteq X$, a decreasing vanishing sequence $(A_i)_{i\in\mathbb{N}}$ of Borel subsets of C, a decreasing sequence $(B_i)_{i\in\mathbb{N}}$ of $(T \restriction C)$-bounded Borel subsets of C, and Borel functions $s_i\colon B_i \to [\mathbb{N}]^{<\mathbb{N}}$ such that, for all $i \in \mathbb{N}$ and $x \in B_i$, the following hold:*

(1) $s_i(x)(0) = 0$ and $s_i(x)(|s_i(x)| - 1) = n_{B_i}^T(x)$.

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(2) For all $k < |s_i(x)| - 1$, exactly one of the following holds:
(a) $T^{s_i(x)(k)}(x) \in A_i$ and $s_i(x)(k+1) = s_i(x)(k) + 1$.
(b) $(R_{s_i(x)(k+1)-s_i(x)(k)}(f, g, T, w) \circ T^{s_i(x)(k)})(x) > (h \circ T^{s_i(x)(k)})(x)$.

In §1, we establish an elementary decomposition result allowing us to assume that $f \times g \times w$ is eventually periodic along the forward orbits of T or T is aperiodic and satisfies a local notion of separability. In §2 and §3, we establish strengthenings of Theorem 1 in both cases. And in §4, we show that Theorem 1 implies Dowker's ratio ergodic theorem.

1. Decomposition

A family $\mathcal{B}$ of subsets of a set X *separates points* if, for all distinct $x, y \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$ but $y \notin B$. We say that a Borel space X is *separable* if there is a countable family of Borel subsets of X that separates points. This easily implies that the equality relation on X is Borel.

Given $n \in \mathbb{Z}^+$ and $T \colon X \to X$, the *T-period n part* of $f \colon X \to Y$ is given by $\mathrm{Per}_n^T(f) = \{x \in X \mid \forall y \in [x]_T^{\rightarrow}\ f(y) = (f \circ T^n)(y)\}$. If X and Y are Borel spaces, Y is separable, and f and T are Borel, then the fact that the class of Borel functions is closed under appropriate compositions and products ensures that $\mathrm{Per}_n^T(f)$ is Borel.

Given a binary relation R on a set X, we say that a family $\mathcal{B}$ of subsets of X *separates R-related points* if, for all distinct $x\ R\ y$, there exists $B \in \mathcal{B}$ such that $x \in B$ but $y \notin B$. When X is a Borel space, we say that R is *separable* if there is a countable family of Borel sets that separates R-related points.

Proposition 1.1. *Suppose that $n \in \mathbb{Z}^+$, X and Y are Borel spaces, Y is separable, $f \colon X \to Y$ and $T \colon X \to X$ are Borel, and $\mathrm{Per}_n^T(f) = \emptyset$. Then T^n has no fixed points and its graph is separable.*

Proof. Fix a countable family $\mathcal{B}$ of Borel subsets of Y that separates points. We need only show that, for all $x \in X$, there exist $B \in \mathcal{B}$ and $k \in \mathbb{N}$ with $x \in (f \circ T^k)^{-1}(B)$ but $T^n(x) \notin (f \circ T^k)^{-1}(B)$. But $x \notin \mathrm{Per}_n^T(f)$, so there exists $k \in \mathbb{N}$ with $(f \circ T^k)(x) \neq (f \circ T^{k+n})(x)$, thus there exists $B \in \mathcal{B}$ with $(f \circ T^k)(x) \in B$ but $(f \circ T^{k+n})(x) \notin B$. ⊠

We say that T is *aperiodic* if its positive powers are fixed-point free. We say that T is *separable* if its positive powers have separable graphs.

Proposition 1.2. *Suppose that X and Y are Borel spaces, Y is separable, $f \colon X \to Y$ and $T \colon X \to X$ are Borel, and $\bigcup_{n \in \mathbb{Z}^+} \mathrm{Per}_n^T(f) = \emptyset$. Then T is aperiodic and separable.*

Proof. By Proposition 1.1. ⊠

2. The aperiodic separable case

The proof of the marker lemma for Borel automorphisms (see [SS88, Lemma 1 of §3]) generalizes beyond the injective and standard cases:

Proposition 2.1. *Suppose that X is a Borel space and $T \colon X \to X$ is an aperiodic separable Borel function. Then there is a decreasing vanishing sequence $(B_i)_{i \in \mathbb{N}}$ of T-complete Borel subsets of X.*

Proof. Set $R = \bigcup_{n \in \mathbb{Z}^+} \mathrm{graph}(T^n)$. Then the separability of T yields a family $\{A_i \mid i \in \mathbb{N}\}$ of Borel subsets of X that separates R-related points. Set $A_s = \bigcap_{i \in s^{-1}(\{0\})} A_i \cap \bigcap_{i \in s^{-1}(\{1\})} \sim A_i$ for all $s \in 2^{<\mathbb{N}}$, let $\leq_i$ denote the lexicographical ordering of 2^i for all $i \in \mathbb{N}$, and define $s_i(x) = \min_{\leq_i} \{s \in 2^i \mid |A_s \cap [x]^{\rightarrow}_T| = \aleph_0\}$ for all $i \in \mathbb{N}$ and $x \in X$. As s_i is T-invariant, the intersection of the set $B'_i = \bigcup_{s \in 2^i} A_s \cap s_i^{-1}(\{s\})$ with each forward orbit of T is infinite. The fact that $A_s = A_{s \frown (0)} \amalg A_{s \frown (1)}$ for all $s \in 2^{<\mathbb{N}}$ ensures that $s_i^{-1}(\{s\}) = s_{i+1}^{-1}(\{s \frown (0)\}) \amalg s_{i+1}^{-1}(\{s \frown (1)\})$ for all $i \in \mathbb{N}$ and $s \in 2^i$, so $B'_{i+1} \subseteq B'_i$ for all $i \in \mathbb{N}$. And the set $B' = \bigcap_{i \in \mathbb{N}} B'_i$ intersects each forward orbit of T in at most one point, for if $k \in \mathbb{Z}^+$ and $x \in B'$, then there exists $i \in \mathbb{N}$ such that $x \in A_i$ and $T^k(x) \notin A_i$, so the fact that $x \in B'_{i+1}$ implies that $T^k(x) \notin B'_{i+1}$, thus $T^k(x) \notin B'$. The sets $B_i = B'_i \setminus B'$ are therefore as desired. ⊠

Set $T^{-\leq i}(Y) = \bigcup_{j \leq i} T^{-j}(Y)$ for all $i \in \mathbb{N}$, $T \colon X \to X$, and $Y \subseteq X$.

Proposition 2.2. *Suppose that X is a Borel space, $T \colon X \to X$ is Borel, and there is a decreasing vanishing sequence $(A_i)_{i \in \mathbb{N}}$ of T-complete Borel sets. Then there is a decreasing vanishing sequence $(B_i)_{i \in \mathbb{N}}$ of T-bounded Borel sets.*

Proof. We can assume that $A_0 = X$. For all $i \in \mathbb{N}$, define $B_i = A_i \cup \bigcup_{j < i} A_j \setminus T^{-\leq i}(A_{j+1})$. To see that $B_{i+1} \subseteq B_i$ for all $i \in \mathbb{N}$, note that $A_{i+1}, A_i \setminus T^{-\leq i+1}(A_{i+1}) \subseteq A_i$ and $A_j \setminus T^{-\leq i+1}(A_{j+1}) \subseteq A_j \setminus T^{-\leq i}(A_{j+1})$ for all $j < i$. To see that $\bigcap_{i \in \mathbb{N}} B_i = \emptyset$, note that if $j \in \mathbb{N}$ and $x \in A_j \setminus A_{j+1}$, then there exists $i > j$ for which $x \in T^{-\leq i}(A_{j+1})$, so $x \notin B_i$. And to see that $X = T^{-\leq i^2}(B_i)$ for all $i \in \mathbb{N}$, note that if $j < i$, then $A_j \subseteq B_i \cup T^{-\leq i}(A_{j+1})$, so $A_j \subseteq T^{-\leq i(i-j)}(B_i)$ by induction on $i - j$. ⊠

The following fact generalizes and strengthens the special case of Theorem 1 where T is aperiodic and separable:

Proposition 2.3. *Suppose that $\ell \in \mathbb{N}$, X is a Borel space, Y is a metric space, $\epsilon \colon X \to (0, \infty)$, $f_n, h_j \colon X \to Y$, and $T \colon X \to X$ are Borel, T is aperiodic and separable, and $h_j(x) \in \overline{\{f_n(x) \mid n \in \mathbb{N}\}}$ for all $j < \ell$ and $x \in X$. Then there are decreasing vanishing sequences $(A_i)_{i \in \mathbb{N}}$ and $(B_i)_{i \in \mathbb{N}}$ of Borel subsets of X and Borel functions $s_{i,j} \colon B_i \to [\mathbb{N}]^{<\mathbb{N}}$*

such that each B_i is T-bounded and, for all $i \in \mathbb{N}$, $j < \ell$, and $x \in B_i$, the following hold:

(1) $s_{i,j}(x)(0) = 0$ *and* $s_{i,j}(x)(|s_{i,j}(x)| - 1) = n^T_{B_i}(x)$.
(2) *For all* $k < |s_{i,j}(x)| - 1$, *exactly one of the following holds:*
   (a) $T^{s_{i,j}(x)(k)}(x) \in A_i$ *and* $s_{i,j}(x)(k+1) = s_{i,j}(x)(k) + 1$.
   (b) $d_Y((f_{s_{i,j}(x)(k+1)-s_{i,j}(x)(k)} \circ T^{s_{i,j}(x)(k)})(x), (h_j \circ T^{s_{i,j}(x)(k)})(x)) < (\epsilon \circ T^{s_{i,j}(x)(k)})(x)$.

Proof. By Propositions 2.1 and 2.2, there is a decreasing vanishing sequence $(B_i)_{i\in\mathbb{N}}$ of T-bounded Borel sets. Then the sequence given by $A_i = \{x \in X \mid \exists j < \ell \forall n < n^T_{B_i}(x)\ d_Y(f_n(x), h_j(x)) \geq \epsilon(x)\}$ is also decreasing and vanishing. For all $i \in \mathbb{N}$, $j < \ell$, and $x \in B_i$, define $s_{i,j}(x)(0) = 0$. Suppose now that $k \in \mathbb{N}$ and $s_{i,j}(x)(k)$ has been defined and is strictly less than $n^T_{B_i}(x)$. If $T^{s_{i,j}(x)(k)}(x) \in A_i$, then define $s_{i,j}(x)(k+1) = s_{i,j}(x)(k)+1$. Otherwise, let $s_{i,j}(x)(k+1)$ be the least natural number $n > s_{i,j}(x)(k)$ with the property that $d_Y((f_{n-s_{i,j}(x)(k)} \circ T^{s_{i,j}(x)(k)})(x), (h_j \circ T^{s_{i,j}(x)(k)})(x)) < (\epsilon \circ T^{s_{i,j}(x)(k)})(x)$. ⊠

3. The eventually periodic case

The function ρ^T_w satisfies an appropriate cocycle identity:

Proposition 3.1. *Suppose that X is a set, $T\colon X \to X$, and $w\colon X \to (0,\infty)$. Then $\forall m, n \in \mathbb{N} \forall x \in X\ \rho^T_w(x, m+n) = \rho^T_w(x, m)\rho^T_w(T^m(x), n)$.*

Proof. We need only note that $\rho^T_w(x, m+n) = \prod_{k<m+n}(w \circ T^k)(x) = (\prod_{k<m}(w \circ T^k)(x))(\prod_{k<n}(w \circ T^{k+m})(x)) = \rho^T_w(x, m)\rho^T_w(T^m(x), n)$. ⊠

The set $\mathcal{R}'(f, g, T, w)(x)$ of limit points of $\{R_n(f, g, T, w)(x) \mid n \in \mathbb{N}\}$ is easily computed in the periodic case:

Proposition 3.2. *Suppose that $n \in \mathbb{Z}^+$, $r \in \mathbb{N}$, X is a set, $f\colon X \to \mathbb{R}$, $g\colon X \to (0,\infty)$, $T\colon X \to X$, $w\colon X \to (0,\infty)$, and $x \in Per^T_n(f \times g \times w)$.*

(1) *If $\rho^T_w(x, n) \leq 1$, then $R_{nq+r}(f, g, T, w)(x) \to R_n(f, g, T, w)(x)$.*
(2) *If $\rho^T_w(x, n) > 1$, then $R_{nq+r}(f, g, T, w)(x) \to (R_n(f, g, T, w) \circ T^r)(x)$.*

Proof. Repeated application of Proposition 3.1 ensures that if $q, s \in \mathbb{N}$, then $\rho^T_w(x, nq+s) = \rho^T_w(x, s)\prod_{p<q}\rho^T_w(T^{np+s}(x), n) = \rho^T_w(x, s)\rho^T_w(x, n)^q$.

It follows that if $h \in \{f, g\}$ and $q \in \mathbb{N}$, then

$$\begin{aligned} &S_{nq+r}(h, T, w)(x) \\ &\quad = \textstyle\sum_{p<q} \sum_{s<n} (h \circ T^{np+s})(x) \rho_w^T(x, np + s) + \\ &\quad\quad \textstyle\sum_{s<r} (h \circ T^{nq+s})(x) \rho_w^T(x, nq + s) \\ &\quad = \textstyle\sum_{p<q} \rho_w^T(x, n)^p \sum_{s<n} (h \circ T^s)(x) \rho_w^T(x, s) + \\ &\quad\quad \rho_w^T(x, n)^q \textstyle\sum_{s<r} (h \circ T^s)(x) \rho_w^T(x, s) \\ &\quad = (\textstyle\sum_{p<q} \rho_w^T(x, n)^p) S_n(h, T, w)(x) + \rho_w^T(x, n)^q S_r(h, T, w)(x). \end{aligned}$$

Case 1: If $\rho_w^T(x, n) < 1$, then $\sum_{p<q} \rho_w^T(x, n)^p \to 1/(1 - \rho_w^T(x, n))$ and $\rho_w^T(x, n)^q \to 0$, so $S_{nq+r}(h, T, w)(x) \to S_n(h, T, w)(x)/(1 - \rho_w^T(x, n))$, thus $R_{nq+r}(f, g, T, w)(x) \to R_n(f, g, T, w)(x)$.

Case 2: If $\rho_w^T(x, n) = 1$, then $\sum_{p<q} \rho_w^T(x, n)^p = q$ and $\rho_w^T(x, n)^q = 1$, so $S_{nq+r}(h, T, w)(x) = qS_n(h, T, w)(x) + S_r(h, T, w)(x)$, in which case $R_{nq+r}(f, g, T, w)(x) \to R_n(f, g, T, w)(x)$.

Case 3: If $\rho_w^T(x, n) > 1$, then set $s_q = \sum_{p<q} \rho_w^T(x, n)^p$ and observe that $\rho_w^T(x, n)^q = (\rho_w^T(x, n) - 1)s_q + 1$, so

$$\begin{aligned} &S_{nq+r}(h, T, w)(x) \\ &\quad = s_q(S_n(h, T, w)(x) + (\rho_w^T(x, n) - 1)S_r(h, T, w)(x)) + S_r(h, T, w)(x) \\ &\quad = s_q(S_n(h, T, w) \circ T^r)(x) + S_r(h, T, w)(x). \end{aligned}$$

But $s_q \to \infty$, thus $R_{nq+r}(f, g, T, w)(x) \to (R_n(f, g, T, w) \circ T^r)(x)$. ⊠

We say that $h \colon X \to Y$ is *eventually T-periodic* if $\bigcup_{n \in \mathbb{Z}^+} \mathrm{Per}_n^T(h)$ is T-complete. The following fact strengthens the special case of Theorem 1 where $f \times g \times w$ is eventually T-periodic:

Proposition 3.3. *Suppose that X is a Borel space, $f \colon X \to \mathbb{R}$, $g \colon X \to (0, \infty)$, $T \colon X \to X$, and $w \colon X \to (0, \infty)$ are Borel, and $f \times g \times w$ is eventually T-periodic. Then there exist a forward T-invariant T-complete Borel set $C \subseteq X$ and a $(T \restriction C)$-bounded Borel set $B \subseteq C$ such that $R_{n_B^T(x)}(f, g, T, w)(x) = \overline{R}(f, g, T, w)(x)$ for all $x \in B$.*

Proof. Define $C = \bigcup_{n \in \mathbb{Z}^+} \mathrm{Per}_n^T(f \times g \times w)$ as well as $n \colon C \to \mathbb{Z}^+$ by $n(x) = \min\{n \in \mathbb{Z}^+ \mid x \in \mathrm{Per}_n^T(f \times g \times w)\}$. By Proposition 3.2, the set $A = \{x \in C \mid R_{n(x)}(f, g, T, w)(x) = \overline{R}(f, g, T, w)(x)\}$ is T-complete. Endow the set $Y = \mathbb{R} \times (0, \infty) \times (0, \infty)$ with the lexicographical ordering, let $\leq_{\mathrm{lex}}$ denote the corresponding lexicographical ordering of $Y^{\mathbb{N}}$, and define $\phi \colon C \to Y^{\mathbb{N}}$ by $\phi(x)(k) = ((f \times g \times w) \circ T^k)(x)$. Then the set $B = \{x \in A \mid \forall y \in A \cap [x]_T^{\rightarrow}\ \phi(x) \leq_{\mathrm{lex}} \phi(y)\}$ is as desired, since $n(x) = n_B^T(x)$ for all $x \in B$. ⊠

Theorem 1 follows from Propositions 1.2, 2.3, and 3.3.

4. The ratio ergodic theorem

Here we show that Theorem 1 implies Dowker's ratio ergodic theorem by using the former to obtain a new proof of Proposition 4.5. The other results of this section are well known and provided for completeness.

Proposition 4.1. *Suppose that X is a set, $f \colon X \to \mathbb{R}$, $T \colon X \to X$, $w \colon X \to (0, \infty)$, $n \in \mathbb{N}$, and $x \in X$. Then $S_{n+1}(f, T, w)(x) = f(x) + w(x)(S_n(h, T, w) \circ T)(x)$.*

Proof. Simply observe that

$$\begin{aligned} S_{n+1}(f, T, w)(x) &= f(x) + \textstyle\sum_{0<k<n+1}(f \circ T^k)(x)\rho_w^T(x, k) \\ &= f(x) + \textstyle\sum_{k<n}(f \circ T^{k+1})(x)\rho_w^T(x, k+1) \\ &= f(x) + w(x)(S_n(f, T, w) \circ T)(x), \end{aligned}$$

since $\rho_w^T(x, k+1) = w(x)\rho_w^T(T(x), k)$ by Proposition 3.1. ⊠

Proposition 4.2. *Suppose that X is a set, $f \colon X \to \mathbb{R}$, $g \colon X \to (0, \infty)$, $T \colon X \to X$, $w \colon X \to (0, \infty)$, and $S_n(g, T, w)(x) \to \infty$ for all $x \in X$. Then $\mathcal{R}'(f, g, T, w)$ is T-invariant.*

Proof. For all $n \in \mathbb{N}$ and $x \in X$, set $\epsilon_n(x) = f(x)/S_n(g, T, w)(x)$ and $r_n(x) = (S_n(g, T, w)(x) - g(x))/S_n(g, T, w)(x)$. Then

$$\begin{aligned} & R_{n+1}(f, g, T, w)(x) \\ &\quad = \tfrac{f(x)+(S_n(f,T,w)\circ T)(x)w(x)}{S_{n+1}(g,T,w)(x)} \\ &\quad = \epsilon_{n+1}(x) + (R_n(f, g, T, w) \circ T)(x)\left(\tfrac{(S_n(g,T,w)\circ T)(x)w(x)}{S_{n+1}(g,T,w)(x)}\right) \\ &\quad = \epsilon_{n+1}(x) + (R_n(f, g, T, w) \circ T)(x)r_{n+1}(x) \end{aligned}$$

by two applications of Proposition 4.1. As $\epsilon_n(x) \to 0$ and $r_n(x) \to 1$, it easily follows that $(\mathcal{R}'(f, g, T, w) \circ T)(x) = \mathcal{R}'(f, g, T, w)(x)$. ⊠

A *Borel measure* on a Borel space X is a measure μ on the Borel subsets of X. Given $T \colon X \to X$ and $w \colon X \to (0, \infty)$, we say that μ is *T-w-invariant* if $\mu(B) = \int_{T^{-1}(B)} w \; d\mu$ for all Borel sets $B \subseteq X$.

Proposition 4.3. *Suppose that $k \in \mathbb{Z}^+$, X is a Borel space, $f \colon X \to \mathbb{R}$, $T \colon X \to X$, and $w \colon X \to (0, \infty)$ are Borel, and μ is a T-w-invariant Borel measure on X for which f is μ-integrable. Then $\int f \; d\mu = \int (f \circ T^k)(x)\rho_w^T(x, k) \; d\mu(x)$.*

Proof. Let ν be the Borel measure on X given by $\nu(B) = \int_B w \; d\mu$. Then $\mu(B) = \int_{T^{-1}(B)} w \; d\mu = \nu(T^{-1}(B))$ for all Borel sets $B \subseteq X$, so $\mu = T_*\nu$, thus $\int g \; d\mu = \int g \; d(T_*\nu) = \int g \circ T \; d\nu = \int (g \circ T)w \; d\mu$ for all

μ-integrable Borel functions $g\colon X \to \mathbb{R}$, hence Proposition 3.1 ensures that if $x \mapsto (f \circ T^k)(x)\rho_w^T(x,k)$ is μ-integrable, then

$$\begin{aligned}\int (f \circ T^k)(x)\rho_w^T(x,k)\ d\mu(x) &= \int (f \circ T^{k+1})(x)\rho_w^T(T(x),k)w(x)\ d\mu(x)\\ &= \int (f \circ T^{k+1})(x)\rho_w^T(x,k+1)\ d\mu(x),\end{aligned}$$

in which case the obvious induction on k yields the desired result. ⊠

We use $\mathbb{1}_Y$ to denote the characteristic function of a set $Y \subseteq X$.

Proposition 4.4. *Suppose that X is a Borel space, $f\colon X \to \mathbb{R}$, $T\colon X \to X$, and $w\colon X \to (0,\infty)$ are Borel, $B \subseteq X$ is a T-bounded Borel set, and μ is a T-w-invariant Borel measure on X for which f is μ-integrable. Then $\int f\ d\mu = \int_B S_{n_B^T(x)}(f,T,w)(x)\ d\mu(x)$.*

Proof. For all $n \in \mathbb{N}$, set $B_n = \bigcup_{1 \leq k \leq n} T^{-k}(B)$. Then

$$\begin{aligned}&\int_{\sim(B \cup B_n)} (f \circ T^n)(x)\rho_w^T(x,n)\ d\mu(x)\\ &\quad= \int_{\sim T^{-1}(B \cup B_n)} (f \circ T^{n+1})(x)\rho_w^T(T(x),n)w(x)\ d\mu(x)\\ &\quad= \int_{\sim T^{-1}(B \cup B_n)} (f \circ T^{n+1})(x)\rho_w^T(x,n+1)\ d\mu(x)\\ &\quad= \int_{\sim B_{n+1}} (f \circ T^{n+1})(x)\rho_w^T(x,n+1)\ d\mu(x)\\ &\quad= \int_{\sim(B \cup B_{n+1})} (f \circ T^{n+1})(x)\rho_w^T(x,n+1)\ d\mu(x) +\\ &\qquad \int_{B \setminus B_{n+1}} (f \circ T^{n+1})(x)\rho_w^T(x,n+1)\ d\mu(x)\end{aligned}$$

by Propositions 3.1 and 4.3. As $\int f\ d\mu = \int_{\sim(B \cup B_0)} f\ d\mu + \int_{B \setminus B_0} f\ d\mu$, the obvious induction ensures that if $n \in \mathbb{N}$, then

$$\begin{aligned}\int f\ d\mu = &\int_{\sim(B \cup B_n)} (f \circ T^n)(x)\rho_w^T(x,n)\ d\mu(x) +\\ &\sum_{k \leq n} \int_{B \setminus B_k} (f \circ T^k)(x)\rho_w^T(x,k)\ d\mu(x).\end{aligned}$$

As B is T-bounded, there exists $n \in \mathbb{N}$ for which $X = B \cup B_n$, so

$$\begin{aligned}\int f\ d\mu &= \sum_{k \leq n} \int_{B \setminus B_k} (f \circ T^k)(x)\rho_w^T(x,k)\ d\mu(x)\\ &= \int_B \sum_{k \leq n} \mathbb{1}_{\sim B_k}(x)(f \circ T^k)(x)\rho_w^T(x,k)\ d\mu(x).\end{aligned}$$

But the latter integrand is $S_{n_B^T(x)}(f,T,w)(x)$ for all $x \in B$. ⊠

A subset of X is *T-wandering* if it intersects every forward orbit of T in at most one point. A Borel measure on X is *T-conservative* if every T-wandering Borel subset of X is null.

Proposition 4.5. *Suppose that X is a Borel space, $f\colon X \to \mathbb{R}$, $g\colon X \to (0,\infty)$, $h\colon X \to \mathbb{R}$, $T\colon X \to X$, and $w\colon X \to (0,\infty)$ are Borel, h is T-invariant, $h(x) \leq \overline{R}(f,g,T,w)(x)$ for all $x \in X$, and μ is a T-conservative T-w-invariant Borel measure on X for which f, g, and gh are μ-integrable. Then $\int f\ d\mu \geq \int gh\ d\mu$.*

Proof. We need only show that $\int f \, d\mu \geq \int gh \, d\mu - \epsilon$ for all $\epsilon > 0$, so we can assume that $h(x) < \overline{R}(f, g, T, w)(x)$ for all $x \in X$. Fix $A_i, B_i, C \subseteq X$ and $s_i \colon B_i \to \mathbb{N}^{<\mathbb{N}}$ satisfying the conclusion of Theorem 1 and $i \in \mathbb{N}$ sufficiently large that $\int_{A_i} |f| + |gh| \, d\mu \leq \epsilon$. Set $A = A_i$, $B = B_i$, $s = s_i$, $K(x) = \{k < |s(x)| - 1 \mid T^{s(x)(k)}(x) \notin A\}$, $N(x) = \bigcup_{k \in K(x)} \{s(x)(k), \ldots, s(x)(k+1) - 1\}$, $D(x) = \{T^n(x) \mid n \in N(x)\}$, and $D'(x) = \{T^n(x) \mid n < n_B^T(x)\} \setminus D(x)$ for all $x \in B$. Then

$$\begin{aligned} \textstyle\int_C f \, d\mu &= \textstyle\int_B S_{n_B^T(x)}(f, T, w)(x) \, d\mu(x) \\ &= \textstyle\int_B S_{n_B^T(x)}(f \mathbb{1}_{D(x)}, T, w)(x) + S_{n_B^T(x)}(f \mathbb{1}_{D'(x)}, T, w)(x) \, d\mu(x) \\ &\geq \textstyle\int_B S_{n_B^T(x)}(f \mathbb{1}_{D(x)}, T, w)(x) - S_{n_B^T(x)}(|f| \mathbb{1}_A, T, w)(x) \, d\mu(x) \\ &= \textstyle\int_B S_{n_B^T(x)}(f \mathbb{1}_{D(x)}, T, w)(x) \, d\mu(x) - \int_A |f| \, d\mu \end{aligned}$$

by Proposition 4.4 and

$$\begin{aligned} &\textstyle\int_B S_{n_B^T(x)}(f \mathbb{1}_{D(x)}, T, w)(x) \, d\mu(x) \\ &\quad = \textstyle\int_B \sum_{k \in K(x)} (S_{s(x)(k+1) - s(x)(k)}(f, T, w) \circ T^{s(x)(k)})(x) \\ &\qquad\qquad \rho_w^T(x, s(x)(k)) \, d\mu(x) \\ &\quad \geq \textstyle\int_B \sum_{k \in K(x)} (h \circ T^{s(x)(k)})(x) (S_{s(x)(k+1) - s(x)(k)}(g, T, w) \circ T^{s(x)(k)})(x) \\ &\qquad\qquad \rho_w^T(x, s(x)(k)) \, d\mu(x) \\ &\quad = \textstyle\int_B \sum_{k \in K(x)} (S_{s(x)(k+1) - s(x)(k)}(gh, T, w) \circ T^{s(x)(k)})(x) \\ &\qquad\qquad \rho_w^T(x, s(x)(k)) \, d\mu(x) \\ &\quad = \textstyle\int_B S_{n_B^T(x)}(gh, T, w)(x) - S_{n_B^T(x)}(gh \mathbb{1}_{D'(x)}, T, w)(x) \, d\mu(x) \\ &\quad \geq \textstyle\int_B S_{n_B^T(x)}(gh, T, w)(x) - S_{n_B^T(x)}(|gh| \mathbb{1}_A, T, w)(x) \, d\mu(x) \\ &\quad = \textstyle\int_C gh \, d\mu - \int_A |gh| \, d\mu \end{aligned}$$

by Proposition 3.1, the T-invariance of h, and Proposition 4.4, in which case $\int_C f \, d\mu \geq \int_C gh \, d\mu - \int_A |f| + |gh| \, d\mu \geq \int_C gh \, d\mu - \epsilon$, so we need only check that C is μ-conull. But the forward T-invariance of C ensures that the set $C'_n = T^{-(n+1)}(C) \setminus T^{-n}(C)$ is T-wandering for all $n \in \mathbb{N}$ and the T-completeness of C implies that $\sim C = \bigcup_{n \in \mathbb{N}} C'_n$. ⊠

Define $R_\infty(f, g, T, w)(x) = \lim_{n \to \infty} R_n(f, g, T, w)(x)$ for all $x \in X$.

Theorem 4.6 (Dowker's ratio ergodic theorem)**.** *Suppose that X is a Borel space, $f \colon X \to \mathbb{R}$, $g \colon X \to (0, \infty)$, $T \colon X \to X$, and $w \colon X \to (0, \infty)$ are Borel, $S_n(g, T, w)(x) \to \infty$ for all $x \in X$, and μ is a T-conservative T-w-invariant Borel measure on X for which f and g are μ-integrable. Then $\int f \, d\mu = \int g R_\infty(f, g, T, w) \, d\mu$.*

Proof. Define $\underline{R}(f,g,T,w)(x) = \liminf_{n\to\infty} R_n(f,g,T,w)(x)$ for all $x \in X$. It is sufficient to show that $\int f\ d\mu \geq \int g\overline{R}(f,g,T,w)\ d\mu$, as the corresponding ostensible weakening of the theorem implies that

$$\begin{aligned}\int f\ d\mu &= -\int -f\ d\mu \\ &\leq -\int g\overline{R}(-f,g,T,w)\ d\mu \\ &= -\int -g\underline{R}(f,g,T,w)\ d\mu \\ &= \int g\underline{R}(f,g,T,w)\ d\mu.\end{aligned}$$

Lemma 4.7. *The function $g\overline{R}(f,g,T,w)$ is μ-integrable.*

Proof. If $\int g|\overline{R}(f,g,T,w)|\ d\mu = \infty$, then there exists $r > 0$ for which $\int |f|\ d\mu < \int g \min\{|\overline{R}(f,g,T,w)|, r\}\ d\mu$. But $\overline{R}(f,g,T,w)$ is T-invariant by Proposition 4.2, so $\int |f|\ d\mu \geq \int g \min\{\overline{R}(|f|,g,T,w), r\}\ d\mu \geq \int g \min\{|\overline{R}(f,g,T,w)|, r\}\ d\mu$ by an application of Proposition 4.5 at $|f|$, g, and $\min\{\overline{R}(|f|,g,T,w), r\}$, a contradiction. ⊠

Lemma 4.7 implies that the Borel set $B = \overline{R}(f,g,T,w)^{-1}(\mathbb{R})$ is μ-conull. As $\overline{R}(f,g,T,w)$ is T-invariant, so too is B and therefore $\overline{R}(f,g,T,w)\mathbb{1}_B$, thus an application of Proposition 4.5 at f, g, and $\overline{R}(f,g,T,w)\mathbb{1}_B$ ensures that $\int f\ d\mu \geq \int g\overline{R}(f,g,T,w)\ d\mu$. ⊠

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