THE FELDMAN–MOORE, GLIMM–EFFROS, AND LUSIN–NOVIKOV THEOREMS OVER QUOTIENTS

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Abstract. We establish generalizations of the Feldman–Moore theorem, the Glimm–Effros dichotomy, and the Lusin–Novikov uniformization theorem from Polish spaces to their quotients by Borel orbit equivalence relations.

Introduction

A topological space is *Polish* if it is separable and admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets. Given an equivalence relation E on a topological space X, we say that a set $B \subseteq X/E$ is *Borel* if $\bigcup B$ is Borel. More generally, given equivalence relations E_i on topological spaces X_i, we say that a set $R \subseteq \prod_{i \in I} X_i/E_i$ is *weakly Borel* if the corresponding set $\tilde{R} = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid ([x_i]_{E_i})_{i \in I} \in R\}$ is Borel. Given equivalence relations E and F on topological spaces X and Y, we say that a partial function $\phi \colon X/E \rightharpoonup Y/F$ is *strongly Borel* if its graph is weakly Borel.

A *σ-ideal* on a set X is a family of subsets of X that is closed under containment and countable unions. When X is a Polish space, we say that an assignment $x \mapsto \mathcal{I}_x$, sending each point of X to a σ-ideal on X, is *strongly Borel-on-Borel* if $\{(x, y) \in X \times Y \mid R_{(x,y)} \in \mathcal{I}_x\}$ is Borel for all Polish spaces Y and Borel sets $R \subseteq (X \times Y) \times X$. A Borel equivalence relation E on a Polish space X is *strongly idealistic* if there is an E-invariant strongly Borel-on-Borel assignment $x \mapsto \mathcal{I}_x$ sending each point in X to a σ-ideal on X for which $[x]_E \notin \mathcal{I}_x$. Following the usual abuse of language, we say that an equivalence relation is *countable* if each of its equivalence classes is countable. The Feldman–Moore theorem ensures that every countable Borel equivalence relation on a Polish space is the orbit equivalence relation induced by a Borel action of a countable discrete group (see [FM77, Theorem 1]), and

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the proof of [Kec92, §1.II.i] shows that every Borel orbit equivalence relation induced by a Borel action of a Polish group on a Polish space is strongly idealistic.

Our goal here is to generalize several basic results underlying the study of countable Borel equivalence relations from Polish spaces to their quotients by strongly idealistic Borel equivalence relations. More precisely, we provide countably-infinite bases of minimal counterexamples to such generalizations.

A *transversal* of an equivalence relation E on X is a set $Y \subseteq X$ that intersects every E-class in exactly one point. An *embedding* of E into an equivalence relation F on a set Y is an injection $\pi \colon X \to Y$ such that $w \mathrel{E} x \iff \pi(w) \mathrel{F} \pi(x)$ for all $w, x \in X$. The *diagonal* on X is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$ and $\mathbb{E}_0$ is the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathrel{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m)$. We use $\mathbb{F}_k$ to denote the index k subequivalence relation of $\mathbb{E}_0$ given by $c \mathrel{\mathbb{F}_k} d \iff \exists n \in \mathbb{N} \forall m \geq n \; \sum_{i<m} c(i) \equiv \sum_{i<m} d(i) \pmod{k}$ for all $k \geq 2$. The following fact generalizes the Glimm–Effros dichotomy for countable Borel equivalence relations from Polish spaces to their quotients by strongly idealistic Borel equivalence relations:

Theorem 1. *Suppose that X is the quotient of a Polish space by a strongly idealistic Borel equivalence relation and E is a countable weakly Borel equivalence relation on X. Then exactly one of the following holds:*

1. *The set X is a countable union of Borel transversals of E.*
2. *There is a strongly Borel embedding of $\mathbb{E}_0/\mathbb{F}$ into E for some $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$.*

The *horizontal sections* of a set $R \subseteq X \times Y$ are the sets of the form $R^y = \{x \in X \mid x \mathrel{R} y\}$ for $y \in Y$, whereas the *vertical sections* are those of the form $R_x = \{y \in Y \mid x \mathrel{R} y\}$ for $x \in X$. The *product* of equivalence relations E on X and F on Y is the equivalence relation on $X \times Y$ given by $(x, y) \mathrel{E \times F} (x', y') \iff (x \mathrel{E} x' \text{ and } y \mathrel{F} y')$. A *rectangular homomorphism* from a binary relation R on $X \times Y$ to a binary relation R' on $X' \times Y'$ is a function of the form $\phi \times \psi$, where $\phi \colon X \to X'$ and $\psi \colon Y \to Y'$, with the property that $(\phi \times \psi)(R) \subseteq R'$. The following fact generalizes the Lusin–Novikov uniformization theorem from Polish spaces to their quotients by strongly idealistic Borel equivalence relations:

Theorem 2. *Suppose that X is the quotient of a Polish space by a Borel equivalence relation, Y is the quotient of a Polish space by a strongly* *idealistic Borel equivalence relation, and* $R \subseteq X \times Y$ *is a weakly Borel set whose vertical sections are countable. Then exactly one of the following holds:*

1. *The set* $\mathrm{proj}_X(R)$ *is Borel and there are strongly Borel functions* $\phi_n \colon \mathrm{proj}_X(R) \to Y$ *for which* $R = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$.
2. *There is an injective strongly Borel rectangular homomorphism from* $\mathbb{E}_0/(\mathbb{E}_0 \times \mathbb{F})$ *to* R *for some* $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$.

Remark 3. In the special case that X is a Polish space, condition (2) cannot hold, so condition (1) always holds.

Remark 4. In the special case that Y is a Polish space, condition (2) cannot hold when $\mathbb{F} \in \{\mathbb{F}_p \mid p \text{ is prime}\}$, so $\mathbb{E}_0/(\mathbb{E}_0 \times \Delta(2^{\mathbb{N}}))$ is the minimal counterexample to condition (1), answering a question that originally arose in a conversation with Kechris.

The *complete equivalence relation* on X is given by $I(X) = X \times X$, and the *disjoint union* of equivalence relations E_0 and E_1 on X is the equivalence relation on $X \times 2$ given by $(x, i)\ E_0 \sqcup E_1\ (y, j) \iff (i = j \text{ and } x\ E_i\ y)$. The following fact generalizes the Feldman–Moore theorem from Polish spaces to their quotients by strongly idealistic Borel equivalence relations:

Theorem 5. *Suppose that* X *is the quotient of a Polish space by a strongly idealistic Borel equivalence relation and* E *is a countable weakly Borel equivalence relation on* X*. Then exactly one of the following holds:*

1. *There is a countable group* Γ *of strongly Borel automorphisms of* X *for which* $E = E_\Gamma^X$.
2. *There is a strongly Borel embedding of* $(\mathbb{E}_0 \times I(2))/(\mathbb{E}_0 \sqcup \mathbb{F})$ *into* E *for some* $\mathbb{F} \in \{\Delta(2^{\mathbb{N}})\} \cup \{\mathbb{F}_p \mid p \text{ is prime}\}$.

In §1, we provide the original classical proof of the $\mathbb{G}_0$ dichotomy. While the underlying argument has been known for some time, it has somehow never appeared in full. We provide it here for future reference, and because our later arguments depend not only upon the $\mathbb{G}_0$ dichotomy, but its proof, as well.

In §2, we establish the special case of Theorem 2 referred to in Remark 3. While it is not strictly necessary to establish this case separately, our argument is substantially simpler than those appearing in the latter sections—to which it serves as a stepping stone—and also yields a more general result.

In §3, we establish versions of the special cases of Theorem 1 where every E-class has cardinality at most k, Theorem 2 where every vertical section of R has cardinality at most k, and Theorem 5 where every E-class has cardinality at most $k+1$. In these special cases, condition (2) can only hold when $\mathbb{F} \in \{\mathbb{F}_p \mid p \leq k\}$.

In §4, we establish analogs of Theorems 1, 2, and 5 in which the first conditions are relaxed to allow for bounded finite error, but the second conditions are strengthened to ensure that $\mathbb{F} = \Delta(2^{\mathbb{N}})$. While it is not difficult to derive these from [CCM16, Theorem 1] and the results of §2, we instead show them using a substantial simplification of the proof of the former. The special case of Theorem 2 referred to in Remark 4 easily follows, and by combining the results of the final three sections, we obtain Theorems 1, 2, and 5.

1. The $\mathbb{G}_0$ dichotomy

Endow $\mathbb{N}$ with the discrete topology and $\mathbb{N}^{\mathbb{N}}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and a subset of a topological space is *co-analytic* if its complement is analytic. Every Polish space is analytic (see, for example, [Kec95, Theorem 7.9]), and Souslin's theorem ensures that a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, [Kec95, Theorem 14.11][1]).

Given a natural number $n \geq 2$, a sequence $(X_i)_{i<n}$, and sets $R \subseteq \prod_{i<n} X_i$ and $Y_i \subseteq X_i$ for all $i < n$, we say that $(Y_i)_{i<n}$ is *R-independent* if $R \cap \prod_{i<n} Y_i = \emptyset$. We identify $\prod_{i<n} X_i$ with $\prod_{i \leq m} X_i \times \prod_{m<i<n} X_i$ for all $m < n$.

Proposition 1.1. *Suppose that $n \geq 2$ is a natural number, $(X_i)_{i<n}$ is a sequence of Hausdorff spaces, $A_i \subseteq X_i$ is analytic for all $i < n$, $R \subseteq \prod_{i<n} X_i$ is analytic, and $(A_i)_{i<n}$ is R-independent. Then there are Borel sets $B_i \subseteq X_i$ such that $A_i \subseteq B_i$ for all $i < n$ and $(B_i)_{i<n}$ is R-independent.*

Proof. It is sufficient to show that if $m < n$, $B_i \subseteq X_i$ is a Borel set such that $A_i \subseteq B_i$ for all $i < m$, and $(B_i)_{i<m} \cup (A_i)_{m \leq i<n}$ is R-independent, then there is a Borel set $B_m \subseteq X_m$ for which $A_m \subseteq B_m$ and $(B_i)_{i \leq m} \cup (A_i)_{m<i<n}$ is R-independent. Towards this end, let proj_i denote the projection function from X^n onto the i^{th} coordinate for all $i < n$. As the class of analytic Hausdorff spaces is closed under continuous images, intersections, products, and Borel subsets (see, for example, [Kec95, Proposition 14.4]), the fact that $R \cap ((\prod_{i<m} B_i) \times X \times (\prod_{m<i<n} A_i)) = R \cap ((\prod_{i \leq m} \text{proj}_i(R)) \times (\prod_{m<i<n} A_i)) \cap ((\prod_{i<m} B_i) \times X^{n-m})$ ensures that

[1] While the results of [Kec95] are stated for Polish spaces, the proofs of those to which we refer go through just as easily in the generality discussed here.

$\mathrm{proj}_m(R \cap ((\prod_{i<m} B_i) \times X \times (\prod_{m<i<n} A_i)))$ is analytic, and since A_m is disjoint from this set, Lusin's separation theorem (see, for example, [Kec95, Theorem 14.7]) yields a Borel set $B_m \subseteq X$ containing A_m and disjoint from $\mathrm{proj}_m(R \cap ((\prod_{i<m} B_i) \times X \times (\prod_{m<i<n} A_i)))$, thus $(B_i)_{i \leq m} \cup (A_i)_{m<i<n}$ is R-independent. ⊠

For all sets X and natural numbers n, we use $(X)^n$ to denote the constant sequence of length n with value X. For all $n \geq 2$, an *n-dimensional dihypergraph* on a set X is an n-ary relation G on X consisting solely of non-constant sequences. We say that a set $Y \subseteq X$ is *G-independent* if $(Y)^n$ is G-independent.

Proposition 1.2. *Suppose that $n \geq 2$, X is a Hausdorff space, G is an analytic n-dimensional dihypergraph on X, and $A \subseteq X$ is a G-independent analytic set. Then there is a G-independent Borel set $B \subseteq X$ for which $A \subseteq B$.*

Proof. By Proposition 1.1, there are Borel sets $B_i \subseteq X$ such that $A \subseteq B_i$ for all $i < n$ and $(B_i)_{i<n}$ is G-independent. Set $B = \bigcap_{i<n} B_i$. ⊠

We use $X^{<\mathbb{N}}$ to denote $\bigcup_{n\in\mathbb{N}} X^n$, (i) to denote the singleton sequence with value i, $\sqsubseteq$ to denote extension, and $\frown$ to denote concatenation of sequences. Following standard practice, we also use $\mathcal{N}_s$ to denote $\{b \in \mathbb{N}^\mathbb{N} \mid s \sqsubseteq b\}$ or $\{c \in 2^\mathbb{N} \mid s \sqsubseteq c\}$ (with the context determining which of the two we have in mind). A *digraph* is a two-dimensional dihypergraph. Fix sequences $\mathfrak{s}_n \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq \mathfrak{s}_n$ and let $\mathbb{G}_0$ denote the digraph on $2^\mathbb{N}$ given by $\mathbb{G}_0 = \{(\mathfrak{s}_n \frown (i) \frown c)_{i<2} \mid c \in 2^\mathbb{N} \text{ and } n \in \mathbb{N}\}$.

Proposition 1.3. *Every $\mathbb{G}_0$-independent set $B \subseteq 2^\mathbb{N}$ with the Baire property is meager.*

Proof. Suppose, towards a contradiction, that B is not meager, and fix a sequence $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s$ (see, for example, [Kec95, Proposition 8.26]) and $n \in \mathbb{N}$ for which $s \sqsubseteq \mathfrak{s}_n$, and let ι be the involution of $\mathcal{N}_{\mathfrak{s}_n}$ given by $\iota(\mathfrak{s}_n \frown (0) \frown c) = \mathfrak{s}_n \frown (1) \frown c$ for all $c \in 2^\mathbb{N}$. As ι is a homeomorphism, it follows that $B \cap \iota(B)$ is comeager in $\mathcal{N}_{\mathfrak{s}_n}$ (see, for example, [Kec95, Exercise 8.45]), so $B \cap \iota(B) \cap \mathcal{N}_{\mathfrak{s}_n \frown (0)} \neq \emptyset$. But $(c, \iota(c)) \in \mathbb{G}_0 \cap (B \times B)$ for all $c \in B \cap \iota(B) \cap \mathcal{N}_{\mathfrak{s}_n \frown (0)}$, contradicting the fact that B is $\mathbb{G}_0$-independent. ⊠

An *I-coloring* of a digraph G on a set X is a function $c \colon X \to I$ such that $c(x) \neq c(y)$ for all $(x, y) \in G$, or equivalently, such that $c^{-1}(\{i\})$ is G-independent for all $i \in I$. A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\phi \colon X \to Y$ for which $(\phi \times \phi)(R) \subseteq S$.

Theorem 1.4 (Kechris–Solecki–Todorcevic)**.** *Suppose that X is a Hausdorff space and G is an analytic digraph on X. Then exactly one of the following holds:*

1. *There is a Borel $\mathbb{N}$-coloring $c\colon X \to \mathbb{N}$ of G.*
2. *There is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G.*

Proof (Miller). To see that the two conditions are mutually exclusive, note that if both hold, then $c \circ \phi$ is a Borel $\mathbb{N}$-coloring of $\mathbb{G}_0$, so there exists $n \in \mathbb{N}$ for which $(c \circ \phi)^{-1}(\{n\})$ is a non-meager $\mathbb{G}_0$-independent Borel set, contradicting Proposition 1.3.

It remains to show that at least one of the two conditions holds. We can clearly assume that G is not empty, so there are continuous surjections $\phi_G\colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow G$ and $\phi_X\colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow \bigcup_{i<2} \mathrm{proj}_i(G)$.

We will recursively construct a decreasing sequence $(B^\alpha)_{\alpha<\omega_1}$ of Borel subsets of X, off of which there are Borel $\mathbb{N}$-colorings of G. We begin by setting $B^0 = X$ and we define $B^\lambda = \bigcap_{\alpha<\lambda} B^\alpha$ for all limit ordinals $\lambda < \omega_1$. To describe the construction of $B^{\alpha+1}$ from B^α, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a\colon 2^{n^a} \to \mathbb{N}^{n^a}$, and $\psi^a_n\colon 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for all $n < n^a$. A *one-step extension* of a is an approximation b such that:

(a) $n^b = n^a + 1$,
(b) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}\ (s \sqsubseteq t \implies \phi^a(s) \sqsubseteq \phi^b(t))$, and
(c) $\forall n < n^a \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}\ (s \sqsubseteq t \implies \psi^a_n(s) \sqsubseteq \psi^b_n(t))$.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma\colon 2^{n^\gamma} \to \mathbb{N}^{\mathbb{N}}$, $\psi^\gamma_n\colon 2^{n^\gamma-(n+1)} \to \mathbb{N}^{\mathbb{N}}$ for all $n < n^\gamma$, and $(\phi_G \circ \psi^\gamma_n)(t) = ((\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (0) \frown t), (\phi_X \circ \phi^\gamma)(\mathbb{s}_n \frown (1) \frown t))$ for all $n < n^\gamma$ and $t \in 2^{n^\gamma-(n+1)}$. We say that γ is *compatible* with a set $X' \subseteq X$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq X'$, and *compatible* with a if:

(i) $n^a = n^\gamma$,
(ii) $\forall t \in 2^{n^a}\ \phi^a(t) \sqsubseteq \phi^\gamma(t)$, and
(iii) $\forall n < n^a \forall t \in 2^{n^a-(n+1)}\ \psi^a_n(t) \sqsubseteq \psi^\gamma_n(t)$.

An approximation a is *X'-terminal* if no configuration is compatible with both X' and a one-step extension of a. Let $A(a, X')$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(\mathbb{s}_{n^a})$, where γ varies over all configurations compatible with a and X'.

Lemma 1.5. *Suppose that $X' \subseteq X$ and a is an X'-terminal approximation. Then $A(a, X')$ is G-independent.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and X', with the property that

$((\phi_X \circ \phi^{\gamma_0})(\mathfrak{s}_{n^a}), (\phi_X \circ \phi^{\gamma_1})(\mathfrak{s}_{n^a})) \in G$. Fix a sequence $d \in \mathbb{N}^\mathbb{N}$ such that $\phi_G(d) = ((\phi_X \circ \phi^{\gamma_0})(\mathfrak{s}_{n^a}), (\phi_X \circ \phi^{\gamma_1})(\mathfrak{s}_{n^a}))$, and let γ be the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(t \frown (i)) = \phi^{\gamma_i}(t)$ for all $i < 2$ and $t \in 2^{n^a}$, $\psi_n^\gamma(t \frown (i)) = \psi_n^{\gamma_i}(t)$ for all $i < 2$, $n < n^a$, and $t \in 2^{n^a-(n+1)}$, and $\psi_{n^a}^\gamma(\emptyset) = d$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is X'-terminal. ⊠

For all B^α-terminal approximations a, Proposition 1.2 yields a G-independent Borel set $B(a, B^\alpha) \supseteq A(a, B^\alpha)$. Let $B^{\alpha+1}$ be the set obtained from B^α by subtracting the union of the sets of the form $B(a, B^\alpha)$, where a varies over all B^α-terminal approximations.

Lemma 1.6. *Suppose that $\alpha < \omega_1$ and a is a non-$B^{\alpha+1}$-terminal approximation. Then a has a non-B^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $B^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(\mathfrak{s}_{n^b}) \in B^{\alpha+1}$, so b is not B^α-terminal. ⊠

Fix $\alpha < \omega_1$ such that the families of B^α- and $B^{\alpha+1}$-terminal approximations coincide, and let a_0 be the unique approximation for which $n^{a_0} = 0$. As $A(a_0, X') = \bigcup_{i<2} \operatorname{proj}_i(G) \cap X'$ for all $X' \subseteq X$, we can assume that a_0 is not B^α-terminal, since otherwise $B^{\alpha+1}$ is disjoint from $\bigcup_{i<2} \operatorname{proj}_i(G)$, so there is a Borel $\mathbb{N}$-coloring of G.

By recursively applying Lemma 1.6, we obtain non-B^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi, \psi_n \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \upharpoonright n)$ and $\psi_n(c) = \bigcup_{m>n} \psi_n^{a_m}(c \upharpoonright (m-(n+1)))$ for all $n \in \mathbb{N}$. Clearly these functions are continuous.

To establish that the function $\pi = \phi_X \circ \phi$ is a homomorphism from $\mathbb{G}_0$ to G, we will show the stronger fact that if $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$, then

$$(\phi_G \circ \psi_n)(c) = ((\phi_X \circ \phi)(\mathfrak{s}_n \frown (0) \frown c), (\phi_X \circ \phi)(\mathfrak{s}_n \frown (1) \frown c)).$$

As $X \times X$ is Hausdorff, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(\mathfrak{s}_n \frown (0) \frown c), (\phi_X \circ \phi)(\mathfrak{s}_n \frown (1) \frown c))$ and V is an open neighborhood of $(\phi_G \circ \psi_n)(c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\phi_X(\mathcal{N}_{\phi^{a_m}(\mathfrak{s}_n \frown (0) \frown s)}) \times \phi_X(\mathcal{N}_{\phi^{a_m}(\mathfrak{s}_n \frown (1) \frown s)}) \subseteq U$ and $\phi_G(\mathcal{N}_{\psi_n^{a_m}(s)}) \subseteq V$ where $s = c \upharpoonright (m-(n+1))$. The fact that a_m is not B^α-terminal yields a configuration γ compatible with a_m. Then $((\phi_X \circ \phi^\gamma)(\mathfrak{s}_n \frown (0) \frown s), (\phi_X \circ \phi^\gamma)(\mathfrak{s}_n \frown (1) \frown s)) \in U$ and $(\phi_G \circ \psi_n^\gamma)(s) \in V$, so $U \cap V \neq \emptyset$. ⊠

Remark 1.7. The apparent use of choice beyond DC in the above argument can be eliminated by first running the simplified version without Proposition 1.2, i.e., by setting $B(a, B^\alpha) = A(a, B^\alpha)$, in order to obtain an upper bound $\alpha' < \omega_1$ on the least ordinal $\alpha < \omega_1$ for which the sets of B^α- and $B^{\alpha+1}$-terminal approximations coincide.

2. Generalizations

Given an equivalence relation E on a set X, the *E-saturation* of a set $Y \subseteq X$ is given by $[Y]_E = \{x \in X \mid \exists y \in Y \; x \mathrel{E} y\}$. We say that an n-dimensional dihypergraph G on X is *E-invariant* if $x \in G \iff y \in G$ for all $x, y \in X^n$ with the property that $\forall i < n \; x(i) \mathrel{E} y(i)$.

Proposition 2.1. *Suppose that $n \geq 2$, X is a Hausdorff space, E is an analytic equivalence relation on X, G is an E-invariant analytic n-dimensional dihypergraph on X, and $A \subseteq X$ is a G-independent analytic set. Then there is an E-invariant G-independent Borel set $B \subseteq X$ for which $A \subseteq B$.*

Proof. Set $A_0 = A$. Given $n \in \mathbb{N}$ and a G-independent analytic set $A_n \subseteq X$, Proposition 1.2 yields a G-independent Borel set $B_n \subseteq X$ containing A_n, and since the class of analytic Hausdorff spaces is closed under continuous images and Borel subsets, it follows that the G-independent set $A_{n+1} = [B_n]_E = \mathrm{proj}_0(E \cap (X \times B_n))$ is analytic. Define $B = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n$. ⊠

Clearly $\mathbb{G}_0 \subseteq \mathbb{E}_0$. The following observation is slightly less obvious:

Proposition 2.2. *The smallest equivalence relation E on $2^{\mathbb{N}}$ containing $\mathbb{G}_0$ is $\mathbb{E}_0$.*

Proof. To see that $\forall c \in 2^{\mathbb{N}} \forall u, v \in 2^n \; u \frown c \mathrel{E} v \frown c$ for all $n \in \mathbb{N}$, observe that if it holds at some $n \in \mathbb{N}$, $c \in 2^{\mathbb{N}}$, and $u, v \in 2^n$, then $u \frown (0) \frown c \mathrel{E} \mathbb{s}_n \frown (0) \frown c \mathrel{\mathbb{G}_0} \mathbb{s}_n \frown (1) \frown c \mathrel{E} v \frown (1) \frown c$, so it holds at $n + 1$. ⊠

We next note another connection between Baire category and $\mathbb{G}_0$:

Proposition 2.3. *Suppose that E and F are equivalence relations on $2^{\mathbb{N}}$ with the Baire property, every E-class is a countable union of $(E \cap F)$-classes, and $F \cap \mathbb{G}_0 = \emptyset$. Then E and F are meager.*

Proof. Suppose, towards a contradiction, that F is not meager. As F has the Baire property, the Kuratowski–Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields a sequence $c \in 2^{\mathbb{N}}$ whose F-class has the Baire property but is not meager, in which case Proposition 1.3 yields a pair $(a, b) \in \mathbb{G}_0 \upharpoonright [c]_F$, contradicting the fact that $F \cap \mathbb{G}_0 = \emptyset$. It therefore follows that F is meager.

The Kuratowski–Ulam theorem now ensures that every F-class is meager, in which case every $(E \cap F)$-class is meager, so every E-class is meager, thus E is meager. ⊠

A *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on a set X to a sequence $(S_i)_{i \in I}$ of binary relations on a set Y is a function $\phi \colon X \to Y$ that is a homomorphism from R_i to S_i for all $i \in I$. In this section, we only need the weakening of the following fact where $(\mathbb{F}_k, \mathbb{E}_0 \setminus \mathbb{F}_k)$ is replaced with $\mathbb{E}_0$:

Proposition 2.4. *Suppose that $k \geq 2$, D is a closed nowhere-dense binary relation on $2^{\mathbb{N}}$, and R is a meager binary relation on $2^{\mathbb{N}}$. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $({\sim}\Delta(2^{\mathbb{N}}), \mathbb{F}_k, \mathbb{E}_0 \setminus \mathbb{F}_k, {\sim}\mathbb{E}_0)$ to $({\sim}D, \mathbb{F}_k, \mathbb{E}_0 \setminus \mathbb{F}_k, {\sim}R)$.*

Proof. Fix a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of dense open symmetric subsets of ${\sim}D$ whose intersection is disjoint from R, as well as $\phi_0 \colon 2^0 \to 2^0$.

Lemma 2.5. *Suppose that $n \in \mathbb{N}$ and $\phi_n \colon 2^n \to 2^{<\mathbb{N}}$. Then there exist $\ell_n > 0$ and $u_n \in 2^{\ell_n} \times 2^{\ell_n}$ with the property that $1 + \sum_{j < \ell_n} u_n(0)(j) \equiv \sum_{j < \ell_n} u_n(1)(j) \pmod{k}$ and $\prod_{i<2} \mathcal{N}_{\phi_{n+1}(t(i) \frown (i))} \subseteq U_n$ for all $t \in 2^n \times 2^n$, where $\phi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ is given by $\phi_{n+1}(t \frown (i)) = \phi_n(t) \frown u_n(i)$ for all $i < 2$ and $t \in 2^n$.*

Proof. Fix an enumeration $(t_j)_{j < 4^n}$ of $2^n \times 2^n$ and $u_{0,n} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$. Given $j < 4^n$ and $u_{j,n} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, fix $u_{j+1,n} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

1. $\forall i < 2 \ u_{j,n}(i) \sqsubseteq u_{j+1,n}(i)$.
2. $\prod_{i<2} \mathcal{N}_{\phi_n(t_j(i)) \frown u_{j+1,n}(i)} \subseteq U_n$.

Then any $\ell_n > 0$ and $u_n \in 2^{\ell_n} \times 2^{\ell_n}$ such that $\forall i < 2 \ u_{4^n,n}(i) \sqsubseteq u_n(i)$ and $1 + \sum_{j < \ell_n} u_n(0)(j) \equiv \sum_{j < \ell_n} u_n(1)(j) \pmod{k}$ are as desired. ⊠

As $\phi_n(t) \sqsubset \phi_{n+1}(t \frown (i))$ for all $i < 2$, $n \in \mathbb{N}$, and $t \in 2^n$, we obtain a continuous function $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by setting $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \upharpoonright n)$ for all $c \in 2^{\mathbb{N}}$. To see that ϕ is a homomorphism from ${\sim}\Delta(2^{\mathbb{N}})$ to ${\sim}D$, note that if $c \in {\sim}\Delta(2^{\mathbb{N}})$, then there exists $n \in \mathbb{N}$ for which $c(0)(n) \neq c(1)(n)$, so $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i) \upharpoonright (n+1))} \subseteq U_n \subseteq {\sim}D$. To see that ϕ is a homomorphism from $(\mathbb{F}_k, \mathbb{E}_0 \setminus \mathbb{F}_k)$ to $(\mathbb{F}_k, \mathbb{E}_0 \setminus \mathbb{F}_k)$, note that if $c \in \mathbb{E}_0$, then there exists $n \in \mathbb{N}$ such that $c(0)(m) = c(1)(m)$ for all $m \geq n$, and setting $\ell = \sum_{j<n} \ell_j$, it follows that $\phi(c(0))(m) = \phi(c(1))(m)$ for all $m \geq \ell$, in which case

$$
\begin{aligned}
& c(0) \ \mathbb{F}_k \ c(1) \\
& \iff \textstyle\sum_{m<n} c(0)(m) \equiv \sum_{m<n} c(1)(m) \pmod{k} \\
& \iff \textstyle\sum_{m<n} \sum_{j<\ell_m} u_m(c(0))(j) \equiv \sum_{m<n} \sum_{j<\ell_m} u_m(c(1))(j) \pmod{k} \\
& \iff \textstyle\sum_{m<\ell} \phi_n(c(0) \upharpoonright n)(m) \equiv \sum_{m<\ell} \phi_n(c(1) \upharpoonright n)(m) \pmod{k} \\
& \iff \phi(c(0)) \ \mathbb{F}_k \ \phi(c(1)).
\end{aligned}
$$

To see that ϕ is a homomorphism from $\sim\mathbb{E}_0$ to $\sim R$, note that if $c \in \sim\mathbb{E}_0$, then there are infinitely many $n \in \mathbb{N}$ with the property that $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i) \upharpoonright (n+1))} \subseteq U_n$, so $(\phi(c(i)))_{i<2} \in \sim R$. ⊠

A *partial transversal* of an equivalence relation E on a set X over a subequivalence relation F is a set $Y \subseteq X$ for which $E \upharpoonright Y = F \upharpoonright Y$.

Theorem 2.6. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X into which $\mathbb{E}_0$ does not continuously embed, F is a Borel equivalence relation on X, and every E-class is a countable union of $(E \cap F)$-classes. Then X is a countable union of $(E \cap F)$-invariant Borel partial transversals of E over $E \cap F$.*

Proof. Note that a set $Y \subseteq X$ is a partial transversal of E over $E \cap F$ if and only if it is independent with respect to the digraph $G = E \setminus F$. Moreover, Proposition 2.1 ensures that every G-independent Borel set is contained in an $(E \cap F)$-invariant G-independent Borel set, so we need only show that there is a Borel $\mathbb{N}$-coloring of G.

Suppose, towards a contradiction, that there is no such coloring. Then Theorem 1.4 yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to G. As the sets $E' = (\phi \times \phi)^{-1}(E)$ and $F' = (\phi \times \phi)^{-1}(F)$ have the Baire property (see, for example, [Kec95, Theorem 21.6]), Proposition 2.3 ensures that E' is meager. As the closed relation $D' = (\phi \times \phi)^{-1}(\Delta(X))$ is contained in E', it is nowhere dense, so Proposition 2.4 yields a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\sim\Delta(2^{\mathbb{N}}), \mathbb{E}_0, \sim\mathbb{E}_0)$ to $(\sim D', \mathbb{E}_0, \sim E')$. As $\mathbb{G}_0 \subseteq E'$, Proposition 2.2 ensures that $\mathbb{E}_0 \subseteq E'$, so $\phi \circ \psi$ is a continuous embedding of $\mathbb{E}_0$ into E, the desired contradiction. ⊠

Remark 2.7. Under the weaker assumption that F is co-analytic, one can run the same argument without Proposition 2.1 to obtain the weaker conclusion in which the Borel partial transversals of E over $E \cap F$ need not be $(E \cap F)$-invariant.

A partial function $\phi \colon X \rightharpoonup Y$ *uniformizes* a set $R \subseteq X \times Y$ if $\mathrm{dom}(\phi) = \mathrm{proj}_X(R)$ and $\mathrm{graph}(\phi) \subseteq R$.

Proposition 2.8. *Suppose that X and Y are Polish spaces, F is a strongly idealistic Borel equivalence relation on Y, and $P \subseteq X \times Y$ is a Borel set whose non-empty vertical sections are F-classes. Then $\mathrm{proj}_X(P)$ is Borel and there is a Borel function $\phi \colon \mathrm{proj}_X(P) \to Y$ that uniformizes P.*

Proof. Fix a witness $y \mapsto \mathcal{J}_y$ to the strong idealisticity of F, and define $\mathcal{I}_x = \mathcal{J}_y$ for all $(x, y) \in P$. Given a Borel set $R \subseteq P$, observe that the set $R' = \{((y,x),z) \in (Y \times X) \times Y \mid x \ P \ y \text{ and } x \ R \ z\}$ is Borel, so the set $S = \{x \in X \mid \exists y \in P_x \ R'_{(y,x)} \in \mathcal{J}_y\}$ is analytic, the set $T = \{x \in X \mid \forall y \in P_x \ R'_{(y,x)} \in \mathcal{J}_y\}$ is co-analytic, and $R_x \in \mathcal{I}_x \iff x \in S \iff x \in T$ for all $x \in \mathrm{proj}_X(P)$. As noted in [Kec, Theorem 18.6*], the proof of [Kec95, Theorem 18.6] can therefore be easily modified to ensure that $\mathrm{proj}_X(P)$ is Borel, as well as to obtain a Borel function $\phi \colon \mathrm{proj}_X(P) \to Y$ that uniformizes P. ⊠

We say that a partial function $T \colon X/E \rightharpoonup X/E$ is *Borel* if preimages of Borel subsets of X/E are Borel.

Proposition 2.9. *Suppose that X is a Polish space, E is a strongly idealistic Borel equivalence relation on X, and $T \colon X/E \rightharpoonup X/E$ is a strongly Borel partial function. Then T is Borel.*

Proof. By Proposition 2.8, the set $D = \bigcup \mathrm{dom}(T)$ is Borel and there is a Borel function $\tilde{T} \colon D \to X$ for which $\mathrm{graph}(T) = [\mathrm{graph}(\tilde{T})]_{\Delta(X) \times E}$. It therefore only remains to note that if $B \subseteq X/E$ is Borel, then $\bigcup T^{-1}(B) = \tilde{T}^{-1}(\bigcup B)$, so $T^{-1}(B)$ is Borel, thus T is Borel. ⊠

The *composition* of partial functions $S \colon Y \rightharpoonup Z$ and $T \colon X \rightharpoonup Y$ is given by $(S \circ T)(x) = z \iff \exists y \in Y \ (T(x) = y \text{ and } S(y) = z)$.

Proposition 2.10. *Suppose that X is a Polish space, E is a strongly idealistic Borel equivalence relation on X, and $S, T \colon X/E \rightharpoonup X/E$ are strongly Borel partial functions. Then $S \circ T$ is strongly Borel.*

Proof. By Proposition 2.8, the sets $C = \bigcup \mathrm{dom}(S)$ and $D = \bigcup \mathrm{dom}(T)$ are Borel and there are Borel functions $\tilde{S} \colon C \to X$ and $\tilde{T} \colon D \to X$ with $\mathrm{graph}(S) = [\mathrm{graph}(\tilde{S})]_{\Delta(X) \times E}$ and $\mathrm{graph}(T) = [\mathrm{graph}(\tilde{T})]_{\Delta(X) \times E}$. It only remains to note that if $x, y \in X$, then $(S \circ T)([x]_E) = [y]_E \iff (x \in D \cap \tilde{T}^{-1}(C) \text{ and } (\tilde{S} \circ \tilde{T})(x) \ E \ y)$, so $S \circ T$ is strongly Borel. ⊠

The *powers* of a partial injection $T \colon X \rightharpoonup X$ are given by $T^0 = \mathrm{id}_X$, $T^{n+1} = T \circ T^n$ for all $n \in \mathbb{N}$, and $T^{-n} = (T^n)^{-1}$ for all $n \in \mathbb{Z}^+$. The *T-saturation* of a set $Y \subseteq X$ is given by $[Y]_T = \bigcup_{n \in \mathbb{Z}} T^n(\mathrm{dom}(T^n) \cap Y)$. The *$T$-orbit* of a point $x \in X$ is given by $[x]_T = [\{x\}]_T$ and the *orbit equivalence relation* induced by T is the equivalence relation on X given by $x \ E_T^X \ y \iff [x]_T = [y]_T$.

Proposition 2.11. *Suppose that X is a Polish space, E is a strongly idealistic Borel equivalence relation on X, and $S \colon X/E \rightharpoonup X/E$ is a strongly Borel partial injection. Then there is a strongly Borel bijection $T \colon X/E \to X/E$ for which $E_S^{X/E} = E_T^{X/E}$.*

Proof. Set $Y = \sim\mathrm{dom}(S^{-1})$ and $Z = \sim\mathrm{dom}(S)$, fix transitive permutations σ of $\mathbb{N}$ and τ of $-\mathbb{N}$, and define $T \colon X/E \to X/E$ by

$$T(x) = \begin{cases} S(x) & \text{if } x \in \sim([Y]_S \cup [Z]_S) \cup ([Y]_S \cap [Z]_S \setminus Z), \\ S^{-n}(x) & \text{if } n \in \mathbb{N} \text{ and } x \in S^n(Y) \cap Z, \\ S^{\sigma(n)-n}(x) & \text{if } n \in \mathbb{N} \text{ and } x \in S^n(Y) \setminus [Z]_S, \text{ and} \\ S^{\tau(n)-n}(x) & \text{if } n \in -\mathbb{N} \text{ and } x \in S^n(Z) \setminus [Y]_S. \end{cases}$$

Propositions 2.9 and 2.10 ensure that T is strongly Borel. The first clause in the definition of T ensures that the orbit equivalence relations of S and T agree on the set of points whose S-orbit has type $\mathbb{Z}$, the first two clauses ensure that the orbit equivalence relations agree on the set of points whose S-orbit is finite, the third clause ensures that the orbit equivalence relations agree on the set of points whose S-orbit has type $\mathbb{N}$, and the fourth clause ensures that the orbit equivalence relations agree on the set of points whose S-orbit has type $-\mathbb{N}$. ⊠

A *reduction* of a binary relation R on a set X to a binary relation S on a set Y is a homomorphism from $(R, \sim R)$ to $(S, \sim S)$.

Theorem 2.12. *Suppose that X and Y are Polish spaces, F is a strongly idealistic Borel equivalence relation on Y, and $R \subseteq X \times Y$ is a Borel set whose vertical sections are countable unions of F-classes. Then $\mathrm{proj}_X(R)$ is Borel and there are Borel maps $\phi_n \colon \mathrm{proj}_X(R) \to Y$ with the property that $\forall x \in \mathrm{proj}_X(R)\ R_x = \bigcup_{n\in\mathbb{N}} [\phi_n(x)]_F$.*

Proof. Note that there is no continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow X \times Y$ of $\mathbb{E}_0$ into $\Delta(X) \times I(Y)$, since otherwise $\mathrm{proj}_X \circ \pi$ would be a continuous reduction of $\mathbb{E}_0$ to $\Delta(X)$, contradicting the well-known fact that every continuous homomorphism from $\mathbb{E}_0$ to $\Delta(X)$ is constant. Theorem 2.6 therefore yields $(\Delta(X) \times F)$-invariant Borel partial transversals $P_n \subseteq R$ of $\Delta(X) \times I(Y)$ over $\Delta(X) \times F$ for which $R = \bigcup_{n\in\mathbb{N}} P_n$. For all $n \in \mathbb{N}$, Proposition 2.8 ensures that the set $D_n = \mathrm{proj}_X(P_n)$ is Borel and yields a Borel function $\psi_n \colon D_n \to Y$ that uniformizes P_n. Then $\mathrm{proj}_X(R) = \bigcup_{n\in\mathbb{N}} D_n$, so $\mathrm{proj}_X(R)$ is Borel, and the functions of the form $\phi_n = \psi_n \cup \bigcup_{k\in\mathbb{N}} \psi_k \restriction (D_k \setminus \bigcup_{j\in k\cup\{n\}} D_j)$ are as desired. ⊠

Remark 2.13. For all $n \in \mathbb{N}$, the set $R_n = [\mathrm{graph}(\phi_n)]_{\Delta(X)\times F} = \{(x,y) \in \mathrm{proj}_X(R) \times Y \mid \phi_n(x)\ F\ y\}$ is Borel, so Theorem 2.12 yields the special case of Theorem 2 referred to in Remark 3.

Remark 2.14. Under the weaker assumption that X and Y are Hausdorff, F is co-analytic, and R is analytic, the above argument still yields a conclusion similar to that of Theorem 2, although the resulting sets neither have full projections nor enjoy the same level of definability.

We say that a subequivalence relation F of an equivalence relation E has *countable index* if every E-class is a countable union of F-classes.

Proposition 2.15. *Suppose that X is a Polish space, E is a Borel equivalence relation on X, F is a countable-index strongly idealistic Borel subequivalence relation of E, and $B \subseteq X$ is an F-invariant Borel set. Then $[B]_E$ is Borel.*

Proof. By Theorem 2.12, there are Borel functions $\phi_n \colon X \to X$ for which $E = \bigcup_{n\in\mathbb{N}}[\mathrm{graph}(\phi_n)]_{\Delta(X)\times F}$. Then $[B]_E = \bigcup_{n\in\mathbb{N}} \phi_n^{-1}(B)$. ⊠

Given an equivalence relation E on a set X, we say that a set $Y \subseteq X$ is *E-complete* if $X = [Y]_E$. A *transversal* of an equivalence relation E over a subequivalence relation F is an E-complete partial transversal of E over F.

Proposition 2.16. *Suppose that X is a topological space, $F \subseteq E$ are equivalence relations on X for which the E-saturation of every F-invariant Borel set is Borel, there is a cover $(B_n)_{n\in\mathbb{N}}$ of X by F-invariant Borel partial transversals of E over F, and $B \subseteq X$ is an F-invariant Borel partial transversal of E over F. Then B is contained in an F-invariant Borel transversal of E over F.*

Proof. Set $B'_0 = B$, recursively define $B'_{n+1} = B'_n \cup (B_n \setminus [B'_n]_E)$ for all $n \in \mathbb{N}$, and observe that the set $B' = \bigcup_{n\in\mathbb{N}} B'_n$ is as desired. ⊠

3. Finite bases

The *support* of a sequence $s \in 2^{<\mathbb{N}}$ is given by $\mathrm{supp}(s) = s^{-1}(\{1\})$.

Proposition 3.1. *Suppose that X is a set, $T \colon X \to X$ is a bijection, and $\phi \colon 2^{\mathbb{N}} \to X$ is a homomorphism from $\mathbb{G}_0$ to the graph of T. Then $\forall n \in \mathbb{N} \forall u, v \in 2^n \forall c \in 2^{\mathbb{N}}\ T^{|\mathrm{supp}(v)|-|\mathrm{supp}(u)|}(\phi(u \frown c)) = \phi(v \frown c)$.*

Proof. Suppose that we have already established the proposition at some $n \in \mathbb{N}$. To see that it holds at $n+1$, observe that if $c \in 2^{\mathbb{N}}$ and $u, v \in 2^n$, then $T^{|\mathrm{supp}(\mathbb{s}_n)|-|\mathrm{supp}(u)|}(\phi(u \frown (0) \frown c)) = \phi(\mathbb{s}_n \frown (0) \frown c)$, $T(\phi(\mathbb{s}_n \frown (0) \frown c)) = \phi(\mathbb{s}_n \frown (1) \frown c)$ since ϕ is a homomorphism, and $T^{|\mathrm{supp}(v)|-|\mathrm{supp}(\mathbb{s}_n)|}(\phi(\mathbb{s}_n \frown (1) \frown c)) = \phi(v \frown (1) \frown c)$. But $|\mathrm{supp}(\mathbb{s}_n)| - |\mathrm{supp}(u)| + 1 + |\mathrm{supp}(v)| - |\mathrm{supp}(\mathbb{s}_n)| = |\mathrm{supp}(v \frown (1))| - |\mathrm{supp}(u \frown (0))|$, so $T^{|\mathrm{supp}(v\frown(1))|-|\mathrm{supp}(u\frown(0))|}(\phi(u \frown (0) \frown c)) = \phi(v \frown (1) \frown c)$. ⊠

An equivalence relation E on a space X is *generically ergodic* if every E-invariant set with the Baire property is meager or comeager.

Proposition 3.2. *Suppose that $k \geq 2$. Then $\mathbb{F}_k$ is generically ergodic.*

Proof. Suppose that $B \subseteq 2^{\mathbb{N}}$ is a non-meager $\mathbb{F}_k$-invariant set with the Baire property, fix a sequence $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s$, and set $n = |s|$. It is sufficient to show that B is comeager in $\mathcal{N}_u$ for all $u \in 2^{k-1+n}$. Towards this end, fix an extension $t \in 2^{k-1+n}$ of s for which $\sum_{i<k-1+n} t(i) \equiv \sum_{i<k-1+n} u(i) \pmod{k}$ and define $\iota \colon \mathcal{N}_t \to \mathcal{N}_u$ by $\iota(t \frown c) = u \frown c$ for all $c \in 2^{\mathbb{N}}$. As ι is a homeomorphism, it follows that $\iota(B)$ is comeager in $\iota(\mathcal{N}_t)$. But the former set is contained in B, and the latter is $\mathcal{N}_u$. ⊠

Proposition 3.3. *Suppose that X is a Baire space, E is an equivalence relation on X with respect to which saturations of meager sets are meager, F is a generically-ergodic subequivalence relation of E, and $B \subseteq X$ is an F-invariant set with the Baire property whose complement is E-complete. Then B is meager.*

Proof. As $[{\sim}B]_E = X$, it follows that $[{\sim}B]_E$ is not meager, so ${\sim}B$ is not meager, thus ${\sim}B$ is comeager. ⊠

We next establish a variant of Proposition 2.15:

Proposition 3.4. *Suppose that $n \in \mathbb{Z}^+$, X is a Hausdorff space, E is an analytic equivalence relation on X, F is a co-analytic equivalence relation on X, E has index n over $E \cap F$, and $B \subseteq X$ is an $(E \cap F)$-invariant Borel set. Then $[B]_E$ is Borel.*

Proof. As saturations of analytic sets with respect to analytic equivalence relations are clearly analytic, it is sufficient to show that if $C \subseteq X$ is an $(E \cap F)$-invariant co-analytic set, then $[C]_E$ is co-analytic. Towards this end, let R be the set of pairs $(x, y) \in X \times X^n$ such that $\forall i < n \ x \ E \ y(i)$, $\forall i < j < n \ \neg y(i) \ F \ y(j)$, and $\forall i < n \ y(i) \notin C$, and note that ${\sim}[C]_E = \mathrm{proj}_X(R)$ and R is analytic, thus so too is ${\sim}[C]_E$. ⊠

A *reduction* of a sequence $(R_i)_{i \in I}$ of binary relations on a set X to a sequence $(S_i)_{i \in I}$ of binary relations on a set Y is a function $\phi \colon X \to Y$ that is a reduction of R_i to S_i for all $i \in I$. An *embedding* is an injective reduction. We next note that the family $\{(\mathbb{E}_0, \mathbb{F}_p) \mid p \text{ is prime}\}$ is a minimal basis for $\{(\mathbb{E}_0, \mathbb{F}_k) \mid k \geq 2\}$ under any quasi-order between Baire-measurable reducibility and continuous embeddability:

Proposition 3.5. *Suppose that $j, k \geq 2$.*

1. *If $j \mid k$, then there is a continuous embedding $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ of $(\mathbb{E}_0, \mathbb{F}_j)$ into $(\mathbb{E}_0, \mathbb{F}_k)$.*
2. *If j and k are relatively prime, then there is no Baire-measurable homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{E}_0 \setminus \mathbb{F}_j, \mathbb{F}_j)$ to $(\mathbb{E}_0 \setminus \mathbb{F}_k, \mathbb{F}_k)$.*

Proof. To see (1), observe that the function $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\phi(c) = \bigoplus_{n \in \mathbb{N}} (c(n))^{k/j}$ is a continuous embedding of $(\mathbb{E}_0, \mathbb{F}_j)$ into $(\mathbb{E}_0, \mathbb{F}_k)$.

To see (2), suppose, towards a contradiction, that there is such a homomorphism $\phi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$. For all $\ell \in \{j, k\}$, define $\psi_\ell \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\psi_\ell((1)^n \frown (0) \frown c) = (1)^n \frown (1) \frown c$ for all $c \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ and $\psi_\ell((1)^\infty) = (0)^{\ell-1} \frown (1)^\infty$, so that ψ_ℓ factors over $\mathbb{F}_\ell$ to a permutation of $2^{\mathbb{N}}/\mathbb{F}_\ell$ whose orbit equivalence relation is $\mathbb{E}_0/\mathbb{F}_\ell$. In particular, it follows that the sets of the form $B_i = \{c \in 2^{\mathbb{N}} \mid \phi(\psi_j(c)) \ \mathbb{F}_k \ \psi_k^i(\phi(c))\}$, for $0 < i < k$, are $\mathbb{F}_j$-invariant and partition $2^{\mathbb{N}}$, so Proposition 3.2 yields a unique i for which B_i is comeager. As $\mathbb{E}_0$-saturations of meager sets are meager, there are comeagerly-many $c \in 2^{\mathbb{N}}$ for which $[c]_{\mathbb{E}_0} \subseteq B_i$. Given any such c, note that $c \ \mathbb{F}_j \ \psi_j^j(c)$, so $\phi(c) \ \mathbb{F}_k \ \psi_k^{ij}(\phi(c))$, thus $k \mid ij$, the desired contradiction. ⊠

Along with Propositions 2.15 and 2.16, the following fact yields the special case of Theorem 1 where there exists $k \in \mathbb{N}$ for which every E-class has cardinality at most k:

Theorem 3.6. *Suppose that $k \geq 2$, X is a Hausdorff space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and every E-class is a union of at most k $(E \cap F)$-classes. Then exactly one of the following holds:*

1. *There is a cover $(B_i)_{i<k}$ of X by $(E \cap F)$-invariant Borel partial transversals of E over $E \cap F$.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ of $(\mathbb{E}_0, \mathbb{F}_p)$ into (E, F) for some prime $p \leq k$.*

Proof. To see that the conditions are mutually exclusive, note that if both hold, then there exists $i < k$ for which $\pi^{-1}(B_i)$ is a non-meager Borel partial transversal of $\mathbb{E}_0$ over $\mathbb{F}_p$, contradicting Proposition 3.3.

To see that at least one of the conditions holds, suppose that we have already established the theorem strictly below k, and define $Y = \{y \in X^k \mid \forall i < j < k \ y(i) \ (E \setminus F) \ y(j)\}$. It is sufficient to show that at least one of the following holds:

(a) There is an $(E \cap F)$-invariant Borel partial transversal $B \subseteq X$ of E over $E \cap F$ for which $\mathrm{proj}_0(Y) \subseteq [B]_E$.
(b) There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow \mathrm{proj}_0(Y)$ of $(\mathbb{E}_0, \mathbb{F}_p)$ into $(E \upharpoonright \mathrm{proj}_0(Y), F \upharpoonright \mathrm{proj}_0(Y))$ for some prime $p \leq k$.

Let Σ be the set of permutations σ of k for which $\sigma(0) \neq 0$, and for each $\sigma \in \Sigma$, let G_σ be the digraph on Y with respect to which two sequences y and z are related if and only if $\forall i < k \ y(\sigma(i)) \ (E \cap F) \ z(i)$.

Lemma 3.7. *Suppose that there are Borel* $\mathbb{N}$*-colorings* $c_\sigma \colon Y \to \mathbb{N}$ *of* G_σ *for all* $\sigma \in \Sigma$*. Then there is an* $(E \cap F)$*-invariant Borel partial transversal* $B \subseteq X$ *of* E *over* $E \cap F$ *for which* $\mathrm{proj}_0(Y) \subseteq [B]_E$*.*

Proof. Note that the function $c_\Sigma \colon Y \to \mathbb{N}^\Sigma$ given by $c_\Sigma(y)(\sigma) = c_\sigma(y)$ is a Borel $\mathbb{N}^\Sigma$-coloring of the digraph $G_\Sigma = \bigcup_{\sigma \in \Sigma} G_\sigma$, and if $Z \subseteq Y$ is a Borel G_Σ-independent set, then $\mathrm{proj}_0(Z)$ is an analytic partial transversal of E over $E \cap F$, so Proposition 2.1 ensures that it is contained in an $(E \cap F)$-invariant Borel partial transversal of E over $E \cap F$, thus there is a cover $(B_n)_{n \in \mathbb{N}}$ of $\mathrm{proj}_0(Y)$ by $(E \cap F)$-invariant Borel partial transversals of E over $E \cap F$. Then Propositions 2.16 and 3.4 yield an $(E \cap F)$-invariant Borel transversal $C \subseteq \mathrm{proj}_0(Y)$ of $E \restriction \mathrm{proj}_0(Y)$ over $(E \cap F) \restriction \mathrm{proj}_0(Y)$, in which case one more application of Proposition 2.1 yields an $(E \cap F)$-invariant Borel partial transversal $B \supseteq C$ of E over $E \cap F$. ⊠

Suppose now that there exists $\sigma \in \Sigma$ for which there is no Borel coloring $c_\sigma \colon Y \to \mathbb{N}$ of G_σ. Then Theorem 1.4 yields a continuous homomorphism $\phi \colon 2^\mathbb{N} \to Y$ from $\mathbb{G}_0$ to G_σ. Define $\phi_0 = \mathrm{proj}_0 \circ \phi$ and $j = |\{\sigma^n(0) \mid n \in \mathbb{Z}\}|$. As the relations $E' = (\phi_0 \times \phi_0)^{-1}(E)$ and $F' = (\phi_0 \times \phi_0)^{-1}(F)$ have the Baire property, Proposition 2.3 ensures that they are meager, and since the closed relation $D' = (\phi_0 \times \phi_0)^{-1}(\Delta(X))$ is contained in E', it is nowhere dense, so Proposition 2.4 yields a continuous homomorphism $\psi \colon 2^\mathbb{N} \to 2^\mathbb{N}$ from $({\sim}\Delta(2^\mathbb{N}), \mathbb{F}_j, \mathbb{E}_0 \setminus \mathbb{F}_j, {\sim}\mathbb{E}_0)$ to $({\sim}D', \mathbb{F}_j, \mathbb{E}_0 \setminus \mathbb{F}_j, {\sim}(E' \cup F'))$. As Proposition 3.1 ensures that ϕ_0 is a homomorphism from $(\mathbb{F}_j, \mathbb{E}_0 \setminus \mathbb{F}_j)$ to $(E \cap F, E \setminus F)$, it follows that $\phi_0 \circ \psi$ is a continuous embedding of $(\mathbb{E}_0, \mathbb{F}_j)$ into (E, F), so it only remains to observe that if p is any prime dividing j, then Proposition 3.5 ensures that there is a continuous embedding of $(\mathbb{E}_0, \mathbb{F}_p)$ into $(\mathbb{E}_0, \mathbb{F}_j)$, and therefore of $(\mathbb{E}_0, \mathbb{F}_p)$ into (E, F). ⊠

A *partial uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a subset of R whose vertical sections are contained in F-classes. Along with Theorem 2.12 and Proposition 2.16, the following fact yields the special case of Theorem 2 where there exists $k \in \mathbb{N}$ for which every vertical section of R has cardinality at most k:

Theorem 3.8. *Suppose that* $k \geq 2$*,* X *and* Y *are Hausdorff spaces,* E *is an analytic equivalence relation on* X*,* F *is a Borel equivalence relation on* Y*, and* $R \subseteq X \times Y$ *is an* $(E \times \Delta(Y))$*-invariant analytic set whose vertical sections are contained in unions of at most* k F*-classes. Then exactly one of the following holds:*

1. *There is a cover* $(R_i)_{i<k}$ *of* R *by* $((E \times F) \restriction R)$*-invariant Borel-in-*$R$ *partial uniformizations of* R *over* F*.*

(2) *There are continuous embeddings* $\pi_X \colon 2^{\mathbb{N}} \hookrightarrow X$ *of* $\mathbb{E}_0$ *into* E *and* $\pi_Y \colon 2^{\mathbb{N}} \hookrightarrow Y$ *of* $\mathbb{F}_p$ *into* F *such that* $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq R$ *for some prime* $p \leq k$.

Proof. To see that the conditions are mutually exclusive, note that if both hold, then there exists $i < k$ for which $\mathrm{proj}_1((\pi_X \times \pi_Y)^{-1}(R_i))$ is a non-meager partial transversal of $\mathbb{E}_0$ over $\mathbb{F}_p$ with the Baire property, contradicting Proposition 3.3.

To see that at least one of the conditions holds, note that $(E \times I(Y)) \restriction R$ is analytic and $(I(X) \times F) \restriction R$ is Borel, and appeal to Theorem 3.6 to see that if condition (1) fails, then there is a continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow R$ of $(\mathbb{E}_0, \mathbb{F}_p)$ into $((E \times I(Y)) \restriction R, (E \times F) \restriction R)$ for some prime $p \leq k$. It follows that the function $\pi'_X = \mathrm{proj}_X \circ \pi$ is a continuous reduction of $\mathbb{E}_0$ to E and the function $\pi'_Y = \mathrm{proj}_Y \circ \pi$ is a continuous homomorphism from $(\mathbb{F}_p, \mathbb{E}_0 \setminus \mathbb{F}_p)$ to $(F, {\sim}F)$, and therefore from $\mathbb{G}_0$ to ${\sim}F$, so Proposition 2.3 ensures that the equivalence relation $F' = (\pi'_Y \times \pi'_Y)^{-1}(F)$ is meager, in which case the closed relations $D'_X = (\pi'_X \times \pi'_X)^{-1}(\Delta(X))$ and $D'_Y = (\pi'_Y \times \pi'_Y)^{-1}(\Delta(Y))$ are nowhere dense, so Proposition 2.4 yields a continuous homomorphism $\pi' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $({\sim}\Delta(2^{\mathbb{N}}), \mathbb{F}_p, \mathbb{E}_0 \setminus \mathbb{F}_p, {\sim}\mathbb{E}_0)$ to $({\sim}(D'_X \cup D'_Y), \mathbb{F}_p, \mathbb{E}_0 \setminus \mathbb{F}_p, {\sim}(\mathbb{E}_0 \cup F'))$, thus the functions $\pi_X = \pi'_X \circ \pi'$ and $\pi_Y = \pi'_Y \circ \pi'$ are continuous embeddings of $\mathbb{E}_0$ into E and $\mathbb{F}_p$ into F. As $\pi(2^{\mathbb{N}}) \subseteq R$, it follows that $(\pi'_X \times \pi'_Y)(\Delta(2^{\mathbb{N}})) \subseteq R$, so the facts that π'_X is a homomorphism from $\mathbb{E}_0$ to E and R is $(E \times \Delta(Y))$-invariant ensure that $(\pi'_X \times \pi'_Y)(\mathbb{E}_0) \subseteq R$, thus the fact that π' is a homomorphism from $\mathbb{E}_0$ to $\mathbb{E}_0$ implies that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq R$. ⊠

Given a binary relation R on X, set $R^{-1} = \{(y, x) \in X \times X \mid x \mathrel{R} y\}$. Given an equivalence relation F on X, we say that R is the *graph of a partial injection* over F if $\forall (x, y), (x', y') \in R \ (x \mathrel{F} x' \iff y \mathrel{F} y')$. Along with Proposition 2.11, the following fact yields the special case of Theorem 5 where there exists $k \in \mathbb{N}$ for which every E-class has cardinality at most $k + 1$:

Theorem 3.9. *Suppose that* $k \geq 2$, X *is a Hausdorff space,* E *is an analytic equivalence relation on* X, F *is a Borel equivalence relation on* X, *and every* E*-class is a union of at most* $k + 1$ $(E \cap F)$*-classes. Then exactly one of the following holds:*

(1) *There is a cover* $(R_{i,j})_{i,j<k}$ *of* $E \setminus F$ *by* $((E \cap F) \times (E \cap F))$*-invariant Borel graphs of partial injections over* $E \cap F$.
(2) *There is a continuous embedding* $\pi \colon 2^{\mathbb{N}} \times 2 \hookrightarrow X$ *of* $(\mathbb{E}_0 \times I(2), \mathbb{E}_0 \sqcup \mathbb{F}_p)$ *into* (E, F) *for some prime* $p \leq k$.

Proof. To see that the conditions are mutually exclusive, observe that if they both hold, then there exist $i, j < k$ with the property that $\{c \in 2^{\mathbb{N}} \mid \pi(c, 0) \; R_{i,j} \; \pi(c, 1)\} \times \{1\}$ is a non-meager partial transversal of $\mathbb{E}_0 \times \Delta(\{1\})$ over $\mathbb{F}_p \times \Delta(\{1\})$ with the Baire property, a contradiction.

To see that at least one of the conditions holds, note first that if there are $((E \cap F) \times F) \restriction (E \setminus F)$-invariant Borel-in-$(E \setminus F)$ partial uniformizations R_j of $E \setminus F$ over F for which $E \setminus F = \bigcup_{j<k} R_j$, then the sets of the form $R_i \cap R_j^{-1}$, where $i, j < k$, are as desired. By Theorem 3.8, we can therefore assume that there are continuous embeddings $\pi_X : 2^{\mathbb{N}} \hookrightarrow X$ of $\mathbb{E}_0$ into $E \cap F$ and $\pi_Y : 2^{\mathbb{N}} \hookrightarrow X$ of $\mathbb{F}_p$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq E \setminus F$ for some prime $p \leq k$. As $\mathbb{E}_0$ has countable index below the equivalence relation $E' = (\pi_X \times \pi_X)^{-1}(E)$, the latter is meager, so the closed subequivalence relation $D' = (\pi_X \times \pi_X)^{-1}(\Delta(X))$ is nowhere dense, thus Proposition 2.4 yields a continuous homomorphism $\pi' : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\sim\Delta(2^{\mathbb{N}}), \mathbb{F}_p, \mathbb{E}_0 \setminus \mathbb{F}_p, \sim\mathbb{E}_0)$ to $(\sim D', \mathbb{F}_p, \mathbb{E}_0 \setminus \mathbb{F}_p, \sim E')$. Define $\pi : 2^{\mathbb{N}} \times 2 \to X$ by $\pi(c, 0) = (\pi_X \circ \pi')(c)$ and $\pi(c, 1) = (\pi_Y \circ \pi')(c)$ for all $c \in 2^{\mathbb{N}}$. ⊠

4. Dichotomies

Let $[X]^n$ denote the family of subsets of X of cardinality n. When X is a topological space, we endow $[X]^n$ with the topology generated by the sets of the form $\{a \in [X]^n \mid \exists f : a \hookrightarrow n \forall x \in a \; x \in U_{f(x)}\}$, where $(U_i)_{i<n}$ is a sequence of open subsets of X. Let $[X]^n_E$ denote the subspace consisting of sets which are contained in a single E-class, and let $[X]^n_{E,F}$ denote the further subspace consisting of such sets which are partial transversals of F. Define $[X]^{<\aleph_0} = \bigcup_{n\in\mathbb{N}} [X]^n$, $[X]^{<\aleph_0}_E = \bigcup_{n\in\mathbb{N}} [X]^n_E$, and $[X]^{<\aleph_0}_{E,F} = \bigcup_{n\in\mathbb{N}} [X]^n_{E,F}$.

The *trace* of a family $\mathscr{A} \subseteq [X]^{<\aleph_0}_E$ on an equivalence class C of E is given by $\mathscr{A} \restriction C = \{a \in \mathscr{A} \mid a \subseteq C\}$. We say that two subsets of X are *F-disjoint* if their F-saturations are disjoint, $\mathscr{A}$ is *F-intersecting* if no two sets in $\mathscr{A}$ are F-disjoint, and $\mathscr{A}$ is *E-locally F-intersecting* if its trace on every E-class is F-intersecting. A set $Y \subseteq X$ *punctures* $\mathscr{A}$ if it intersects every set in $\mathscr{A}$. For each non-empty set $a' \in [X]^{<\aleph_0}_E$, define $[a', \mathscr{A}]_F = \{[a]_F \mid a \in \mathscr{A} \text{ and } a' \subseteq [a]_F\}$.

A *partial quasi-transversal* of an equivalence relation E over a subequivalence relation F is a set $Y \subseteq X$ for which there exists $k \in \mathbb{N}$ such that every $(E \restriction Y)$-class is contained in a union of at most k F-classes. The following fact is essentially a special case of [CCM16, Proposition 2.3.1]; we provide the proof for the reader's convenience:

Proposition 4.1. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and $\mathscr{A} \subseteq [X]^{<\aleph_0}_{E,F}$ is an E-locally F-intersecting analytic family of sets of bounded finite cardinality. Then there is an $(E \cap F)$-invariant Borel partial quasi-transversal $B \subseteq X$ of E over $E \cap F$ puncturing $\mathscr{A}$.*

Proof. Recursively define $f(n) = n^{n+1} + \sum_{0<k<n} f(k)$ for all $n \in \mathbb{Z}^+$. We will show that if every set in $\mathscr{A}$ has cardinality at most n, then there is an $(E \cap F)$-invariant Borel set $B \subseteq X$ puncturing $\mathscr{A}$ such that every $(E \restriction B)$-class is contained in a union of at most $f(n)$ $(E \cap F)$-classes.

We proceed by induction on n. The base case $n = 1$ follows from an application of Proposition 2.1 to the equivalence relation $E \cap F$ and the digraph $E \setminus F$, so suppose that $n \geq 2$ and we have established the proposition strictly below n. We will construct analytic families $\mathscr{A}_k \subseteq \mathscr{A}$ and $\mathscr{A}'_k \subseteq [X]^k_{E,F}$, as well as $(E \cap F)$-invariant Borel sets $B_k \subseteq X$, such that:

1. $\forall k < n$ $\mathscr{A}_k = \{a \in \mathscr{A}_{k+1} \mid a \cap B_{k+1} = \emptyset\}$.
2. $\forall 1 \leq k \leq n$ $\mathscr{A}'_k = \{a' \in [X]^k_{E,F} \mid |[a', \mathscr{A}_k]_{E \cap F}| > n^{n-k}\}$.
3. $\forall 1 \leq k \leq n$ B_k punctures $\mathscr{A}'_k$.
4. $\forall 1 \leq k \leq n$ E has index at most $f(k)$ over $E \cap F$ on B_k.

We proceed by reverse recursion, beginning with $\mathscr{A}_n = \mathscr{A}$, $\mathscr{A}'_n = \emptyset$, and $B_n = \emptyset$. Suppose now that $0 < k < n$ and we have already found $\mathscr{A}_{k+1}$, $\mathscr{A}'_{k+1}$, and B_{k+1}. Conditions (1) and (2) then yield $\mathscr{A}_k$ and $\mathscr{A}'_k$.

Lemma 4.2. *Suppose that $a' \in \mathscr{A}'_k$ and $x \in [a']_E \setminus [a']_F$. Then the family $[a' \cup \{x\}, \mathscr{A}_k]_{E \cap F}$ has cardinality at most $n^{n-(k+1)}$.*

Proof. We can clearly assume that $[a' \cup \{x\}, \mathscr{A}_k]_{E \cap F} \neq \emptyset$. Given any set $a \in [a' \cup \{x\}, \mathscr{A}_k]_{E \cap F}$, condition (1) ensures that $a \cap B_{k+1} = \emptyset$, thus $(a' \cup \{x\}) \cap B_{k+1} = \emptyset$. Condition (3) therefore implies that $a' \cup \{x\} \notin \mathscr{A}'_{k+1}$, so $|[a' \cup \{x\}, \mathscr{A}_{k+1}]_{E \cap F}| \leq n^{n-(k+1)}$ by condition (2). As condition (1) also ensures that $\mathscr{A}_k \subseteq \mathscr{A}_{k+1}$, the lemma follows. ⊠

Lemma 4.3. *Suppose that $a' \in \mathscr{A}'_k$ and $b \subseteq [a']_E \setminus [a']_F$ has cardinality at most n. Then there exists $a \in \mathscr{A}_k$ whose $(E \cap F)$-saturation contains a' and is disjoint from b.*

Proof. Lemma 4.2 ensures that $|[a' \cup \{x\}, \mathscr{A}_k]_{E \cap F}| \leq n^{n-(k+1)}$ for all $x \in b$, so $[a', \mathscr{A}_k]_{E \cap F}$ contains at most n^{n-k} sets intersecting b, thus condition (2) implies that $[a', \mathscr{A}_k]_{E \cap F}$ contains a set disjoint from b. ⊠

Lemma 4.4. *The family $\mathscr{A}'_k$ is E-locally F-intersecting.*

Proof. Suppose, towards a contradiction, that there are $(E \cap F)$-disjoint sets $a', b' \in \mathscr{A}'_k$ with the property that $[a']_E = [b']_E$. Then Lemma 4.3

yields $b \in \mathscr{A}_k$ whose $(E \cap F)$-saturation contains b' and is disjoint from a', as well as $a \in \mathscr{A}_k$ whose $(E \cap F)$-saturation contains a' and is disjoint from b, contradicting the fact that $\mathscr{A}$ is E-locally F-intersecting. ⊠

As the induction hypothesis yields an $(E \cap F)$-invariant Borel set $B_k \subseteq X$ puncturing $\mathscr{A}'_k$ on which E has index at most $f(k)$ over $E \cap F$, this completes the recursive construction. Define $A_0 = \bigcup \mathscr{A}_0$.

Lemma 4.5. *Suppose that $x \in A_0$. Then $A_0 \cap [x]_E$ is contained in a union of at most n^{n+1} $(E \cap F)$-classes.*

Proof. Fix $a_0 \in \mathscr{A}_0$ for which $x \in a_0$, and note that if $y \in a_0$, then $|\{[a]_{E \cap F} \mid a \in \mathscr{A}_0 \text{ and } y \in [a]_{E \cap F}\}| \leq n^{n-1}$, in which case $\bigcup\{[a]_{E \cap F} \mid a \in \mathscr{A}_0 \text{ and } y \in [a]_{E \cap F}\}$ is a union of at most n^n $(E \cap F)$-classes. But every element of $A_0 \cap [x]_E$ is contained in a set of this form, so $A_0 \cap [x]_E$ is contained in a union of at most n^{n+1} $(E \cap F)$-classes. ⊠

By applying Proposition 2.1 to the equivalence relation $E \cap F$ and the dihypergraph $G = \{x \in X^{n^{n+1}+1} \mid \forall i < j \leq n^{n+1} \; x(i) \; (E \setminus F) \; y(j)\}$, we obtain an $(E \cap F)$-invariant Borel set $B_0 \supseteq A_0$ with the property that every $(E \upharpoonright B_0)$-class is contained in a union of at most n^{n+1} $(E \cap F)$-classes. It only remains to note that if $a \in \mathscr{A}$, then there is a least $k \leq n$ such that $a \in \mathscr{A}_k$, in which case $a \cap B_k$ is non-empty, so the set $B = \bigcup_{k \leq n} B_k$ punctures $\mathscr{A}$. ⊠

A *partial quasi-transversal* of an equivalence relation E on X is a partial quasi-transversal of E over $\Delta(X)$. Theorem 1 is a consequence of Propositions 2.15 and 2.16, Theorem 3.6, and the following:

Theorem 4.6. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and every E-class is a countable union of $(E \cap F)$-classes. Then exactly one of the following holds:*

1. *There is a cover $(B_n)_{n \in \mathbb{N}}$ of X by $(E \cap F)$-invariant Borel partial quasi-transversals of E over $E \cap F$.*
2. *There exists a continuous embedding $\pi \colon 2^{\mathbb{N}} \hookrightarrow X$ of $(\mathbb{E}_0, \Delta(2^{\mathbb{N}}))$ into (E, F).*

Proof. To see that the conditions are mutually exclusive, note that if both hold, then there exists $n \in \mathbb{N}$ for which $\pi^{-1}(B_n)$ is a non-meager Borel partial quasi-transversal of $\mathbb{E}_0$. But the proof of the well-known fact that every partial transversal of $\mathbb{E}_0$ with the Baire property is meager works just as well to show that every partial quasi-transversal of $\mathbb{E}_0$ with the Baire property is meager, a contradiction.

To see that at least one of the conditions holds, we can assume that $E \nsubseteq F$, so there are continuous surjections $\phi_{E \setminus F} \colon \mathbb{N}^{\mathbb{N}} \twoheadrightarrow E \setminus$

F and $\phi_X \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow X$. We will recursively define a decreasing sequence $(B^\alpha)_{\alpha<\omega_1}$ of Borel subsets of X whose complements are countable unions of $(E \cap F)$-invariant Borel partial quasi-transversals of E over $E \cap F$. We begin by setting $B^0 = X$. For all limit ordinals $\lambda < \omega_1$, we set $B^\lambda = \bigcap_{\alpha<\lambda} B^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a triple of the form $a = (n^a, \phi^a, (\psi^a_n)_{n<n^a})$, where $n^a \in \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{n^a}$, and $\psi^a_n \colon I(2^n) \times 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for all $n < n^a$. A *one-step extension* of a is an approximation b for which:

(1) $n^b = n^a + 1$.
(2) $\forall s \in 2^{n^a} \forall t \in 2^{n^b}\ (s \sqsubseteq t \implies \phi^a(s) \sqsubseteq \phi^b(t))$.
(3) $\forall n < n^a \forall r \in I(2^n) \forall s \in 2^{n^a-(n+1)} \forall t \in 2^{n^b-(n+1)}$
$(s \sqsubseteq t \implies \psi^a_n(r, s) \sqsubseteq \psi^b_n(r, t))$.

A *configuration* is a triple of the form $\gamma = (n^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$, where $n^\gamma \in \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^\mathbb{N}$, $\psi^\gamma_n \colon I(2^n) \times 2^{n^\gamma-(n+1)} \to \mathbb{N}^\mathbb{N}$ for all $n < n^\gamma$, and $(\phi_{E\setminus F} \circ \psi^\gamma_n)(r, s) = ((\phi_X \circ \phi^\gamma)(r(i) \frown (i) \frown s))_{i<2}$ for all $n < n^\gamma$, $r \in I(2^n)$, and $s \in 2^{n^\gamma-(n+1)}$. We say that γ is *compatible* with a set $X' \subseteq X$ if $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq X'$, and *compatible* with a if:

(i) $n^a = n^\gamma$.
(ii) $\forall s \in 2^{n^a}\ \phi^a(s) \sqsubseteq \phi^\gamma(s)$.
(iii) $\forall n < n^a \forall r \in I(2^n) \forall s \in 2^{n^a-(n+1)}\ \psi^a_n(r, s) \sqsubseteq \psi^\gamma_n(r, s)$.

An approximation a is *X'-terminal* if no configuration is compatible with both X' and a one-step extension of a. Let $\mathscr{A}(a, X')$ denote the family of sets of the form $[(\phi_X \circ \phi^\gamma)(2^{n^a})]_{E\cap F}$, where γ varies over all configurations compatible with a and X'.

Lemma 4.7. *Suppose that $X' \subseteq X$ and a is an X'-terminal approximation. Then $\mathscr{A}(a, X')$ is E-locally F-intersecting.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and X', such that $(\phi_X \circ \phi^{\gamma_0})(2^{n^a})$ and $(\phi_X \circ \phi^{\gamma_1})(2^{n^a})$ are F-disjoint sets contained in the same E-class. Fix $f \colon I(2^{n^a}) \to \mathbb{N}^\mathbb{N}$ such that $(\phi_{E\setminus F} \circ f)(r) = ((\phi_X \circ \phi^{\gamma_i})(r(i)))_{i<2}$ for all $r \in I(2^{n^a})$, and let γ be the configuration given by $n^\gamma = n^a + 1$, $\phi^\gamma(s \frown (i)) = \phi^{\gamma_i}(s)$ for all $i < 2$ and $s \in 2^{n^a}$, $\psi^\gamma_n(r, s \frown (i)) = \psi^{\gamma_i}_n(r, s)$ for all $i < 2$, $n < n^a$, $r \in I(2^n)$, and $s \in 2^{n^a-(n+1)}$, and $\psi^\gamma_{n^a}(r, \emptyset) = f(r)$ for all $r \in I(2^{n^a})$. Then γ is compatible with a one-step extension of a, contradicting the fact that a is X'-terminal. ⊠

Proposition 4.1 and Lemma 4.7 ensure that if a is B^α-terminal, then there is an $(E \cap F)$-invariant Borel partial quasi-transversal $B(a, B^\alpha)$ of E over $E \cap F$ puncturing $\mathscr{A}(a, B^\alpha)$. Let $B^{\alpha+1}$ be the set obtained from B^α by subtracting the union of the sets of the form $B(a, B^\alpha)$, where a varies over all B^α-terminal approximations.

Lemma 4.8. *Suppose that $\alpha < \omega_1$ and a is a non-$B^{\alpha+1}$-terminal approximation. Then a has a non-B^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $B^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma)(2^{n^b}) \subseteq B^{\alpha+1}$, so b is not B^α-terminal. ⊠

Fix $\alpha < \omega_1$ such that the families of B^α- and $B^{\alpha+1}$-terminal approximations coincide, and let a_0 denote the unique approximation for which $n^{a_0} = 0$. As $\mathscr{A}(a_0, X') = \{[x]_{E \cap F} \mid x \in X'\}$ for all $X' \subseteq X$, we can assume that a_0 is not B^α-terminal, since otherwise $B^{\alpha+1} = \emptyset$, so X is a countable union of $(E \cap F)$-invariant Borel partial quasi-transversals of E over $E \cap F$.

By recursively applying Lemma 4.8, we obtain non-B^α-terminal one-step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Define $\phi \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi^{a_n}(c \restriction n)$, as well as $\psi_n \colon I(2^n) \times 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_n(r, c) = \bigcup_{m > n} \psi_n^{a_m}(r, c \restriction (m - (n+1)))$ for all $n \in \mathbb{N}$. Clearly these functions are continuous.

Lemma 4.9. *The function $\phi_X \circ \phi$ is a homomorphism from $\mathbb{E}_0 \setminus \Delta(2^\mathbb{N})$ to $E \setminus F$.*

Proof. We will show that if $c \in 2^\mathbb{N}$, $n \in \mathbb{N}$, and $r \in I(2^n)$, then

$$(\phi_{E \setminus F} \circ \psi_n)(r, c) = ((\phi_X \circ \phi)(r(i) \frown (i) \frown c))_{i<2}.$$

As $X \times X$ is Hausdorff, it is sufficient to show that if U is an open neighborhood of $((\phi_X \circ \phi)(r(i) \frown (i) \frown c))_{i<2}$ and V is an open neighborhood of $(\phi_{E \setminus F} \circ \psi_n)(r, c)$, then $U \cap V \neq \emptyset$. Towards this end, fix $m > n$ such that $\prod_{i<2} \phi_X(\mathcal{N}_{\phi^{a_m}(r(i) \frown (i) \frown s)}) \subseteq U$ and $\phi_{E \setminus F}(\mathcal{N}_{\psi_n^{a_m}(r,s)}) \subseteq V$, where $s = c \restriction (m - (n+1))$. As a_m is not B^α-terminal, there is a configuration γ compatible with it. Then $((\phi_X \circ \phi^\gamma)(r(i) \frown (i) \frown s))_{i<2} \in U$ and $(\phi_{E \setminus F} \circ \psi_n^\gamma)(r, s) \in V$, thus $U \cap V \neq \emptyset$. ⊠

Set $\phi' = \phi_X \circ \phi$. As the equivalence relations $E' = (\phi' \times \phi')^{-1}(E)$ and $F' = (\phi' \times \phi')^{-1}(F)$ have the Baire property, Proposition 2.3 ensures that they are meager, in which case the closed equivalence relation $D' = (\phi' \times \phi')^{-1}(\Delta(X))$ is nowhere dense, so Proposition 2.4 yields a continuous homomorphism $\psi \colon 2^\mathbb{N} \to 2^\mathbb{N}$ from $({\sim}\Delta(2^\mathbb{N}), \mathbb{E}_0, {\sim}\mathbb{E}_0)$ to $({\sim}D', \mathbb{E}_0, {\sim}(E' \cup F'))$. As ϕ' is a homomorphism from $\mathbb{G}_0$ to E, it follows that $\mathbb{G}_0 \subseteq E'$, so Proposition 2.2 ensures that $\mathbb{E}_0 \subseteq E'$, thus $\phi' \circ \psi$ is a continuous embedding of $(\mathbb{E}_0, \Delta(2^\mathbb{N}))$ into (E, F). ⊠

Remark 4.10. The apparent use of choice beyond DC in the above argument can be eliminated by first running the analog of the argument using the weakening of Proposition 4.1 without any definability constraints on the partial quasi-transversal puncturing the family (which can be proven in the same manner, but without using Proposition 2.1), in order to obtain an upper bound $\alpha' < \omega_1$ on the least ordinal $\alpha < \omega_1$ for which the sets of B^α- and $B^{\alpha+1}$-terminal approximations coincide.

A *partial quasi-uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a set $S \subseteq R$ for which there exists $k \in \mathbb{N}$ such that every vertical section of S is contained in a union of at most k F-classes. Theorem 2 is a consequence of Proposition 2.16, Theorems 2.12 and 3.8[2], and the following:

Theorem 4.11. *Suppose that X and Y are Hausdorff spaces, E is an analytic equivalence relation on X, F is a Borel equivalence relation on Y, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant analytic set whose vertical sections are contained in countable unions of F-classes. Then exactly one of the following holds:*

1. *There is a cover $(R_n)_{n \in \mathbb{N}}$ of R by $((E \times F) \restriction R)$-invariant Borel-in-$R$ partial quasi-uniformizations of R over F.*
2. *There are continuous embeddings $\pi_X \colon 2^\mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0$ into E and $\pi_Y \colon 2^\mathbb{N} \hookrightarrow Y$ of $\Delta(2^\mathbb{N})$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq R$.*

Proof. To see that the conditions are mutually exclusive, note that if both hold, then there exists $n \in \mathbb{N}$ for which $\mathrm{proj}_1((\pi_X \times \pi_Y)^{-1}(R_n))$ is a non-meager partial quasi-transversal of $\mathbb{E}_0$ with the Baire property, a contradiction.

To see that at least one of the conditions holds, observe that $(E \times I(Y)) \restriction R$ is analytic and $(I(X) \times F) \restriction R$ is Borel, and appeal to Theorem 4.6 to see that if condition (1) fails, then there is a continuous embedding $\pi \colon 2^\mathbb{N} \hookrightarrow R$ of $(\mathbb{E}_0, \Delta(2^\mathbb{N}))$ into $((E \times I(Y)) \restriction R, (E \times F) \restriction R)$. It follows that the function $\pi'_X = \mathrm{proj}_X \circ \pi$ is a continuous reduction of $\mathbb{E}_0$ to E and the function $\pi'_Y = \mathrm{proj}_Y \circ \pi$ is a continuous homomorphism from $\mathbb{E}_0 \setminus \Delta(2^\mathbb{N})$ to $\sim F$, and therefore from $\mathbb{G}_0$ to $\sim F$, so Proposition 2.3 ensures that the relation $F' = (\pi'_Y \times \pi'_Y)^{-1}(F)$ is meager, in which case the closed relations $D'_X = (\pi'_X \times \pi'_X)^{-1}(\Delta(X))$ and $D'_Y = (\pi'_Y \times \pi'_Y)^{-1}(\Delta(Y))$ are nowhere dense, so Proposition 2.4 yields a continuous homomorphism $\pi' \colon 2^\mathbb{N} \to 2^\mathbb{N}$ from $(\sim\Delta(2^\mathbb{N}), \mathbb{E}_0, \sim\mathbb{E}_0)$ to $(\sim(D'_X \cup D'_Y), \mathbb{E}_0, \sim(\mathbb{E}_0 \cup F'))$, thus the functions $\pi_X = \pi'_X \circ \pi'$ and $\pi_Y = \pi'_Y \circ \pi'$ are continuous embeddings of $\mathbb{E}_0$ into E and $\Delta(2^\mathbb{N})$ into

[2]Theorem 3.8 can be replaced with the usual Lusin–Novikov uniformization theorem to establish the special case referred to in Remark 4.

F. As $\pi(2^{\mathbb{N}}) \subseteq R$, it follows that $(\pi'_X \times \pi'_Y)(\Delta(2^{\mathbb{N}})) \subseteq R$, so the facts that π'_X is a homomorphism from $\mathbb{E}_0$ to E and R is $(E \times \Delta(Y))$-invariant ensure that $(\pi'_X \times \pi'_Y)(\mathbb{E}_0) \subseteq R$, thus the fact that π' is a homomorphism from $\mathbb{E}_0$ to $\mathbb{E}_0$ implies that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq R$. ⊠

We say that a set $R \subseteq X \times X$ is the *graph of a partial quasi-function* over an equivalence relation F on X if there exists $k \in \mathbb{N}$ such that $\forall (x_i, y_i)_{i<k} \in R^k \ (\forall i, j < k \ x_i \ F \ x_j \implies \exists i < j < k \ y_i \ F \ y_j)$. We say that R is the *graph of a partial quasi-injection* if both R and R^{-1} have this property. Theorem 5 is a consequence of Proposition 2.11, Theorem 3.9, and the following:

Theorem 4.12. *Suppose that X is a Hausdorff space, E is an analytic equivalence relation on X, F is a Borel equivalence relation on X, and every E-class is a countable union of $(E \cap F)$-classes. Then exactly one of the following holds:*

1. *There is a cover $(R_n)_{n\in\mathbb{N}}$ of $E \setminus F$ by $((E \cap F) \times (E \cap F))$-invariant Borel-in-$E$ graphs of partial quasi-injections over $E \cap F$.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \times 2 \hookrightarrow X$ of $(\mathbb{E}_0 \times I(2), \mathbb{E}_0 \sqcup \Delta(2^{\mathbb{N}}))$ into (E, F).*

Proof. To see that the conditions are mutually exclusive, note that if both hold, then there exists $n \in \mathbb{N}$ with the property that $\{c \in 2^{\mathbb{N}} \mid \pi(c, 0) \ R_n \ \pi(c, 1)\} \times \{1\}$ is a non-meager partial quasi-transversal of $\mathbb{E}_0 \times \Delta(\{1\})$ with the Baire property, a contradiction.

To see that at least one of the conditions holds, note first that if there are $((E \cap F) \times F) \restriction (E \setminus F)$-invariant Borel-in-$E$ partial quasi-uniformizations R_n of E over $E \cap F$ for which $E = \bigcup_{n\in\mathbb{N}} R_n$, then the sets of the form $R_m \cap R_n^{-1}$, where $m, n \in \mathbb{N}$, are as desired. By Theorem 4.11, we can therefore assume that there are continuous embeddings $\pi_X \colon 2^{\mathbb{N}} \hookrightarrow X$ of $\mathbb{E}_0$ into E and $\pi_Y \colon 2^{\mathbb{N}} \hookrightarrow X$ of $\Delta(2^{\mathbb{N}})$ into F such that $(\pi_X \times \pi_Y)(\mathbb{E}_0) \subseteq E \setminus F$. As $\mathbb{E}_0$ has countable index below the equivalence relation $E' = (\pi_X \times \pi_X)^{-1}(E)$, the latter is meager, so the closed subequivalence relation $D' = (\pi_X \times \pi_X)^{-1}(\Delta(X))$ is nowhere dense, thus Proposition 2.4 yields a continuous homomorphism $\pi' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $({\sim}\Delta(2^{\mathbb{N}}), \mathbb{E}_0, {\sim}\mathbb{E}_0)$ to $({\sim}D', \mathbb{E}_0, {\sim}E')$. Define $\pi \colon 2^{\mathbb{N}} \times 2 \to X$ by $\pi(c, 0) = (\pi_X \circ \pi')(c)$ and $\pi(c, 1) = (\pi_Y \circ \pi')(c)$ for all $c \in 2^{\mathbb{N}}$. ⊠

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