

THE OPEN DIHYPERGRAPH DICHOTOMY AND THE SECOND LEVEL OF THE BOREL HIERARCHY

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ABSTRACT. We show that several dichotomy theorems concerning the second level of the Borel hierarchy are special cases of the \aleph_0 -dimensional generalization of the open graph dichotomy, which itself follows from the usual proof(s) of the perfect set theorem. Under the axiom of determinacy, we obtain the generalizations of these results from analytic to separable metric spaces. We also consider connections between cardinal invariants and the chromatic numbers of the corresponding dihypergraphs.

INTRODUCTION

A topological space is *Polish* if it is second countable and completely metrizable. A subset of a topological space is *Borel* if it is in the smallest σ -algebra containing the open sets, F_σ if it is a countable union of closed sets, G_δ if it is a countable intersection of open sets, and $\mathbf{\Delta}_2^0$ if it is both F_σ and G_δ . A *D-ary relation* on X is a subset of X^D . A *homomorphism* from a D -ary relation R on X to a D -ary relation S on Y is a function $\pi: X \rightarrow Y$ such that $\pi^D(R) \subseteq S$, a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, and an *embedding* of R into S is an injective reduction of R to S . The existence of a continuous-embeddability-wise-minimal Borel subset of a Polish space that is not F_σ was established in [Hur28].

A set Z *separates* a set X from a set Y if $X \subseteq Z$ and $Y \cap Z = \emptyset$. A subset of a product $\prod_{d \in D} X_d$ is a *hyperrectangle* if it is of the form $\prod_{d \in D} Y_d$, where $Y_d \subseteq X_d$ for all $d \in D$. When $D = 2$, we say that such a set is a *rectangle*. A *D-dimensional dihypergraph* on X is a D -ary relation H on X disjoint from the set $\Delta^D(X)$ of constant sequences. When $D = 2$, we say that a symmetric such set is a *graph*, and use $\Delta(X)$ to denote $\Delta^D(X)$. The *complete D-dimensional hypergraph* on X is the complement of $\Delta^D(X)$, and a set $Y \subseteq X$ is *H-independent* if $H \upharpoonright Y = \emptyset$. A κ -*coloring* of a D -dimensional dihypergraph H on X is

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a homomorphism from H to the complete D -dimensional hypergraph on a set Y of cardinality κ , or equivalently, a function $\pi: X \rightarrow Y$ such that $\pi^{-1}(\{y\})$ is H -independent for all $y \in Y$. The *chromatic number* of H is the least cardinal $\chi(H)$ for which there is a $\chi(H)$ -coloring of H . Given a pointclass Γ of subsets of topological spaces, a function $\pi: X \rightarrow Y$ is Γ -*measurable* if $\pi^{-1}(V) \in \Gamma$ for all open sets $V \subseteq Y$. A characterization of the circumstances under which a Borel subset of a product of two Polish spaces can be separated from another subset by a countable union of closed rectangles was given in [LZ14], as was a characterization of the circumstances under which a Borel graph on a Polish space admits a Δ_2^0 -measurable \aleph_0 -coloring. These results were obtained by adapting a classical proof of Hurewicz's original dichotomy theorem. Despite their simplicity, the underlying arguments rely upon finite injury, and do not yield the theorem in its natural generality. Moreover, the resulting minimal objects in some sense fail to be canonical, in that they depend on an arbitrary parameter. Analogous characterizations were obtained at the next level of the Borel hierarchy, although the underlying arguments were quite intricate, as were the corresponding minimal objects. The question as to whether such results hold at still higher levels of the Borel hierarchy remains open.

A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. A function is *Borel* if it is Borel-measurable, and *Baire class one* if it is F_σ -measurable. We say that a function $\pi: X \rightarrow Y$ is σ -*continuous with Γ witnesses* if X is a union of countably-many sets in Γ on which π is continuous. We write $\forall^\infty n \in N \phi(n)$ to indicate that $\phi(n)$ holds for all but finitely many $n \in N$. The *eventual domination quasi-order* on $\mathbb{N}^{\mathbb{N}}$ is given by $\mathbf{c} \leq^* \mathbf{d} \iff \forall^\infty n \in \mathbb{N} \mathbf{c}(n) \leq \mathbf{d}(n)$, and the *dominating number* is the least cardinal \mathfrak{d} for which there is a set $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality \mathfrak{d} such that $\forall \mathbf{c} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{d} \in \mathcal{F} \mathbf{c} \leq^* \mathbf{d}$. In [JR82], it was shown that a function from an analytic metric space to a separable metric space is Δ_2^0 -measurable if and only if it is σ -continuous with closed witnesses. In [Sol98], this was derived from the fact that there is a two-element basis, consisting of non- Δ_2^0 -measurable functions, for the family of Baire-class-one functions that are not σ -continuous with closed witnesses. As a corollary, it was also shown that if a Baire-class-one function $\pi: X \rightarrow Y$ is not σ -continuous with closed witnesses, then \mathfrak{d} is the least cardinal κ for which X is the union of κ -many closed sets on which π is continuous. These results were established using ad-hoc recursive constructions reminiscent of those underlying the Lecomte–Zeleny results. Despite having received quite a bit of attention, the

question as to whether these results generalize in their most natural form to higher levels of the Borel hierarchy remains open.

A natural reaction to the aforementioned shortcomings is to seek simpler proofs which more readily adapt to Borel sets of higher complexity. Here we use a straightforward generalization of the perfect set theorem to provide a simple unified explanation of the Kechris–Louveau–Woodin generalization of Hurewicz’s original dichotomy theorem (see [KLW87]), the level-two results of Lecomte–Zeleny, the Jayne–Rogers theorem, the generalizations of these results from analytic to separable metric spaces under the axiom of determinacy, and the generalization of Solecki’s result concerning \mathfrak{d} to Borel functions. Unfortunately, the question as to whether such arguments do indeed generalize to higher levels of the Borel hierarchy remains open.

The underlying dichotomy. We say that a subset of a topological space has a property P of topological spaces if it has property P when equipped with the subspace topology. The *perfect set theorem* for Γ is the statement that for all $X \in \Gamma$, either $|X| \leq \aleph_0$ or there is a continuous injection of $2^{\mathbb{N}}$ into X . In [Sou17], this simplest of descriptive set-theoretic dichotomy theorems was established for the pointclass of analytic subsets of Hausdorff spaces. In [Dav64], it was generalized to the pointclass of all subsets of analytic Hausdorff spaces under the axiom of determinacy.

The *open graph dichotomy* for Γ is the statement that for all $X \in \Gamma$ and open graphs G on X , either $\chi(G) \leq \aleph_0$ or there is a continuous homomorphism from the *complete graph* $\mathbb{K}_{2^{\mathbb{N}}} = \sim\Delta(2^{\mathbb{N}})$ on $2^{\mathbb{N}}$ to G . In [Fen93], this generalization of the perfect set theorem was established for the pointclass of analytic subsets of Hausdorff spaces, as was its generalization to the pointclass of all subsets of analytic Hausdorff spaces under the axiom of determinacy. This was achieved by showing that the usual proofs of the perfect set theorem also yield the open graph dichotomy, giving a sense in which the latter is also among the simplest descriptive set-theoretic dichotomy theorems.

The *box topology* on a product $\prod_{d \in D} X_d$ of topological spaces is the topology generated by the sets of the form $\prod_{d \in D} U_d$, where $U_d \subseteq X_d$ is open for all $d \in D$. Given partial functions $s, t: \mathbb{N} \rightarrow D$, we write $s \sqsubseteq t$ to indicate that $s = t \upharpoonright \text{dom}(s)$, and we define $\mathcal{N}_t = \{\mathbf{d} \in D^{\mathbb{N}} \mid t \sqsubseteq \mathbf{d}\}$. We use (d) to denote the sequence of length one with value d , and we use $s \frown t$ to denote concatenation of s and t . The *box-open D -dimensional dihypergraph dichotomy* for Γ , or $\text{OGD}^D(\Gamma)$, is the statement that for all $X \in \Gamma$ and box-open D -dimensional dihypergraphs H on

X , either $\chi(H) \leq \aleph_0$ or there is a continuous homomorphism from the D -dimensional dihypergraph $\mathbb{H}_{D^{\aleph}} = \bigcup_{t \in D^{<\aleph}} \prod_{d \in D} \mathcal{N}_{t \smallfrown (d)}$ on D^{\aleph} to H .

Organization. In §1, we establish the box-open \aleph_0 -dimensional dihypergraph dichotomy for the pointclass of analytic subsets of Hausdorff spaces, as well as its generalization to the pointclass of all subsets of analytic Hausdorff spaces under the axiom of determinacy. As in [Fen93], we achieve this by showing that the usual proofs of the perfect set theorem easily adapt, giving a sense in which even the latter is among the simplest descriptive set-theoretic dichotomy theorems.

A subset of a topological space is K_σ if it is a union of countably-many compact sets. In §2, we give a first glimpse into how topological properties can be codified into dihypergraphs by showing that the Kechris–Saint Raymond generalizations of Hurewicz’s characterization of the circumstances under which a Polish space is K_σ (see [Kec77, SR75]) are special cases of the box-open \aleph_0 -dimensional dihypergraph dichotomy.

In §3, we establish the basic properties of partial compactifications of \mathbb{N}^{\aleph} arising from the box-open \aleph_0 -dimensional dihypergraph dichotomy.

In §4, we show that a special case of the box-open \aleph_0 -dimensional dihypergraph dichotomy yields a characterization of the circumstances under which an analytic subset of a D -fold product of metric spaces can be separated from another subset by a countable union of closed hyperrectangles. As a corollary, we obtain a characterization of the circumstances under which a D -dimensional dihypergraph on an analytic metric space has a Δ_2^0 -measurable \aleph_0 -coloring. We also obtain the generalizations in which analyticity is weakened to separability under the axiom of determinacy. The Kechris–Louveau–Woodin and level-two Lecomte–Zeleny theorems follow from the special cases of these results where $D = 1$ and $D = 2$.

In §5, we show that a special case of the box-open \aleph_0 -dimensional dihypergraph dichotomy yields the Jayne–Rogers characterization of the circumstances under which a function from an analytic metric space to a separable metric space is σ -continuous with closed witnesses. We also obtain the generalization in which analyticity is weakened to separability under the axiom of determinacy.

We use $\mathbb{H}'_{\aleph^{\aleph}}$ to denote the \aleph -dimensional dihypergraph on \aleph^{\aleph} given by $\mathbb{H}'_{\aleph^{\aleph}} = \bigcup_{(\mathbf{d}, t) \in (\aleph^{\aleph})^{\aleph} \times \aleph^{<\aleph}} \prod_{n \in \aleph} \mathcal{N}_{t \smallfrown (\mathbf{d}(n))}$, where $(M)^{\aleph}$ denotes the set of injective elements of M^{\aleph} . The *covering number* of an ideal \mathcal{I} on a set X is the least cardinal $\text{cov}(\mathcal{I})$ for which X is the union of $\text{cov}(\mathcal{I})$ -many sets in \mathcal{I} . The *cofinality* of an ideal \mathcal{I} on a set X is the least cardinal $\text{cof}(\mathcal{I})$ for which there is a set $\mathcal{J} \subseteq \mathcal{I}$ of cardinality $\text{cof}(\mathcal{I})$

with the property that $\forall I \in \mathcal{I} \exists J \in \mathcal{J} I \subseteq J$. We use \mathcal{M} to denote the σ -ideal on \mathbb{R} consisting of the meager sets, and \mathcal{N} to denote the σ -ideal on $[0, 1]$ consisting of the Lebesgue null sets. The *bounding number* is the least cardinal \mathfrak{b} for which there is a set $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality \mathfrak{b} such that $\forall \mathbf{c} \in \mathbb{N}^{\mathbb{N}} \exists \mathbf{d} \in \mathcal{F} \mathbf{d} \not\leq^* \mathbf{c}$. In §6, we note that $\chi(\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}) = \text{cov}(\mathcal{M})$ and $\chi(\mathbb{H}'_{\mathbb{N}^{\mathbb{N}}}) = \mathfrak{d}$. In conjunction with the box-open \aleph_0 -dimensional dihypergraph dichotomy, the latter fact yields the promised generalization of Solecki's theorem. We also show that if $2 < D < \aleph_0$, then $\chi(\mathbb{H}_{D^{\mathbb{N}}})$ is at least $\mathfrak{b} \cdot \text{cov}(\mathcal{N})$, consistently strictly below \mathfrak{d} , and consistently strictly above $\text{cof}(\mathcal{N})$.

Assumptions. We work in the base theory $\text{ZF} + \text{DC}$ throughout, with the exception of the final section, where we work in ZFC so as to keep our language as transparent as possible.

1. THE BOX-OPEN D -DIMENSIONAL DIHYPERGRAPH DICHOTOMY

In this section, we establish two instances of the box-open D -dimensional dihypergraph dichotomy.

Theorem 1.1. *Suppose that D is a discrete space of cardinality at least two, X is an analytic Hausdorff space, and H is a box-open D -dimensional dihypergraph on X . Then exactly one of the following holds:*

- (1) *There is an \aleph_0 -coloring of H .*
- (2) *There is a continuous homomorphism from $\mathbb{H}_{D^{\mathbb{N}}}$ to H .*

Proof. To see that the two conditions are mutually exclusive, it is sufficient to show that there is no \aleph_0 -coloring of $\mathbb{H}_{D^{\mathbb{N}}}$. Towards this end, suppose that $X \subseteq D^{\mathbb{N}}$ and $c: X \rightarrow \mathbb{N}$ is an \aleph_0 -coloring of $\mathbb{H}_{D^{\mathbb{N}}} \upharpoonright X$, recursively find $d_n \in D$ such that $n \notin c(\mathcal{N}_{(d_m)_{m \leq n}})$ for all $n \in \mathbb{N}$, and observe that $(d_n)_{n \in \mathbb{N}} \notin c^{-1}(\mathbb{N})$, thus $X \neq D^{\mathbb{N}}$.

To see that at least one of the conditions hold, we can assume that $X \neq \emptyset$, in which case there is a continuous surjection $\pi: \mathbb{N}^{\mathbb{N}} \rightarrow X$. By replacing H with its pullback through π , we can assume that $X = \mathbb{N}^{\mathbb{N}}$. Set $S = \{s \in \mathbb{N}^{<\mathbb{N}} \mid H \upharpoonright \mathcal{N}_s \text{ has an } \aleph_0\text{-coloring}\}$ and $Y = \sim \bigcup_{s \in S} \mathcal{N}_s$. Note that if $s \in \sim S$, then there is no \aleph_0 -coloring of $H \upharpoonright (\mathcal{N}_s \cap Y)$, so there exists $(y_d)_{d \in D} \in H \upharpoonright (\mathcal{N}_s \cap Y)$, thus the fact that H is box open yields a sequence $(s_d)_{d \in D} \in (\sim S)^D$ of proper extensions of s such that $\prod_{d \in D} \mathcal{N}_{s_d} \subseteq H$. It follows that if there is no \aleph_0 -coloring of H , then there is a function $f: D^{<\mathbb{N}} \rightarrow \sim S$ such that:

- (a) $\forall d \in D \forall t \in D^{<\mathbb{N}} f(t) \sqsubset f(t \frown (d))$.
- (b) $\forall t \in D^{<\mathbb{N}} \prod_{d \in D} \mathcal{N}_{f(t \frown (d))} \subseteq H$.

Condition (a) ensures that we obtain a continuous function $\phi: D^{\mathbb{N}} \rightarrow Y$ by setting $\phi(\mathbf{d}) = \bigcup_{n \in \mathbb{N}} f(\mathbf{d} \upharpoonright n)$, and condition (b) implies that ϕ is a homomorphism from $\mathbb{H}_{D^{\mathbb{N}}}$ to H . \square

As in [Fen93], the same argument yields the natural generalization to κ -Souslin Hausdorff spaces, and a Cantor–Bendixson-style derivative can be used to avoid the need for copious amounts of choice. However, in order to establish the box-open D -dimensional dihypergraph dichotomy for all subsets of analytic metric spaces under the axiom of determinacy, the natural analog of the game considered in [Fen93, §3] is insufficient, as it requires the first player to play $|D|$ -many natural numbers in each round. We next show that a slowed-down version of this game can be used instead.

Given a box-open \mathbb{N} -dimensional dihypergraph H on $\mathbb{N}^{\mathbb{N}}$ and a set $X \subseteq \mathbb{N}^{\mathbb{N}}$, consider the ω -length two-player game $G(X, H)$ whose n^{th} round consists of the first player playing a sequence $s_n \in \mathbb{N}^{<\mathbb{N}}$ with the property that $i_m = 1 \implies s_m \sqsubset s_n$ for all $m < n$, and then the second playing a natural number $i_n < 2$. There are two types of runs of $G(X, H)$, depending on whether the set $N = \{n \in \mathbb{N} \mid i_n = 1\}$ is finite or infinite. In the former case, the first player wins if and only if $\prod_{n \in \mathbb{N}} \mathcal{N}_{s_{\max(N)+1+n}} \subseteq H$. In the latter, the first player wins if and only if $\bigcup_{n \in N} s_n \in X$.

Proposition 1.2. *Suppose that H is a box-open \mathbb{N} -dimensional dihypergraph on $\mathbb{N}^{\mathbb{N}}$ and $X \subseteq \mathbb{N}^{\mathbb{N}}$.*

- (1) *The first player has a winning strategy in $G(X, H)$ if and only if there is a continuous homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to $H \upharpoonright X$.*
- (2) *The second player has a winning strategy in $G(X, H)$ if and only if there is an \aleph_0 -coloring of $H \upharpoonright X$.*

Proof. Let S denote the set of sequences $s \in 2^{<\mathbb{N}}$ that do not end in zero. Let $[\mathbb{N}]^{\mathbb{N}}$ denote the set of strictly increasing elements of $\mathbb{N}^{\mathbb{N}}$, define $\pi_{\mathbb{N}^{\mathbb{N}}, [\mathbb{N}]^{\mathbb{N}}}: \mathbb{N}^{\mathbb{N}} \rightarrow [\mathbb{N}]^{\mathbb{N}}$ by $\pi_{\mathbb{N}^{\mathbb{N}}, [\mathbb{N}]^{\mathbb{N}}}(b)(n) = n + \sum_{m \leq n} b(m)$, and define $\pi_{[\mathbb{N}]^{\mathbb{N}}, 2^{\mathbb{N}}}: [\mathbb{N}]^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $\pi_{[\mathbb{N}]^{\mathbb{N}}, 2^{\mathbb{N}}}(b) = \mathbf{1}_{b(\mathbb{N})}$. Then the function $\pi_{\mathbb{N}^{\mathbb{N}}, 2^{\mathbb{N}}} = \pi_{[\mathbb{N}]^{\mathbb{N}}, 2^{\mathbb{N}}} \circ \pi_{\mathbb{N}^{\mathbb{N}}, [\mathbb{N}]^{\mathbb{N}}}$ is a homeomorphism from $\mathbb{N}^{\mathbb{N}}$ to the space $C = \{c \in 2^{\mathbb{N}} \mid |\text{supp } c| = \aleph_0\}$, in addition to being an isomorphism of $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ with the restriction of the box-open \mathbb{N} -dimensional dihypergraph $H_{2^{\mathbb{N}}} = \bigcup_{s \in S} \prod_{n \in \mathbb{N}} \mathcal{N}_{s \frown (0)^n \frown (1)}$ to C .

Suppose that τ is a winning strategy for the first player in $G(X, H)$, and define $\phi: C \rightarrow \mathbb{N}^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n \in \text{supp } c} \tau(c \upharpoonright n)$. To see that ϕ is a homomorphism from $H_{2^{\mathbb{N}}} \upharpoonright C$ to H , note that if $(c_n)_{n \in \mathbb{N}} \in H_{2^{\mathbb{N}}} \upharpoonright C$, then there exists $s \in S$ for which $(c_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{N}_{s \frown (0)^n \frown (1)}$, in which case $(\phi(c_n))_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{N}_{\tau(s \frown (0)^n)}$, so the fact that the first player wins

runs of $G(X, H)$ of the first type ensures that $(\phi(c_n))_{n \in \mathbb{N}} \in H$. The fact that the first player wins runs of $G(X, H)$ of the second type implies that $\phi(C) \subseteq X$.

Conversely, suppose that $\phi: C \rightarrow X$ is a continuous homomorphism from $H_{2^{\mathbb{N}}} \upharpoonright C$ to H . For each non-empty sequence $s \in S$, let s^- denote the immediate predecessor of s . We will recursively construct functions $\sigma: 2^{<\mathbb{N}} \rightarrow S$ and $\tau: 2^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that:

- (a) $\forall s \in S \prod_{n \in \mathbb{N}} \mathcal{N}_{\tau(s \frown (0)^n)} \subseteq H$.
- (b) $\forall s \in 2^{<\mathbb{N}} \phi(\mathcal{N}_{\sigma(s)}) \subseteq \mathcal{N}_{\tau(s)}$.
- (c) $\forall n \in \mathbb{N} \forall s \in S \setminus \{\emptyset\} \tau(s^-) \sqsubset \tau(s \frown (0)^n)$.
- (d) $\forall n \in \mathbb{N} \forall s \in S \setminus \{\emptyset\} \sigma(s^-) \frown (0)^n \frown (1) \sqsubseteq \sigma(s \frown (0)^n)$.

Suppose that $s \in S$ and we have already found $\sigma(r)$ and $\tau(r)$ for all $r \sqsubset s$. If $s = \emptyset$, then set $u_s = v_s = \emptyset$, and otherwise define $u_s = \sigma(s^-)$ and $v_s = \tau(s^-)$. As $(u_s \frown (0)^n \frown (1)^\infty)_{n \in \mathbb{N}} \in H_{2^{\mathbb{N}}} \upharpoonright C$, there are strict extensions $\tau(s \frown (0)^n) \sqsubset \phi(u_s \frown (0)^n \frown (1)^\infty)$ of v_s such that $\prod_{n \in \mathbb{N}} \mathcal{N}_{\tau(s \frown (0)^n)} \subseteq H$, as well as positive integers $k_{n,s}$ such that $\phi(\mathcal{N}_{u_s \frown (0)^n \frown (1)^{k_{n,s}}}) \subseteq \mathcal{N}_{\tau(s \frown (0)^n)}$ for all $n \in \mathbb{N}$. We complete the construction by setting $\sigma(s \frown (0)^n) = u_s \frown (0)^n \frown (1)^{k_{n,s}}$ for all $n \in \mathbb{N}$. To see that τ is a winning strategy for the first player in $G(X, H)$, note that the first player wins runs of $G(X, H)$ of the first type by condition (a), whereas the other conditions ensure that if $c \in C$, then $\phi(\bigcup_{n \in \text{supp } c} \sigma(c \upharpoonright n)) = \bigcup_{n \in \text{supp } c} \tau(c \upharpoonright n)$, thus the first player wins runs of $G(X, H)$ of the second type.

Suppose now that τ is a winning strategy for the second player in $G(X, H)$. Let T denote the set of partial runs of $G(X, H)$ against τ for which $(\tau(t \upharpoonright \{0, \dots, n\}))_{n < |t|} \in S$, and associate with each $t \in T$ the set $X_t = \{x \in X \mid t \neq \emptyset \implies t(|t| - 1) \sqsubseteq x\}$. If $(x_n)_{n \in \mathbb{N}} \in H \upharpoonright X_t$, then there are sequences $t_n \sqsubset x_n$ with the property that $\prod_{n \in \mathbb{N}} \mathcal{N}_{t_n} \subseteq H$ and $t \neq \emptyset \implies \forall n \in \mathbb{N} t(|t| - 1) \sqsubset t_n$, so the fact that the second player wins runs of $G(X, H)$ of the first type therefore yields $n \in \mathbb{N}$ for which $t \frown (t_m)_{m \leq n} \in T$ and $x_n \in X_{t \frown (t_m)_{m \leq n}}$. In particular, it follows that the sets of the form $X_t \setminus \bigcup_{t \sqsubset u, u \in T} X_u$ are H -independent. As the fact that the second player wins runs of $G(X, H)$ of the second type ensures that every $x \in X$ appears in a set of the latter form, it follows that there is an \aleph_0 -coloring of $H \upharpoonright X$.

Conversely, suppose that $c: X \rightarrow \mathbb{N}$ is an \aleph_0 -coloring of $H \upharpoonright X$, and let τ be the strategy for the second player in $G(X, H)$ in which 0 is played in the n^{th} round of the game if and only if $c^{-1}(\{k_n\}) \cap \mathcal{N}_{s_n} \neq \emptyset$, where $k_n = |\{m < n \mid i_m = 1\}|$. The fact that the sets of the form $c^{-1}(\{k\})$ are H -independent ensures that the second player wins runs

of $G(X, H)$ of the first type, while the fact that $X \subseteq c^{-1}(\mathbb{N})$ implies that the second player wins runs of $G(X, H)$ of the second type. \square

In particular, we obtain the following.

Theorem 1.3 (AD). *Suppose that Y is an analytic Hausdorff space, H is a box-open \mathbb{N} -dimensional dihypergraph on Y , and $X \subseteq Y$. Then exactly one of the following holds:*

- (1) *There is an \aleph_0 -coloring of $H \upharpoonright X$.*
- (2) *There is a continuous homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to $H \upharpoonright X$.*

Proof. As noted in the proof of Theorem 1.1, conditions (1) and (2) are mutually exclusive, so it is sufficient to show that at least one of them holds. We can also assume that $Y \neq \emptyset$, in which case there is a continuous surjection $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow Y$. By replacing H and X with their pullbacks through π , we can assume that $Y = \mathbb{N}^{\mathbb{N}}$. But AD and Proposition 1.2 ensure that one of the two conditions holds. \square

2. K_σ SETS

Given a topological space X , let H_X denote the \mathbb{N} -dimensional dihypergraph on X consisting of all injective sequences $(x_n)_{n \in \mathbb{N}}$ of elements of X with no convergent subsequence.

Proposition 2.1. *Suppose that X is a metric space.*

- (1) *The dihypergraph H_X is box open.*
- (2) *There is an \aleph_0 -coloring of the restriction of H_X to a set $Y \subseteq X$ if and only if Y is contained in a K_σ subset of X .*
- (3) *A continuous function $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ is a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_X if and only if it is an injective closed map.*

Proof. To see (1), note that if $(x_n)_{n \in \mathbb{N}} \in H_X$, then there exist positive real numbers $\epsilon_n \rightarrow 0$ such that $\rho_X(x_m, x_n) \geq 2\epsilon_n$ for all natural numbers $m \neq n$, in which case $\prod_{n \in \mathbb{N}} \mathcal{B}_X(x_n, \epsilon_n) \subseteq H_X$.

To see (2), it is sufficient to observe that a set $Y \subseteq X$ is H_X -independent if and only if its closure is compact.

To see (3), note first that if ϕ is an injective closed map, then the fact that each sequence $(\mathbf{d}_n)_{n \in \mathbb{N}} \in \mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ is an injective enumeration of a closed discrete set ensures that the same holds of $(\phi(\mathbf{d}_n))_{n \in \mathbb{N}}$. Conversely, suppose that ϕ is a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_X . The fact that H_X consists solely of injective sequences easily implies that ϕ is injective. To see that ϕ is a closed map, it is sufficient to show that every sequence $(\mathbf{d}_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\phi(\mathbf{d}_n))_{n \in \mathbb{N}}$ converges has a convergent subsequence. If there exists $\mathbf{d} \in \mathbb{N}^{\mathbb{N}}$ such that $\mathbf{d}_n(i) < \mathbf{d}(i)$ for all $i, n \in \mathbb{N}$, then the compactness of $\prod_{i \in \mathbb{N}} \mathbf{d}(i)$

yields the desired subsequence. So suppose, towards a contradiction, that there does not exist such a \mathbf{d} . Then there is a least $k \in \mathbb{N}$ for which $\{\mathbf{d}_n(k) \mid n \in \mathbb{N}\}$ is infinite. By passing to a subsequence, we can assume that for all distinct $m, n \in \mathbb{N}$, the sequences \mathbf{d}_m and \mathbf{d}_n differ from one another for the first time on their k^{th} coordinates. By passing to a further subsequence, we can assume that $(\mathbf{d}_n)_{n \in \mathbb{N}}$ is a subsequence of an element of $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$, so $(\phi(\mathbf{d}_n))_{n \in \mathbb{N}}$ is a subsequence of an element of H_X , contradicting the fact that it converges. \square

In particular, we obtain the following.

Theorem 2.2 ($\text{OGD}^{\mathbb{N}}(\Gamma)$). *Suppose that X is a metric space and $Y \subseteq X$ is in Γ . Then exactly one of the following holds:*

- (1) *The set Y is contained in a K_σ subset of X .*
- (2) *There is a closed continuous injection $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow X$ with the property that $\phi(\mathbb{N}^{\mathbb{N}}) \subseteq Y$.*

Proof. This follows from $\text{OGD}^{\mathbb{N}}(\Gamma)$ and Proposition 2.1. \square

Theorems 1.1 and 1.3 together with the special cases of Theorem 2.2 where Γ is either the pointclass of analytic subsets of metric spaces or the pointclass of all subsets of analytic metric spaces yield the Kechris–Saint Raymond generalizations of Hurewicz’s characterization of the circumstances under which a Polish space is K_σ .

3. PARTIAL COMPACTIFICATIONS

For each topological space X and discrete set $D \subseteq X$, endow the set $\text{Ext}_X(D^{\mathbb{N}}) = D^{\mathbb{N}} \cup \{t \frown (x) \mid t \in D^{<\mathbb{N}} \text{ and } x \in \sim D\}$ with the topology generated by the sets $\mathcal{N}_{t,U} = \{\mathbf{d} \in \text{Ext}_X(D^{\mathbb{N}}) \mid t \sqsubset \mathbf{d} \text{ and } \mathbf{d}(|t|) \in U\}$, where $t \in D^{<\mathbb{N}}$ and $U \subseteq X$ is open. As we shall later see, such spaces arise naturally in applications of the box-open D -dimensional dihypergraph dichotomy.

Proposition 3.1. *Suppose that X is a compact space and $D \subseteq X$ is discrete and open. Then $\text{Ext}_X(D^{\mathbb{N}})$ is compact.*

Proof. Set $\mathcal{U} = \{\mathcal{N}_{t,U} \mid t \in D^{<\mathbb{N}} \text{ and } U \subseteq X \text{ is open}\}$, and for each sequence $t \in D^{<\mathbb{N}}$ and family $\mathcal{V} \subseteq \mathcal{U}$, let \mathcal{V}_t denote the family of open sets $V \subseteq X$ for which $\mathcal{N}_{t,V} \in \mathcal{V}$. The main observation is as follows:

Lemma 3.2. *Suppose that $t \in D^{<\mathbb{N}}$ and $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t,X}$. If for all $d \in D$ there is a finite set $\mathcal{F}_d \subseteq \mathcal{V}$ covering $\mathcal{N}_{t \frown (d), X}$, then there is a finite set $\mathcal{F} \subseteq \mathcal{V}$ covering $\mathcal{N}_{t,X}$.*

Proof. We can assume that there do not exist $s \sqsubset t$ and an open set $U \subseteq X$ for which $t(|s|) \in U$ and $\mathcal{N}_{s,U} \in \mathcal{V}$. Then there is a finite set $\mathcal{F} \subseteq \mathcal{V}_t$ covering $\sim D$, in which case the set $F = \sim \bigcup \mathcal{F}$ is compact and contained in D , thus finite. But $\mathcal{F} \cup \bigcup_{d \in F} \mathcal{F}_d$ covers $\mathcal{N}_{t,X}$. \square

Now observe that if $\mathcal{V} \subseteq \mathcal{U}$ is a cover of $\text{Ext}_X(D^{\mathbb{N}})$ with no finite subcover, then by recursively applying the contrapositive of Lemma 3.2, we obtain a sequence $\mathbf{d} \in D^{\mathbb{N}}$ such that for no $n \in \mathbb{N}$ is there a finite subset of \mathcal{V} covering $\mathcal{N}_{\mathbf{d}|n,X}$, contradicting the fact that $\mathbf{d} \in \bigcup \mathcal{V}$. \square

Proposition 3.3. *Suppose that X is a (complete) ultrametric space and $D \subseteq X$ is discrete and open. Then $\text{Ext}_X(D^{\mathbb{N}})$ has a compatible (complete) ultrametric.*

Proof. By replacing ρ_X with $\rho_X/(1 + \rho_X)$, we can assume that $\rho_X < 1$. Fix real numbers $\epsilon_d > 0$ such that $\sup_{d \in D} \epsilon_d < 1$ and $\mathcal{B}(d, \epsilon_d) = \{d\}$ for all $d \in D$, and define $\rho: \text{Ext}_X(D^{\mathbb{N}}) \times \text{Ext}_X(D^{\mathbb{N}}) \rightarrow \mathbb{R}$ by

$$\rho(\mathbf{c}, \mathbf{d}) = \begin{cases} 0 & \text{if } \mathbf{c} = \mathbf{d} \text{ and} \\ \rho_X(\mathbf{c}(n), \mathbf{d}(n)) \prod_{m < n} \epsilon_{\mathbf{c}(m)} & \text{otherwise,} \end{cases}$$

where $n = n(\mathbf{c}, \mathbf{d})$ is the least natural number for which $\mathbf{c}(n) \neq \mathbf{d}(n)$.

To establish that ρ is an ultrametric, it is sufficient to show that if $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \text{Ext}_X(D^{\mathbb{N}})$ are distinct, then $\rho(\mathbf{b}, \mathbf{d}) \leq \max\{\rho(\mathbf{b}, \mathbf{c}), \rho(\mathbf{c}, \mathbf{d})\}$. Setting $n = \max\{n(\mathbf{b}, \mathbf{c}), n(\mathbf{c}, \mathbf{d})\}$, there are three cases to check:

- (1) If $n(\mathbf{b}, \mathbf{d}) < n$, then $\rho(\mathbf{b}, \mathbf{d}) \in \{\rho(\mathbf{b}, \mathbf{c}), \rho(\mathbf{c}, \mathbf{d})\}$.
- (2) If $n(\mathbf{b}, \mathbf{d}) = n$, then $n(\mathbf{b}, \mathbf{c}) = n(\mathbf{b}, \mathbf{d}) = n(\mathbf{c}, \mathbf{d})$, so

$$\begin{aligned} \rho(\mathbf{b}, \mathbf{d}) &= \rho_X(\mathbf{b}(n), \mathbf{d}(n)) \prod_{m < n} \epsilon_{\mathbf{b}(m)} \\ &\leq \max\{\rho_X(\mathbf{b}(n), \mathbf{c}(n)), \rho_X(\mathbf{c}(n), \mathbf{d}(n))\} \prod_{m < n} \epsilon_{\mathbf{b}(m)} \\ &= \max\{\rho(\mathbf{b}, \mathbf{c}), \rho(\mathbf{c}, \mathbf{d})\}. \end{aligned}$$

- (3) If $n(\mathbf{b}, \mathbf{d}) > n$, then $\rho(\mathbf{b}, \mathbf{d}) < \prod_{m \leq n} \epsilon_{\mathbf{b}(m)} \leq \rho(\mathbf{b}, \mathbf{c})$.

To see that the topology generated by ρ is contained in that of $\text{Ext}_X(D^{\mathbb{N}})$, suppose that $\mathbf{d} \in \text{Ext}_X(D^{\mathbb{N}})$ and $\epsilon > 0$. If $\mathbf{d} \in D^{\mathbb{N}}$, then there exists $n \in \mathbb{N}$ sufficiently large that $\prod_{m < n} \epsilon_{\mathbf{d}(m)} \leq \epsilon$, in which case $\mathcal{N}_{\mathbf{d}|n,X}$ is an open neighborhood of \mathbf{d} contained in $\mathcal{B}(\mathbf{d}, \epsilon)$. Otherwise, there exist $t \in D^{<\mathbb{N}}$ and $x \in \sim D$ for which $\mathbf{d} = t \frown (x)$, in which case $\mathcal{N}_{t, \mathcal{B}_X(x, \epsilon)}$ is an open neighborhood of \mathbf{d} contained in $\mathcal{B}(\mathbf{d}, \epsilon)$.

To see that the topology of $\text{Ext}_X(D^{\mathbb{N}})$ is contained in that generated by ρ , suppose that $t \in D^{<\mathbb{N}}$, $U \subseteq X$ is open, and $\mathbf{d} \in \mathcal{N}_{t,U}$, and fix $0 < \epsilon < 1$ such that $\mathcal{B}_X(\mathbf{d}(|t|), \epsilon) \subseteq U$. Then $\mathcal{B}(\mathbf{d}, \epsilon \prod_{n < |t|} \epsilon_{t(n)}) \subseteq \mathcal{N}_{t,U}$.

To see that the completeness of ρ_X yields that of ρ , suppose that $(\mathbf{d}_k)_{k \in \mathbb{N}}$ is an injective Cauchy sequence of elements of $\text{Ext}_X(D^{\mathbb{N}})$. If

there exists $\mathbf{d} \in D^{\mathbb{N}}$ with the property that $\mathbf{d}_k(n) \rightarrow \mathbf{d}(n)$ for all $n \in \mathbb{N}$, then $\mathbf{d}_k \rightarrow \mathbf{d}$. Otherwise, there is a sequence $t \in D^{<\mathbb{N}}$ of maximal length such that $\mathbf{d}_k(n) \rightarrow t(n)$ for all $n < |t|$. By passing to a subsequence of $(\mathbf{d}_k)_{k \in \mathbb{N}}$, we can assume that $n(\mathbf{d}_j, \mathbf{d}_k) = |t|$, thus $\rho(\mathbf{d}_j, \mathbf{d}_k) = \rho_X(\mathbf{d}_j(|t|), \mathbf{d}_k(|t|)) \prod_{n < |t|} \epsilon_{t(n)}$, for all distinct $j, k \in \mathbb{N}$. It follows that $(\mathbf{d}_k(|t|))_{k \in \mathbb{N}}$ is Cauchy, and therefore convergent, thus $(\mathbf{d}_k)_{k \in \mathbb{N}}$ is convergent as well. \square

Let $\text{Cnvg}(X)$ denote the set of convergent sequences $(x_n)_{n \in \mathbb{N}}$ of elements of X , and let $\text{Cnvg}_X(D^{\mathbb{N}})$ denote the \mathbb{N} -ary relation on $D^{\mathbb{N}}$ given by $\text{Cnvg}_X(D^{\mathbb{N}}) = \bigcup_{(\mathbf{d}, t) \in (\text{Cnvg}(X) \cap D^{\mathbb{N}}) \times D^{<\mathbb{N}}} \prod_{n \in \mathbb{N}} \mathcal{N}_{t \frown (\mathbf{d}(n))}$. In our applications of the box-open D -dimensional dihypergraph dichotomy, the following fact will yield extensions of homomorphisms between dihypergraphs.

Proposition 3.4. *Suppose that X and Y are metric spaces, $D \subseteq X$ is dense, discrete, and open, and $\phi: D^{\mathbb{N}} \rightarrow Y$ is a continuous homomorphism from $\text{Cnvg}_X(D^{\mathbb{N}})$ to $\text{Cnvg}(Y)$. Then there is a continuous extension of ϕ to $\text{Ext}_X(D^{\mathbb{N}})$.*

Proof. Given a point $x \in X$, we say that a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets of X converges to x , or $X_n \rightarrow x$, if for every open neighborhood $U \subseteq X$ of x , there exists $n \in \mathbb{N}$ for which $\bigcup_{m \geq n} X_m \subseteq U$.

Lemma 3.5. *Suppose that $t \in D^{<\mathbb{N}}$ and $x \in \sim D$. Then there exists $y_{t,x} \in Y$ such that $\mathbf{d}(n) \rightarrow x \implies \phi(\mathcal{N}_{t \frown (\mathbf{d}(n))}) \rightarrow y_{t,x}$ for all $\mathbf{d} \in D^{\mathbb{N}}$.*

Proof. Note that if $\mathbf{d} \in D^{\mathbb{N}}$, $\mathbf{d}(n) \rightarrow x$, and $y_n \in \phi(\mathcal{N}_{t \frown (\mathbf{d}(n))})$ for all $n \in \mathbb{N}$, then there exists $y_{t,x} \in Y$ for which $y_n \rightarrow y_{t,x}$. If there exists $\mathbf{c} \in D^{\mathbb{N}}$ such that $\mathbf{c}(n) \rightarrow x$ and $\phi(\mathcal{N}_{t \frown (\mathbf{c}(n))}) \not\rightarrow y_{t,x}$, then there exist an infinite set $N \subseteq \mathbb{N}$, an open neighborhood $V \subseteq Y$ of $y_{t,x}$, and points $y_n \in \phi(\mathcal{N}_{t \frown (\mathbf{c}(n))}) \setminus V$ for all $n \in N$. By thinning down N , we can assume that it is co-infinite. Set $\mathbf{b} = (\mathbf{c} \upharpoonright N) \cup (\mathbf{d} \upharpoonright \sim N)$, and observe that $\mathbf{b}(n) \rightarrow x$ but $(\phi(\mathbf{b}(n)))_{n \in \mathbb{N}}$ does not converge, a contradiction. \square

To see that the extension given by $\bar{\phi}(t \frown (x)) = y_{t,x}$ is continuous, suppose that $\mathbf{d} \in \text{Ext}_X(D^{\mathbb{N}})$ and $V \subseteq Y$ is an open neighborhood of $\bar{\phi}(\mathbf{d})$, and fix an open neighborhood $W \subseteq Y$ of $\bar{\phi}(\mathbf{d})$ whose closure is contained in V . If $\mathbf{d} \in D^{\mathbb{N}}$, then the continuity of ϕ yields $n \in \mathbb{N}$ for which $\phi(\mathcal{N}_{\mathbf{d} \upharpoonright n}) \subseteq W$, in which case $\bar{\phi}(\mathcal{N}_{\mathbf{d} \upharpoonright n, X}) \subseteq \bar{\phi}(\mathcal{N}_{\mathbf{d} \upharpoonright n}) \subseteq \bar{W} \subseteq V$. Otherwise, there exists $t \in D^{<\mathbb{N}}$ and $x \in \sim D$ for which $\mathbf{d} = t \frown (x)$, so Lemma 3.5 yields an open neighborhood $U \subseteq X$ of x for which $\phi(\mathcal{N}_{t,U} \cap D^{\mathbb{N}}) \subseteq W$, in which case $\bar{\phi}(\mathcal{N}_{t,U}) \subseteq \bar{\phi}(\mathcal{N}_{t,U}) \subseteq \bar{W} \subseteq V$. \square

4. COUNTABLE UNIONS OF HYPERRECTANGLES

Given sequences $(X_d)_{d \in D}$ and $(Y_d)_{d \in D}$ of topological spaces, a *hyperrectangular homomorphism* from a pair (R_X, S_X) of subsets of $\prod_{d \in D} X_d$ to a pair (R_Y, S_Y) of subsets of $\prod_{d \in D} Y_d$ is a function ϕ of the form $\prod_{d \in D} \phi_d$, where $\phi_d: \text{proj}_d(R_X \cup S_X) \rightarrow Y_d$ for all $d \in D$, such that $\phi(R_X) \subseteq R_Y$ and $\phi(S_X) \subseteq S_Y$.

Let $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$ denote the one-point compactification of \mathbb{N} , and $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ the D -dimensional dihypergraph on $\text{Ext}_{D \times \mathbb{N}_*}((D \times \mathbb{N})^{\mathbb{N}})$ consisting of all sequences $(t \frown ((d, \infty)))_{d \in D}$, where $t \in (D \times \mathbb{N})^{<\mathbb{N}}$.

Given a sequence $(X_d)_{d \in D}$ and disjoint sets $R, S \subseteq \prod_{d \in D} X_d$, let $H_{R,S}$ denote the $(D \times \mathbb{N})$ -dimensional dihypergraph on R consisting of all sequences $((x_{c,d,n})_{c \in D})_{(d,n) \in D \times \mathbb{N}}$ of elements of R with the property that $\bar{x}_d = \lim_{n \rightarrow \infty} x_{d,d,n}$ exists for all $d \in D$ and $(\bar{x}_d)_{d \in D} \in S$.

Proposition 4.1. *Suppose that D is a discrete set, $(X_d)_{d \in D}$ is a sequence of metric spaces, and $R, S \subseteq \prod_{d \in D} X_d$ are disjoint.*

- (1) *The $(D \times \mathbb{N})$ -dimensional dihypergraph $H_{R,S}$ is box open.*
- (2) *There is an \aleph_0 -coloring of $H_{R,S}$ if and only if there is a countable union of closed hyperrectangles separating R from S .*
- (3) *There is a continuous homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}}$ to $H_{R,S}$ if and only if there is a continuous hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S) .*

Proof. To see (1), note that if $((x_{c,d,n})_{c \in D})_{(d,n) \in D \times \mathbb{N}} \in H_{R,S}$, $\epsilon_n \rightarrow 0$, and $U_{d,n} = \{(x_c)_{c \in D} \in R \mid \rho_{X_d}(x_d, x_{d,d,n}) < \epsilon_n\}$ for all $(d, n) \in D \times \mathbb{N}$, then $\prod_{(d,n) \in D \times \mathbb{N}} U_{d,n} \subseteq H_{R,S}$.

To see (2), note first that if $Q \subseteq R$ and $(\bar{x}_d)_{d \in D} \in \overline{\prod_{d \in D} \text{proj}_d(Q)}$, then there are sequences $(x_{d,n})_{n \in \mathbb{N}}$ of elements of $\text{proj}_d(Q)$ such that $x_{d,n} \rightarrow \bar{x}_d$ for all $d \in D$, so there are sequences $(x_{c,d,n})_{c \in D} \in Q$ such that $x_{d,d,n} = x_{d,n}$ for all $(d, n) \in D \times \mathbb{N}$, thus $x_{d,d,n} \rightarrow \bar{x}_d$ for all $d \in D$. It follows that if Q is $H_{R,S}$ -independent, then $\overline{\prod_{d \in D} \text{proj}_d(Q)}$ and S are disjoint, so if $c: R \rightarrow \mathbb{N}$ is an \aleph_0 -coloring of $H_{R,S}$, then the union of the closed hyperrectangles $\prod_{d \in D} \text{proj}_d(c^{-1}(\{n\}))$ separates R from S . Conversely, suppose that F_d is a closed subset of X_d for all $d \in D$ and $S \cap \prod_{d \in D} F_d = \emptyset$. If $((x_{c,d,n})_{c \in D})_{(d,n) \in D \times \mathbb{N}}$ is a sequence of elements of $\prod_{c \in D} F_c$ such that $\bar{x}_d = \lim_{n \rightarrow \infty} x_{d,d,n}$ exists for all $d \in D$, then $(\bar{x}_d)_{d \in D} \in \prod_{d \in D} F_d$, so $(\bar{x}_d)_{d \in D} \notin S$, thus $R \cap \prod_{d \in D} F_d$ is $H_{R,S}$ -independent. Hence if there is a countable union of closed hyperrectangles separating R from S , then there is an \aleph_0 -coloring of $H_{R,S}$.

To see (3), suppose first that $\prod_{d \in D} \phi_d$ is a continuous hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S) , and

define $\phi: (D \times \mathbb{N})^{\mathbb{N}} \rightarrow \prod_{d \in D} X_d$ by $\phi(\mathbf{d}) = (\phi_d(\mathbf{d}))_{d \in D}$. Clearly ϕ is continuous and $\phi((D \times \mathbb{N})^{\mathbb{N}}) = (\prod_{d \in D} \phi_d)(\Delta^D((D \times \mathbb{N})^{\mathbb{N}})) \subseteq R$. To see that ϕ is a homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}}$ to $H_{R,S}$, we need only note that $(\lim_{n \rightarrow \infty} \phi_d(\mathcal{N}_{t \frown ((d,n))}))_{d \in D} = (\phi_d(t \frown ((d, \infty))))_{d \in D}$ for all $t \in (D \times \mathbb{N})^{<\mathbb{N}}$, since $(\prod_{d \in D} \phi_d)(\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}) \subseteq S$. Conversely, suppose that $\phi: (D \times \mathbb{N})^{\mathbb{N}} \rightarrow R$ is a continuous homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}}$ to $H_{R,S}$. Then each of the functions $\text{proj}_d \circ \phi$ is a continuous homomorphism from $\text{Cnvg}_{\mathcal{G}(D \times \mathbb{N}) \cup \{(d, \infty)\}}((D \times \mathbb{N})^{\mathbb{N}})$ to $\text{Cnvg}(X_d)$, so Proposition 3.4 yields continuous extensions $\phi_d: \text{Ext}_{(D \times \mathbb{N}) \cup \{(d, \infty)\}}((D \times \mathbb{N})^{\mathbb{N}}) \rightarrow X_d$ for all $d \in D$, and clearly $\prod_{d \in D} \phi_d$ is a hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S) . \square

In particular, we obtain the following.

Theorem 4.2 (OGD $^{\mathbb{N}}(\Gamma)$). *Suppose that D is a countable discrete set, $(X_d)_{d \in D}$ is a sequence of metric spaces, $R, S \subseteq \prod_{d \in D} X_d$ are disjoint, and $R \in \Gamma$. Then exactly one of the following holds:*

- (1) *There is a countable union of closed hyperrectangles separating R from S .*
- (2) *There is a continuous hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to (R, S) .*

Proof. This follows from Proposition 4.1. \square

Theorems 1.1 and 1.3 together with the special cases of Theorem 4.2 where $D = 1$ and Γ is either the pointclass of analytic subsets of metric spaces or the pointclass of all subsets of analytic metric spaces easily yield the Kechris–Louveau–Woodin generalizations of Hurewicz’s characterization of the circumstances under which two sets can be separated by an F_{σ} set.

Theorem 1.1 and the special case of Theorem 4.2 where $D = 2$ and Γ is the pointclass of analytic subsets of metric spaces easily yield the Lecomte–Zeleny characterization of the circumstances under which an analytic subset of the plane can be separated from another subset of the plane by a countable union of closed rectangles. Theorem 1.3 and the special case of Theorem 4.2 where $D = 2$ and Γ is the pointclass of all subsets of analytic metric spaces easily yield the generalization in which analyticity is weakened to separability under the axiom of determinacy.

Proposition 4.3. *Suppose that D is a discrete set of cardinality at least two, X is a metric space, and H is a D -dimensional dihypergraph on X .*

- (1) *There is a countable union of closed hyperrectangles separating $\Delta^D(X)$ from H if and only if there is a Δ_2^0 -measurable \aleph_0 -coloring of H .*
- (2) *There is a continuous hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to $(\Delta^D(X), H)$ if and only if there is a continuous homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H .*

Proof. To see (1), note first that if $(F_{d,n})_{(d,n) \in D \times \mathbb{N}}$ is a sequence of closed subsets of X for which $\bigcup_{n \in \mathbb{N}} \prod_{d \in D} F_{d,n}$ separates $\Delta^D(X)$ from H , then the H -independent closed sets $\bigcap_{d \in D} F_{d,n}$ cover X , so there is a Δ_2^0 -measurable \aleph_0 -coloring of H . Conversely, if there is a Δ_2^0 -measurable \aleph_0 -coloring of H , then there is a cover $(F_n)_{n \in \mathbb{N}}$ of X by H -independent closed sets, in which case $\bigcup_{n \in \mathbb{N}} F_n^D$ is a countable union of closed hyperrectangles separating $\Delta^D(X)$ from H .

To see (2), note first that if $\phi: \text{Ext}_{D \times \mathbb{N}^*}((D \times \mathbb{N})^{\mathbb{N}}) \rightarrow X$ is a homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H , and ϕ_d is the restriction of ϕ to $\text{Ext}_{(D \times \mathbb{N}) \cup \{(d, \infty)\}}((D \times \mathbb{N})^{\mathbb{N}})$ for all $d \in D$, then $\prod_{d \in D} \phi_d$ is a hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to $(\Delta^D(X), H)$. Conversely, observe that if $\prod_{d \in D} \phi_d$ is a continuous hyperrectangular homomorphism from $(\Delta^D((D \times \mathbb{N})^{\mathbb{N}}), \mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty})$ to $(\Delta^D(X), H)$, then the function $\phi = \bigcup_{d \in D} \phi_d$ is a continuous homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H . \square

In particular, we obtain the following.

Theorem 4.4 (OGD $^{\mathbb{N}}(\Gamma)$). *Suppose that D is a countable discrete set of cardinality at least two, X is a metric space in Γ , and H is a D -dimensional dihypergraph on X . Then exactly one of the following holds:*

- (1) *There is a Δ_2^0 -measurable \aleph_0 -coloring of H .*
- (2) *There is a continuous homomorphism from $\mathbb{H}_{(D \times \mathbb{N})^{\mathbb{N}}, \infty}$ to H .*

Proof. This follows from Propositions 4.1 and 4.3. \square

Theorem 1.1 and the special case of Theorem 4.4 where $D = 2$ and Γ is the pointclass of analytic subsets of metric spaces easily yield the Lecomte–Zeleny characterization of the circumstances under which a graph on an analytic metric space has a Δ_2^0 -measurable \aleph_0 -coloring. Theorem 1.3 and the special case of Theorem 4.4 where $D = 2$ and Γ is the pointclass of all subsets of analytic metric spaces easily yield the generalization in which analyticity is weakened to separability under the axiom of determinacy.

5. SIGMA-CONTINUOUS FUNCTIONS WITH CLOSED WITNESSES

Given a function $\pi: X \rightarrow Y$, let H_π denote the \mathbb{N} -dimensional dihypergraph on $\text{graph}(\pi)$ consisting of all sequences $((x_n, y_n))_{n \in \mathbb{N}}$ of elements of $\text{graph}(\pi)$ with the property that $\bar{x} = \lim_{n \rightarrow \infty} x_n$ exists but $\pi(\bar{x}) \notin \overline{\{y_n \mid n \in \mathbb{N}\}}$.

Proposition 5.1. *Suppose that X and Y are metric spaces and $\pi: X \rightarrow Y$.*

- (1) *The dihypergraph H_π is box open.*
- (2) *There is an \aleph_0 -coloring of H_π if and only if π is σ -continuous with closed witnesses.*
- (3) *There is a continuous homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_π if and only if there is a continuous function $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$ with the property that $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to a closed set and $(\pi \circ \psi) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous.*

Proof. To see (1), note that if $((x_n, y_n))_{n \in \mathbb{N}} \in H_\pi$ and $\bar{x} = \lim_{n \rightarrow \infty} x_n$, then there exists $\epsilon > 0$ such that $\rho_Y(\pi(\bar{x}), y_n) \geq 2\epsilon$ for all $n \in \mathbb{N}$. Fix positive real numbers $\epsilon_n \rightarrow 0$, and observe that the intersection of $\text{graph}(\pi)$ with $\prod_{n \in \mathbb{N}} \mathcal{B}_X(x_n, \epsilon_n) \times \mathcal{B}_Y(y_n, \epsilon)$ is contained in H_π .

To see (2), note first that if $F \subseteq X$ is a closed set on which π is continuous, then $\text{graph}(\pi) \cap (F \times Y)$ is H_π -independent, so if π is σ -continuous with closed witnesses, then there is a Borel \aleph_0 -coloring of H_π . Conversely, if $R \subseteq \text{graph}(\pi)$ and $(\bar{x}_n)_{n \in \mathbb{N}}$ is a convergent sequence of elements of the closure of $\text{proj}_X(R)$, then there are sequences $(x_{m,n})_{m \in \mathbb{N}}$ of elements of $\text{proj}_X(R)$ such that $x_{m,n} \rightarrow \bar{x}_n$ for all $n \in \mathbb{N}$. If R is H_π -independent, then $\pi(x_{m,n}) \rightarrow \pi(\bar{x}_n)$ for all $n \in \mathbb{N}$, so there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\rho_X(x_{f(n),n}, \bar{x}_n) \rightarrow 0$ and $\rho_Y(\pi(x_{f(n),n}), \pi(\bar{x}_n)) \rightarrow 0$. It follows that $(x_{f(n),n})_{n \in \mathbb{N}}$ converges, so the fact that R is H_π -independent ensures that $\pi(\lim_{n \rightarrow \infty} x_{f(n),n}) = \lim_{n \rightarrow \infty} \pi(x_{f(n),n})$, in which case $\pi(\lim_{n \rightarrow \infty} \bar{x}_n) = \lim_{n \rightarrow \infty} \pi(\bar{x}_n)$, thus $\pi \upharpoonright \text{proj}_X(R)$ is continuous. In particular, it follows that if there is an \aleph_0 -coloring of H_π , then π is σ -continuous with closed witnesses.

To see (3), note first that if $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$ is continuous, $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to a closed set, $(\pi \circ \psi) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous, and $(\mathbf{d}_n)_{n \in \mathbb{N}} \in \mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$, then $(\pi \circ \psi)(\lim_{n \rightarrow \infty} \mathbf{d}_n) \notin \overline{\{(\pi \circ \psi)(\mathbf{d}_n) \mid n \in \mathbb{N}\}}$, so the continuity of ψ ensures that $((\psi(\mathbf{d}_n), (\pi \circ \psi)(\mathbf{d}_n)))_{n \in \mathbb{N}} \in H_\pi$, thus $(\psi \upharpoonright \mathbb{N}^{\mathbb{N}}) \times ((\pi \circ \psi) \upharpoonright \mathbb{N}^{\mathbb{N}})$ is a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_π .

Conversely, suppose that $\phi: \mathbb{N}^{\mathbb{N}} \rightarrow \text{graph}(\pi)$ is a continuous homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_π , and set $\phi_X = \text{proj}_X \circ \phi$ and $\phi_Y = \text{proj}_Y \circ \phi$. As the definition of H_π ensures that ϕ_X is a homomorphism from $\text{Cnvg}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}})$ to $\text{Cnvg}(X)$, Proposition 3.4 yields a continuous extension $\bar{\phi}_X: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$.

If there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $\phi_Y \upharpoonright \mathcal{N}_t$ is constant, then the function $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$ given by $\psi(\mathbf{d}) = \overline{\phi_X}(t \frown \mathbf{d})$ is continuous and $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to a singleton, so we can assume that

$$(\dagger) \quad \forall t \in \mathbb{N}^{<\mathbb{N}} \quad \phi_Y(\mathcal{N}_t) \text{ is infinite.}$$

If there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $(\pi \circ \overline{\phi_X})(\mathcal{N}_{t, \mathbb{N}^*} \setminus \mathbb{N}^{\mathbb{N}})$ is finite, then by appealing to (\dagger) and extending t , we can ensure that $\overline{\phi_Y(\mathcal{N}_t)}$ and $(\pi \circ \overline{\phi_X})(\mathcal{N}_{t, \mathbb{N}^*} \setminus \mathbb{N}^{\mathbb{N}})$ are disjoint, so the function $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$ given by $\psi(\mathbf{d}) = \overline{\phi_X}(t \frown \mathbf{d})$ is continuous, $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to $\overline{\phi_Y(\mathcal{N}_t)}$, and $(\pi \circ \psi)(\mathbf{d}) = \phi_Y(t \frown \mathbf{d})$ for all $\mathbf{d} \in \mathbb{N}^{\mathbb{N}}$. We can therefore additionally assume that

$$(\ddagger) \quad \forall t \in \mathbb{N}^{<\mathbb{N}} \quad (\pi \circ \overline{\phi_X})(\mathcal{N}_{t, \mathbb{N}^*} \setminus \mathbb{N}^{\mathbb{N}}) \text{ is infinite.}$$

Fix positive real numbers $\epsilon_n \rightarrow 0$, as well as an enumeration $(t_k)_{k \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_j \sqsubseteq t_k \implies j \leq k$ for all $j, k \in \mathbb{N}$. We say that a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets of X is *closed and discrete* if there are no convergent sequences in $\prod_{n \in \mathbb{N}} X_n$. We will recursively construct functions $f: \mathbb{N}^{<\mathbb{N}} \rightarrow \mathbb{N}^{<\mathbb{N}}$ and $g: \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\} \rightarrow \mathbb{N}^{<\mathbb{N}}$ such that, for all $k \in \mathbb{N}$, the following conditions hold:

- (a) $\forall n \in \mathbb{N} \quad f(t_k) \sqsubset g(t_k \frown (n))$.
- (b) $\forall m \neq n \quad g(t_k \frown (m))(|f(t_k)|) \neq g(t_k \frown (n))(|f(t_k)|)$.
- (c) $(\phi_Y(\mathcal{N}_{g(t_k \frown (n))}))_{n \in \mathbb{N}}$ is either closed-and-discrete or convergent.
- (d) $g(t_{k+1}) \sqsubseteq f(t_{k+1})$.
- (e) $\forall j \leq k \quad (\pi \circ \overline{\phi_X})(f(t_j) \frown (\infty)) \notin \overline{\phi_Y(\mathcal{N}_{f(t_{k+1})})}$.
- (f) $\forall j \leq k \quad (\pi \circ \overline{\phi_X})(f(t_{k+1}) \frown (\infty)) \neq \lim_{n \rightarrow \infty} \phi_Y(\mathcal{N}_{g(t_j \frown (n))})$.

We begin by setting $f(\emptyset) = \emptyset$.

Suppose now that $k \in \mathbb{N}$ and we have already found $f(t_k)$, as well as $g(t_j \frown (n))$ for all $j < k$ and $n \in \mathbb{N}$, and fix extensions $\mathbf{d}_{k,n} \in \mathbb{N}^{\mathbb{N}}$ of $f(t_k) \frown (n)$ for all $n \in \mathbb{N}$. Then there is an injective sequence $(i_{k,n})_{n \in \mathbb{N}}$ of natural numbers such that $(\{\phi_Y(\mathbf{d}_{k,i_{k,n}})\})_{n \in \mathbb{N}}$ is either closed-and-discrete or convergent, and continuity yields extensions $g(t_k \frown (n)) \sqsubset \mathbf{d}_{k,i_{k,n}}$ of $f(t_k) \frown (i_{k,n})$ with the property that $\phi_Y(\mathcal{N}_{g(t_k \frown (n))}) \subseteq \mathcal{B}_Y(\phi_Y(\mathbf{d}_{k,i_{k,n}}), \epsilon_n)$ for all $n \in \mathbb{N}$. By (\dagger) , there is an extension $\mathbf{d}_k \in \mathbb{N}^{\mathbb{N}}$ of $g(t_{k+1})$ such that $\phi_Y(\mathbf{d}_k) \neq (\pi \circ \overline{\phi_X})(f(t_j) \frown (\infty))$ for all $j \leq k$, and therefore an extension $u_k \sqsubset \mathbf{d}_k$ of $g(t_{k+1})$ such that $(\pi \circ \overline{\phi_X})(f(t_j) \frown (\infty)) \notin \overline{\phi_Y(\mathcal{N}_{u_k})}$ for all $j \leq k$. By (\ddagger) , there is an extension $f(t_{k+1})$ of u_k satisfying condition (f), thereby completing the recursive construction.

Conditions (a), (b), and (d) ensure that the function $\phi_X \circ \phi'$ is a homomorphism from $\text{Cnvg}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}})$ to $\text{Cnvg}(X)$, where $\phi': \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ is given by $\phi'(\mathbf{d}) = \bigcup_{n \in \mathbb{N}} f(\mathbf{d} \upharpoonright n)$. Proposition 3.4 therefore yields a

continuous extension $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$. As $(\pi \circ \psi) \upharpoonright \mathbb{N}^{\mathbb{N}} = \phi_Y \circ \phi'$, it only remains to show that $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to a closed set.

Suppose, towards a contradiction, that there exists $k \in \mathbb{N}$ with the property that $(\pi \circ \psi)(t_k \smallfrown (\infty)) \in \overline{(\pi \circ \psi)(\mathbb{N}^{\mathbb{N}})}$. As conditions (a), (b), and (d) ensure that

$$\begin{aligned} (\pi \circ \psi)(t_k \smallfrown (\infty)) &= \pi(\lim_{n \rightarrow \infty} (\phi_X \circ \phi')(\mathcal{N}_{t_k \smallfrown (n)})) \\ &= \pi(\lim_{n \rightarrow \infty} \phi_X(\mathcal{N}_{f(t_k \smallfrown (n))})) \\ &= (\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)), \end{aligned}$$

it follows that $(\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)) \in \overline{\phi_Y(\mathbb{N}^{\mathbb{N}})}$. As condition (e) ensures that $(\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)) \notin \overline{\phi_Y(\mathcal{N}_{f(t_j)})}$ for all $j > k$, and condition (f) implies that if $i < k$ and $(\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)) \in \overline{\phi_Y(\mathcal{N}_{f(t_i)})}$ then there exists $j > i$ for which $(\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)) \in \overline{\phi_Y(\mathcal{N}_{f(t_j)})}$, it follows that $(\pi \circ \overline{\phi}_X)(f(t_k) \smallfrown (\infty)) \in \overline{\phi_Y(\mathcal{N}_{f(t_k)})} \setminus \bigcup_{n \in \mathbb{N}} \overline{\phi_Y(\mathcal{N}_{f(t_k \smallfrown (n))})}$. Conditions (a), (b), and (d) therefore yield extensions $\mathbf{c}_n \in \mathbb{N}^{\mathbb{N}}$ of $f(t_k) \smallfrown (n)$ such that $\pi(\lim_{n \rightarrow \infty} \phi_X(\mathbf{c}_n)) \in \{(\pi \circ \phi_X)(\mathbf{c}_n)\}$, contradicting the fact that ϕ is a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H_π . \square

In particular, we obtain the following.

Theorem 5.2 ($\text{OGD}^{\mathbb{N}}(\Gamma)$). *Suppose that X and Y are separable metric spaces, $X \in \Gamma$, and $\pi: X \rightarrow Y$ is Borel. Then exactly one of the following holds:*

- (1) *The function π is σ -continuous with closed witnesses.*
- (2) *There is a continuous function $\psi: \text{Ext}_{\mathbb{N}^*}(\mathbb{N}^{\mathbb{N}}) \rightarrow X$ with the property that $\pi \circ \psi$ is a reduction of $\mathbb{N}^{\mathbb{N}}$ to a closed set and $(\pi \circ \psi) \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous.*

Proof. This follows from Proposition 5.1. \square

Theorem 1.1 and the special case of Theorem 5.2 where Γ is the pointclass of analytic subsets of metric spaces easily yield the Jayne–Rogers characterization of the circumstances under which a function from an analytic metric space to a separable metric space is σ -continuous with closed witnesses. Theorem 1.3 and the special case of Theorem 5.2 where Γ is the pointclass of all subsets of analytic metric spaces easily yield the generalization in which analyticity is weakened to separability under the axiom of determinacy.

6. CARDINAL INVARIANTS AND CHROMATIC NUMBERS

In this section, we explore connections between cardinal invariants and the chromatic numbers of box-open dihypergraphs.

Proposition 6.1. *The chromatic number of $\mathbb{H}'_{\mathbb{N}^{\mathbb{N}}}$ is \mathfrak{d} .*

Proof. This follows from the fact that a subset of $\mathbb{N}^{\mathbb{N}}$ is $\mathbb{H}'_{\mathbb{N}^{\mathbb{N}}}$ -independent if and only if its closure is compact. \square

We say that an \mathbb{N} -dimensional dihypergraph is *hereditary* if it is closed under subsequences.

Proposition 6.2 ($\text{OGD}^{\mathbb{N}}(\Gamma)$). *Suppose that X is a Hausdorff space in Γ and H is a box-open hereditary \mathbb{N} -dimensional dihypergraph on X . Then either $\chi(H) \leq \aleph_0$ or $\chi(H) \geq \mathfrak{d}$.*

Proof. This follows from Proposition 6.1 and the fact that every homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to H is a homomorphism from $\mathbb{H}'_{\mathbb{N}^{\mathbb{N}}}$ to H . \square

As every analytic space is trivially the union of \mathfrak{d} -many compact sets, Theorem 1.1 and the special case of Proposition 6.2 where Γ is the pointclass of analytic subsets of metric spaces and H is of the form H_X ensure that if an analytic subset of a metric space is not contained in a K_σ set, then \mathfrak{d} is the least cardinal κ for which it is contained in a union of κ -many compact sets.

As every analytic space is trivially the union of \mathfrak{d} -many closed sets, Theorem 1.1 and the special case of Proposition 6.2 where Γ is the pointclass of analytic subsets of metric spaces, H is of the form $H_{R,S}$, and $D = 1$ ensure that if an analytic subset of a metric space cannot be separated from another set by an F_σ set, then \mathfrak{d} is the least cardinal κ for which it can be separated from the other set by a union of κ -many closed sets.

As every analytic graph of a function is trivially the union of \mathfrak{d} -many compact sets that are graphs of functions, Theorem 1.1 and the special case of Proposition 6.2 where Γ is the pointclass of analytic subsets of metric spaces and H is of the form H_π ensure that if a Borel function π from an analytic metric space to a separable metric space is not σ -continuous with closed witnesses, then \mathfrak{d} is the least cardinal κ for which X is the union of κ -many closed sets on which π is continuous.

The special case of the last fact for Baire-class-one functions is due to Solecki (see [Sol98]). He established this by noting that the two elements of his basis for Baire-class-one functions that are not σ -continuous have the desired property. The above argument shows that this more sophisticated basis theorem is unnecessary to obtain the desired result; one need only note that σ -continuity with closed witnesses can be characterized using a box-open hereditary \mathbb{N} -dimensional dihypergraph.

We next turn our attention to the computation of the chromatic number of $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ itself. For each partial function $f: D^{<\mathbb{N}} \rightarrow D$, define $\mathbf{D}_f = \{\mathbf{d} \in D^{\mathbb{N}} \mid \forall n \in \mathbb{N} (\mathbf{d} \upharpoonright n \in \text{dom}(f) \implies \mathbf{d}(n) \neq f(\mathbf{d} \upharpoonright n))\}$.

Proposition 6.3. *Suppose that D is a set of cardinality at least two. Then $\chi(\mathbb{H}_{D^{\mathbb{N}}}) = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq D^{D^{<\mathbb{N}}}$ and $D^{\mathbb{N}} = \bigcup_{f \in \mathcal{F}} \mathbf{D}_f\}$.*

Proof. This follows from the fact that a set $X \subseteq D^{\mathbb{N}}$ is $\mathbb{H}_{D^{\mathbb{N}}}$ -independent if and only if there is a function $f: D^{<\mathbb{N}} \rightarrow D$ for which $X \subseteq \mathbf{D}_f$. \square

We now establish an analog of Proposition 6.2 without the assumption that H is hereditary.

Proposition 6.4 ($\text{OGD}^D(\Gamma)$). *Suppose that D is a countable set of cardinality at least two, X is a Hausdorff space in Γ , and H is a box-open D -dimensional dihypergraph on X . Then either $\chi(H) \leq \aleph_0$ or $\chi(H) \geq \text{cov}(\mathcal{M})$.*

Proof. It is sufficient to show that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) \geq \text{cov}(\mathcal{M})$, which follows from Proposition 6.3 and the observation that if $f: D^{<\mathbb{N}} \rightarrow D$, then \mathbf{D}_f is meager with respect to the usual topology on $D^{\mathbb{N}}$. \square

Alternatively, one can obtain the above result by noting that if H is a box-open D -dimensional dihypergraph on a Hausdorff space, \mathcal{I}_H is the σ -ideal generated by the family of closed H -independent sets, and $\chi(H) > \aleph_0$, then $\chi(H) = \text{cov}(\mathcal{I}_H)$.

Conversely, given a σ -ideal \mathcal{I} on a topological space X , let $H_{\mathcal{I}}$ be the \mathbb{N} -dimensional dihypergraph on X consisting of all sequences $(x_n)_{n \in \mathbb{N}}$ of elements of X for which $\overline{\{x_n \mid n \in \mathbb{N}\}} \notin \mathcal{I}$.

Proposition 6.5. *Suppose that X is a separable metric space and \mathcal{I} is a σ -ideal on X generated by a family of closed subsets of X .*

- (1) *If \mathcal{I} covers X , then $H_{\mathcal{I}}$ is box open.*
- (2) *If $\text{cov}(\mathcal{I}) > \aleph_0$, then $\chi(H_{\mathcal{I}}) = \text{cov}(\mathcal{I})$.*

Proof. To see (1), suppose that $(x_n)_{n \in \mathbb{N}} \in H_{\mathcal{I}}$, fix positive real numbers $\epsilon_n \rightarrow 0$, and observe that if $(y_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{B}_X(x_n, \epsilon_n)$, then $\overline{\{x_n \mid n \in \mathbb{N}\}} \subseteq \{x_n \mid n \in \mathbb{N}\} \cup \overline{\{y_n \mid n \in \mathbb{N}\}}$, thus $(y_n)_{n \in \mathbb{N}} \in H_{\mathcal{I}}$.

To see (2), note first that every $H_{\mathcal{I}}$ -independent set is in \mathcal{I} , so $\text{cov}(\mathcal{I}) \leq \chi(H_{\mathcal{I}})$. Conversely, as every set in \mathcal{I} is contained in the union of countably-many closed sets in \mathcal{I} , the fact that every closed set in \mathcal{I} is $H_{\mathcal{I}}$ -independent ensures that $\chi(H_{\mathcal{I}}) \leq \text{cov}(\mathcal{I}) \cdot \aleph_0 = \text{cov}(\mathcal{I})$. \square

As a corollary, we also obtain an analog of Proposition 6.1 without the assumption that H is hereditary.

Proposition 6.6. *The chromatic number of $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ is $\text{cov}(\mathcal{M})$.*

Proof. Proposition 6.4 ensures that $\chi(\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}) \geq \text{cov}(\mathcal{M})$. As the open dihypergraph dichotomy yields a homomorphism from $\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}$ to $H_{\mathcal{M}}$, Proposition 6.5 implies that $\chi(\mathbb{H}_{\mathbb{N}^{\mathbb{N}}}) \leq \text{cov}(\mathcal{M})$. \square

Stronger bounds can be obtained when D is finite. In the special case that $D = 2$, this is trivial.

Proposition 6.7 ($\text{OGD}(\Gamma)$). *Suppose that X is a Hausdorff space in Γ and G is an open graph on X . Then either $\chi(G) \leq \aleph_0$ or $\chi(G) = \mathfrak{c}$.*

Proof. It is sufficient to show that $\chi(\mathbb{H}_{2^{\mathbb{N}}}) = \mathfrak{c}$, which follows from the fact that $\mathbb{H}_{2^{\mathbb{N}}} = \mathbb{K}_{2^{\mathbb{N}}}$. \square

The case that $2 < |D| < \aleph_0$ is substantially subtler, and related to the adaptive global prediction numbers considered in [Bla10, §10]. Our first observation is a straightforward analog of Proposition 6.4.

Proposition 6.8 ($\text{OGD}^D(\Gamma)$). *Suppose that D is a finite set of cardinality at least two, X is a Hausdorff space in Γ , and H is an open D -dimensional dihypergraph on X . Then either $\chi(H) \leq \aleph_0$ or $\chi(H) \geq \text{cov}(\mathcal{N})$.*

Proof. It is sufficient to show that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) \geq \text{cov}(\mathcal{N})$, which follows from Proposition 6.3 and the observation that if $f: D^{<\mathbb{N}} \rightarrow D$ and μ is a strictly positive probability measure on D , then \mathbf{D}_f is $\mu^{\mathbb{N}}$ -null. \square

We next establish an analog of the main result of [Bre03]. We say that a partial function $u: \mathbb{N} \rightarrow D$ is a *D -Silver condition* if its domain is co-infinite, and we say that a set $X \subseteq D^{\mathbb{N}}$ is *D -Silver null* if every D -Silver condition $u: \mathbb{N} \rightarrow D$ extends to a D -Silver condition $v: \mathbb{N} \rightarrow D$ for which \mathcal{N}_v is disjoint from X .

Theorem 6.9. *Suppose that D is a finite set of cardinality at least two, $X \subseteq D^{\mathbb{N}}$, and $\chi(\mathbb{H}_{D^{\mathbb{N}}} \upharpoonright X) < \mathfrak{b}$. Then X is D -Silver null.*

Proof. By Proposition 6.3, it is sufficient to show that if $\mathcal{F} \subseteq D^{D^{<\mathbb{N}}}$ has cardinality strictly less than \mathfrak{b} , then $\bigcup_{f \in \mathcal{F}} \mathbf{D}_f$ is D -Silver null. By a straightforward recursive construction of length $|D|$, we need only show that for all $d \in D$, every D -Silver condition $u: \mathbb{N} \rightarrow D$ extends to a D -Silver condition $v: \mathbb{N} \rightarrow D$ with the property that

$$\forall f \in \mathcal{F} \forall \mathbf{d} \in \mathbf{D}_f \cap \mathcal{N}_v \forall^\infty n \in \sim\text{dom}(v) \ d \neq f(\mathbf{d} \upharpoonright n).$$

It is clearly sufficient to handle the special case that $u = \emptyset$.

For all $f \in \mathcal{F}$, fix a function $g_f: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $t \in D^n$, if there exists $i \in \mathbb{N}$ for which $d = f(t \frown (d)^i)$, then there is such an $i < g_f(n)$. Fix a function $g: \mathbb{N} \rightarrow \mathbb{N}$ eventually dominating g_f for all $f \in \mathcal{F}$, set $h'(0) = 0$, and recursively define $h(n) = g(h'(n) + 1)$ and $h'(n+1) = h'(n) + 1 + h(n)$ for all $n \in \mathbb{N}$. Let v be the function on $\sim\{h'(n+1) \mid n \in \mathbb{N}\}$ with constant value d .

To see that v is as desired, note first that if $\mathbf{d} \in \mathcal{N}_v$, then a straightforward inductive argument reveals that $\mathbf{d} \upharpoonright h'(n+1)$ is the concatenation of $\mathbf{d} \upharpoonright (h'(n)+1)$ and $(d)^{h(n)}$, for all $n \in \mathbb{N}$. It follows that if $f \in \mathcal{F}$, $\mathbf{d} \in \mathbf{D}_f$, and $n \in \mathbb{N}$ is sufficiently large that $g_f(h'(n)+1) \leq g(h'(n)+1)$, then $d \neq f(\mathbf{d} \upharpoonright h'(n+1))$. \square

Theorem 6.10 ($\text{OGD}^D(\Gamma)$). *Suppose that D is a finite set of cardinality at least two, X is a Hausdorff space in Γ , and H is an open D -dimensional dihypergraph on X . Then either $\chi(H) \leq \aleph_0$ or $\chi(H) \geq \mathfrak{b}$.*

Proof. It is sufficient to show that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) \geq \mathfrak{b}$, which follows from Theorem 6.9. \square

We now establish an analog of the main result of [Kam00].

Theorem 6.11. *It is consistent that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) > \text{cof}(\mathcal{N})$ for every finite set D of cardinality at least two.*

Proof. Given a natural number $D \geq 2$, we say that a subtree T of $D^{<\mathbb{N}}$ is D -perfect if every element of T has an extension $t \in T$ with the property that $\forall d \in D \ t \frown (d) \in T$. Let S_D denote the set of such trees. Let \leq denote the partial order on S_D with respect to which $S \leq T$ if and only if $S \subseteq T$, and define $\mathbb{S}_D = (S_D, \leq)$. When $D = 2$, this is just Sacks forcing, and the usual proof that the latter is proper and has the Sacks property works just as well for every \mathbb{S}_D .

Lemma 6.12. *Suppose that $D \geq 2$ is a natural number, G is \mathbb{S}_D -generic over V , \mathbf{d} is the unique branch through $\bigcap G$, and $f: D^{<\mathbb{N}} \rightarrow D$ is in V . Then there exists $n \in \mathbb{N}$ for which $\mathbf{d}(n) = f(\mathbf{d} \upharpoonright n)$.*

Proof. This follows from the fact that there are \leq -densely many trees $T \in S_D$ for which there exists $t \in D^{<\mathbb{N}}$ such that $t \frown (f(t))$ is \sqsubseteq -comparable with every element of T . \square

Suppose now that CH holds in V , and fix a sequence $(D_\beta)_{\beta < \omega_2}$ of natural numbers that are at least two for which every such natural number appears cofinally often, as well as a countable support iteration $(\mathbb{P}_\alpha, \dot{Q}_\beta)_{\alpha \leq \omega_2, \beta < \omega_2}$ such that $\Vdash_{\mathbb{P}_\beta} \dot{Q}_\beta = (\mathbb{S}_{D_\beta})^{V^{\mathbb{P}_\beta}}$ for all $\beta < \omega_2$.

As \mathbb{P}_{ω_2} has the Sacks property, it follows that $\text{cof}(\mathcal{N}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$ (see, for example, [Bla10, §11.5]). To see that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) > \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$ for all natural numbers $D \geq 2$, note that if $\mathcal{F} \subseteq D^{D^{<\mathbb{N}}}$ is in $V^{\mathbb{P}_{\omega_2}}$ and has cardinality at most \aleph_1 in $V^{\mathbb{P}_{\omega_2}}$, then there exists $\alpha < \omega_2$ for which $\mathcal{F} \in V^{\mathbb{P}_\alpha}$. Fix $\beta \geq \alpha$ for which $D_\beta = D$, note that $D^{\mathbb{N}} \not\subseteq \bigcup_{f \in \mathcal{F}} \mathbf{D}_f$ in $V^{\mathbb{P}_{\beta+1}}$ by Lemma 6.12, and appeal to Proposition 6.3. \square

Theorem 6.13 ($\text{OGD}^D(\Gamma)$). *It is consistent that whenever D is a finite set of cardinality at least two, X is a Hausdorff space in Γ , and H is an open D -dimensional dihypergraph on X , either $\chi(H) \leq \aleph_0$ or $\chi(H) > \text{cof}(\mathcal{N})$.*

Proof. It is sufficient to establish the consistency of $\chi(\mathbb{H}_{D^{\mathbb{N}}}) > \text{cof}(\mathcal{N})$ for all finite sets D of cardinality at least two, which follows from Theorem 6.11. \square

We finally establish a consistent upper bound as well.

Theorem 6.14. *It is consistent that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) < \mathfrak{d}$ for every finite set D of cardinality at least three.*

Proof. Given a natural number $D \geq 3$, let P_D denote the set of pairs (\mathbf{D}, f) with the property that $f: D^{<n} \rightarrow D$ for some $n \in \mathbb{N}$, $\mathbf{D} \subseteq \mathbf{D}_f$, and $\mathbf{c} \upharpoonright n \neq \mathbf{d} \upharpoonright n$ for all distinct $\mathbf{c}, \mathbf{d} \in \mathbf{D}$. Let \leq denote the partial order on P_D with respect to which $(\mathbf{C}, f) \leq (\mathbf{D}, g)$ if and only if $\mathbf{C} \subseteq \mathbf{D}$ and $f \sqsubseteq g$, and define $\mathbb{P}_D = (P_D, \leq)$.

Lemma 6.15. *Suppose that $n \in \mathbb{N}$ and $f: D^{<n} \rightarrow D$. Then any two elements of P_D of the form (\mathbf{C}, f) and (\mathbf{D}, f) are \leq -compatible.*

Proof. Fix $m \geq n$ sufficiently large that $c \upharpoonright m \neq d \upharpoonright m$ for all distinct $c \in \mathbf{C}$ and $d \in \mathbf{D}$. As $D \geq 3$, there exists $g: D^{<m} \rightarrow D$ such that $\mathbf{C} \cup \mathbf{D} \subseteq \mathbf{D}_g$ and $f \sqsubseteq g$, in which case $(\mathbf{C} \cup \mathbf{D}, g)$ is a common \leq -extension of (\mathbf{C}, f) and (\mathbf{D}, f) . \square

Fix a sequence $(D_n)_{n \in \mathbb{N}}$ of natural numbers that are at least three for which every such natural number appears cofinally often, and let $\mathbb{P} = (P, \leq)$ denote the finite support product of $(\mathbb{P}_{D_n})_{n \in \mathbb{N}}$. Lemma 6.15 ensures that each of the partial orders \mathbb{P}_D is σ -linked, thus so too is \mathbb{P} .

Lemma 6.16. *Suppose that $D \geq 3$ is a natural number, G is \mathbb{P} -generic over V , and $\mathbf{d} \in (D^{\mathbb{N}})^V$. Then there exists $k \in \mathbb{N}$ such that $D_k = D$ and $\mathbf{d}(n) \neq f(\mathbf{d} \upharpoonright n)$ for all $n \in \mathbb{N}$, where $f = \bigcup \{f_k \mid (\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \in G\}$.*

Proof. This follows from the fact that there are \leq -densely many sequences $(\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \in P$ for which there exists $k \in \mathbb{N}$ such that $D = D_k$ and $\mathbf{d} \in \mathbf{D}_k$. \square

Lemma 6.17. *Suppose that $\dot{\mathbf{d}}$ is a \mathbb{P} -name for an element of $\mathbb{N}^{\mathbb{N}}$, $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, and $(f_n)_{n \in \mathbb{N}}$ is in the finite support product of $(D_n^{<k_n})_{n \in \mathbb{N}}$. Then $\forall i \in \mathbb{N} \exists j \in \mathbb{N} \forall (\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \in P (\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \Vdash_{\mathbb{P}} \dot{\mathbf{d}}(i) \geq j$.*

Proof. Suppose, towards a contradiction, that there exist $i \in \mathbb{N}$ and $(\mathbf{D}_{j,n}, f_n)_{n \in \mathbb{N}} \in P$ such that $(\mathbf{D}_{j,n}, f_n)_{n \in \mathbb{N}} \Vdash_{\mathbb{P}} \dot{\mathbf{d}}(i) \geq j$ for all $j \in \mathbb{N}$.

By passing to a subsequence, we can assume that there are sequences $(I_n)_{n \in \mathbb{N}}$ of finite sets and $(\mathbf{d}_{i,j,n})_{(i,j,n) \in I_n \times \mathbb{N} \times \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ such that $\mathbf{D}_{j,n} = \{\mathbf{d}_{i,j,n} \mid i \in I_n\}$ for all $j, n \in \mathbb{N}$ and $\mathbf{d}_{i,n} = \lim_{j \rightarrow \infty} \mathbf{d}_{i,j,n}$ exists for all $i, n \in \mathbb{N}$. Set $\mathbf{D}_n = \{\mathbf{d}_{i,n} \mid i \in I_n\}$. Then $(\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \in P$, so there exist $k \in \mathbb{N}$ and an extension of $(\mathbf{D}_n, f_n)_{n \in \mathbb{N}} \in P$ for which $(\mathbf{D}'_n, f'_n)_{n \in \mathbb{N}} \Vdash_{\mathbb{P}} \dot{\mathbf{d}}(i) = k$. As $(\mathbf{D}_{j,n}, f'_n)_{n \in \mathbb{N}} \in P$ for all sufficiently large $j \in \mathbb{N}$, Lemma 6.15 ensures that $(\mathbf{D}_{j,n}, f_n)_{n \in \mathbb{N}}$ and $(\mathbf{D}'_n, f'_n)_{n \in \mathbb{N}}$ are compatible for all sufficiently large $j > k$, the desired contradiction. \square

Suppose now that $\mathfrak{d} > \aleph_1$ in V , and fix a finite support iteration $(\mathbb{P}_\alpha, \dot{Q}_\beta)_{\alpha < \omega_1, \beta < \omega_1}$ such that $\Vdash_{\mathbb{P}_\beta} \dot{Q}_\beta = \mathbb{P}^{V^{\mathbb{P}_\beta}}$ for all $\beta < \omega_1$.

Proposition 6.3 and Lemma 6.16 easily imply that $\chi(\mathbb{H}_{D^{\mathbb{N}}}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_1}}$. To see that $\mathfrak{d} > \aleph_1$ in $V^{\mathbb{P}_{\omega_1}}$, note that if $(\dot{\mathbf{d}}_\alpha)_{\alpha < \omega_1} \in V$ is a sequence of \mathbb{P}_{ω_1} -names for elements of $\mathbb{N}^{\mathbb{N}}$, then Lemma 6.17 yields a sequence $(\mathbf{d}_\alpha)_{\alpha < \omega_1} \in V$ of elements of $\mathbb{N}^{\mathbb{N}}$ such that $\Vdash_{\mathbb{P}_{\omega_1}} \dot{\mathbf{d}}_\alpha \leq^* \mathbf{d}_\alpha$ for all $\alpha < \omega_1$. The fact that $\mathfrak{d} > \aleph_1$ in V then yields $\mathbf{d} \in (\mathbb{N}^{\mathbb{N}})^V$ such that $\mathbf{d} \not\leq^* \mathbf{d}_\alpha$ for all $\alpha < \omega_1$, so $\Vdash_{\mathbb{P}_{\omega_1}} \mathbf{d} \not\leq^* \dot{\mathbf{d}}_\alpha$ for all $\alpha < \omega_1$. \square

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