MEANS ON EQUIVALENCE RELATIONS

BY

ALEXANDER S. KECHRIS*

Department of Mathematics, California Institute of Technology
1200 East California Boulevard 253-37, Pasadena, CA 91125
(e-mail: kechris@caltech.edu)

AND

BENJAMIN D. MILLER†

Department of Mathematics, University of California
520 Portola Plaza, Los Angeles, CA 90095-1555
(e-mail: bdm@math.ucla.edu)

ABSTRACT

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. We show that if there is a Borel assignment of means to the equivalence classes of E, then E is smooth. We also show that if there is a Baire measurable assignment of means to the equivalence classes of E, then E is generically smooth.

1. Introduction

A **mean** on a countable set S is a positive linear functional $\varphi : \ell^\infty(S) \to \mathbb{C}$ such that $\varphi(\mathbb{1}) = 1$, where $\mathbb{1}$ denotes the constant function on S with value 1. Means provide a way of associating an average value with each element of $\ell^\infty(S)$.

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. An **assignment of means** is a map which associates with each equivalence class $[x]_E$ a mean $\varphi_{[x]_E}$ on $[x]_E$. Assignments of means provide a way of associating an assignment of average values $x \mapsto \varphi_{[x]_E}(f_x)$ with each assignment of functions $x \mapsto f_x \in \ell^\infty([x]_E)$.

An assignment of functions $x \mapsto f_x \in \ell^\infty([x]_E)$ is **Borel** if the corresponding function $f : E \to \mathbb{C}$, given by $f(x, y) = f_x(y)$, is Borel. Given a family $\Gamma \subseteq \mathscr{P}(X)$

Not yet received.

*Research partially supported by NSF Grants DMS-9987437 and DMS-0455285.

†Research partially supported by NSF VIGRE Grant DMS-0502315.

of subsets of X, we say that a function $F : X \to \mathbb{C}$ is **Γ-measurable** if

$$\forall U \subseteq \mathbb{C} \text{ open } (F^{-1}(U) \in \Gamma).$$

We say that an assignment of means $[x]_E \mapsto \varphi_{[x]_E}$ is **Γ-measurable** if for every Borel assignment of functions $x \mapsto f_x \in \ell^\infty([x]_E)$, the corresponding assignment of average values $x \mapsto \varphi_{[x]_E}(f_x)$ is Γ-measurable.

Suppose that μ is a (Borel) probability measure on X. An equivalence relation E is **μ-amenable** if it admits a μ-measurable assignment of means. This notion has played an important role in ergodic theory over the last few decades. In Connes-Feldman-Weiss [2], it is shown that the existence of such assignments is equivalent to the existence of a μ-conull Borel set $B \subseteq X$ such that $E|B$ is of the form $\bigcup_{n\in\mathbb{N}} F_n$, where $F_0 \subseteq F_1 \subseteq \cdots$ is an increasing sequence of finite Borel equivalence relations. That is, the equivalence relation $E|B$ is **hyperfinite**. From the point of view of descriptive set theory, the hyperfinite equivalence relations are the simplest non-trivial equivalence relations (see, for example, Jackson-Kechris-Louveau [7]). Thus, the result of Connes-Feldman-Weiss [2] says that μ-amenability characterizes those countable Borel equivalence relations which are μ-almost everywhere no more complicated than the simplest non-trivial sort of equivalence relation.

Kaimanovich has asked what happens to the notion of μ-amenability if the family Γ of μ-measurable subsets of X is replaced with the family of Borel subsets of X, and whether hyperfiniteness in the Borel context (in fact, even whether the condition of 1-amenability as in Jackson-Kechris-Louveau [7]) implies the existence of a Borel assignment of means. In light of the result of Connes-Feldman-Weiss [2], one might think that this modified notion should characterize hyperfiniteness. However, it turns out that this is far from the truth. Our main goal is to describe which equivalence relations admit Borel assignments of means, as well as which equivalence relations admit Baire measurable assignments of means.

In §2, we use an argument of Adams [1] to show that if E is a meager-preserving countable Borel equivalence relation which admits a Baire measurable assignment of means, then every treeing of E is of a certain form. In §3, we introduce the object which lies at the heart of our argument — the generic n-regular treeing of an aperiodic countable Borel equivalence relation. In §4, we show that if E is a meager-preserving, generically non-smooth countable Borel equivalence relation, then the generic 3-regular treeing of E from §3 is not of the form described in §2.

In §5, we obtain our main results. We say that E is **smooth** if there is a Borel set $B \subseteq X$ which contains exactly one point of every E-class, E is **generically smooth** if there is a comeager Borel set $C \subseteq X$ such that $E|C$ is smooth, and E is **generically non-smooth** if for every non-meager Borel set $B \subseteq X$, the equivalence relation $E|B$ is not smooth.

Theorem: *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is generically smooth.*
2. *E admits a Baire measurable assignment of means.*

Coupled with the Glimm-Effros dichotomy, this yields the following:

Theorem: *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is smooth.*
2. *E admits a Borel assignment of means.*

We also note that strong set-theoretic assumptions can be employed to obtain similar results when the family Γ of Borel subsets of X is substantially enlarged.

2. Adams's argument

In this section, we use the argument of Adams [1] to show that every treeing of a meager-preserving countable Borel equivalence relation which admits a Baire measurable assignment of means must be of a certain form.

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. The **E-saturation** of a Borel set $B \subseteq X$ is given by

$$[B]_E = \{x \in X : \exists y \in B \ (xEy)\}.$$

We say E is **meager-preserving** if the E-saturations of meager sets are meager. (Such equivalence relations are sometimes called generic — we reserve this term, however, to refer only to properties which occur comeagerly often.) A set $B \subseteq X$ is an **E-complete section** if $X = [B]_E$, and **E-invariant** if $B = [B]_E$.

A **forest** is a graph whose connected components are trees. A **treeing** of E is a Borel forest $\mathscr{T} \subseteq E$ whose connected components coincide with the equivalence classes of E. A function $f : X \to X$ is **aperiodic** if

$$\forall x \in X \, \forall n > 0 \ (x \neq f^n(x)).$$

Associated with every such function is an **induced forest**, given by

$$\mathscr{T}_f = \{(x, y) \in X \times X : f(x) = y \text{ or } f(y) = x\}.$$

We say that a Borel forest $\mathscr{T}$ on X is **directable** if it is of the form $\mathscr{T}_f$, for some aperiodic Borel function $f : X \to X$. A set $B \subseteq X$ is **$\mathscr{T}$-convex** if the vertex set of every $\mathscr{T}$-path which begins and ends in B is contained in B. Such a set is **$\mathscr{T}$-linear** if each point of B has at most two $\mathscr{T}$-neighbors within B.

Proposition 2.1 (essentially Adams): *Suppose that X is a Polish space, E is a meager-preserving countable Borel equivalence relation on X which admits a Baire measurable assignment of means, and $\mathscr{T}$ is a locally finite treeing of E. Then there are disjoint E-invariant Borel sets $A_1, A_2 \subseteq X$ such that:*

1. *$A_1 \cup A_2$ is comeager.*
2. *$\mathscr{T}|A_1$ is directable.*
3. *$E|A_2$ admits a $\mathscr{T}$-linear Borel complete section.*

Proof: We say that an assignment of sets $x \mapsto S_x \subseteq [x]_E$ is **Borel** if the corresponding set $S = \{(x, y) \in X \times X : y \in S_x\}$ is Borel. Given a family $\Gamma \subseteq \mathscr{P}(X)$, we say that an assignment of finitely additive probability measures $[x]_E \mapsto \mu_{[x]_E} \in P([x]_E)$ is **Γ-measurable** if for every Borel assignment of sets $x \mapsto S_x \subseteq [x]_E$, the corresponding assignment $x \mapsto \mu_{[x]_E}(S_x)$ is Γ-measurable.

It is clear that every Γ-measurable assignment of means gives rise to a Γ-measurable assignment of finitely additive probability measures. For our purposes, it will be more convenient to work with the latter, so fix a Baire measurable assignment of finitely additive probability measures $[x]_E \mapsto \mu_{[x]_E}$.

For each $x \in X$, let $\mathscr{T}_{\hat{x}}$ denote the forest obtained from $\mathscr{T}|[x]_E$ by deleting all edges of the form (y, z), where $x \in \{y, z\}$. Let $E_{\hat{x}}$ be the equivalence relation on $[x]_E$ whose equivalence classes are the connected components of $\mathscr{T}_{\hat{x}}$. By Feldman-Moore [3], there is a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$, where E_Γ^X is the **orbit equivalence relation** associated with Γ,

$$x E_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y).$$

For each $\gamma \in \Gamma$, define $F_\gamma : X \to [0, 1]$ by

$$F_\gamma(x) = \mu_{[x]_E}([\gamma \cdot x]_{E_{\hat{x}}}).$$

As the assignment $x \mapsto [\gamma \cdot x]_{E_{\hat{x}}}$ is Borel, it follows that F_γ is Baire measurable.

Lemma 2.2: *There is a comeager E-invariant Borel set $C \subseteq X$ such that*

$$\forall \gamma \in \Gamma \ (F_\gamma | C \text{ is Borel}).$$

Proof: For each $\gamma \in \Gamma$, fix a comeager Borel set $C_\gamma \subseteq X$ on which $F_\gamma|C_\gamma$ is Borel. As E is meager-preserving, it follows that the E-invariant Borel set

$$C = X \setminus \bigcup_{\gamma \in \Gamma} [X \setminus C_\gamma]_E$$

is also comeager. As $C \subseteq \bigcap_{\gamma \in \Gamma} C_\gamma$, it follows that each $F_\gamma|C$ is Borel. ∎

By replacing X with the set $C \subseteq X$ of Lemma 2.2, we may assume that each of the functions F_γ is Borel.

A **partial transversal** of E is a set $B \subseteq X$ which intersects every equivalence class of E in at most one point. A **transversal** is a partial transversal which is also an E-complete section, and E is **smooth** if it admits a Borel transversal.

Lemma 2.3: *The restriction of E to the set*

$$A = \Big\{ x \in X : \exists y \in [x]_E \ \Big(\mu_{[x]_E}(\{y\}) > 0 \Big) \Big\}$$

is smooth.

Proof: Define $B \subseteq A$ by

$$B = \Big\{ x \in A : \forall y \in [x]_E \ \Big(\mu_{[x]_E}(\{y\}) \leq \mu_{[x]_E}(\{x\}) \Big) \Big\}.$$

It is clear that B intersects each equivalence class of $E|A$ in a non-empty, finite set. Fix a Borel linear ordering $\leq$ of X, set

$$C = \{ x \in B : \forall y \in B \cap [x]_E \ (y \leq x) \},$$

and observe that C is a Borel transversal of $E|A$, thus $E|A$ is smooth. ∎

As partial transversals constitute the simplest examples of $\mathscr{T}$-linear sets, we may therefore assume that $\forall x \in X \ (\mu_{[x]_E}(\{x\}) = 0)$. Define $R \subseteq \mathscr{T}$ by

$$R = \Big\{ (x, y) \in \mathscr{T} : \forall z \in [x]_E \ \Big(\mu_{[x]_E}([z]_{E_{\hat{x}}}) \leq \mu_{[x]_E}([y]_{E_{\hat{x}}}) \Big) \Big\},$$

and note that R is Borel, since

$$R = \bigcup_{\gamma \in \Gamma} \Big\{ (x, \gamma \cdot x) \in \mathscr{T} : \forall \delta \in \Gamma \ \Big(F_\delta(x) \leq F_\gamma(x) \Big) \Big\}.$$

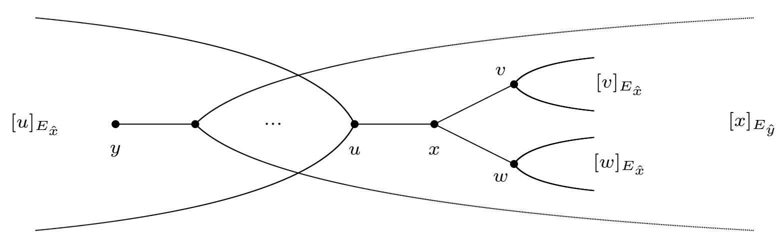

Figure 1: If $|R_x| \geq 3$ and $y \neq x$, then $|R_y| = 1$.

Lemma 2.4: *The set* $\{x \in X : |R_x| \geq 3\}$ *is a partial transversal of* E.

Proof: It is enough to show that if $|R_x| \geq 3$, then

$$\forall y \in [x]_E \ (x \neq y \Rightarrow |R_y| = 1).$$

Fix $y \in [x]_E$ with $x \neq y$, and find distinct points $u, v, w \in \mathscr{T}_x$ such that $y \in [u]_{E_{\hat{x}}}$ and $v, w \in R_x$ (see Figure 1). Then $[v]_{E_{\hat{x}}} \cup [w]_{E_{\hat{x}}} \subseteq [x]_{E_{\hat{y}}}$ and every other equivalence class of $E_{\hat{y}}$ is contained in $[u]_{E_{\hat{x}}}$. As

$$\mu_{[x]_E}([u]_{E_{\hat{x}}}) < \mu_{[x]_E}([v]_{E_{\hat{x}}}) + \mu_{[x]_E}([w]_{E_{\hat{x}}}),$$

it follows that $[x]_{E_{\hat{y}}}$ is the unique $E_{\hat{y}}$-class of maximal measure, so $|R_y| = 1$. ∎

Again appealing to the fact that partial transversals are $\mathscr{T}$-linear sets, we may now assume that $\forall x \in X \ (|R_x| \leq 2)$.

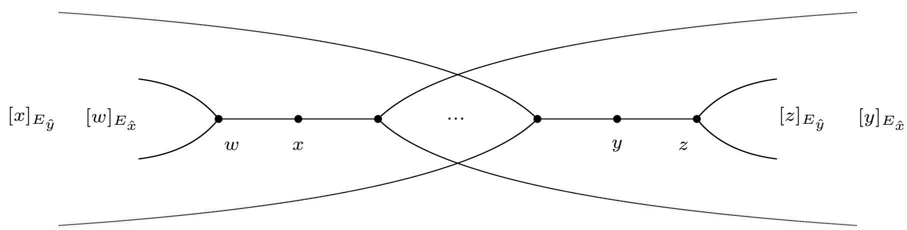

Figure 2: If $[x]_{E_{\hat{y}}}$ is not of maximal $\mu_{[x]_E}$-measure, then $|R_x| = 1$.

Lemma 2.5: *The set* $A_2 = \{x \in X : |R_x| = 2\}$ *is* $\mathscr{T}$*-linear.*

Proof: It is enough to show that A_2 is $\mathscr{T}$-convex. The main point is as follows:

Sublemma 2.6: *Suppose that* $y \in [x]_E \setminus \{x\}$ *and there exists* $z \in [x]_E$ *such that*

$$\mu_{[x]_E}([z]_{E_{\hat{y}}}) > \mu_{[x]_E}([x]_{E_{\hat{y}}}).$$

Then $|R_x| = 1$.

Proof: Suppose that w is a $\mathscr{T}$-neighbor of x which is not $\mathscr{T}_{\hat{x}}$-connected to y. Now, observe that $[w]_{E_{\hat{x}}} \subseteq [x]_{E_{\hat{y}}}$ and $[z]_{E_{\hat{y}}} \subseteq [y]_{E_{\hat{x}}}$ (see Figure 2), thus

$$\begin{aligned} \mu_{[x]_E}([y]_{E_{\hat{x}}}) &\geq \mu_{[x]_E}([z]_{E_{\hat{y}}}) \\ &> \mu_{[x]_E}([x]_{E_{\hat{y}}}) \\ &\geq \mu_{[x]_E}([w]_{E_{\hat{x}}}), \end{aligned}$$

so $[y]_{E_{\hat{x}}}$ is the unique $E_{\hat{x}}$-class of maximal measure, hence $|R_x| = 1$. ∎

To see that A_2 is $\mathscr{T}$-convex, suppose $x, z \in A_2$ and $y \in X$ lies along the $\mathscr{T}$-path from x to z. By Sublemma 2.6, both x and z lie in an equivalence class of $E_{\hat{y}}$ of maximal measure. It follows that $|R_y| = 2$, thus $y \in A_2$. ∎

Thus, we may assume that $\forall x \in X \ (|R_x| = 1)$. Let f be the Borel function which associates with each point $x \in X$ the unique element of R_x.

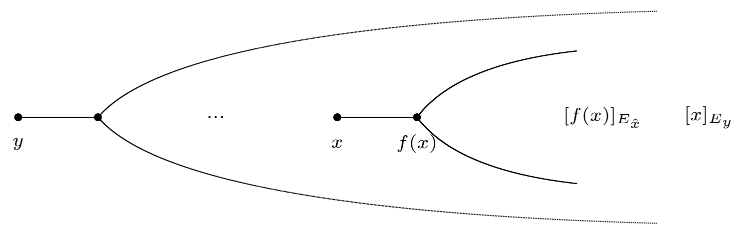

Figure 3: If $y \notin \{x, f(x)\}$, then $d_{\mathscr{T}}(f(y), \{x, f(x)\}) < d_{\mathscr{T}}(y, \{x, f(x)\})$.

Lemma 2.7: *The restriction of* E *to* $\{x \in X : \exists y \in [x]_E \ (f^2(y) = y)\}$ *is smooth.*

Proof: It is enough to show that the set

$$A = \{x \in X : f^2(x) = x\}$$

intersects each equivalence class of E in at most two points. Fix $x \in A$. We will show that for every $y \in [x]_E$ which lies outside of the set $\{x, f(x)\}$, there exists $n \in \mathbb{N}$ such that $f^n(y) \in \{x, f(x)\}$. Letting $d_{\mathscr{T}}$ denote the graph metric, it is enough to show that for each $y \in [x]_E$ which lies outside of the set $\{x, f(x)\}$,

$$d_{\mathscr{T}}(f(y), \{x, f(x)\}) < d_{\mathscr{T}}(y, \{x, f(x)\}).$$

That is, we must show that $[x]_{E_{\hat{y}}}$ is of maximal measure. By reversing the roles of $x, f(x)$ if necessary, we may assume that the unique path from x to y avoids $f(x)$. It then follows that every equivalence class of $E_{\hat{y}}$ other than that of x is contained in $[y]_{E_{\hat{x}}}$ and $[f(x)]_{E_{\hat{x}}} \subseteq [x]_{E_{\hat{y}}}$ (see Figure 3), thus $[x]_{E_{\hat{y}}}$ is the unique equivalence class of $E_{\hat{y}}$ of maximal measure. ∎

By appealing once more to the fact that partial transversals are $\mathscr{T}$-linear, we may now assume that $f^2(x) \neq x$, for all $x \in X$. As $\mathscr{T}$ is a forest, it follows that f is aperiodic. The following fact completes the proof of the proposition:

Lemma 2.8: $\mathscr{T} = \mathscr{T}_f$.

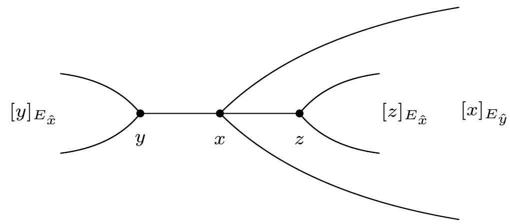

Figure 4: If $(x, y) \in \mathscr{T}$ and $f(x) \neq y$, then $f(y) = x$.

Proof: Clearly $\mathscr{T}_f \subseteq \mathscr{T}$, so we must show that whenever $(x, y) \in \mathscr{T}$, either $f(x) = y$ or $f(y) = x$. Suppose that $z = f(x)$ is distinct from y. Then $[z]_{E_{\hat{x}}}$ is the unique equivalence class of $E_{\hat{x}}$ of maximal measure, every equivalence class of $E_{\hat{y}}$ other than that of x is contained in $[y]_{E_{\hat{x}}}$, and $[z]_{E_{\hat{x}}} \subseteq [x]_{E_{\hat{y}}}$ (see Figure 4). It follows that $[x]_{E_{\hat{y}}}$ is the unique equivalence class of $E_{\hat{y}}$ of maximal measure, thus $f(y) = x$. ∎

Remark 2.9: As in §3 of Jackson-Kechris-Louveau [7], the assumption that $\mathscr{T}$ is locally finite is unnecessary in the statement of Proposition 2.1. Although we will have no need for this generalization, it certainly could be used to extend the results of the upcoming sections to the generic ($\aleph_0$-regular) treeing of E.

3. Generic treeings

An equivalence relation E is **aperiodic** if every equivalence class of E is infinite. In this section, we introduce a parameterized collection of attempts at building an n-regular treeing of an aperiodic countable Borel equivalence relation, for $n \geq 2$. We show that the generic such attempt successfully produces an n-regular treeing of the restriction of the equivalence relation to an invariant comeager Borel set.

Fix a natural number $n \geq 2$. We say that a treeing of E is **n-regular** if all of its vertices have exactly n neighbors. A **finite partial n-regular treeing** of E is a Borel forest $\mathscr{T} \subseteq E$ whose connected components are finite and whose vertices have at most n neighbors. (It is rare for such treeings to literally be of finite cardinality!) The **equivalence relation induced by** $\mathscr{T}$ is given by

$$xE_{\mathscr{T}}y \Leftrightarrow (x, y \text{ lie in the same connected component of } \mathscr{T}),$$

and a **one-step proper extension** of $\mathscr{T}$ is a pair $(x, y) \in E \setminus E_{\mathscr{T}}$ such that both x, y are of $\mathscr{T}$-vertex degree strictly less than n. We will use $\Phi_{\mathscr{T}}$ to denote the standard Borel space of all such extensions of $\mathscr{T}$.

A **coloring** of a graph $\mathscr{G}$ on X is a function $c : X \to Y$ such that

$$\forall x, y \in X \ \Big((x, y) \in \mathscr{G} \Rightarrow c(x) \neq c(y) \Big).$$

The **Borel chromatic number** of $\mathscr{G}$ is the cardinality of the smallest Polish space Y for which there is a Borel coloring $c : X \to Y$ (see Kechris-Solecki-Todorcevic [11] for a detailed study of this notion). Associated with each finite partial n-regular treeing $\mathscr{T}$ of E is the graph $\mathscr{G}_{\mathscr{T}}$ on $\Phi_{\mathscr{T}}$, whose vertices are one-step proper extensions of $\mathscr{T}$ and whose edges consist of pairs $((x, y), (x', y'))$ of distinct one-step proper extensions of $\mathscr{T}$ for which

$$([x]_{E_{\mathscr{T}}} \cup [y]_{E_{\mathscr{T}}}) \cap ([x']_{E_{\mathscr{T}}} \cup [y']_{E_{\mathscr{T}}}) \neq \emptyset.$$

Note that if $\Phi \subseteq \Phi_{\mathscr{T}}$ is a Borel subset of $\Phi_{\mathscr{T}}$ and no two elements of Φ are $\mathscr{G}_{\mathscr{T}}$-neighbors, then the graph obtained by adding Φ to $\mathscr{T}$ is again a finite partial n-regular treeing of E.

Proposition 3.1: *The graph $\mathscr{G}_{\mathscr{T}}$ has countable Borel chromatic number.*

Proof: We will produce a Borel coloring $c : \Phi_{\mathscr{T}} \to \mathbb{N}^{<\mathbb{N}}$. We use $[E]^{<\infty}$ to denote the standard Borel space of all finite sets $S \subseteq X$ such that

$$\forall x, y \in S \ (xEy).$$

Associated with this space is a graph $\mathscr{G}$ on $[E]^{<\infty}$, given by

$$\mathscr{G} = \{(S, T) \in [E]^{<\infty} \times [E]^{<\infty} : S \neq T \text{ and } S \cap T \neq \emptyset\}.$$

Lemma 3.2: *The graph $\mathscr{G}$ has countable Borel chromatic number.*

Proof: By Feldman-Moore [3], there are Borel involutions $\iota_n : X \to X$ such that

$$E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(\iota_n).$$

Let $\leq$ be a Borel linear ordering of X, and given $S \in [E]^{<\infty}$, let $x_1^{(S)}, \ldots, x_{|S|}^{(S)}$ be the $\leq$-increasing enumeration of S. Define $c : \Phi \to \mathbb{N}^{<\mathbb{N}}$ by letting $c(S)$ be the unique sequence $\langle k_{ij} \rangle_{1 \leq i,j \leq |S|}$ such that

$$\forall 1 \leq i, j \leq |S| \ \Big(k_{ij} = \min\{k \in \mathbb{N} : \iota_k \cdot x_i^{(S)} = x_j^{(S)}\} \Big).$$

Now suppose, towards contradiction, that c is not a coloring. Fix $(S, T) \in \mathscr{G}$ such that $c(S) = c(T)$, put $n = |S| = |T|$, and fix $i, j < n$ such that

$$x_i^{(S)} = x_j^{(T)}.$$

Then

$$\begin{aligned} i < j &\Leftrightarrow x_i^{(S)} <_X x_j^{(S)} \\ &\Leftrightarrow x_i^{(S)} <_X \iota_{k_{ij}}(x_i^{(S)}) \\ &\Leftrightarrow x_j^{(T)} <_X \iota_{k_{ij}}(x_j^{(T)}) \\ &\Leftrightarrow x_j^{(T)} <_X x_i^{(T)} \\ &\Leftrightarrow j < i, \end{aligned}$$

thus $i = j$, so $x_i^{(S)} = x_i^{(T)}$. It follows that for all $m < n$,

$$\begin{aligned} x_m^{(S)} &= \iota_{k_{im}}(x_i^{(S)}) \\ &= \iota_{k_{im}}(x_i^{(T)}) \\ &= x_m^{(T)}, \end{aligned}$$

thus $S = T$, which contradicts our assumption that $(S, T) \in \mathscr{G}$. ∎

Fix a Borel coloring $c_0 : [E]^{<\infty} \to \mathbb{N}$ of $\mathscr{G}$, fix a Borel linear ordering $\leq$ of X, and define $c_1 : X \times X \to \mathbb{N} \times \mathbb{N}$ by

$$c_1(x, y) = (i, j),$$

where x is the i^{th} element of the $\leq$-increasing enumeration of $[x]_{E_\mathscr{T}} \cup [y]_{E_\mathscr{T}}$, and y is the j^{th} element of the $\leq$-increasing enumeration of $[x]_{E_\mathscr{T}} \cup [y]_{E_\mathscr{T}}$. Now define

$$c(x, y) = \Big(c_0([x]_{E_\mathscr{T}} \cup [y]_{E_\mathscr{T}}), c_1(x, y)\Big),$$

and suppose, towards a contradiction, that there exists $((x, y), (x', y')) \in \mathscr{G}$ such that $c(x, y) = c(x', y')$. As $c_0([x]_{E_\mathscr{T}} \cup [y]_{E_\mathscr{T}}) = c_0([x']_{E_\mathscr{T}} \cup [y']_{E_\mathscr{T}})$, it follows that

$$[x]_{E_\mathscr{T}} \cup [y]_{E_\mathscr{T}} = [x']_{E_\mathscr{T}} \cup [y']_{E_\mathscr{T}}.$$

As $c_1(x, y) = c_1(x', y')$, it follows that $(x, y) = (x', y')$, a contradiction. ∎

Next, we define a family of finite partial n-regular treeings $\mathscr{T}_s$ of E, for $s \in \mathbb{N}^{<\mathbb{N}}$. We begin by putting $\mathscr{T}_\emptyset = \emptyset$. Given $\mathscr{T}_s$, set $E_s = E_{\mathscr{T}_s}$, $\Phi_s = \Phi_{\mathscr{T}_s}$, and $\mathscr{G}_s = \mathscr{G}_{\mathscr{T}_s}$, and fix a Borel coloring $c_s : \Phi_s \to \mathbb{N}$ of $\mathscr{G}_s$. Then, for each $k \in \mathbb{N}$, set

$$\mathscr{T}_{s^\frown k} = \mathscr{T}_s \cup \{(x, y) \in \Phi_s : c_s(x, y) = k \text{ or } c_s(y, x) = k\}.$$

(Here we use $s^\frown k$ to denote the concatenation of s with the singleton sequence $\langle k \rangle$.) Note that $\mathscr{T}_{s^\frown k}$ is a partial n-regular treeing of E whose connected components each consist of at most two connected components of $\mathscr{T}_s$, and are therefore finite. Once the recursion is complete, we associate with each $\alpha \in \mathbb{N}^\mathbb{N}$ the forest

$$\mathscr{T}_\alpha = \bigcup_{n \in \mathbb{N}} \mathscr{T}_{\alpha | n},$$

as well as the associated equivalence relation $E_\alpha = E_{\mathscr{T}_\alpha}$, and the set

$$C_\alpha = \{x \in X : \mathscr{T}_\alpha | [x]_E \text{ is an } n\text{-regular tree}\}.$$

PROPOSITION 3.3: *Suppose that X is a Polish space and E is an aperiodic countable Borel equivalence relation on X. Then for comeagerly many $\alpha \in \mathbb{N}^\mathbb{N}$, the set C_α is comeager.*

Proof: By Feldman-Moore [3], there is a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$. We must show that

$$\forall^* \alpha \in \mathbb{N}^\mathbb{N}\, \forall^* x \in X \ (\mathscr{T}_\alpha | [x]_E \text{ is an } n\text{-regular tree}).$$

By the theorem of Kuratowski-Ulam (see, for example, §8 of Kechris [9]), it is enough to show that for all $x \in X$,

$$\forall^* \alpha \in \mathbb{N}^\mathbb{N} \ (\mathscr{T}_\alpha | [x]_E \text{ is an } n\text{-regular tree}).$$

It is therefore sufficient to verify the following two lemmas:

LEMMA 3.4: $\forall^* \alpha \in \mathbb{N}^\mathbb{N}$ ($\mathscr{T}_\alpha | [x]_E$ *is connected*).

Proof: It is enough to show that for all $\gamma, \delta \in \Gamma$ and $s \in 2^{<\mathbb{N}}$, there exists $t \supseteq s$ such that $\gamma \cdot x E_t \delta \cdot x$, as this implies that the set of $\alpha \in \mathbb{N}^\mathbb{N}$ for which $\gamma \cdot x E_\alpha \delta \cdot x$ contains a dense open set, thus the set of $\alpha \in \mathbb{N}^{<\mathbb{N}}$ for which $\mathscr{T}_\alpha | [x]_E$ is connected contains a countable intersection of dense open sets. Suppose that $(\gamma \cdot x, \delta \cdot x) \notin E_s$. As the connected components of $\mathscr{T}_s$ are finite, there exists $y \in [\gamma \cdot x]_{E_s}$ and $z \in [\delta \cdot x]_{E_s}$ which are of $\mathscr{T}_s$-vertex degree strictly less than n. It follows that the pair (y, z) is a one-step proper extension of $\mathscr{T}_s$, thus there exists $k \in \mathbb{N}$ such that $y E_{s^\frown k} z$, so $\gamma \cdot x E_{s^\frown k} \delta \cdot x$. Hence, $t = s^\frown k$ is as desired. ∎

LEMMA 3.5: $\forall^* \alpha \in \mathbb{N}^\mathbb{N}$ ($\mathscr{T}_\alpha | [x]_E$ *is n-regular*).

Proof: It is enough to show that for all $\gamma \in \Gamma$, $m \leq n$, and $s \in 2^{<\mathbb{N}}$, there exists $t \supseteq s$ such that $\deg_{\mathscr{T}_t}(\gamma \cdot x) \geq m$, as this implies that the set of $\alpha \in \mathbb{N}^\mathbb{N}$ for which $\deg_{\mathscr{T}_\alpha}(\gamma \cdot x) = n$, thus the set of $\alpha \in \mathbb{N}^{<\mathbb{N}}$ for which $\mathscr{T}_\alpha | [x]_E$ is n-regular contains a countable intersection of dense open sets. We proceed by induction on m. Of course, the case $m = 0$ is trivial, so it is enough to show that if $m < n$ and $\deg_{\mathscr{T}_s}(\gamma \cdot x) = m$, then there exists $t \supseteq s$ such that $\deg_{\mathscr{T}_t}(\gamma \cdot x) = m + 1$. The aperiodicity of E coupled with the fact that the connected components of $\mathscr{T}_s$ are finite ensures that there exists $y \in [x]_E \setminus [\gamma \cdot x]_{E_s}$ such that $\deg_{\mathscr{T}_s}(y) < n$. It follows that the pair (x, y) is a one-step proper extension of $\mathscr{T}_s$, thus there exists $k \in \mathbb{N}$ such that $(x, y) \in \mathscr{T}_{s^\frown k}$. Hence, $t = s^\frown k$ is as desired. ∎

Remark 3.6: For each $\alpha \in \mathbb{N}^\mathbb{N}$, the union of the equivalence relations

$$E_{\alpha_0} \subseteq E_{\alpha_0\alpha_1} \subseteq E_{\alpha_0\alpha_1\alpha_2} \subseteq \cdots$$

is E_α. It follows from Proposition 3.3 that for comeagerly many $\alpha \in \mathbb{N}^\mathbb{N}$, the equivalence relation $E|C_\alpha$ is hyperfinite. The fact that every countable Borel equivalence relation is generically hyperfinite was originally shown in Hjorth-Kechris [5], and extends results of Sullivan-Weiss-Wright [13] and Woodin. See also §12 of Kechris-Miller [10].

4. Generic treeings of non-smooth equivalence relations

In this section, we show that the generic 3-regular treeing of a meager-preserving, generically non-smooth equivalence relation does not satisfy the conclusion of Proposition 2.1.

Recall that an equivalence relation E is **generically non-smooth** if for every non-meager Borel set $B \subseteq X$, the equivalence relation $E|B$ is not smooth.

Proposition 4.1: *Suppose X is a Polish space and E is a meager-preserving, generically non-smooth countable Borel equivalence relation on X. Then for comeagerly many $\alpha \in \mathbb{N}^\mathbb{N}$ and every E-invariant non-meager Borel set $B \subseteq X$, $\mathscr{T}_\alpha|B$ is undirectable.*

Proof: Fix a countable open basis $\mathscr{B}$ for X. By Feldman-Moore [3], there is a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. In order to implement a category argument similar to that used in the proof of Proposition 3.3, we must first describe the conclusion that we wish to draw in terms of countably many conditions which depend only on $\mathscr{B}$ and Γ:

Lemma 4.2: *Suppose that $A \subseteq X$ is an E-invariant non-meager Borel set and $\mathscr{T}$ is a directable treeing of $E|A$. Then there exists $U \in \mathscr{B}$ and $\gamma \in \Gamma$ such that*

$$\forall^* x \in U \, \forall x_0, x_1 \in U \cap [x]_E \, \left(d_\mathscr{T}(\gamma \cdot x_0, \gamma \cdot x_1) \leq d_\mathscr{T}(x_0, x_1)\right).$$

Proof: Fix an aperiodic Borel $f : A \to A$ which induces $\mathscr{T}$, fix $\gamma \in \Gamma$ such that

$$B = \{x \in A : f(x) = \gamma \cdot x\}$$

is non-meager, fix $U \in \mathscr{B}$ such that B is comeager in U, and observe that the set

$$C = X \setminus [U \setminus B]_E$$

is comeager, since E is meager-preserving. As $\mathscr{T} = \mathscr{T}_f$, it follows that

$$\forall x \in A \, \forall x_0, x_1 \in [x]_E \, \left(d_{\mathscr{T}_f}(f(x_0), f(x_1)) \leq d_{\mathscr{T}_f}(x_0, x_1)\right).$$

As $U \cap C \subseteq B \cap C \subseteq A \cap C$, the lemma follows. ∎

It is therefore enough to show that for every $U \in \mathscr{B}$ and $\gamma \in \Gamma$,

$$\forall^* \alpha \in \mathbb{N}^{\mathbb{N}} \, \forall^* x \in U \, \exists x_0, x_1 \in U \cap [x]_E \, \left(d_{\mathscr{T}_\alpha}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_\alpha}(x_0, x_1) \right).$$

By the theorem of Kuratowski-Ulam, it is enough to show that

$$\forall^* x \in U \, \forall^* \alpha \in \mathbb{N}^{\mathbb{N}} \, \exists x_0, x_1 \in U \cap [x]_E \, \left(d_{\mathscr{T}_\alpha}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_\alpha}(x_0, x_1) \right).$$

Lemma 4.3: *Suppose that X is a Polish space, E is a generically non-smooth countable Borel equivalence relation on X, and $A \subseteq X$ is Borel. Then*

$$\forall^* x \in X \ (A \cap [x]_E = \emptyset \text{ or } |A \cap [x]_E| = \infty).$$

Proof: Define $B \subseteq X$ by

$$B = \{x \in X : 0 < |A \cap [x]_E| < \infty\},$$

fix a Borel linear ordering $\leq$ of X, and observe that the set

$$C = \{x \in A \cap B : \forall y \in A \cap [x]_E \ (x \leq y)\}$$

is a Borel transversal of $E|B$, thus B is meager, and the lemma follows. ∎

Thus, it is enough to show that if $|U \cap [x]_E| = \infty$, then

$$\forall^* \alpha \in \mathbb{N}^{\mathbb{N}} \, \exists x_0, x_1 \in U \cap [x]_E \, \left(d_{\mathscr{T}_\alpha}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_\alpha}(x_0, x_1) \right).$$

It is therefore sufficient to verify the following lemma:

Lemma 4.4: *For all $s \in \mathbb{N}^{<\mathbb{N}}$, there exists $v \supseteq s$ and $x_0, x_1 \in U \cap [x]_E$ such that the points $x_0, x_1, \gamma \cdot x_0, \gamma \cdot x_1$ are pairwise $\mathscr{T}_v$-connected, and*

$$d_{\mathscr{T}_v}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_v}(x_0, x_1).$$

Proof: Set $x_0 = x$, and find $t \supseteq s$ such that $x_0 E_t \gamma \cdot x_0$. As $|U \cap [x]_E| = \infty$, there exists $x_1 \in U \cap [x]_E$ which lies outside of the finite set $[x_0]_{E_t} \cup \gamma^{-1}([x_0]_{E_t})$. In particular, it follows that neither x_1 nor $\gamma \cdot x_1$ lies in $[x_0]_{E_t}$, thus there is an extension $u \supseteq t$ such that $x_1 E_u \gamma \cdot x_1$ but $(x_0, x_1) \notin E_u$ (see Figure 5).

Let x_i' be the $\mathscr{T}_u$-neighbor of x_i which is $d_{\mathscr{T}_u}$-closest to $\gamma \cdot x_i$, let $\mathscr{T}_i$ be the forest obtained from $\mathscr{T}_u$ by deleting the edge (x_i, x_i'), and fix $y_i \in [x_i]_{E_{\mathscr{T}_i}}$ such that $\deg_{\mathscr{T}_u}(y_i) < n$. Then (y_0, y_1) is a one-step proper extension of $\mathscr{T}_u$, thus there exists $v \supseteq u$ such that $(y_0, y_1) \in \mathscr{T}_v$. It follows that the points $x_0, x_1, \gamma \cdot x_0, \gamma \cdot x_1$ are pairwise $\mathscr{T}_v$-connected, and $d_{\mathscr{T}_v}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_v}(x_0, x_1)$. ∎

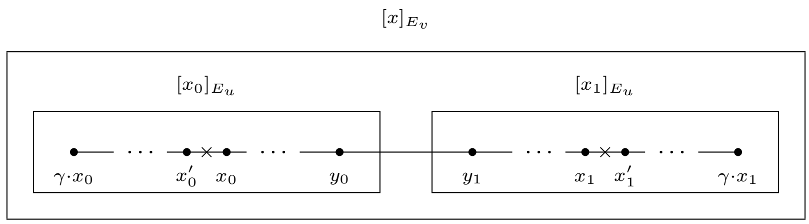

Figure 5: Finding an extension $v \supseteq s$ such that $d_{\mathscr{T}_v}(\gamma \cdot x_0, \gamma \cdot x_1) > d_{\mathscr{T}_v}(x_0, x_1)$.

Remark 4.5: The directability of Borel forests can be characterized in terms of a Glimm-Effros style dichotomy. See Hjorth-Miller [6] for more on this.

Next, we prove a similar fact about the existence of (non-trivial) convex sets:

Proposition 4.6: *Suppose X is a Polish space and E is a meager-preserving, generically non-smooth countable Borel equivalence relation on X. Then for comeagerly many $\alpha \in \mathbb{N}^\mathbb{N}$ and every non-meager E-invariant Borel set $B \subseteq X$, the forest $\mathscr{T}_\alpha | B$ admits no convex Borel complete, co-complete section.*

Proof: Fix a countable open basis $\mathscr{B}$ for X. Again, we begin by describing the conclusion that we wish to draw in terms of countably many conditions which depend only on $\mathscr{B}$. We say that $B \subseteq X$ is **generically non-trivial** if the set

$$[B]_E \cap [X \setminus B]_E = \{x \in X : \emptyset \subsetneq B \cap [x]_E \subsetneq [x]_E\}$$

is non-meager.

Lemma 4.7: *Suppose that $A \subseteq X$ is a non-meager E-invariant Borel set and $\mathscr{T}$ is a treeing of $E|A$. If $E|A$ admits a $\mathscr{T}$-convex Borel complete, co-complete section, then there is a set $U \in \mathscr{B}$ with generically non-trivial convex $\mathscr{T}$-closure.*

Proof: Suppose that $B \subseteq X$ is a $\mathscr{T}$-convex Borel complete, co-complete section for $E|A$. Then B is non-meager, since A is non-meager and E is meager-preserving. Fix $U \in \mathscr{B}$ such that B is comeager in U, and observe that

$$C = A \setminus [U \setminus B]_E$$

is comeager in U, since E is meager-preserving. As $U \cap C \subseteq B \cap C$, it follows that the convex $\mathscr{T}$-closure of U is generically non-trivial. ∎

Thus, it is enough to show that for all $U \in \mathscr{B}$,

$$\forall^* \alpha \in \mathbb{N}^\mathbb{N} \, \forall^* x \in X \, \left(U_\alpha \cap [x]_E = \emptyset \text{ or } (X \setminus U_\alpha) \cap [x]_E = \emptyset\right),$$

where U_α is the convex $\mathscr{T}_\alpha$-closure of U. By Feldman-Moore [3], there is a countable group Γ of Borel automorphisms such that $E = E_\Gamma^X$. By Lemma 4.3, the set

$$A = \{x \in U : |U \cap [x]_E| = \infty\}$$

is comeager in U, thus by the theorem of Kuratowski-Ulam, it is enough to show

$$\forall x \in A \, \forall \gamma \in \Gamma \, \forall^* \alpha \in \mathbb{N}^\mathbb{N} \, \left(\gamma \cdot x \in U_\alpha\right).$$

Fix $x \in A$ and $\gamma \in \Gamma$. It only remains to check the following:

LEMMA 4.8: *For all $s \in \mathbb{N}^{<\mathbb{N}}$, there exists $u \supseteq s$ such that $\gamma \cdot x \in U_u$, where U_u is the convex $\mathscr{T}_u$-closure of U.*

Proof: As $x \in U$, we may assume that $\gamma \cdot x \neq x$. Fix an extension $t \supseteq s$ such that $x E_{\mathscr{T}_t} \gamma \cdot x$. Let x' be the $\mathscr{T}_t$-neighbor of $\gamma \cdot x$ which is $d_{\mathscr{T}_t}$-closest to x, let $\mathscr{T}$ be the forest obtained from $\mathscr{T}_t$ by deleting the edge $(\gamma \cdot x, x')$, and find $y_0 \in [\gamma \cdot x]_{E_\mathscr{T}}$ such that $\deg_{\mathscr{T}_t}(y_0) < n$. As $|U \cap [x]_E| = \infty$, there exists $z \in U \cap [x]_E$ which lies outside of $[x]_{E_{\mathscr{T}_t}}$.

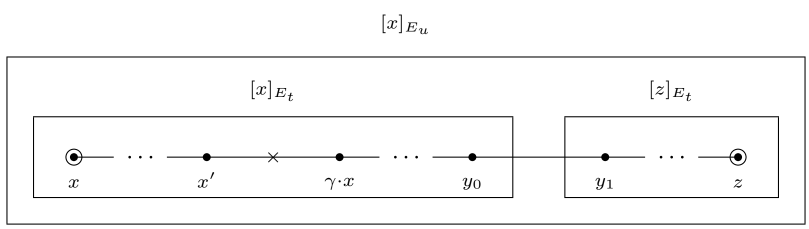

Figure 6: Finding an extension $u \supseteq s$ such that $\gamma \cdot x \in U_u$.

Fix $y_1 \in [z]_{E_t}$ such that $\deg_{\mathscr{T}_t}(y_1) < n$ (see Figure 6). As (y_0, y_1) is a one-step proper extension of $\mathscr{T}_t$, it follows that there is an extension $u \supseteq t$ such that $(y_0, y_1) \in \mathscr{T}_u$. As $\gamma \cdot x$ lies along the unique injective $\mathscr{T}_u$-path from x to z, it follows that $\gamma \cdot x$ is in the convex $\mathscr{T}_u$-closure of U. ∎

REMARK 4.9: The existence of $\mathscr{T}$-linear Borel complete, co-complete sections can be characterized in terms of a Glimm-Effros style dichotomy. See Miller-Rosendal [12] for more on this.

5. Assignments of means

In this section, we finally prove our main results on assignments of means.

Recall that an equivalence relation E is **generically smooth** if there is a comeager Borel set $C \subseteq X$ such that $E|C$ is smooth.

THEOREM 5.1: *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is generically smooth.*
2. *There is a comeager E-invariant Borel set $C \subseteq X$ on which $E|C$ admits a Borel assignment of means.*
3. *E admits a Baire measurable assignment of means.*

Proof: To see (1) $\Rightarrow$ (2), fix a comeager E-invariant Borel set $C \subseteq X$ on which $E|C$ is smooth, let $B \subseteq C$ be a Borel transversal of $E|C$, and let $s : C \to C$ be the (Borel) function which associates with each point $x \in C$ the unique element of $B \cap [x]_E$. The map which assigns to each equivalence class $[x]_E$ the mean

$$\varphi_{[x]_E}(f) = f \circ s(x)$$

is as desired.

To see (2) $\Rightarrow$ (3), suppose $[x]_{E|C} \mapsto \varphi_{[x]_{E|C}}$ is a Borel assignment of means. By AC, this can be extended to an assignment of means $[x]_E \mapsto \psi_{[x]_E}$. For every Borel assignment of functions $x \mapsto f_x \in \ell^\infty([x]_E)$ and open $U \subseteq \mathbb{C}$, the sets

$$\{x \in C : \varphi_{[x]_E}(f_x) \in U\} \text{ and } \{x \in X : \psi_{[x]_E}(f_x) \in U\}$$

have meager symmetric difference. As the former set is Baire measurable, so too is the latter. It follows that the assignment $x \mapsto \psi_{[x]_E}$ is Baire measurable.

It only remains to show (3) $\Rightarrow$ (1). Suppose, towards a contradiction, that E admits a Baire measurable assignment of means, but E is not generically smooth.

LEMMA 5.2: *There is an open $U \subseteq X$ such that $E|U$ is generically non-smooth.*

Proof: Fix a countable open basis $\mathscr{B}$ for X. Now suppose, towards a contradiction, that for each $U \in \mathscr{B}$, the equivalence relation $E|U$ is not generically non-smooth, and find non-meager Borel sets $B_U \subseteq U$ such that $E|B_U$ is smooth. It follows that the restriction of E to the comeager Borel set

$$B = \bigcup_{U \in \mathscr{B}} B_U$$

is smooth, which contradicts the fact that E is not generically smooth. ∎

Fix such an open set $U \subseteq X$. Next, we will need the following fact:

LEMMA 5.3 (WOODIN): *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a dense G_δ set $C \subseteq X$ such that $E|C$ is meager-preserving.*

Proof: Fix a countable open basis $\mathscr{B}$ for X. By Feldman-Moore [3], there is a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$. For each pair $(\gamma, U) \in \Gamma \times \mathscr{B}$ for which it is possible, fix a Borel set $B_{(\gamma,U)} \subseteq U$ such that

$$B_{(\gamma,U)} \text{ is comeager in } U \text{ and } \gamma^{-1}(B_{(\gamma,U)}) \text{ is meager.}$$

Each of the maps $\gamma^{-1}(B_{(\gamma,U)}) \xrightarrow{\gamma} B_{(\gamma,U)}$ witnesses that E is not meager-preserving. We now remove these witnesses by restricting our attention a dense, G_δ set

$$C \subseteq X \setminus \bigcup_{\gamma,U} \gamma^{-1}(B_{(\gamma,U)}).$$

Suppose, towards a contradiction, that there is a meager Borel set $A \subseteq C$ such that $[A]_{E|C}$ is non-meager in C, and therefore non-meager in X. Set $C_\gamma = C \cap \gamma^{-1}(C)$, and note that

$$[A]_{E|C} = \bigcup_{\gamma\in\Gamma} \gamma(A \cap C_\gamma).$$

In particular, it follows that there exists $\gamma \in \Gamma$ such that $\gamma(A \cap C_\gamma)$ is non-meager, thus comeager in U, for some $U \in \mathscr{B}$. Then $B_{(\gamma,U)}$ exists and $B_{(\gamma,U)} \cap \gamma(A \cap C_\gamma)$ is comeager in U, thus $\gamma^{-1}(B_{(\gamma,U)}) \cap A \neq \emptyset$, which contradicts the fact that $A \subseteq C$, and completes the proof of the lemma. ∎

Now fix a G_δ set $C \subseteq U$ which is dense in U, such that the equivalence relation $F = E|C$ is meager-preserving. As F is generically non-smooth, it follows from the results of §4 that the conclusion of Proposition 2.1 does not hold for the generic 3-regular treeing of F. Thus, to draw out the desired contradiction, it only remains to show the following:

Lemma 5.4: *F admits a Baire measurable assignment of means.*

Proof: By Feldman-Moore [3], there is a countable group $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms such that $E = E_\Gamma^X$. For each $x \in [C]_E$, let $n(x) = \min\{n \in \mathbb{N} : \gamma_n \cdot x \in C\}$, and associate with each $f : [x]_F \to [x]_F$ the map $f^* : [x]_E \to [x]_E$ given by $f^*(x) = f(\gamma_{n(x)} \cdot x)$. Then $\psi_{[x]_F}(f) = \varphi_{[x]_E}(f^*)$ defines a Baire measurable assignment of means to the equivalence classes of F. ∎

We say that a set $B \subseteq X$ is **globally Baire measurable** if for every Polish space Y and Borel injection $\pi : Y \to X$, the set $\pi^{-1}(B)$ is Baire measurable. (This notion should be thought of as an analog of universal measurability.)

Theorem 5.5: *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. E is smooth.

2. E admits a Borel assignment of means.

3. E admits a globally Baire measurable assignment of means.

Proof: To see (1) $\Rightarrow$ (2), fix a Borel transversal $B \subseteq X$ of E, and let $s : X \to X$ be the Borel function which associates with each point $x \in X$ the unique element of $B \cap [x]_E$. The map which assigns to each equivalence class $[x]_E$ the mean

$$\varphi_{[x]_E}(f) = f \circ s(x)$$

is as desired.

As (2) $\Rightarrow$ (3) is obvious, only $\neg(1) \Rightarrow \neg(3)$ remains. By the Glimm-Effros dichotomy (see, for example, Harrington-Kechris-Louveau [4]), there is a continuous injection $\pi : 2^{\mathbb{N}} \to X$ such that

$$\forall x, y \in 2^{\mathbb{N}} \ (x E_0 y \Leftrightarrow \pi(x) E \pi(y)),$$

where E_0 is the equivalence relation on $2^{\mathbb{N}}$ given by

$$x E_0 y \Leftrightarrow \exists N \in \mathbb{N} \, \forall n \geq N \ (x_n = y_n).$$

Now suppose, towards a contradiction, that E admits a globally Baire measurable assignment of means. Setting $B = \pi(2^{\mathbb{N}})$, it follows as in the proof of Lemma 5.4 that $E|B$ admits a globally Baire measurable assignment of means. Pulling back through π, it then follows that E_0 admits a Baire measurable assignment of means. As E_0 is generically non-smooth, this contradicts Theorem 5.1. ∎

Remark 5.6: Under CH, the existence of a universally measurable assignment of means is equivalent to the μ-hyperfiniteness of E with respect to every Borel probability measure on X (see Kechris [8]).

Recall that a set $A \subseteq X$ is **analytic**, or $\mathbf{\Sigma^1_1}$, if it is of the form

$$A = \{x \in X : \exists y \in \mathbb{N}^{\mathbb{N}} \ ((x, y) \in B)\},$$

where $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$ is Borel. A set $C \subseteq X$ is **co-analytic**, or $\mathbf{\Pi^1_1}$, if it is the complement of an analytic set. A set $A \subseteq X$ is $\mathbf{\Sigma^1_{n+1}}$ if it is of the form

$$A = \{x \in X : \exists y \in \mathbb{N}^{\mathbb{N}} \ ((x, y) \in B)\},$$

where $B \subseteq X \times \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi^1_n}$, and $A \subseteq X$ is $\mathbf{\Pi^1_{n+1}}$ if it is the complement of a $\mathbf{\Sigma^1_{n+1}}$ set. A set $P \subseteq X$ is **projective** if it is $\mathbf{\Sigma^1_n}$, for some $n \in \mathbb{N}$.

One of the successes of modern descriptive set theory has been the resolution of various classical questions about the projective hierarchy via determinacy axioms (which transcend ZFC). In particular, it follows from the axiom of Projective Determinacy (PD) that every projective subset of a Polish space is Baire measurable. Theorem 5.5 therefore implies the following:

THEOREM 5.7 (PD): *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is smooth.*

2. *E admits a projective assignment of means.*

By employing still stronger set-theoretic hypotheses, we can show that there are even weaker notions of measurability with respect to which only smooth equivalence relations admit measurable assignments of means.

Along similar lines, Theorem 5.5 can be used to see that in certain models of $\mathrm{ZF} + \mathrm{DC}$ in which the axiom of choice fails, non-smooth equivalence relations cannot admit assignments of means whatsoever. Let BP abbreviate the statement that every subset of a Polish space is Baire measurable.

THEOREM 5.8 ($\mathrm{ZF} + \mathrm{DC} + \mathrm{BP}$): *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is smooth.*

2. *E admits an assignment of means.*

ACKNOWLEDGEMENTS: We would like to thank Vadim Kaimanovich for asking the question that led to this paper, Greg Hjorth for many interesting conversations along the way, and the anonymous referee for suggesting the current proof of Lemma 5.4 as a replacement for our original incorrect argument, and also for making several other useful suggestions.

References

[1] S. Adams. Trees and amenable equivalence relations. **Ergodic Theory Dynam. Systems**, **10 (1)**, (1990), 1–14

[2] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relation is generated by a single transformation. **Ergodic Theory Dynamical Systems**, **1 (4)**, (1981), 431–450 (1982)

[3] J. Feldman and C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. **Trans. Amer. Math. Soc.**, **234 (2)**, (1977), 289–324

[4] L. Harrington, A. Kechris, and A. Louveau. A Glimm-Effros dichotomy for Borel equivalence relations. **J. Amer. Math. Soc.**, **3 (4)**, (1990), 903–928

[5] G. Hjorth and A. Kechris. Analytic equivalence relations and Ulm-type classifications. **J. Symbolic Logic**, **60 (4)**, (1995), 1273–1300

[6] G. Hjorth and B. Miller. Ends of graphed equivalence relations, II (2005). Preprint

[7] S. Jackson, A. Kechris, and A. Louveau. Countable Borel equivalence relations. **J. Math. Log.**, **2 (1)**, (2002), 1–80

[8] A. Kechris. Amenable versus hyperfinite Borel equivalence relations. **J. Symbolic Logic**, **58 (3)**, (1993), 894–907

[9] A. Kechris. **Classical descriptive set theory**, volume 156 of **Graduate Texts in Mathematics**. Springer-Verlag, New York (1995)

[10] A. Kechris and B. Miller. **Topics in orbit equivalence**, volume 1852 of **Lecture Notes in Mathematics**. Springer-Verlag, Berlin (2004)

[11] A. Kechris, S. Solecki, and S. Todorčević. Borel chromatic numbers. **Adv. Math.**, **141 (1)**, (1999), 1–44

[12] B. Miller and C. Rosendal. Descriptive Kakutani equivalence (2005). In preparation

[13] D. Sullivan, B. Weiss, and J. Wright. Generic dynamics and monotone complete C^*-algebras. **Trans. Amer. Math. Soc.**, **295 (2)**, (1986), 795–809