MEASURABLE PERFECT MATCHINGS FOR ACYCLIC LOCALLY COUNTABLE BOREL GRAPHS

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Abstract. We characterize the structural impediments to the existence of Borel perfect matchings for acyclic locally countable Borel graphs admitting a Borel selection of finitely many ends from their connected components. In particular, this yields the existence of Borel matchings for such graphs of degree at least three. As a corollary, it follows that acyclic locally countable Borel graphs of degree at least three generating μ-hyperfinite equivalence relations admit μ-measurable matchings. We establish the analogous result for Baire measurable matchings in the locally finite case, and provide a counterexample in the locally countable case.

Introduction

A *graph* on a set X is an irreflexive symmetric subset G of $X \times X$. An *involution* is a permutation which is its own inverse, and a *matching* of G is an involution of a subset of X whose graph is contained in G. Such a matching is *perfect* if its domain is X itself.

A *G-path* is a sequence $(x_i)_{i \leq n}$ such that $x_i \mathrel{G} x_{i+1}$, for all $i < n$. We say that G is *connected* if there is a G-path between any two points of X. More generally, the equivalence relation *generated* by a graph G on X is the smallest equivalence relation on X containing G, and the *connected components* of G are the equivalence classes $[x]_G$ of this relation. A graph G is *acyclic* if there is at most one injective G-path between any two points. When G is acyclic, the *G-distance* between two points of the same connected component of G is one less than the number of points along the unique injective G-path between them. A *tree* is an acyclic connected graph.

A straightforward recursive analysis yields a characterization of the existence of perfect matchings for acyclic graphs. Here we consider

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the substantially more subtle question of the existence of measurable perfect matchings for acyclic definable graphs.

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is *Borel* if it is in the σ-algebra generated by the underlying topology. A *standard Borel space* is a set X equipped with the family of Borel sets associated with a Polish topology on X. Every subset of a standard Borel space inherits the σ-algebra consisting of its intersection with each Borel subset of the original space; this restriction is again standard Borel exactly when the subset in question is Borel (see, for example, [Kec95, Corollary 13.4 and Theorem 15.1]). A function between standard Borel spaces is *Borel* if pre-images of Borel sets are Borel. We will take being Borel as our notion of definability.

The *G-degree* of a point y is given by $\deg_G(y) = |\{x \in X \mid x \mathrel{G} y\}|$. A graph is *locally countable* if every point has countable G-degree, and *locally finite* if every point has finite G-degree. A graph is *n-regular* if every point has G-degree n. We say that a graph has *degree at least* n if every point has G-degree at least n. The existence of perfect matchings can be reduced to the case of graphs of degree at least two (modulo a minor caveat in the Borel setting).

A *G-ray* is a sequence $(x_n)_{n\in\mathbb{N}}$ with the property that $x_n \mathrel{G} x_{n+1}$, for all $n \in \mathbb{N}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ has *G-degree two on even indices* if $\deg_G(x_{2n}) = 2$, for all $n \in \mathbb{N}$. Note that if G has degree at least three, then there are no such sequences.

When G is acyclic, we say that injective G-rays $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are *end equivalent* if there exist $i, j \in \mathbb{N}$ with $x_{i+n} = y_{j+n}$, for all $n \in \mathbb{N}$. We say that a set $\mathcal{X} \subseteq X^{\mathbb{N}}$ *selects* a finite non-empty set of ends from every connected component of G if $\mathcal{X} \cap [x]_G^{\mathbb{N}}$ is a finite non-empty union of end-equivalence classes, for all $x \in X$.

Theorem A. *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X of degree at least two, and there is a Borel set selecting a finite non-empty set of ends from every connected component of G. Then there is a Borel set $B \subseteq X$ such that:*

1. *The restriction of G to B is two-regular.*
2. *Every connected component of G contains at most one connected component of the restriction of G to B.*
3. *No two points of B of G-degree at least three have odd G-distance from one another.*
4. *There is a Borel perfect matching of G off of B.*

In particular, if there are no injective G-rays of G-degree two on even indices, then the set B is empty, thus G has a Borel perfect matching.

There are well-known examples of acyclic two-regular Borel graphs which do not have Borel perfect matchings, and a result of Marks yields acyclic n-regular Borel graphs which do not have Borel perfect matchings, for all natural numbers $n \geq 3$ (see [Mar, Theorem 1.5]).

A *Borel probability measure* on a Polish space X is a function μ, assigning to each Borel set $B \subseteq X$ an element of $[0, 1]$, with the property that $\mu(X) = 1$ and $\mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$, for every sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint Borel subsets of X. A Borel set $B \subseteq X$ is *μ-null* if $\mu(B) = 0$, and *μ-conull* if its complement is μ-null.

Following the usual abuse of language, we say that an equivalence relation is *countable* if its classes are countable, and *finite* if its classes are finite. We say that a countable Borel equivalence relation on a standard Borel space is *hyperfinite* if it is the union of an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations, and *μ-hyperfinite* if there is a μ-conull invariant Borel set on which it is hyperfinite.

A well-known result of Adams and Jackson-Kechris-Louveau (see [JKL02, Lemma 3.21]) ensures that if G is an acyclic locally countable Borel graph on X, then the equivalence relation generated by G is μ-hyperfinite if and only if there is a μ-conull G-invariant Borel set on which there is a Borel set selecting a finite non-empty set of ends from every connected component of G that has injective G-rays. Theorem A therefore yields the following corollary.

Theorem B. *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X of degree at least two and with no injective G-rays of G-degree two on even indices, and μ is a Borel probability measure on X for which the equivalence relation generated by G is μ-hyperfinite. Then there is a μ-conull G-invariant Borel set on which G has a Borel perfect matching.*

By a result of Lyons-Nazarov, a wide class of regular bipartite Borel graphs (notably including any bipartite Cayley graphing of the Bernoulli shift action of a nonamenable group) admit μ-measurable matchings (see [LN11]). The general case of μ-measurable matchings for (not necessarily bipartite) Cayley graphings of Bernoulli shifts of nonamenable groups is discussed in [CL].

A subset of a Polish space is *meager* if it is a countable union of nowhere dense sets, and *comeager* if its complement is meager. In contrast with the measure-theoretic setting, a well-known result of Hjorth-Kechris implies that every countable Borel equivalence relation is hyperfinite on a comeager invariant Borel set (see, for example, [KM04, Theorem 12.1]). However, there are acyclic locally finite Borel graphs of degree at least two which do not admit Borel sets selecting a finite non-empty set of ends on any comeager invariant Borel set (see, for example, the graph $\mathscr{T}_0$ of [HM09]). Nevertheless, an entirely different approach yields an analog of Theorem B in this context.

Theorem C. *Suppose that X is a Polish space and G is an acyclic locally finite Borel graph on X of degree at least two and with no injective G-rays of G-degree two on even indices. Then there is a comeager G-invariant Borel set on which G has a Borel perfect matching.*

However, we provide an example of an $\aleph_0$-regular Borel graph which does not have a Borel perfect matching on a comeager invariant Borel set. Some rather general sufficient conditions for the existence of Baire measurable matchings of graphs are presented in [KM] and [MU].

The paper is organized as follows. In §1, we mention a pair of elementary facts concerning matchings outside of the definable context. In §2, we establish Theorem A. And in §3, we establish Theorems B and Theorem C, and describe the example mentioned above.

1. Matchings using choice

Here we establish a pair of elementary facts, whose proofs will later prove useful in the definable setting.

Clearly the existence of a perfect matching for a graph on a non-empty set necessitates that the graph in question has degree at least one. The following observation allows one to focus upon graphs of degree at least two.

Proposition 1.1. *Suppose that X is a set and G is a graph on X. Then there is a set $Y \subseteq X$, on which G has degree at least two, with the property that if there is a matching ι of G whose domain contains $X \setminus Y$, then $X \setminus Y$ is an ι-invariant set on which every other such matching agrees with ι.*

Proof. The *G-boundary* of a set $Y \subseteq X$, denoted by $\partial_G(Y)$, is the set of points in Y which are G-related to at least one point outside of Y. Let α denote the supremum of the ordinals β for which there exists $x \in X$ such that $|\beta| \leq |[x]_G|$, and recursively define a decreasing sequence $(X^\beta)_{\beta \leq \alpha}$ of subsets of X by setting $X^0 = X$, $X^\lambda = \bigcap_{\beta<\lambda} X^\beta$, $X^{\lambda+2n+1} = \{x \in X^{\lambda+2n} \mid \deg_{G \restriction X^{\lambda+2n}}(x) \geq 2\}$, and $X^{\lambda+2n+2} = X^{\lambda+2n+1} \setminus \partial_{G \restriction X^{\lambda+2n}}(X^{\lambda+2n} \setminus X^{\lambda+2n+1})$, for all limit ordinals λ and natural numbers n such that the corresponding indices are at most α.

Set $Y = X^\alpha$. As $X^\alpha = X^{\alpha+1}$, it follows that $G \restriction Y$ has degree at least two. And a straightforward transfinite induction shows that if ι is a matching of G whose domain contains $X \setminus Y$, then $\iota(x)$ is the unique G-neighbor of x in $X^{\lambda+2n}$ for all limit ordinals λ, natural numbers n, and $x \in X^{\lambda+2n} \setminus X^{\lambda+2n+1}$, whereas $\iota(x)$ is the unique G-neighbor of x in $X^{\lambda+2n} \setminus X^{\lambda+2n+1}$ for all limit ordinals λ, natural numbers n, and $x \in X^{\lambda+2n+1} \setminus X^{\lambda+2n+2}$. ⊠

In the absence of definability requirements, the following completes the analysis of the existence of perfect matchings for acyclic graphs.

Proposition 1.2. *Suppose that X is a set and G is an acyclic graph on X of degree at least one whose connected components have at most one point of G-degree exactly one. Then G has a perfect matching.*

Proof. A *transversal* of an equivalence relation is a set intersecting every equivalence class in exactly one point. We will recursively define a sequence $(X_n)_{n \in \mathbb{N}}$ of pairwise disjoint subsets of X, as well as a sequence $\phi_n \colon X_{2n} \to X_{2n+1}$ of functions whose graphs are contained in G, with the property that the graph $G_n = G \restriction (X \setminus \bigcup_{m<2n} X_m)$ has degree at least one and X_{2n} is a transversal of the equivalence relation generated by G_n, containing every point of G_n-degree exactly one.

We begin by fixing a transversal $X_0 \subseteq X$ of the equivalence relation generated by G, containing every point of G-degree exactly one. Suppose now that $n \in \mathbb{N}$ and we have already found $(X_m)_{m \leq 2n}$. Fix a function $\phi_n \colon X_{2n} \to X \setminus \bigcup_{m \leq 2n} X_m$ whose graph is contained in G, and set $X_{2n+1} = \phi_n(X_{2n})$ and $X_{2n+2} = \partial_G(X \setminus \bigcup_{m \leq 2n+1} X_m)$.

Set $\phi = \bigcup_{n \in \mathbb{N}} \phi_n$, and observe that the involution $\iota = \phi \cup \phi^{-1}$ is a perfect matching of G. ⊠

2. Borel matchings

We will frequently employ the following well-known fact.

Theorem 2.1 (Lusin-Novikov)**.** *Suppose that X and Y are Polish spaces and $R \subseteq X \times Y$ is a Borel set whose vertical sections are all countable. Then $\mathrm{proj}_X(R)$ is Borel and there are Borel functions $\phi_n \colon \mathrm{proj}_X(R) \to Y$ such that $R = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$.*

Proof. See, for example, [Kec95, Theorem 18.10]. ⊠

There is a natural analog of Proposition 1.1 in the Borel setting.

Proposition 2.2. *Suppose that X is a Polish space and G is a locally finite Borel graph on X. Then there is a Borel set $B \subseteq X$, on which G has degree at least two, with the property that if there is a matching ι of G whose domain contains $X \setminus B$, then the latter is an ι-invariant set on which every other such matching agrees with ι. In particular, it follows that the restriction of every such matching to $X \setminus B$ is Borel.*

Proof. Following the proof of Proposition 1.1, our assumption that G is locally finite ensures that $X_\alpha = X_\omega$, and Theorem 2.1 implies that the set $B = X_\omega$ is Borel.

Again by Theorem 2.1, there are Borel functions $\phi_n \colon X \to X$ such that the equivalence relation generated by G is $\bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$. We say that a function $i \colon \mathbb{N} \to \mathbb{N}$ *codes* a function on the connected component of x off of B if $\phi_{i(m)}(x) = \phi_{i(n)}(x)$ and neither is in B, for all $m, n \in \mathbb{N}$ such that $\phi_m(x) = \phi_n(x)$ and neither is in B. The corresponding function $\iota \colon [x]_G \setminus B \to [x]_G \setminus B$ is then given by

$$\iota(y) = z \iff \exists m, n \in \mathbb{N} \ (y = \phi_m(x),\ z = \phi_n(x), \text{ and } i(m) = n).$$

Note that if ι is a matching of G on $X \setminus B$, then $\iota(y) = z$ if and only if there is a function $i \colon \mathbb{N} \to \mathbb{N}$, coding a matching of G on the connected component of x off of B, which sends y to z. As the set of $(x, i) \in X \times I$ for which i codes a matching of G on the connected component of x off of B is Borel, as is the set of $(i, m, n, x, y, z) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \times X \times X \times X$ for which $y = \phi_m(x)$, $z = \phi_n(x)$, and $i(m) = n$, it follows that the graph of ι is *analytic*, in the sense that it is the image of a Borel subset of a standard Borel space under a Borel function. As functions between Polish spaces with analytic graphs are Borel (see, for example, [Kec95, Theorem 14.12]), it follows that ι is Borel. ⊠

Remark 2.3. Proposition 1.1 also has an analog in the more general setting of locally countable Borel graphs, granting that we slightly relax the requirement that the matching is Borel. To be specific, in this case one can check that the set $A = X_\alpha$ is analytic, and that the graph of the unique matching of $G \restriction (X \setminus A)$ is both relatively analytic and co-analytic. Here it is worth noting that our other results generalize to Borel graphs on analytic sets. It is also worth noting that, off of a meager or μ-null G-invariant Borel set, this yields the full conclusion of Proposition 2.2.

Proposition 1.2 also has a natural analog in the Borel setting, albeit only when the equivalence relation generated by the graph in question is particularly simple. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ such that $x_1 \mathrel{E} x_2 \iff \pi(x_1) \mathrel{F} \pi(x_2)$, for all $x_1, x_2 \in X$. A Borel equivalence relation on a Polish space is *smooth* if it is Borel reducible to equality on a Polish space.

Proposition 2.4. *Suppose that X is a Polish space and G is an acyclic locally countable Borel graph on X of degree at least one, each of whose connected components have at most one point of G-degree exactly one,* *whose induced equivalence relation is smooth. Then G has a Borel perfect matching.*

Proof. Following the proof of Proposition 1.2, Theorem 2.1 ensures that we can choose the functions ϕ_n and the sets X_n to be Borel, in which case the corresponding matching is also Borel. ⊠

Beyond the smooth case, the purely combinatorial and definable settings are quite different. The graph *generated* by a function $f\colon X \to X$ is the graph G_f on X with respect to which two distinct points are related if f sends one to the other. A function $f\colon X \to X$ is *aperiodic* if f^n is fixed-point free, for all $n > 0$.

Proposition 2.5 (Laczkovich)**.** *There is a Polish space X and an aperiodic Borel automorphism $T\colon X \to X$ with the property that G_T does not have a Borel perfect matching.*

Proof. We say that a Borel probability measure μ on X is T*-quasi-invariant* if $\mu(B) = 0 \iff \mu(T(B)) = 0$, for all Borel sets $B \subseteq X$. And we say that a Borel probability measure μ on X is T*-ergodic* if $\mu(B) \in \{0, 1\}$, for all T-invariant Borel sets $B \subseteq X$.

An I*-coloring* of a graph G on X is a function $c\colon X \to I$ with the property that $\forall (x, y) \in G\ c(x) \neq c(y)$. It is easy to see that G_T has a Borel perfect matching if and only if G_T has a Borel two-coloring, or equivalently, if there is a Borel set $B \subseteq X$ such that B and $T(B)$ partition X. And the latter is ruled out by the existence of a Borel probability measure on X which is T-quasi-invariant and T^2-ergodic.

As Lebesgue measure is well known to be ergodic and quasi-invariant with respect to irrational rotations of the circle, it follows that the latter do not have Borel (or even Lebesgue measurable) matchings. ⊠

Remark 2.6. The above argument goes through just as well using Baire category in lieu of Lebesgue measure.

Remark 2.7. The existence of such measures (or topologies with corresponding notions of Baire category) for graphs generated by aperiodic Borel automorphisms is, in fact, equivalent to the inexistence of Borel perfect matchings. This follows from a dichotomy theorem of Louveau's (see, for example, [Mil12, Theorem 15]).

Let $\mathbb{E}_0$ denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$x \mathrel{\mathbb{E}_0} y \iff \exists n \in \mathbb{N} \forall m \geq n\ x(m) = y(m).$$

The Harrington-Kechris-Louveau $\mathbb{E}_0$ dichotomy ensures that, under Borel reducibility, this is the minimal non-smooth Borel equivalence relation (see [HKL90, Theorem 1.1]). Arguments of Dougherty-Jackson-Kechris can be used to show that a countable Borel equivalence relation on a Polish space is hyperfinite if and only if it is Borel reducible to $\mathbb{E}_0$ (see, for example, [DJK94, Theorem 1]), and Slaman-Steel and Weiss have noted that every Borel automorphism of a Polish space generates a hyperfinite Borel equivalence relation (see, for example, [DJK94, Theorem 5.1]). In particular, it follows that the smoothness of the equivalence relation in Proposition 2.4 cannot be weakened.

Among graphs generated by aperiodic Borel functions, there are essentially no further examples without Borel matchings. The *tail equivalence relation* on X induced by a function $f \colon X \to X$ is given by

$$x \; E_t(f) \; y \iff \exists m, n \in \mathbb{N} \; f^m(x) = f^n(y).$$

Note that if G is the graph generated by f, then $E_t(f)$ is the equivalence relation generated by G.

The *injective part* of f is the set $\{x \in X \mid f \restriction [x]_{E_t(f)} \text{ is injective}\}$. When f is countable-to-one, Theorem 2.1 ensures that the injective part of f is a Borel set on which f is a Borel automorphism.

Proposition 2.8. *Suppose that X is a Polish space and $f \colon X \to X$ is a countable-to-one Borel surjection. Then G_f has a Borel perfect matching, off of the injective part of f.*

Proof. By Theorem 2.1, there is a Borel function $g \colon X \to X$ such that $(f \circ g)(x) = x$, for all $x \in X$. Off of the $E_t(f)$-saturation of $\bigcap_{n \in \mathbb{N}} g^n(X)$, the involution agreeing with g on $\bigcup_{n \in \mathbb{N}} g^{2n}(X \setminus g(X))$ and with f on $\bigcup_{n \in \mathbb{N}} g^{2n+1}(X \setminus g(X))$ is a Borel perfect matching of G_f. So it only remains to produce a Borel perfect matching of G_f on the $E_t(f)$-saturation of the set

$$B = \{x \in \textstyle\bigcap_{n \in \mathbb{N}} g^n(X) \mid x \text{ is not in the injective part of } f\}.$$

Towards this end, define $A = \{x \in B \mid |f^{-1}(x)| \geq 2\}$. As Proposition 2.4 allows us to throw out an $E_t(f)$-invariant Borel set on which $E_t(f)$ is smooth, we can assume that for all $x \in B$, there exist $m, n \in \mathbb{N}$ such that $(f \restriction B)^{-m}(x), f^n(x) \in A$. For each $x \in A$, let $n(x)$ denote the least positive natural number n for which $f^n(x) \in A$, and define $A' = \{x \in A \mid n(x) \text{ is odd}\}$. We then obtain a Borel perfect matching of G_f on $B \setminus A'$ by associating $f^{2i+1}(x)$ with $f^{2i+2}(x)$ for all $i \in \mathbb{N}$ and $x \in A'$ with $2i + 2 < n(x)$, as well as $f^{2i}(x)$ with $f^{2i+1}(x)$ for all $i \in \mathbb{N}$ and $x \in A \setminus A'$ with $2i+1 < n(x)$. As the equivalence relation generated by the restriction of G_f to $A' \cup ([B]_{E_t(f)} \setminus B)$ is smooth, Proposition 2.4 yields an extension to a Borel perfect matching of G_f on $[B]_{E_t(f)}$. ⊠

We now turn our attention to another class of Borel graphs without Borel perfect matchings. The *line-and-point graph* associated with a graph G on X is the graph on the disjoint union of X with the set $E = \{\{x, y\} \mid x \; G \; y\}$ of *unordered edges of* G, in which two elements of $X \cup E$ are related if one of them is in X, one of them is in E, and the former is an element of the latter.

Note that if G is a Borel graph on a standard Borel space, then the set of unordered edges of G inherits a standard Borel structure from G, thus the line-and-point graph of G can also be viewed as a Borel graph on a standard Borel space.

Observe also that if a graph has an injective ray, then its line-and-point graph has an injective ray of degree two on even indices. Together with the following proposition, this is part of the motivation for focusing on graphs without such rays in our later results.

Proposition 2.9. *Suppose that X is a Polish space and G is an acyclic Borel graph on X. Then G is generated by an aperiodic Borel function if and only if its line-and-point graph has a Borel perfect matching.*

Proof. If $f \colon X \to X$ is an aperiodic function generating G, then the fact that f is fixed-point free ensures that $\{x, f(x)\}$ is an unordered edge of G for all $x \in X$, and the fact that f^2 is fixed-point free ensures that the involution ι associating x with $\{x, f(x)\}$ is injective. As the fact that f generates G ensures that ι is surjective, it is necessarily a perfect matching of the line-and-point graph of G.

Conversely, if ι is a perfect matching of the line-and-point graph of G, then the function f, sending each point x to the unique point y with the property that $\iota(x) = \{x, y\}$, generates G. The definition of f ensures that both f and f^2 are fixed-point free, and the acyclicity of G ensures that f^n is fixed-point free for all $n > 2$, thus f is aperiodic. ⊠

Remark 2.10. One can drop the acyclicity of G in the statement of Proposition 2.9 by weakening the hypothesis that G is generated by an aperiodic Borel function to the hypothesis that G is generated by a Borel function for which both f and f^2 are fixed-point free.

Remark 2.11. When G is an acyclic locally countable Borel graph of degree at least two, the hypothesis that G is generated by an aperiodic Borel function is equivalent to the apparently weaker hypothesis that G is generated by a Borel function.

We next consider the combinatorially simplest examples of Borel graphs which are not induced by Borel functions.

Proposition 2.12. *There is a Polish space X and an acyclic two-regular Borel graph G on X which is not induced by a Borel function, thus there is such a graph which does not have a Borel perfect matching.*

Proof. The graph $\mathscr{L}_0$ of [HM09] yields an example of an acyclic two-regular Borel graph on a Polish space which is not induced by a Borel function. Proposition 2.9 then ensures that the corresponding line-and-point graph has no Borel perfect matching (and clearly it is not induced by a Borel function, since the restriction of the square of an aperiodic such function to $2 \times 2^{\mathbb{N}}$ would generate $\mathscr{L}_0$). ⊠

Remark 2.13. It is not difficult to verify that the fact we used in the parenthetical remark above is far more general. Namely, a Borel graph on a Polish space is generated by a Borel function if and only if its line-and-point graph is generated by a Borel function.

Among graphs for which such combinatorially simple graphs can be isolated in a Borel fashion, there are essentially no further examples without Borel matchings.

Proposition 2.14. *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X of degree at least two, and there is a Borel set $B \subseteq X$ such that:*

(1) *The restriction of G to B is two-regular.*
(2) *Every connected component of G contains exactly one connected component of the restriction of G to B.*
(3) *Every connected component of G contains two points in B, with G-degree at least three, having odd G-distance from one another.*

Then there is a Borel perfect matching of G.

Proof. Let A denote the set of points of B of G-degree at least three, and let A' denote the set of initial points of injective G-paths whose initial and terminal points are in A, whose other points are not in A, and along which there are an even number of points. As Proposition 2.4 allows us to throw out a G-invariant Borel set on which the equivalence relation generated by G is smooth, we can assume that for all $x \in B$, there are points of A' on either *side* of x, in the sense that for both G-neighbors y of x in B, there is an injective G-path of the form $(x, y, \ldots)$ whose terminal point is in A'. By Theorem 2.1, there is a Borel set $A'' \subseteq A'$ consisting of exactly one point from every pair of points in A' between which there is an injective G-path whose other points are not in A' and along which there are an odd number of points. Then there is a Borel perfect matching of the restriction of G to $A'' \cup (B \setminus A')$, and Proposition 2.4 yields an extension of the latter to a Borel perfect matching of G. ⊠

We can now establish the main result of this section.

Theorem 2.15. *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X of degree at least two, and there is a Borel set selecting a finite non-empty set of ends from every connected component of G. Then there is a Borel set $B \subseteq X$ such that:*

(1) *The restriction of G to B is two-regular.*
(2) *Every connected component of G contains at most one connected component of the restriction of G to B.*
(3) *No two points of B of G-degree at least three have odd G-distance from one another.*
(4) *There is a Borel perfect matching of G off of B.*

In particular, if there are no injective G-rays of G-degree two on even indices, then the set B is empty, thus G has a Borel perfect matching.

Proof. Fix a Borel set $\mathcal{B} \subseteq [X]^{\mathbb{N}}$ selecting a finite non-empty set of ends from every connected component of G. By Theorem 2.1, we can assume that the set $\mathcal{B}$ selects exactly n ends from every connected component of G, for some $n > 0$.

If $n = 1$, then G is generated by the Borel function $f \colon X \to X$ associating to each point x its unique G-neighbor y for which there is an injective G-ray in $\mathcal{B}$ of the form $(x, y, \ldots)$, in which case Proposition 2.8 allows us to take B to be the injective part of f.

If $n = 2$, then let A denote the set of all points x with two distinct G-neighbors y for which there are injective G-rays in $\mathcal{B}$ of the form $(x, y, \ldots)$. Proposition 2.14 allows us to take B to be the set of x in A for which there do not exist $y, z \in A \cap [x]_G$ of G-degree at least three having odd G-distance from one another.

If $n > 2$, then [JKL02, Lemma 3.19] ensures that the equivalence relation generated by G is smooth, in which case Proposition 2.4 allows us to take B to be the empty set. ⊠

3. Measurable matchings

We begin this section with a fact which, despite being quite well known, seems not to have previously appeared in the form we require.

Proposition 3.1 (Adams, Jackson-Kechris-Louveau)**.** *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X, and μ is a Borel probability measure on X for which the equivalence relation generated by G is μ-hyperfinite. Then there are G-invariant Borel sets $A, B \subseteq X$ such that:*

(1) *The equivalence relation generated by G is smooth on A.*

(2) *There is a Borel set selecting a finite non-empty set of ends from every connected component of the restriction of G to B.*
(3) *The set $A \cup B$ is μ-conull.*

Proof. This follows from the proof of [JKL02, Lemma 3.21]. ⊠

Remark 3.2. Although unnecessary for our arguments, it is worth noting that [JKL02, Lemma 3.19] allows us to strengthen condition (2) in Proposition 3.1 to the existence of a Borel set selecting one or two ends from every connected component of the restriction of G to B.

Remark 3.3. It is also worth noting that, using a fairly straightforward metamathematical argument, Proposition 3.1 can also be established from [JKL02, Lemma 3.21] itself, as opposed to its proof. But this approach seems rather needlessly roundabout.

As a corollary, we obtain the following.

Theorem 3.4. *Suppose that X is a Polish space, G is an acyclic locally countable Borel graph on X of degree at least two and with no injective G-rays of G-degree two on even indices, and μ is a Borel probability measure on X for which the equivalence relation generated by G is μ-hyperfinite. Then there is a μ-conull G-invariant Borel set on which G has a Borel perfect matching.*

Proof. Let A and B denote the G-invariant Borel sets whose existence is granted by Proposition 3.1. Proposition 2.4 yields a Borel perfect matching of G on A, and Proposition 2.15 yields a Borel perfect matching of G on B, thus there is a Borel perfect matching of G on $A \cup B$. ⊠

In the context of Baire category, we obtain the analogous result for locally finite graphs.

Theorem 3.5. *Suppose that X is a Polish space and G is an acyclic locally finite Borel graph on X of degree at least two and with no injective G-rays of G-degree two on even indices. Then there is a comeager G-invariant Borel set on which G has a Borel perfect matching.*

Proof. Let $\mathcal{X}$ denote the set of pairs (S, T) of finite subsets of X, where $S \subseteq T$ and both are contained in a connected component of G. This set inherits a standard Borel structure from X. Let $\mathcal{G}$ denote the graph on $\mathcal{X}$ given by

$$\mathcal{G} = \{((S, T), (S', T')) \in \mathcal{X} \times \mathcal{X} \mid (S, T) \neq (S', T') \text{ and } T \cap T' \neq \emptyset\}.$$

By [CM, Proposition 3], there is a Borel $\mathbb{N}$-coloring c of $\mathcal{G}$.

We will now define a decreasing sequence $(X_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of Borel subsets of X such that the graph $G \restriction X_s$ has degree at least two and no injective $(G \restriction X_s)$-ray has $(G \restriction X_s)$-degree two on even indices, for all $s \in \mathbb{N}^{<\mathbb{N}}$. We will simultaneously produce an increasing sequence $(\iota_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of Borel matchings of G such that the domain of ι_s is $X \setminus X_s$, for all $s \in \mathbb{N}^{<\mathbb{N}}$.

Once we have constructed these, for each $p \in \mathbb{N}^\mathbb{N}$ we will define $X_p = \bigcap_{n \in \mathbb{N}} X_{p \restriction n}$ and $\iota_p \colon X \setminus X_p \to X \setminus X_p$ by $\iota_p(x) = \iota_{p \restriction n}(x)$, where $n \in \mathbb{N}$ is sufficiently large that $x \in X \setminus X_{p \restriction n}$. As each ι_p is necessarily a Borel matching of G with domain $X \setminus X_p$, it will only remain to show that the details of our construction ensure the existence of $p \in \mathbb{N}^\mathbb{N}$ for which the saturation of X_p with respect to the equivalence relation generated by G is meager.

We begin by setting $X_\emptyset = X$ and $\iota_\emptyset = \emptyset$. Suppose now that we have already defined X_s and ι_s. Let $\mathcal{X}_s$ denote the set of pairs $(S, T) \in \mathcal{X}$ which satisfy the following conditions:

(1) The inclusion $\partial_{G \restriction X_s}(X_s \setminus S) \subseteq T \subseteq X_s$ holds.
(2) The graph $G \restriction (X_s \setminus S)$ has degree at least two.
(3) No injective $(G \restriction (X_s \setminus S))$-path passing through both a point in $\partial_{G \restriction X_s}(X_s \setminus S)$ and a point in $\partial_{G \restriction X_s}(T)$ has $(G \restriction (X_s \setminus S))$-degree two on even indices.
(4) There is a perfect matching of $G \restriction S$.

We will extend ι_s by adding perfect matchings of the graphs $G \restriction S$ in a Borel fashion, using the fact that the sets T provide buffers preventing these new matchings from interacting with one another, at least among pairs (S, T) on which our coloring c of $\mathcal{G}$ is constant.

For each $i \in \mathbb{N}$, define $X_{s \frown (i)} \subseteq X_s$ by

$$X_{s \frown (i)} = X_s \setminus \bigcup \{S \mid \exists T \ ((S, T) \in \mathcal{X}_s \text{ and } c(S, T) = i)\}.$$

Theorem 2.1 ensures that these sets are Borel.

Lemma 3.6. *Suppose that $i \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$. Then $G \restriction X_{s \frown (i)}$ has degree at least two.*

Proof. Suppose that $x \in X_{s \frown (i)}$. If x is not in $\partial_{G \restriction X_s}(X_{s \frown (i)})$, then $\deg_{G \restriction X_{s \frown (i)}}(x) = \deg_{G \restriction X_s}(x) \geq 2$. If x is in $\partial_{G \restriction X_s}(X_{s \frown (i)})$, then there exists $(S, T) \in \mathcal{X}_s$ such that $c(S, T) = i$ and x is in $\partial_{G \restriction X_s}(X_s \setminus S)$. Condition (1) ensures that $x \in T$, and since c is an $\mathbb{N}$-coloring of $\mathcal{G}$, it follows that (S, T) is the unique such pair. Condition (2) therefore implies that $\deg_{G \restriction X_{s \frown (i)}}(x) = \deg_{G \restriction (X_s \setminus S)}(x) \geq 2$. ⊠

Lemma 3.7. *Suppose that $i \in \mathbb{N}$ and $s \in \mathbb{N}^{<\mathbb{N}}$. Then there is no injective $(G \restriction X_{s \frown (i)})$-ray of $(G \restriction X_{s \frown (i)})$-degree two on even indices.*

Proof. If $(x_k)_{k\in\mathbb{N}}$ is an injective $(G \restriction X_{s\frown(i)})$-ray of $(G \restriction X_{s\frown(i)})$-degree two on even indices, then condition (3) ensures that $x_k \notin \partial_{G\restriction X_s}(X_{s\frown(i)})$ for all $k \in \mathbb{N}$, thus $(x_k)_{k\in\mathbb{N}}$ is a $(G \restriction X_s)$-ray of $(G \restriction X_s)$-degree two on even indices, a contradiction. ⊠

Condition (4) ensures that ι_s extends to a Borel matching $\iota_{s\frown(i)}$ of G with domain $X \setminus X_{s\frown(i)}$.

As noted earlier, it only remains to show that there exists $p \in \mathbb{N}^\mathbb{N}$ for which the saturation of X_p with respect to the equivalence relation generated by G is meager. We will establish the stronger fact that $\forall^* p \in \mathbb{N}^\mathbb{N} \forall^* x \in X \; [x]_G \cap X_p = \emptyset$. Note that $\{(p, x) \in \mathbb{N}^\mathbb{N} \times X \mid x \in X_p\}$ is Borel. By the Kuratowski-Ulam Theorem (see, for example, [Kec95, Theorem 8.41]), it is enough to show that $\forall x \in X \forall^* p \in \mathbb{N}^\mathbb{N} \; x \notin X_p$. For this, it is enough to show that $\forall s \in \mathbb{N}^{<\mathbb{N}} \forall x \in X_s \exists i \in \mathbb{N} \; x \notin X_{s\frown(i)}$.

Towards this end, suppose that $s \in \mathbb{N}^{<\mathbb{N}}$ and $x \in X_s$.

Lemma 3.8. *There is a finite set $S \subseteq X_s$ such that $x \in S$, $G \restriction (X_s \setminus S)$ has degree at least two, and $G \restriction S$ has a perfect matching.*

Proof. We say that a set $Y \subseteq X$ is *G-connected* if the graph $G \restriction Y$ is connected. We will recursively construct increasing sequences $(\iota_k)_{k\in\mathbb{N}}$ of matchings of G and $(S_k)_{k\in\mathbb{N}}$ of finite G-connected subsets of X_s containing x such that the domain of ι_k is S_k, for all $k \in \mathbb{N}$. We begin by fixing $y \in X_s$ for which $x \; G \; y$, and setting $\iota_0 = (x \; y)$ and $S_0 = \{x, y\}$. Given ι_k and S_k, observe that for each connected component C of $G \restriction (X_s \setminus S_k)$, there is at most one point $z \in C$ such that $|C \cap G_z| = 1$. Let ι_{k+1} denote the minimal extension of ι_k to an involution which associates every such z with the unique element of $C \cap G_z$, and let S_{k+1} denote the domain of ι_{k+1}. This completes the recursive construction.

Set $\iota = \bigcup_{k\in\mathbb{N}} \iota_k$ and $S = \bigcup_{k\in\mathbb{N}} S_k$. Clearly x is in S, the restriction of G to $X_s \setminus S$ has degree at least two, and ι is a perfect matching of $G \restriction S$, so it only remains to show that S is finite. But if S is infinite, then we can recursively construct an injective $(G \restriction X_s)$-ray $(x_{2k})_{k\in\mathbb{N}}$ with the property that $S \cap [x_{2k}]_{G\restriction(X_s \setminus S_k)}$ is infinite and x_{2k+1} is the unique G-neighbor of x_{2k} in $X_s \setminus S_k$, for all $k \in \mathbb{N}$. But then $(x_k)_{k\in\mathbb{N}}$ has $(G \restriction X_s)$-degree two on even indices, a contradiction. ⊠

Lemma 3.9. *There is a finite set $T \subseteq X_s$, with $S \cup \partial_{G\restriction X_s}(X_s \setminus S) \subseteq T$, such that no injective $(G \restriction (X_s \setminus S))$-path passing through both a point in $\partial_{G\restriction X_s}(X_s \setminus S)$ and a point in $\partial_{G\restriction X_s}(T)$ has $(G \restriction (X_s \setminus S))$-degree two on even indices.*

Proof. It is enough to show that for all $z \in X_s \setminus S$, there exists $n \in \mathbb{N}$ such that there is no injective $(G \restriction (X_s \setminus S))$-path beginning at z, having

$(G \upharpoonright (X_s \setminus S))$-degree two on even indices, and along which there are n points. Towards this end, observe that if there are arbitrarily long injective $(G \upharpoonright (X_s \setminus S))$-paths of $(G \upharpoonright (X_s \setminus S))$-degree two on even indices beginning at some point x_0, then we can recursively choose $x_n \notin \{x_m \mid m < n\}$ such that there are arbitrarily long injective $(G \upharpoonright (X_s \setminus S))$-paths of $(G \upharpoonright (X_s \setminus S))$-degree two on even indices extending $(x_k)_{k \le n}$, in which case $(x_k)_{k \in \mathbb{N}}$ is an injective $(G \upharpoonright (X_s \setminus S))$-ray of $(G \upharpoonright (X_s \setminus S))$-degree two on even indices, a contradiction. ⊠

As $(S, T) \in \mathcal{X}_s$, it follows that $i = c(S, T)$ is as desired. ⊠

We close the paper by noting that the above result fails in the more general locally countable setting.

Theorem 3.10. *There is a Polish space X and an acyclic $\aleph_0$-regular Borel graph G on X which does not have a Borel perfect matching on a comeager Borel set.*

Proof. We will find Polish spaces X and Y and a Borel set $R \subseteq X \times Y$, whose horizontal and vertical sections are countably infinite, such that for no comeager Borel set $C \subseteq X$ is there a Borel injection $\phi \colon C \to Y$ whose graph is contained in R. For such R, let G_R denote the graph on the disjoint union of X and Y in which two points are related if one of them is in X, one of them is in Y, and the corresponding pair is in R. As long as we are able to simultaneously ensure that G_R is acyclic, it will have the desired properties.

Towards this end, we will recursively define $R_n \subseteq S_n \subseteq 2^n \times 2^n$. The sets R_n will provide increasingly precise approximations to the set R we seek, whereas the sets S_n will provide restrictions on the construction aimed at ruling out the existence of injections whose graphs are contained in R. We will obtain R_{n+1} and S_{n+1} from the sets

$$R'_{n+1} = \{(u \frown (i), v \frown (i)) \mid i < 2 \text{ and } u \ R_n \ v\}$$

and

$$S'_{n+1} = \{(u \frown (i), v \frown (j)) \mid i, j < 2 \text{ and } u \ R_n \ v\}$$

by either adding a pair to the former or subtracting pairs from the latter, depending on whether n is even or odd. Define a function $\mathrm{proj}_0 \colon 2^{<\mathbb{N}} \times 2^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by setting $\mathrm{proj}_0(u, v) = u$. In order to ensure that the construction can continue, we will proceed in such a fashion that for all $n \in \mathbb{N}$, the following conditions hold:

(1) $\forall u \in 2^n \exists v \in 2^n \ u \ S_n \ v$.
(2) $\forall v \in 2^n \exists u \in 2^n \setminus \mathrm{proj}_0(R_n) \ u \ S_n \ v$.

We will describe the exact fashion in which this is accomplished in terms of sequences $u_n, v_n \in 2^n$, for $n \in \mathbb{N}$. We can already define the sequences of the form u_{2n} and v_{2n+1}. In fact, these need only be chosen in such a fashion that the corresponding sets $\{u_{2n} \mid n \in \mathbb{N}\}$ and $\{v_{2n+1} \mid n \in \mathbb{N}\}$ are *dense* in $2^{<\mathbb{N}}$, in the sense that

$$\forall t \in 2^{<\mathbb{N}} \exists m, n \in \mathbb{N}\ t \sqsubseteq u_{2m}, v_{2n+1}.$$

The remaining sequences will be chosen during the construction.

We begin by setting $R_0 = \emptyset$ and $S_0 = \{(\emptyset, \emptyset)\}$. Suppose now that $n \in \mathbb{N}$ and we have already found R_{2n} and S_{2n} satisfying conditions (1) and (2). By the former, there exists $v_{2n} \in 2^{2n}$ such that $u_{2n}\ S_{2n}\ v_{2n}$. It then follows that the sets

$$R_{2n+1} = R'_{2n+1} \cup \{(u_{2n} \frown (0), v_{2n} \frown (1))\}$$

and $S_{2n+1} = S'_{2n+1}$ satisfy conditions (1) and (2) as well. By the latter, there exists $u_{2n+1} \in 2^{2n+1} \setminus \mathrm{proj}_0(R_{2n+1})$ such that $u_{2n+1}\ S_{2n+1}\ v_{2n+1}$. It then follows that the sets $R_{2n+2} = R'_{2n+2}$ and

$$S_{2n+2} = S'_{2n+2} \setminus \{(u_{2n+1} \frown (0), v) \mid v \neq v_{2n+1} \frown (1)\}$$

also satisfy conditions (1) and (2). This completes the recursive construction. Define $R \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$R = \{(u \frown x, v \frown x) \mid n \in \mathbb{N},\ u\ R_n\ v, \text{ and } x \in 2^{\mathbb{N}}\}.$$

A simple induction using the definition of R_n reveals that each of the graphs G_{R_n} is acyclic, from which it follows that so too is G_R.

Another simple induction utilizing the density of $\{u_{2n} \mid n \in \mathbb{N}\}$ and $\{v_{2n+1} \mid n \in \mathbb{N}\}$ along with the definition of R_n and S_n ensures that the sets $U_n = \{x \in 2^{\mathbb{N}} \mid |R_x| \geq n\}$ and $V_n = \{y \in 2^{\mathbb{N}} \mid |R^y| \geq n\}$ are dense and open. Fix homeomorphisms ϕ_n of $2^{\mathbb{N}}$ with the property that $\mathbb{E}_0 = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(\phi_n)$, and note that the set $Z = \bigcap_{m,n \in \mathbb{N}} \phi_m^{-1}(U_n \cap V_n)$ is a countable intersection of dense open sets, so the subspace topology on Z is Polish (see, for example, [Kec95, Theorem 3.11]), and a subset of Z is comeager if and only if it is comeager when viewed as a subset of $2^{\mathbb{N}}$. Set $X = Y = Z$, and observe that when viewed as a subset of $X \times Y$, every horizontal and vertical section of R is countably infinite.

Suppose, towards a contradiction, that there is a comeager Borel set $C \subseteq 2^{\mathbb{N}}$ for which there is a Borel injection $\phi \colon C \to 2^{\mathbb{N}}$ whose graph is contained in R. As the definitions of R_n and G ensure that

$$\mathrm{graph}(\phi) \subseteq \bigcup_{n \in \mathbb{N}} \{(u_{2n} \frown (0) \frown x, v_{2n} \frown (1) \frown x) \mid x \in 2^{\mathbb{N}}\},$$

there exists $n \in \mathbb{N}$ for which the set of $x \in 2^{\mathbb{N}}$ with the property that $\phi(u_{2n} \frown (0) \frown x) = v_{2n} \frown (1) \frown x$ is non-meager. By localization (see, for example, [Kec95, Proposition 8.26]), there exists $r \in 2^{<\mathbb{N}}$ such that $\phi(u_{2n} \frown (0) \frown r \frown x) = v_{2n} \frown (1) \frown r \frown x$, for comeagerly many $x \in 2^{\mathbb{N}}$. Fix $m \geq n$ with $v_{2n} \frown (1) \frown r \sqsubseteq v_{2m+1}$, noting that u_{2m+1} is incompatible with $u_{2n} \frown (0)$ and ϕ necessarily sends sequences beginning with $u_{2m+1} \frown (0)$ to sequences beginning with $v_{2m+1} \frown (1)$. Again appealing to the above restriction on the graph of ϕ imposed by the definitions of R_n and G, there exists $\ell > m$ for which $u_{2m+1} \frown (0) \sqsubseteq u_{2\ell}$ and the set of $x \in 2^{\mathbb{N}}$ with the property that $\phi(u_{2\ell} \frown (0) \frown x) = v_{2\ell} \frown (1) \frown x$ is non-meager. By one more appeal to localization, there exists $s \in 2^{<\mathbb{N}}$ with the property that $\phi(u_{2\ell} \frown (0) \frown s \frown x) = v_{2\ell} \frown (1) \frown s \frown x$, for comeagerly many $x \in 2^{\mathbb{N}}$. As $u_{2m+1} \frown (0) \sqsubseteq u_{2\ell}$, it follows that $v_{2m+1} \frown (1) \sqsubseteq v_{2\ell}$. And since $v_{2n} \frown (1) \frown r \sqsubseteq v_{2m+1}$, it follows that $v_{2n} \frown (1) \frown r \sqsubseteq v_{2\ell}$. Fix $t \in 2^{<\mathbb{N}}$ such that $v_{2\ell} \frown (1) \frown s = v_{2n} \frown (1) \frown r \frown t$, and observe that

$$\begin{aligned}\phi(u_{2\ell} \frown (0) \frown s \frown x) &= v_{2\ell} \frown (1) \frown s \frown x \\ &= v_{2n} \frown (1) \frown r \frown t \frown x \\ &= \phi(u_{2n} \frown (0) \frown r \frown t \frown x),\end{aligned}$$

for comeagerly many $x \in 2^{\mathbb{N}}$. As u_{2m+1} is incompatible with $u_{2n} \frown (0)$ and extended by $u_{2\ell}$, it follows that $u_{2\ell}$ is incompatible with $u_{2n} \frown (0)$, thus $u_{2\ell} \frown (0) \frown s \frown x$ and $u_{2n} \frown (0) \frown r \frown t \frown x$ are distinct sequences with the same image under ϕ, a contradiction. ⊠

Remark 3.11. The same idea can, in fact, be used to rule out the existence of a comeager Borel set $C \subseteq 2^{\mathbb{N}}$ for which there is a finite-to-one Borel function $\phi \colon C \to Y$ whose graph is contained in R.

Remark 3.12. It is tempting to try to strengthen the conclusion of Theorem 3.10 to show that any Borel matching of G has meager domain, but this is impossible. Indeed, any locally countable Borel graph has a countable Borel edge coloring [KST99, Proposition 4.10], and at least one of these colors must use a non-meager set of vertices.

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