AN EMBEDDING THEOREM OF \mathbb{E}_0 WITH MODEL THEORETIC APPLICATIONS

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ABSTRACT. We provide a new criterion for embedding \mathbb{E}_0 , and apply it to equivalence relations in model theory. This generalize the results of the authors and Pierre Simon on the Borel cardinality of Lascar strong types equality, and Newelski's results about pseudo F_{σ} groups.

1. INTRODUCTION

Given two topological spaces X and X' and two equivalence relations E and E' respectively on X and X', we say that E is Borel reducible to E' if there is a Borel map f from X to X' such that $x \ E \ y \iff f(x) \ E' \ f(y)$ for all $x, y \in X$. The quasi-order of Borel reducibility of Borel equivalence relations on Polish spaces is a well-studied object in descriptive set theory, and enjoys a number of remarkable properties. One of them is given by the Harrington-Kechris-Louveau dichotomy, which asserts that a Borel equivalence relation is either smooth (Borel reducible to equality on 2^{ω}) or at least as complicated as \mathbb{E}_0 (eventual equality on 2^{ω}). In other words, \mathbb{E}_0 is the first non-smooth Borel equivalence relation.

In Section 2, we provide a new criterion for being non-smooth. We also translate this criterion to another context, that of strong Choquet spaces.

In the majority of Section 3, we apply this criterion to bounded invariant equivalence relation in model theory.

Suppose T is a complete first order theory and \mathfrak{C} a κ -saturated model for some large κ . If E is an equivalence relation on \mathfrak{C}^{α} which is a countable union of \emptyset -type definable sets U_n (i.e., U_n is definable by intersection of parameter free formulas), we say that it is bounded when the number of classes is smaller than κ . We call E a bounded invariant pseudo F_{σ} equivalence relation. Such relations appear naturally in model theory, and include the finest bounded invariant equivalence relation: equality of Lascar strong types $= \equiv_L^{\alpha}$. It turns out that if the T and α are countable, one can interpret E as an (honest) F_{σ} equivalence relation on a compact Polish space in a very natural way, which equips E with a well defined Borel cardinality. This was done for Lascar strong types in [KPS12], where many examples are computed, but in fact works for any E. It is explained in details in Subsections 3.2.2 and 3.2.3.

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If E is an invariant bounded pseudo F_{σ} equivalence relation, we can assume by compactness that there are \emptyset -type definable sets U_n which are reflexive, symmetric and $U_n \circ U_n \subseteq U_{n+1}$ with $E = \bigcup_{n < \omega} U_n$. Such a sequence $\langle U_n | n < \omega \rangle$ is called a *normal form* of E.

In [New03, Corollary 1.12], Newelski proved that if E is an invariant pseudo F_{σ} equivalence relation on \mathfrak{C}^{α} with normal form $\langle U_n | n < \omega \rangle$, and X is a type definable set, all its elements have the same type over \emptyset , then either $E \upharpoonright X = U_n$ for some n or $|X/E| \ge 2^{\aleph_0}$.

(*) Equivalently, if there is some $x \in X$ such that $E \upharpoonright [x]_E$ is not already $U_n \upharpoonright [x]_E$ for some n then $|X/E| \ge 2^{\aleph_0}$.

He continued to prove [New03, Theorem 3.1] that if H is an \emptyset -type definable group and $G \leq H$ is generated by countably many sets V_n , each \emptyset -type definable, then G is type definable iff Gis generated by finitely many V_n 's in finitely many steps and if G is not type definable then $[H:G] \geq 2^{\aleph_0}$. In that case, if moreover T is small (has only countably many types over \emptyset) and H consists of finite tuples, then [H:G] is unbounded. Let X = H and $E = E_G^H$ be the orbit equivalence relation of the action of G on H (so it is an invariant pseudo F_σ equivalence relation).

(**) In this language this is equivalent to: if for some $x \in X$, $E \upharpoonright [x]_E$ is not already generated by finitely many of the V_n 's in finitely many steps, then $|X/E| \ge 2^{\aleph_0}$.

An important example of such a pair (G, H) is (G, G_{\emptyset}^{000}) where G is \emptyset -type definable and G_{\emptyset}^{000} is the minimal \emptyset -invariant bounded index subgroup. See [Gis11] for more.

In [KMS13] the authors dealt with the case where X was a KP-strong type and $E \equiv_L^{\alpha}$ The main result there is that if E is not trivial on X then it is non-smooth. This went through a stronger theorem [KMS13, Main Theorem A] that stated that:

(***) If Y is a pseudo G_{δ} , \equiv_{L}^{α} -invariant subset of \mathfrak{C}^{α} and for some $x \in Y$, $[x]_{\equiv_{L}^{\alpha}}$ has unbounded Lascar diameter, then $\equiv_{L}^{\alpha} \upharpoonright Y$ is non-smooth. "Unbounded Lascar diameter" means exactly that it is not the case that $\equiv_{L}^{\alpha} \upharpoonright [x]_{\equiv_{L}^{\alpha}} = U_{n} \upharpoonright [x]_{\equiv_{L}^{\alpha}}$ for some n, where $U_{n}(a, b)$ is the type saying that the Lascar distance between a and b is at most n. (This is a normal form for \equiv_{L}^{α} .)

Here we try to generalize (*), (**) and (***) in a uniform way using the results from Section 2. So the idea is to prove, in each case (when everything is countable), that if Y is a pseudo G_{δ} , *E*-invariant and for some $x \in Y$, $E \upharpoonright [x]_E$ is not already $U_n \upharpoonright [x]_E$ for some n, then $E \upharpoonright X$ is not smooth.

While we do not successfully generalize (*), we do prove it if there is a subgroup of Aut (\mathfrak{C}) which acts nicely on $[x]_E$, for instance when it is transitive on this class and preserves all classes. This is done in Subsection 3.2.5, and includes also (***) (the subgroup in that case is Aut $f_L(\mathfrak{C})$). (**) is successfully generalized and moreover stated for group actions (with an extra technical assumption called "shiftiness" which holds in the case where the action is free).

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2. A sufficient condition for embedding \mathbb{E}_0

2.1. Preliminaries.

Definition 2.1. Suppose X and Y are topological spaces, and E and F are Borel equivalence relations on X and Y. We say that a function $f : X \to Y$ is a *reduction* of E to F if for all $x_0, x_1 \in X, (x_0, x_1) \in E$ iff $(f(x_0), f(x_1)) \in F$.

- (1) We say that E is Borel reducible to F, denoted by $E \leq_B F$, when there is a Borel reduction $f: X \to Y$ of E to F.
- (2) We write $E \sqsubseteq_c F$ when there is a continuous injective reduction $f: X \to Y$ of E to F.
- (3) We say that E and F are Borel bi-reducible, denoted by $E \sim_B F$, when $E \leq_B F$ and $F \leq_B E$.
- (4) We write $E <_B F$ to mean that $E \leq_B F$ but $E \not\sim_B F$.

Example 2.2. For a Polish space X, the relations $\Delta(X)$ denotes equality on X. Then $\Delta(1) <_B \Delta(2) <_B \ldots <_B \Delta(\omega) <_B \Delta(2^{\omega})$.

Definition 2.3. We say that E is smooth iff $E \leq_B \Delta(2^{\omega})$.

Fact 2.4. [Sil80] (Silver dichotomy) For all Borel equivalence relations E on Polish spaces, $E \leq_B \Delta(\omega)$ or $\Delta(2^{\omega}) \sqsubseteq_c E$. It follows that $\Delta(2^{\omega})$ is the successor of $\Delta(\omega)$.

Fact 2.5. Closed equivalence relations are smooth.

Example 2.6. Let \mathbb{E}_0 be the following equivalence relation on the Cantor space 2^{ω} : $(\eta, \nu) \in \mathbb{E}_0$ iff there exists some $n < \omega$ such that for all m > n, $\eta(m) = \nu(m)$.

Fact 2.7. The relation \mathbb{E}_0 is non-smooth.

In addition, we have the following dichotomy:

Fact 2.8. [HKL90] (Harrington-Kechris-Louveau dichotomy) For every Borel equivalence relation E on a Polish space either $E \leq_B \Delta(2^{\omega})$ (i.e., E is smooth) or $\mathbb{E}_0 \sqsubseteq_c E$. It follows that \mathbb{E}_0 is the successor of $\Delta(2^{\omega})$.

2.2. The ideal embedding theorem. Suppose that X is a topological space. Associated with each family \mathcal{U} of open subsets of X is the corresponding family $\mathcal{I}_{\mathcal{U}}$ of subsets of X given by $F \in \mathcal{I}_{\mathcal{U}} \iff \forall U \in \mathcal{U} \exists$ open $V \supseteq F \ V \cup U \in \mathcal{U}$.

Equivalently,

Remark 2.9. $F \in \mathcal{I}_{\mathcal{U}} \iff \forall U \in \mathcal{U} \exists V \in \mathcal{U} \ F \cup U \subseteq V.$

Proposition 2.10. Suppose that X is a topological space and \mathcal{U} is a family of open subsets of X. Then $\mathcal{I}_{\mathcal{U}}$ is an ideal. *Proof.* To see that $\mathcal{I}_{\mathcal{U}}$ is downward closed, note that if $F \in \mathcal{I}_{\mathcal{U}}$ and $F' \subseteq F$, then for each $U \in \mathcal{U}$, there exists an open set $V \supseteq F \supseteq F'$ with the property that $V \cup U \in \mathcal{U}$, thus $F' \in \mathcal{I}_{\mathcal{U}}$.

To see that $\mathcal{I}_{\mathcal{U}}$ is closed under finite unions, note that if $F, F' \in \mathcal{I}_{\mathcal{U}}$, then for each $U \in \mathcal{U}$, there exists an open set $V \supseteq F$ with $V \cup U \in \mathcal{U}$, so there exists an open set $V' \supseteq F'$ with $(V \cup V') \cup U \in \mathcal{U}$, thus $F \cup F' \in \mathcal{I}_{\mathcal{U}}$.

Proposition 2.11. Suppose that X is a topological space, Γ is a group of homeomorphisms of X, $Y \subseteq X$ is Γ -invariant, and \mathcal{U} is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ with $Y \subseteq \Delta \cdot U$. Then for all $\mathcal{I}_{\mathcal{U}}$ -positive sets $F \subseteq X$ and all open sets $W \supseteq F$, there is a finite set $\Delta \subseteq \Gamma$ such that whenever I is a finite set, $\langle F_i | i \in I \rangle$ is a finite sequence of subsets of X whose union contains F, and $\langle \lambda_i | i \in I \rangle$ is a sequence of elements of Γ , there exists $\delta \in \Delta$ and $i \in I$ for which $\overline{\delta \cdot W} \cap \lambda_i \cdot F_i$ is $\mathcal{I}_{\mathcal{U}}$ -positive.

Proof. We will use Remark 2.9. Fix $U \in \mathcal{U}$ such that for no $V \in \mathcal{U}$ is $F \cup U \subseteq V$. Then there is a finite set $\Delta \subseteq \Gamma$ with $Y \subseteq \Delta \cdot (U \cup W)$. In light of Proposition 2.10, it is sufficient to show that there is no finite set I, sequence $\langle F_i | i \in I \rangle$ of subsets of X whose union contains F, and sequence $\langle \lambda_i | i \in I \rangle$ such that $\overline{\Delta \cdot W} \cap \bigcup_{i \in I} \lambda_i \cdot F_i \in \mathcal{I}_{\mathcal{U}}$.

Suppose, towards a contradiction, that there is such a triple. Then $Y \setminus \overline{\Delta \cdot W} \subseteq \Delta \cdot U$, so $\sim \overline{\Delta \cdot W} \cup \bigcup_{i \in I} \lambda_i \cdot U \in \mathcal{U}$. Fix $V \in \mathcal{U}$ with $\bigcup_{i \in I} \lambda_i \cdot (U \cup F_i) \subseteq (\overline{\Delta \cdot W} \cap \bigcup_{i \in I} \lambda_i \cdot F_i) \cup (\sim \overline{\Delta \cdot W} \cup \bigcup_{i \in I} \lambda_i \cdot U) \subseteq V$. Then $\bigcup_{i \in I} \lambda_i^{-1} \cdot V \in \mathcal{U}$ and $F \cup U \subseteq \bigcup_{i \in I} \lambda_i^{-1} \cdot V$, a contradiction.

Theorem 2.12. Suppose that X is a complete metric space, Γ is a group of homeomorphisms of X, $Y \subseteq X$ is Γ -invariant, \mathcal{U} is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ such that $Y \subseteq \Delta \cdot U$, $\langle R_n | n \in \mathbb{N} \rangle$ is an increasing sequence of reflexive symmetric closed subsets of $X \times X$, and there is a compact $\mathcal{I}_{\mathcal{U}}$ -positive set $K \subseteq X$ with the following properties:

- (a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_n^{(4)} \gamma \cdot x.$
- (b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in \Gamma \cdot K \ x \ R_n \ \gamma \cdot x.$

Then for some $x \in K$ there is a continuous injective homomorphism $\phi: 2^{\omega} \to \overline{\Gamma \cdot x}$ from $(\mathbb{E}_0, \sim \mathbb{E}_0)$ into $(E_{\Gamma}^X, \sim \bigcup_{n \in \mathbb{N}} R_n)$.

Proof. Let \mathcal{V} denote the family of open sets $V \subseteq X$ containing compact $\mathcal{I}_{\mathcal{U}}$ -positive subsets of K. We recursively construct $V_n \in \mathcal{V}$ and $\gamma_n \in \Gamma$, from which we define $\gamma_s = \prod_{i < n} \gamma_i^{s(i)}$ for $s \in 2^{<\omega}$, so as to ensure that at stage n of the construction, the following conditions are satisfied:

- (1) $\forall m < n \ \overline{V_{m+1}} \cup \gamma_m \cdot \overline{V_{m+1}} \subseteq V_m.$
- (2) $\forall m < n \forall s \in 2^{m+1} \operatorname{diam} (\gamma_s \cdot V_{m+1}) \leq 1/m.$
- (3) $\forall m < n \forall s, t \in 2^m ((\gamma_s \cdot V_{m+1}) \times (\gamma_t \gamma_m \cdot V_{m+1})) \cap R_m = \emptyset.$

We begin by setting $V_0 = X$.

Suppose now that $n \in \mathbb{N}$ and we have found V_n and $\langle \gamma_i | i < n \rangle$. Fix an $\mathcal{I}_{\mathcal{U}}$ -positive compact set $L \subseteq K$ contained in V_n , as well as an open set $W \supseteq L$ for which $\overline{W} \subseteq V_n$. By Proposition 2.11, there is a finite set $\Delta \subseteq \Gamma$ such that whenever I is a finite set, $\langle L_i | i \in I \rangle$ is a sequence of compact sets whose union is L, and $\langle \lambda_i | i \in I \rangle$ is a sequence of elements of Γ , there are $\delta \in \Delta$ and $i \in I$ for which $\overline{\delta \cdot W} \cap \lambda_i \cdot L_i$ is $\mathcal{I}_{\mathcal{U}}$ -positive. Condition (b) yields $m \ge n$ such that $\forall x \in \Gamma \cdot K \forall \gamma \in$ $\{\gamma_s | s \in 2^n\} \cup \Delta^{-1} x R_m \gamma \cdot x$. In particular, it follows that:

(*) If $x \in K$, $\lambda \in \Gamma$ and $\neg x \ R_m^{(4)} \ \lambda \cdot x$, then for no $\delta \in \Delta$ and $s, t \in 2^n$ is it the case that $\gamma_s \cdot x \ R_m \ \gamma_t \delta^{-1} \lambda \cdot x$.

Thus condition (a) yields a finite set I, a sequence $\langle L_i | i \in I \rangle$ of compact subsets of X whose union is L, and a sequence $\langle \lambda_i | i \in I \rangle$ of elements of Γ with

$$\forall \delta \in \Delta \forall i \in I \forall s, t \in 2^n \ (\gamma_s \cdot L_i \times \gamma_t \delta^{-1} \lambda_i \cdot L_i) \cap R_m = \emptyset.$$

Fix $\delta \in \Delta$ and $i \in I$ such that $\overline{\lambda_i^{-1} \delta \cdot W} \cap L_i$ is $\mathcal{I}_{\mathcal{U}}$ -positive, and define $\gamma_n = \delta^{-1} \lambda_i$. Then $\forall s, t \in 2^n ((\gamma_s \cdot L_i) \times (\gamma_t \gamma_n \cdot L_i)) \cap R_m = \emptyset$. Proposition 2.10 ensures that by replacing L_i with a compact $\mathcal{I}_{\mathcal{U}}$ -positive subset of $\overline{\lambda_i^{-1} \delta \cdot W} \cap L_i$, we can assume that $\forall s \in 2^{n+1} \operatorname{diam} (\gamma_s \cdot L_i) < 1/n$. It follows that there is an open set $V_{n+1} \subseteq X$ containing L_i such that $\overline{V_{n+1}} \cup (\gamma_n \cdot \overline{V_{n+1}}) \subseteq V_n$, $\forall s \in 2^{n+1} \operatorname{diam} (\gamma_s \cdot V_{n+1}) \leq 1/n$, and $\forall s, t \in 2^n ((\gamma_s \cdot V_{n+1}) \times (\gamma_t \gamma_n \cdot V_{n+1})) \cap R_n = \emptyset$. This completes the recursive construction.

Conditions (1) and (2) ensure that we obtain a continuous function $\phi: 2^{\omega} \to X$ by insisting that $\{\phi(c)\} = \bigcap_{n \in \mathbb{N}} \gamma_{c \upharpoonright n} \cdot V_n$ for all $c \in 2^{\omega}$. To see that ϕ is a homomorphism from \mathbb{E}_0 to E_{Γ}^X , suppose that $c \in 2^{\omega}$, $k \in \mathbb{N}$, and $s \in 2^k$, and observe that

$$\left\{\gamma_s \cdot \phi((0)^k \frown c)\right\} = \bigcap_{n \in \mathbb{N}} \gamma_s \gamma_{((0)^k \frown c) \upharpoonright n} \cdot V_n = \left\{\phi(s \frown c)\right\}.$$

To see that ϕ is an injective homomorphism from $\sim \mathbb{E}_0$ to $\sim \bigcup_{n \in \mathbb{N}} R_n$, suppose that $c, d \in 2^{\omega}$, $n \in \mathbb{N}$, c(n) = 0, and d(n) = 1, and observe that $\phi(c) \in \gamma_{c \restriction n} \cdot V_{n+1}$ and $\phi(d) \in \gamma_{d \restriction n} \gamma_n \cdot V_{n+1}$, in which case condition (3) ensures that $\neg \phi(c) R_n \phi(d)$. Finally, set $x = \phi((0)^{\infty})$ and note that $\phi[2^{\omega}] \subseteq \overline{\Gamma \cdot x}$ and $x \in K$.

We give a slight variant of Theorem 2.12, adding an extra assumption.

Theorem 2.13. Suppose that $X, \Gamma, Y, \mathcal{U}, \langle R_n | n \in \mathbb{N} \rangle$ are as in Theorem 2.12. Suppose that there is a compact $\mathcal{I}_{\mathcal{U}}$ -positive set $K \subseteq X$ with the following properties:

- (a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_n^{(2)} \gamma \cdot x.$
- (b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in K \ x \ R_n \ \gamma \cdot x$. Note that this condition is weaker than (b) in Theorem 2.12.
- (c) $\forall \gamma \in \Gamma \forall x, y \in \Gamma \cdot K \forall n \in \mathbb{N} \ x \ R_n \ y \Rightarrow \gamma \cdot x \ R_n \ \gamma \cdot y.$

Then for some $x \in K$ there is a continuous injective homomorphism $\phi: 2^{\omega} \to \overline{\Gamma \cdot x}$ from $(\mathbb{E}_0, \sim \mathbb{E}_0)$ into $(E_{\Gamma}^X, \sim \bigcup_{n \in \mathbb{N}} R_n)$.

Proof. The proof is parallel to the proof of Theorem 2.12, reading the same up to (*), but we choose m so that $\forall x \in K \forall \gamma_1, \gamma_2 \in \{\gamma_s \mid s \in 2^n\} \forall \delta \in \Delta \ x \ R_m \ \delta \gamma_1^{-1} \gamma_2 \cdot x$. By (c), we get:

(**) If $x \in K$, $\lambda \in \Gamma$ and $\neg x R_m^{(2)} \lambda \cdot x$, then for no $\delta \in \Delta$ and $s, t \in 2^n$ is it the case that $\gamma_s \cdot x R_m \gamma_t \delta^{-1} \lambda \cdot x$.

The rest of the proof is exactly the same.

2.3. Choquet spaces. The proof of Theorem 2.12 easily goes through in the context of strong Choquet spaces.

Definition 2.14. The *Choquet game* on a topological space X is a two player game in ω -many rounds. In round n, player A chooses a nonempty open set $U_n \subseteq V_{n-1}$ (where $V_{-1} = X$), and player B responds by choosing a nonempty open subset $V_n \subseteq U_n$. Player B wins if the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty.

The strong Choquet game is similar: in round n player A chooses an open set $U_n \subseteq V_{n-1}$ and $x_n \in U_n$, and player B responds by choosing an open set $V_n \subseteq U_n$ containing x_n . Again, player B wins when the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty.

A topological space X is a *(strong) Choquet space* if player B has a winning strategy in every (strong) Choquet game.

It is easy to see that:

Example 2.15. Every Polish space is strong Choquet.

But for our purposes, we shall need the following example:

Example 2.16. If X is compact (not necessarily Hausdorff) and has a basis consisting of clopen sets then it is strong Choquet.

Proof. In round n, player B will choose a clopen set $x_n \in V_n \subseteq U_n$. By compactness, the intersection $\bigcap \{V_n \mid n < \omega\}$ is not empty. \Box

Fact 2.17. (see e.g., [KMS13]) If X is strong Choquet and $\emptyset \neq U \subseteq X$ is G_{δ} , then U is also strong Choquet.

Theorem 2.18. Suppose that X is a regular strong Choquet space, Γ is a group of homeomorphisms of X, $Y \subseteq X$ is Γ -invariant, \mathcal{U} is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ such that $Y \subseteq \Delta \cdot U$, $\langle R_n | n \in \mathbb{N} \rangle$ is an increasing sequence of reflexive symmetric closed subsets of $X \times X$, and there is a compact $\mathcal{I}_{\mathcal{U}}$ -positive set $K \subseteq X$ with the following properties:

- (a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_n^{(4)} \gamma \cdot x.$
- (b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in \Gamma \cdot K \ x \ R_n \ \gamma \cdot x.$

Then there is a map $\phi: 2^{\omega} \to \mathcal{P}(X)$ such that for every $y, z \in 2^{\omega}$:

- $\phi(y)$ is a nonempty closed G_{δ} subset of X.
- If $z \mathbb{E}_0 y$ then there is some $\gamma \in \Gamma$ such that $\gamma \cdot \phi(z) = \phi(y)$.
- If $\sim z \mathbb{E}_0 y$ then $(\phi(y) \times \phi(z)) \subseteq \sim \bigcup_{n \in \mathbb{N}} R_n$.

In particular, there is homomorphism $\phi: 2^{\omega} \to X$ from $(\mathbb{E}_0, \sim \mathbb{E}_0)$ into $(E_{\Gamma}^X, \sim \bigcup_{n \in \mathbb{N}} R_n)$.

Moreover, if X is compact, then we can choose ϕ so that its range is $\overline{\Gamma \cdot x}$ for some $x \in K$.

Proof. Fix a winning strategy for Player B in the strong Choquet game. The main point is that in the construction done in the proof of Theorem 2.12, instead of decreasing the diameter of the open sets, we choose them according to the strategy. So in addition to choosing V_n and γ_n , we also choose points $x_n \in X$ and open sets $U_n \in \mathcal{V}$ (the family of open sets containing compact $\mathcal{I}_{\mathcal{U}}$ -positive subsets of K) such that $x_n \in U_n \subseteq V_n$, and the new construction will satisfy:

- (1) $\forall m < n \ \overline{U_m} \cup \gamma_m \cdot \overline{U_m} \subseteq V_m$.
- (2) $\forall m < n \forall s, t \in 2^m ((\gamma_s \cdot U_m) \times (\gamma_t \gamma_m \cdot U_m)) \cap R_m = \emptyset.$
- (3) $\forall m < n \text{ and } \forall s \in 2^{m+1}, \gamma_s \cdot V_{m+1}$ is contained in an open set which is played according to Player B's strategy in the Strong Choquet game in which Player A plays $\langle (\gamma_{s \restriction i+1} \cdot U_i, \gamma_{s \restriction i+1} \cdot x_i) | i \leq m \rangle$ and Player B plays according to his strategy.

For the construction, we follow the proof of Theorem 2.12, and note that:

Claim. Suppose L is an $I_{\mathcal{U}}$ -positive compact set contained in some open set U, and suppose Δ is a finite subset of Γ . Furthermore, suppose that for any $\gamma \in \Delta$, $\gamma \cdot U$ is contained in an open set which is chosen by Player B in some finite strong Choquet play according to his strategy. Then, there is some $x \in L$ such that if Player A plays $(\gamma \cdot U, \gamma \cdot x)$ then there is a set $V \in \mathcal{V}$ contained in U such that $\gamma \cdot V$ is contained in Player B's response for all $\gamma \in \Delta$.

Proof. Indeed, for each point $x \in L$, let Player A play $(\gamma \cdot U, \gamma \cdot x)$, and let $\gamma \cdot U_{\gamma,x}$ be Player B's response. Let $U_x = \bigcap_{\gamma \in \Delta} U_{x,\gamma}$, and let U'_x be such that $x \in U'_x$ and $\overline{U'_x} \subseteq U_x$. By compactness and by Proposition 2.10, for some $x \in L$, $\overline{U'_x} \cap L$ is $I_{\mathcal{U}}$ -positive. Let $V = U_x$.

Now we let U_n be the set denoted V_{n+1} in the proof of Theorem 2.12 (without the condition on the diameter), and proceed using the claim. Finally, we let $\phi(c) = \bigcap_{n \in \mathbb{N}} \gamma_{c \upharpoonright n} \cdot V_n$.

For the moreover part, choose any $x \in K \cap \phi((0)^{\omega})$, and note that by compactness $\overline{\Gamma} \cdot x \cap \phi(c) \neq \emptyset$.

We also have an analog to Theorem 2.13, which we state briefly.

Theorem 2.19. Suppose that $X, \Gamma, Y, \mathcal{U}, \langle R_n | n \in \mathbb{N} \rangle$ are as in Theorem 2.18. Suppose that there is a compact $\mathcal{I}_{\mathcal{U}}$ -positive set $K \subseteq X$ satisfying the assumptions of Theorem 2.13.

Then the conclusion of Theorem 2.18 hold.

Problem 2.20. All the applications we found use a weak version of Theorem 2.12, i.e., we apply it with (a) replaced by: $\forall n \in \mathbb{N} \exists \gamma \in \Gamma \forall x \in K \neg x R_n^{(4)} \gamma \cdot x$. Is there an interesting application that uses the full strength of the theorem?

3. Applications

3.1. Application to compact group actions. Most of our applications will be model theoretic, but we start with a simple topological one.

Corollary 3.1. Suppose that G is a compact topological group and that $\cdot : G \times X \to X$ is a continuous action on a complete metric space X. Let H a subgroup of G, and suppose $H = \bigcup_{n < \omega} V_n$ where V_n are closed subsets of G, $e \in V_n^{-1} = V_n$, $V_n^2 \subseteq V_{n+1}$. Then, if there is some $x \in X$ such that $H \cdot x \neq V_n \cdot x$ for all $n < \omega$, then $\mathbb{E}_0 \sqsubseteq_c E_H^X$. If not and X is Polish then E_H^X is smooth.

Proof. First assume that there is such an $x \in X$. Let $Y = H \cdot x$, $R_n = \{(x, y) \in X^2 \mid \exists h \in V_n (y = h \cdot x)\}$, $\Gamma = H$ and $K = \{x\}$. All the conditions of Theorem 2.12 but the condition that K is $I_{\mathcal{U}}$ -positive hold trivially (note that R_n is closed by the compactness of V_n). To show that K is $I_{\mathcal{U}}$ -positive it is enough to see that for any open $x \in V \subseteq X$, there is some finite $\Delta \subseteq H$ such that $\Delta \cdot V \supseteq Y$. Suppose not. Recursively choose $h_n \in H$ for $n < \omega$ such that $h_n \cdot x \notin \bigcup_{i < n} h_i \cdot V$.

Let $\kappa = |G|^+$, and let $L = G^{\kappa}$ equipped with the product topology (so it is compact). For a finite $s \subseteq \kappa$, let $F_s = \left\{ \eta \in L \ | \ \forall i \in s \left(\eta \left(i \right) \cdot x \notin \bigcup_{j < i, j \in s} \eta \left(j \right) \cdot V \right) \right\}$. By the construction above, this is a closed nonempty set. By compactness, there is some $\eta \in \bigcap_{s \subseteq \kappa, |s| < \omega} F_s$. In particular $\eta : \kappa \to G$ is injective — contradiction.

Now assume that X is Polish and that there is no such x but E_H^X is not smooth. By assumption, for all $x \in X$, $H \cdot x = V_n \cdot x$ for some $n < \omega$ and as G is compact it follows that all classes are compact, so also G_{δ} . But a Borel equivalence relation E with G_{δ} classes on a Polish space X must be smooth. Otherwise Fact 2.8 gives us a continuous embedding of \mathbb{E}_0 into E, and it follows that every \mathbb{E}_0 class is G_{δ} . But \mathbb{E}_0 classes are also dense — contradiction.

Corollary 3.2. If G is a compact complete metric group, and H is an F_{σ} subgroup, then either H is closed (in which case, if G is Polish, E_H^G is smooth), or $\mathbb{E}_0 \sqsubseteq_c E_H^G$.

3.2. Applications to model theory. In applying Theorem 2.12 or any of its variations, we need to find the space X, the set Y, the group Γ , the closed sets R_n and the compact $I_{\mathcal{U}}$ -positive set K. In all our applications, X will be some subspace of $S_{\alpha}(M)$ for model M, invariant under E,

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Y will be the projection of some E-class C, R_n will be the projections of U_n (from E's normal form), Γ will be some group of homeomorphisms of X, which is either induced by automorphisms of the model M or by a type definable model theoretic group and K will be the projection of some type of the form $U_n(x, a)$. The main point is to show that K is $I_{\mathcal{U}}$ -positive, which we will call here "proper".

3.2.1. Preliminaries. We briefly introduce our notation, which is fully explained in [KMS13].

- T is a complete (perhaps many sorted) first order theory.
- α is some ordinal.
- $S_{\alpha}(A)$ is the Stone space of complete α -types over A, which comes equipped with a compact Hausdorff topology, and $L_{\alpha}(A)$ is the set of formulas in the first α variables.
- \mathfrak{C} is a monster model of T a κ -saturated, κ -homogeneous model where κ is a big cardinal.
- All parameter sets and models considered will be *small* (i.e., of cardinality less than κ) subsets and elementary substructures of \mathfrak{C} .
- \equiv is equality of types, \equiv_L^{α} is equality of Lascar strong types of α -tuples (if A is a small set, then \equiv_A denotes types equality over A, etc.).
- Aut (C/A) is the group of automorphisms of C that fix A pointwise, and an A-invariant subset of C^α is one invariant under the action of this group.
- A subset $X \subseteq \mathfrak{C}^{\alpha}$ is *pseudo closed* if X is type definable over some small set. A *pseudo* open set is a complement of a pseudo closed set. *Pseudo* G_{δ} sets and *pseudo* F_{σ} sets are defined in the obvious way.
- If $Y \subseteq \mathfrak{C}^{\alpha}$ is some set, and M some model then $Y_M = \{p \in S_{\alpha}(M) \mid \exists a \in Y (p = \operatorname{tp}(a/M))\}$. This is also denoted by $S_M(Y)$.

We also recall the notion of an indiscernible sequence:

Definition 3.3. Let A be a small set. Let (I, <) be some linearly ordered set. A sequence $\bar{a} = \langle a_i | i \in I \rangle \in (\mathfrak{C}^{\alpha})^I$ is called A-indiscernible (or indiscernible over A) if for all $n < \omega$, every increasing n-tuple from \bar{a} realizes the same type over A. When A is omitted, it is understood that $A = \emptyset$.

Also recall:

Fact 3.4.

(1) [TZ12, Lemma 5.1.3] Let $(I, <_I)$, $(J, <_J)$ be small linearly ordered sets, and let A be some small set. Suppose $\overline{b} = \langle b_j | j \in J \rangle$ is some sequence of elements from \mathfrak{C}^{α} . Then there exists an indiscernible sequence $\overline{a} = \langle a_i | i \in I \rangle \in (\mathfrak{C}^{\alpha})^I$ such that:

- For any $n < \omega$ and $\varphi \in L_{\alpha \cdot n}$, if $\mathfrak{C} \models \varphi(b_{j_0}, \ldots, b_{j_{n-1}})$ for every $j_0 <_J \ldots <_J j_{n-1}$ from J then $\mathfrak{C} \models \varphi(a_{i_0}, \ldots, a_{i_{n-1}})$ for every $i_0 <_I \ldots <_I i_{n-1}$ from I.
- (2) [Ker07, proof of Proposition 3.1.4] If M is a small model and $a \equiv_M b$, then there is an indiscernible sequence $\bar{c} = \langle c_i | i < \omega \rangle$ such that both $a \frown \bar{c}$ and $b \frown \bar{c}$ are indiscernible.
- 3.2.2. Equivalence relations on \mathfrak{C}^{α} .

Definition 3.5. An equivalence relation E on a set X is called *bounded* if $|X/E| < \kappa$.

See [KMS13, Remark 1.12] for a discussion of bounded invariant equivalence relations.

Suppose that A is some small set, $X \subseteq \mathfrak{C}^{\alpha}$ is type definable over A, and that E is some \emptyset -invariant relation on $\mathfrak{C}^{\alpha \cdot 2}$ such that $E \upharpoonright X$ is a bounded equivalence relation on X.

Definition 3.6. Let $M \supseteq A$ be any model. For $p, q \in S_X(M)$, we write $p \in E^M q$ iff $\exists a \models p, b \models q (a \in b)$.

Note that this does not depend on the choice of representatives, i.e.,:

Proposition 3.7. For $p, q \in S_X(M)$, $p \in E^M q$ iff $\forall a \models p, \forall b \models q (a \in b)$.

Proof. Since E is bounded, $\equiv_{L,A}^{\alpha}$ refines it on X, so if $a \equiv_M b$ for $a, b \in X$ then $a \in b$.

Remark 3.8. Suppose $Y \subseteq X$ is pseudo G_{δ} . For a model M, Y_M is not necessarily G_{δ} . But in case Y is $\equiv_{L,A}^{\alpha}$ -invariant and $A \subseteq M$, it is. Indeed, $\mathfrak{C}^{\alpha} \setminus Y$ is pseudo F_{σ} , and so $(\mathfrak{C}^{\alpha} \setminus Y)_M$ is F_{σ} . But since \equiv_M refines $\equiv_{L,A}^{\alpha}$, $(\mathfrak{C}^{\alpha} \setminus Y)_M \cap Y_M = \emptyset$. In addition, if A, T and α are countable, Y is pseudo closed and $\equiv_{L,A}^{\alpha}$ -invariant, then Y_M is G_{δ} , so Y is pseudo G_{δ} . In fact, in that case Y is type definable over M.

Assume that E is pseudo F_{σ} . This is equivalent to saying that there are \emptyset -type definable sets $U_n \subseteq \mathfrak{C}^{\alpha \cdot 2}$ for $n < \omega$ such that $E = \bigcup \{U_n \mid n < \omega\}$ (this follows by compactness, as E is \emptyset -invariant). In this case the set $U_n^M = \pi (U_{n,M}) \subseteq S_\alpha (M)^2$ is closed (where $\pi : S_{\alpha \cdot 2} (M) \rightarrow$ $S_\alpha (M)^2$ is the projection) and hence $E_M = \bigcup \{U_n^M \mid n < \omega\}$ is F_{σ} . We assume that the sequence $\langle U_n \mid n < \omega \rangle$ is in *normal form*, i.e., U_0 contains the diagonal Δ_X , U_n is symmetric and:

$$U_n \circ U_n \upharpoonright X = \left\{ (a, b) \in X^2 \mid \exists c \in X \ (a, c) \in U_n \land (c, b) \in U_n \right\} \subseteq U_{n+1}.$$

So the U_n are increasing on X.

Definition 3.9. Suppose $Y \subseteq X$ is E invariant. We say E is strongly closed on Y if there exists some $n < \omega$ such that $E \upharpoonright Y = Y^2 \cap U_n$. Note that this may depend on the choice of the U_n 's.

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3.2.3. Countable language. Suppose T and α are countable. In this setting we will translate our relation E into an F_{σ} relation on X_M , as was done in [KPS12].

For a countable model $A \subseteq M$, $S_{\alpha}(M)$ is Polish and if Y is as in Remark 3.8 then Y_M is a Polish space (every G_{δ} set is), and similarly to [KMS13, Proposition 1.41] (with the same proof as there) we have:

Proposition 3.10. Fix a pseudo G_{δ} set $Y \subseteq X$, such that Y is E-invariant. Then for any two models $A \subseteq M, N$ we have:

$$E^M \upharpoonright Y_M \sim_B E^N \upharpoonright Y_N.$$

So with this assumption and Proposition 3.10, we can refer to the Borel cardinality of the F_{σ} equivalence relation $E \upharpoonright Y$ without specifying the model.

3.2.4. Countable or uncountable language. Let T be any complete first order theory and α any ordinal.

Definition 3.11. We say that a set $Y \subseteq \mathfrak{C}^{\alpha}$ for some small α is *pseudo strong Choquet* if Y_M is strong Choquet for all M.

Example 3.12. If $Y \subseteq \mathfrak{C}^{\alpha}$ is pseudo closed or pseudo G_{δ} and \equiv_L^{α} -invariant, then by Remark 3.8 and Proposition 2.17 it is pseudo strong Choquet.

Remark 3.13. For countable T and α , "pseudo strong Choquet" is the correct analog of pseudo G_{δ} for \equiv_{L}^{α} -invariant sets. This follows from [Kec95, Theorem 8.17].

3.2.5. Invariant equivalence relations with a nice automorphism group. Let C be some subset of X. Suppose that $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$.

- **Definition 3.14.** (1) A formula $\varphi \in L_{\alpha}(\mathfrak{C})$ is said to be *C*-generic if finitely many translates of φ under the action of Γ cover *C*.
 - (2) The formula φ is said to be *C*-weakly generic if there is a non-*C*-generic formula $\psi \in L_{\alpha}(\mathfrak{C})$ such that $\varphi \lor \psi$ is *C*-generic.
 - (3) A partial type $p \subseteq L_{\alpha}(\mathfrak{C})$ is said to be C-generic (C-weakly generic) if all its formulas are.
 - (4) A partial type $p \subseteq L_{\alpha}(\mathfrak{C})$ which is closed under conjunctions is said to be *C*-proper if there is a non-*C*-generic formula ψ such that for all $\varphi \in p, \varphi \lor \psi$ is *C*-generic. In general, p is *C*-proper when its closure under finite conjunctions is.

For the most part we will omit C from the notation.

For $n < \omega$, let $p_n(x, y)$ be the type defining U_n .

Proposition 3.15. Suppose that Γ is *C*-transitive: for all $a, b \in C$ there is some $\sigma \in \Gamma$ such that $\sigma(a) = b$. Then, for some $n < \omega$ and for all $a \in C$, $p_n(x, a)$ is proper. Moreover, there is a formula $\psi(x, y)$ such that $\psi(x, a)$ is the non-generic formula that witnesses this.

Proof. First observe that if $p_n(x, a)$ is proper for some $a \in C$, $\psi(x, a)$ witnesses this and $b \in C$, then $p_n(x, b)$ is proper with $\psi(x, b)$ witnessing it. So fix some $a \in C$.

Note that if $\psi(x, a)$ is not generic, then we can construct inductively a sequence $a_i \in C$ for $i < \omega$ such that $\neg \psi(a_i, a_j)$ for j < i: let $a_0 = a$, and for n + 1, let $\sigma_0, \ldots, \sigma_n \in \Gamma$ be such that $\sigma_i(a) = a_i$ (so $\sigma_0 = \mathrm{id}$) and let $a_{n+1} \not\models \bigvee_{i \leq n} \sigma(\psi(x, a)) = \bigvee_{i \leq n} \psi(x, a_i)$. By Ramsey and compactness (Fact 3.4), there is an A-indiscernible sequence $\langle b_i | i < \omega \rangle$ in X with the property that $\neg \psi(b_i, b_j)$ for j < i. Here we used the fact that X is type definable.

Now suppose that for no $n < \omega$ is $p_n(x, a)$ proper. This allows us to inductively construct formulas $\varphi_n(x, a) \in p_n(x, a)$ such that $\bigvee_{i < n} \varphi_n$ is not generic. By the remark above and compactness, there is an A-indiscernible sequence $\langle b_i | i < \omega \rangle$ in X such that for all $n < \omega$ and $j < i < \omega$, $\neg \varphi_n(b_i, b_j)$. But this means that $(b_i, b_j) \notin U_n$ for all $j < i < \omega$ and $n < \omega$, so $\neg E(b_i, b_j)$. By compactness, we may increasing the length of the sequence to any length, contradicting the fact that E is bounded on X.

Now assume that C is Γ invariant, and fix some $a \in C$. By taking a countable union of models M_i and a countable union of subsets Γ_i of Γ , we can find a model M of size $|A| + |L| + |\alpha|$ containing A and a subgroup $\Gamma^* \leq \Gamma$ of the size $|\alpha| + |L|$ such that:

- (1) $\{a\} \cup A \subseteq M$.
- (2) For all $\sigma \in \Gamma^*$, $\sigma(M) = M$ setwise.
- (3) If φ is a formula over *a* which is generic, then there are finitely many elements from Γ^* which witness this.

Recall that the Stone space $S_{\alpha}(M)$ has a natural topology in which basic open sets are of the form $[\varphi] = \{p \in S_{\alpha}(M) | \varphi \in p\}$. When r is a partial type, i.e., a consistent set of formulas over M, we denote by [r] the set $\{p \in S_{\alpha}(M) | r \subseteq p\}$. This set is compact.

By (2) above, Γ^* is a group of homeomorphisms of $S_{\alpha}(M)$.

Lemma 3.16. Suppose $[a]_E \subseteq C \subseteq Y \subseteq X$ is Γ invariant and that Γ is *C*-transitive. Let \mathcal{U} be the family of open sets $U \subseteq Y_M$ for which there is no finite set $\Delta \subseteq \Gamma^*$ with $C_M \subseteq \Delta \cdot U$ (all in the induced Stone space topology). Then for some $n < \omega$, the compact set $[p_n(x, a)] \subseteq Y_M$ is $I_{\mathcal{U}}$ -positive.

Proof. By Proposition 3.15, for some $n < \omega$, $p_n(x, a)$ is proper. By (3) above, if a formula φ over M is generic then $[\varphi] \cap Y_M \notin \mathcal{U}$ and the converse also holds. Unwinding the definitions, the proposition is clear.

Assume now that $E \upharpoonright C$ is not strongly closed, that Γ is *C*-transitive and that $C = [a]_E$. Since U_n is \emptyset -invariant and Γ is *C*-transitive, this means that for any $n < \omega$, there is some $b \in C$ such that $(a, b) \notin U_n$. By enlarging Γ^* and M, we may assume:

(4) For all $n < \omega$ there is $\sigma \in \Gamma^*$ such that $(a, \sigma(a)) \notin U_n$.

We are now ready to state our result:

Theorem 3.17. Assume that $T, A \subseteq \mathfrak{C}$ and α are countable. Suppose that:

- (1) $X \subseteq \mathfrak{C}^{\alpha}$ is some type definable set over A.
- (2) E is a pseudo F_{σ} \emptyset -invariant equivalence relation on X with normal form $\langle U_n | n < \omega \rangle$ and E is bounded on X.
- (3) $C \subseteq X$ is an E class, and $E \upharpoonright C$ is not strongly closed (with respect to $\langle U_n \mid n < \omega \rangle$).
- (4) $C \subseteq Y \subseteq X$ is pseudo G_{δ} and E invariant.
- (5) $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ is C-transitive, and preserves all E-classes (in particular, it preserves X).

Then $E \upharpoonright Y$ is not smooth (see Proposition 3.10).

Proof. Keeping the notation from above, this follows directly from Theorem 2.13 with X there being Y_M (note that it is Γ^* invariant by assumptions (4) and (5) and that it is Polish by (4) and Remark 3.8), Γ there being Γ^* here, Y there being C_M here, R_n there being $U_n^M \upharpoonright Y_M$ here and K there being $[p_k(x, a)]$ for some $k < \omega$, chosen by Proposition 3.15 (note that as Y contains C, Y_M contains $[p_k(x, a)]$, so it is compact). By assumption (5), Theorem 2.13's E_{Γ}^X is contained in $E_M \upharpoonright Y_M$, so checking that the conditions of this theorem hold will suffice:

By Lemma 3.16, K is $I_{\mathcal{U}}$ -positive.

Condition (a) there follows from assumption (3) here. Note that if $p \in [p_m(x, a)]$, $q R_n p$ and $b \models q$ then $(a, b) \in U_{\max\{n,m\}+1}$ (because there is some $b' \models q, c \models p$ such that $(b', c) \in U_n$, but $(c, a) \in U_m$ so $(b', a) \in U_{\max\{n,m\}+1}$ but $b \equiv_a b'$). So $q \in [p_{\max\{n,m\}+1}(x, a)]$. From this computation it follows that if $p \in [p_k(x, a)]$ and $q R_n^{(2)} p$ for some $n \ge k$ then for all $b \models q$, $(a, b) \in U_{n+2}$. So if $\sigma \in \Gamma^*$ is such that $(a, \sigma(a)) \notin U_{n+3}$ for $n \ge k$, then for all $p \in [p_k(x, a)]$, $(p, \sigma(p)) \notin R_n^{(2)}$ (because for $b \models p$, $(\sigma(a), \sigma(b)) \in U_k$).

Condition (b) there follows similarly. As Γ preserves E classes, there is some $n < \omega$ such that $(a, \sigma(a)) \in U_n$. So if $p \in [p_k(x, a)]$, then for all $b \models p$, $(\sigma(b), b) \in U_{\max\{n, k\}+3}$.

Condition (c) there follows from the fact that $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ and that for all $n < \omega$, U_n is \emptyset -invariant.

Theorem 3.18. Let T, A and α be of any (small) size. Then under the same conditions as Theorem 3.17 replacing (4) with:

(4) $C \subseteq Y \subseteq X$ is pseudo strong Choquet and E invariant.

 $E \upharpoonright Y$ has at least 2^{\aleph_0} classes.

Proof. Follows similarly from Theorem 2.19 as in the proof of Theorem 3.17.

Theorem 3.19. Suppose T, A and α are countable, and the same assumptions as in Theorem 3.17 hold, except (4) and (5) which we replace by:

- (4) $C \subseteq Y \subseteq X$ is pseudo G_{δ} and Γ invariant.
- (5) $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ is *C*-transitive, and for all $\sigma \in \Gamma$ there is some $n < \omega$ such that for all $c \in C$, $(c, \sigma(c)) \in U_n$.

Then $E \upharpoonright Y$ is not smooth.

Proof. To prove this theorem we could use either Theorem 2.12 or Theorem 2.13 similarly to the proof of Theorem 3.17. The conditions there hold, but since Γ may not preserve E classes, it is not clear that E_{Γ}^X is contained in $E^M \upharpoonright Y_M$. To solve this problem, we note that for any $x \in K$ (which is just $[p_k(x, a)]$ for some $a \in C, k < \omega$), $E_{\Gamma}^X \upharpoonright \overline{\Gamma \cdot x}$ is contained in $E^M \upharpoonright Y_M$, and recall that the the image of the embedding ϕ of either Theorem 2.12 or Theorem 2.13 is into $\overline{\Gamma \cdot x}$ for some $x \in K$.

Indeed, fix some $p \in [p_k(x, a)]$ and $\sigma \in \Gamma^*$, and let $n < \omega$ correspond to (5). Then for any $q \in \overline{\Gamma^* \cdot p}$, $(\sigma(q), q) \in U_n^M$ as this is a closed condition.

As above we give a general analog (using Theorem 2.18 or Theorem 2.19). Unfortunately, in this case, being pseudo strong Choquet is not enough in order to prove the theorem since we do not know that the range of ϕ can be chosen to be $\overline{\Gamma \cdot x}$.

Theorem 3.20. Let T, A and α be of any (small) size. Then under the same conditions as Theorem 3.19 replacing (4) with:

(4) $C \subseteq Y \subseteq X$ is pseudo closed and Γ invariant.

 $E \upharpoonright Y$ has at least 2^{\aleph_0} classes.

Corollary 3.21. For $E \equiv \equiv_L^{\alpha}$, the group Aut $f_L(\mathfrak{C})$ satisfies both the condition of Theorem 3.17 and Theorem 3.19, and so [KMS13, Main Theorems A and B] both follow directly.

In addition, [KMS13, Fact 1.1] has an obvious analog (at least in the countable case) for the cases described in Theorem 3.17 and Theorem 3.19. In particular, in these cases, an E class is closed iff it is G_{δ} iff E is strongly closed on it, and if T is small and α is finite then all classes are closed.

We can also deduce that [New03, Corollary 1.12] hold for the cases described above (both for countable and uncountable languages), which begs the question:

Problem 3.22. Do our result extend to any \emptyset -invariant F_{σ} relation?

Remark 3.23. One of the properties of \equiv_L is that if $a \equiv_M b$ for some model M, then $d_L(a,b) \leq 2$ where d_L is the Lascar metric. An analog for E and its normal form would be that for some $n < \omega$, if $M \supseteq A$ and $a \equiv_M b$ then $(a,b) \in U_n$. This has no reason to hold in general. However, if Γ is C-transitive then for some $n < \omega$ and all M and Γ^* as in (1)–(3) above, there is a nonempty Γ^* invariant closed subset $S \subseteq S_X(M)$ such that for any $p \in S \cap C_M$, if $b, c \models p$ then $(b, c) \in U_{n+1}$. Moreover, it is dense in the following sense: for every $b \in C$, there is some $c \in C$ such that $\operatorname{tp}(c/M) \in S$ and $(b, c) \in U_n$.

Indeed, let $n < \omega$ be such that $p_n(x, a)$ is proper for all $a \in C$, and let $\psi(x, y)$ be the formula that witnesses this (see Proposition 3.15). Let $a \in C$, M and Γ^* be as in (1)–(3). Let S be the set of types $[\{\neg \psi(x, \sigma(a)) \mid \sigma \in \Gamma^*\}]$. This is obviously closed and Γ^* -invariant. Suppose $p \in S \cap C_M$ and $b, c \models p$. We will show that $(b, c) \in U_{n+1}$, i.e., $(b, c) \models p_{n+1}$. Let $\xi(x, y) \in p_{n+1}$, and let $\chi(x, y) \in p_n$ be such that $\chi(x, y) \land \chi(z, y) \rightarrow \xi(x, z)$. Since $\chi(x, a) \lor \psi(x, a)$ is generic, for some $\sigma \in \Gamma^*$, $b, c \models \chi(x, \sigma(a)) \lor \psi(x, \sigma(a))$, but by the definition of $S, b, c \models \chi(x, \sigma(a))$. It follows that $\xi(b, c)$ holds.

We also need to show the denseness property. Fix some $b \in C$. It is enough to show that the set $\{\neg \psi(x, \sigma(a)) \mid \sigma \in \Gamma^*\} \cup p_n(x, b)$ is consistent. Suppose not, so for some $\xi(x, y) \in p_n(x, y)$ and some finite $\Delta \subseteq \Gamma^*$, $\xi(x, b) \to \bigvee_{\sigma \in \Delta} \psi(x, \sigma(a))$. Since Γ is *C*-transitive, for some $\tau \in \Gamma$, $\tau(b) = a$, so $\xi(x, a)$ implies $\bigvee_{\sigma \in \Delta} \psi(x, \tau \circ \sigma(a))$. But then $\bigvee_{\sigma \in \Delta} \psi(x, \tau \circ \sigma(a)) \lor \psi(x, a)$ is generic — contradiction.

This observation could have been used in the proof of e.g., Theorem 3.17, using $S \cap Y_M$ as our Polish space.

3.2.6. Definable and type definable group action.

Definition 3.24. For an ordinal β , (H, \cdot) is a *type definable group* contained in \mathfrak{C}^{β} when H is type definable and the multiplication is type definable.

Suppose (H, \cdot) is a type definable group over \emptyset . Let G be an \emptyset -invariant pseudo F_{σ} subgroup. In this case G has a normal form: $G = \bigcup \{V_n \mid m < \omega\}$ where V_n is \emptyset -type definable, $\{e\} \in V_n, V_n$ is symmetric $(V_n = V_n^{-1})$, and $V_n^{\cdot 2} \subseteq V_{n+1}$.

Suppose that $X \subseteq \mathfrak{C}^{\alpha}$ is \emptyset -type definable and that * is an \emptyset -type definable group action of H on X. In particular, the orbit equivalence relation of the action E_H^X is a closed invariant equivalence relation on X and E_G^X is an \emptyset -invariant pseudo F_{σ} equivalence relation in the sense discussed in the previous subsection, with normal form defined by:

$$U_n = \{(a, b) \in X \times X \mid \exists g \in V_n \ (g \ast a = b)\}$$

Definition 3.25. To simplify notation, we call such a tuple $\overline{D} = (\alpha, \beta, G, H, \langle V_n, U_n | n < \omega \rangle, \cdot, X, *)$ an F_{σ} action. If E_G^X is bounded, we call \overline{D} a bounded F_{σ} action.

Example 3.26. For an \emptyset -type definable group $G \subseteq \mathfrak{C}^{\alpha}$, G_{\emptyset}^{000} is defined as the smallest bounded index invariant subgroup of G and it is generated by the set $\{a^{-1} \cdot b \mid a \equiv_L^{\alpha} b, a, b \in G\}$. So, letting $W_n = \{a^{-1} \cdot b \mid d_L(a, b) \leq n\}$ where d_L is the Lascar distance, we see that $G = \bigcup_{n < \omega} V_n$ where $V_n = \{\prod_{i < n} c_i^{\pm 1} \mid c_i \in W_n\}$. See [Gis11] for more. So $(\alpha, \alpha, G_{\emptyset}^{000}, G, \langle V_n, U_n \mid n < \omega \rangle, \cdot, G, \cdot)$ is a bounded F_{σ} action.

We shall need a technical assumption that seems necessary for this approach to work.

Definition 3.27. We say that $a \in X$ is *shifty* if one of the following holds:

- (1) (*Right* shifty) For every $k < \omega$ there exists $n = n_k < \omega$ such that for any $g_1, g_2 \in H$ if $(g_1 * a, g_2 * a) \in U_k$ then $((g_1 \cdot g_2^{-1}) * a, a) \in U_n$ or:
- (2) (Left shifty) For every $k < \omega$ there exists $n = n_k < \omega$ such that for any $g_1, g_2 \in H$ if $(g_1 * a, g_2 * a) \in U_k$ then $((g_1^{-1} \cdot g_2) * a, a) \in U_n$ and if $((g_1^{-1} \cdot g_2) * a, a) \in U_k$ then $(g_1 * a, g_2 * a) \in U_n$.

Remark 3.28. In both cases, we may safely assume that $n_k \ge k$.

Example 3.29. Suppose $a \in X$ and $\operatorname{stab}_H(a) \leq H$. Then a is right shifty.

Proof. Let $k < \omega$ be given and let n = k. If $g_2 * a = (h \cdot g_1) * a$ for $h \in V_n$ then $(g_2^{-1} \cdot h \cdot g_1) * a = a$ and since stab_H (a) is normal, $(h \cdot g_1 \cdot g_2^{-1}) * a = a$ so $(g_1 \cdot g_2^{-1}) * a = h^{-1} * a$. As V_n is symmetric, we are done.

Example 3.30. Suppose that for every $k < \omega$ there exists $n < \omega$ such that for any $c, d \in G * a$ and $g \in H$, if $(c, d) \in U_k$ then $(g * c, g * d) \in U_n$. Then a is left shifty. This happens for instance when V_n is definable for all $n < \omega$ and G is a normal subgroup of H.

Proof. If V_n is definable, then by compactness for every $k < \omega$ there is some $n < \omega$ such that for all $g \in H$, $gV_kg^{-1} \subseteq V_n$. So if there is some $h \in V_k$ such that c = h * d, then $g * c = (g \cdot h) * d$, but $g \cdot h = h' \cdot g$ for $h' \in V_n$ so $(g * c, g * d) \in U_n$.

Lemma 3.31. Suppose $\varphi(x, y)$ is some formula where x comes from the first α variables. Then there is a formula $\psi(x', y, z)$ with x' coming from the first α variables such that for every $g \in H$, and any $a \in \mathfrak{C}^{\lg(y)}$, $g * (\varphi(\mathfrak{C}^{\alpha}, a) \cap X) = (\psi(\mathfrak{C}^{\alpha}, a, g) \cap X)$.

Proof. If α , and β were finite, so that * and \cdot were definable, then we could just define $\psi(x, y, z) = \varphi(z^{-1} \cdot x, y)$ (so x = x'). Otherwise, it is a standard compactness argument. Note that we need that both X and H are closed.

Lemma 3.31 defines an action of H on sets of the form $X \cap \varphi(\mathfrak{C}^{\alpha})$. In order to ease notation, we will write $g * \varphi$ instead of $g * (\varphi(\mathfrak{C}^{\alpha}) \cap X)$. This induces a natural action of H on the set of types in X. If dcl (A) = A, then $H \cap A$ (and also $G \cap A$) is a subgroup of H, and so it acts naturally by homeomorphisms on $S_X(A)$ (with the usual Stone topology). In that case, for any $g \in H \cap A$, $c \models p$ iff $g * c \models g * p$.

Fix an E_G^X -class $C \subseteq X$. Similarly to Definition 3.14, we define C-generic and C-weakly generic formulas and C-proper types, replacing the action of an automorphism group Γ by the action of G on $L_{\alpha}(\mathfrak{C})$ (note: G and not H). We omit the details, since it is exactly as above.

For $n < \omega$, let $p_n \subseteq L_\beta(\emptyset)$ be the partial types defining V_n and let $q_n(x, a)$ be the partial type saying $x \in X \land \exists g \in V_n (g * a = x)$.

Lemma 3.32. Suppose $a \in X$ is shifty. Then, for some $n < \omega$, $q_n(x, a)$ is a G * a-proper type.

Proof. The proof uses the same basic idea as in Lemma 3.15, but one has to be a bit careful.

Assume first that a is right shifty. Suppose π_* is the partial type defining * and that π_X is the type defining X. We may assume that these types, as well as p_n and q_n are closed under conjunctions. First we need to establish the following:

Claim. For each $k < \omega$ there is some $n < \omega$ such that for all formulas $\varphi \in p_n$, $\theta \in \pi_*$ there are formulas $\psi \in p_k$ and $\theta' \in \pi_*$ such that for every $g_1, g_2 \in H$, if

$$\exists z \left(\psi\left(z\right) \land \theta'\left(z, g_1 \ast a, g_2 \ast a\right)\right)$$

then

$$\exists z \left(\varphi(z) \land \theta\left(z, a, \left(g_1 \cdot g_2^{-1}\right) \ast a\right)\right).$$

Proof of claim. Let $k < \omega$ be given, and let $n < \omega$ be the corresponding number from Definition 3.27. Then the following is inconsistent: there are $g_1, g_2 \in H$ such that $g_2 * a \in V_k * (g_1 * a)$ but $(g_1 \cdot g_2^{-1}) * a \notin V_n * a$. Applying compactness, we are done. Note that these formulas may depend on a (but not on g_1, g_2).

Assume that for all $n < \omega$, q_n is not proper. For each $k < \omega$, let $n_k < \omega$ be the corresponding number from the claim.

Since q_{n_k} is not proper for all $k < \omega$, we can find formulas $\varphi_k \in p_{n_k}$ and $\theta_k \in \pi_*$ such that $\bigvee_{k < m} \psi'_k$ is not generic for all $m < \omega$ where $\psi'_k(x) = \exists y (\varphi_k(y) \land \theta_k(y, a, x))$. (note: a formula in q_{n_k} generally looks like $\psi'_k \land \tau$ for $\tau \in \pi_X$, but this does not matter for genericity.)

For each $k < \omega$, the claim provides formulas $\psi_k \in p_k$ and $\theta'_k \in \pi_*$ such that:

If $g_1, g_2 \in H$ and $g_2^{-1} * a \notin g_1^{-1} * \psi'_k$ then $\neg \exists z (\psi_k (z) \land \theta'_k (z, g_1 * a, g_2 * a))$. Note that this latter condition implies that $(g_1 * a, g_2 * a) \notin U_k$.

Fix some $n < \omega$ and let $\psi' = \bigvee_{k < n} \psi'_k$. Since ψ' is not generic, there is a sequence $\langle g_i \in G | i < \omega \rangle$ such that $g_i^{-1} * a \notin g_j^{-1} * \psi'$ for j < i. This means that $(g_j * a, g_i * a) \notin U_k$ for k < n, and for each k, this is because of ψ_k and θ'_k . Note that although $V_k \subseteq V_{k+1}$, we do not get that ψ_k implies ψ_{k+1} , so we really need to keep all the formulas.

Now, by compactness we can find a sequence $\langle a_i \in X | i < \omega \rangle$ such that for all $j < i < \omega$, $(a_j, a_i) \notin U_k$ for all $k < \omega$ (and each time because of the same formulas). By Ramsey and compactness (Fact 3.4) we may assume that this sequence is indiscernible. But this is a contradiction to our assumption that the action is bounded.

If a is left shifty, the proof is exactly the same, replacing g_i^{-1} by g_i .

Recall that if $E_G^X \upharpoonright G * a$ is not strongly closed for some $a \in X$ (see Definition 3.9), then for all $n < \omega$ there are $g_1, g_2 \in G$ such that $(g_1 * a, g_2 * a) \notin U_{n+1}$. But then either $(a, g_1 * a) \notin U_n$ or $(a, g_2 * a) \notin U_n$. So we may always assume that $g_1 = e$.

Theorem 3.33. Suppose T is a complete countable first-order theory, α, β countable ordinals. Suppose that $(\alpha, \beta, G, H, \langle V_n, U_n | n < \omega \rangle, \cdot, X, *)$ is a bounded F_{σ} action and suppose $Y \subseteq \mathfrak{C}^{\alpha}$ is a pseudo G_{δ} set contained in X which is E_G^X invariant. If for some shifty $a \in Y$, $E_G^X \upharpoonright G * a$ is not strongly closed, then $E_G^X \upharpoonright Y$ is non-smooth.

Proof. This follows from Theorem 2.12, just like the proof of Theorem 3.17. By Lemma 3.32, for some $k < \omega$, $q_k(x, a)$ is G * a-proper, and this is witnessed by some non-generic formula ψ . Construct recursively a countable model M such that:

- (1) $a \in M$ and ψ is over M.
- (2) If $\varphi \in L_{\alpha}(M)$ is G * a-generic, then for some $\Delta \subseteq G \cap M$, $\Delta * \varphi$ contains G * a.
- (3) For all $n < \omega$, there is some $g \in G \cap M$ such that $(a, g * a) \notin U_n$.

In the language of Theorem 2.12, X is Y_M , Γ is $G \cap M$, Y is $(G * a)_M$, R_n is $U_n^M \cap Y_M^2$ and K is the compact set $[q_k(x, a)]$. The fact that $[q_k(x, a)]$ is $I_{\mathcal{U}}$ -positive follows from (2) and the fact that $q_k(x, a)$ is proper (see the proof of Lemma 3.16).

Condition (a) follows from (3) above: if $(a, g * a) \notin U_N$ for N big enough, then for all $p \in Y_M$ containing $q_k(x, a), (g * p, p) \notin U_n^{M,(4)}$. We illustrate this: if $(p, g * p) \in U_n^{M,(2)}$, then for some $q \in X_M, (p,q) \in U_n^M$ and $(q, g * p) \in U_n^M$. So for some $b_1, b_2 \models p$ and $c_1, c_2 \models q$, $(b_1, c_1) \in$ $U_n, (c_2, g * b_2) \in U_n$. Since $(b_1, a) \in U_k$, and $b_1 \equiv_a b_2, (b_1, b_2) \in U_{k+1}$. Similarly, $(c_1, c_2) \in$ $U_{\max\{n,k\}+2}$. It follows that $(b_2, g * b_2) \in U_{\max\{k,n\}+4}$. Suppose a is right shifty. As $b_2 \in G * a$ and since a is right shifty, we get that $(a, g * a) \in U_N$ for some N. If a is left shifty, then as $(a, b_2) \in U_k, (g * a, g * b_2) \in U_{n_k}$ for some large n_k , so $(a, g * a) \in U_{n_k+2}$.

Condition (b) is trivial, since any $g \in G \cap M$ belongs to some V_n .

Problem 3.34. Is shiftiness of *a* necessary?

Corollary 3.35. With the same assumptions of Theorem 3.33, if the action of G is free (if g * x = h * x then h = g) then either $G = V_n$ for some $n < \omega$, in which case E_G^X is strongly closed so smooth or $E_G^X \upharpoonright Y$ is non-smooth.

Proof. Note that by assumption, every $a \in X$ is right shifty (since $\operatorname{stab}_H(a) = e$ is a normal subgroup). Now, if $E_G^X \upharpoonright Y$ is smooth, then by Theorem 3.33, for every $a \in Y$, $E \upharpoonright G * a$ is strongly closed. So for some $a \in Y$ and $n < \omega$ for all $b \in Y$, $a E_G^X b$ iff $(a, b) \in U_n$. Since the action is free, it follows that then $G = V_n$.

Theorem 3.36. Suppose T is a complete first-order theory, α, β small ordinals. Suppose that $(\alpha, \beta, G, H, \langle V_n, U_n | n < \omega \rangle, \cdot, X, *)$ is a bounded F_{σ} action and suppose $Y \subseteq \mathfrak{C}^{\alpha}$ is a pseudo a strong Choquet set contained in X which is E_G^X invariant. Suppose also that for some <u>shifty</u> $a \in Y, E_G^X \upharpoonright G * a$ is not strongly closed. Then $|Y/E_G^X| \geq 2^{\aleph_0}$.

Proof. Follows similarly from Theorem 2.18.

We can also recover Newelski's results [New03, Theorem 3.1] about groups generated by countably many type definable sets over \emptyset .

Corollary 3.37. Let T be any first order theory and α any small ordinal. Suppose that (H, \cdot) is a \emptyset -type definable group such that $H \subseteq \mathfrak{C}^{\alpha}$. Suppose that $G \leq H$ is a subgroup which is generated by countably many sets V_n for $n < \omega$ which are \emptyset -type definable. Suppose that $G \leq H_0 \leq H$ is a subgroup which is pseudo G_{δ} or pseudo closed (or even pseudo strong Choquet). Assume also that [H:G] is bounded. Then:

- (1) If G is pseudo G_{δ} or pseudo closed or even pseudo strong Choquet then G is pseudo closed and in fact generated by finitely many of the sets V_n in finitely many steps.
- (2) If G is not pseudo closed then $[H_0:G] \ge 2^{\aleph_0}$.
- (3) If T and α are countable then either G is pseudo closed or the equivalence relation $E_G^{H_0}$ on H_0 of being in the same coset modulo G is not smooth.
- (4) If T is small and α is finite then G is pseudo closed.
- (5) If we remove the assumption that [H:G] is bounded, we still get (1) for pseudo closed G.

Proof. We may assume that V_n is symmetric $(V_n = V_n^{-1})$, $V_0 = \{1_H\}$, and $V_n^{\cdot 2} \subseteq V_{n+1}$. Consider the action of G on H by left multiplication and the orbit equivalence relation E_G^H on H. Then by Theorem 3.36 with X = H, Y = G, $\alpha = \beta$, $a = e_G$ we get (1). Applying it again with $X = H, Y = H_0$ we get (2). (3) follows from Corollary 3.35.

(4) Suppose not. Since T is small, the set $S_{\alpha}(\emptyset)$ is countable. Thus every subset of it is G_{δ} , in particular the set

$$Q = \{q \in S_{\alpha}(\emptyset) \mid \forall b \models q \ (b \in G)\}.$$

But then G is pseudo G_{δ} so it is pseudo closed by (1).

(5) Note that in that case G is type definable over \emptyset , so we can replace H by G.

Corollary 3.38. Suppose H is a definable group, and G an \emptyset -invariant subgroup which is a union of countably many type definable sets. Then if $[H:G] < \infty$ then G is definable.

Proof. By Corollary 3.37 (2), G must be type definable over \emptyset . But then its complement is also type definable since it is a finite union of type definable sets, so it is definable by compactness. \Box

Example 3.39. (with Pierre Simon) Theorem 3.33 does not hold in a very strong sense, if the group G is only \emptyset -invariant and not pseudo F_{σ} . More precisely there is a countable theory T where Corollary 3.38 fails.

Let T be the theory of an infinite dimensional vector space over \mathbb{F}_2 in the language $\{+, 0\}$. Add predicates U_n to the language and add axioms saying that U_n are independent subspaces of co-dimension 1 (independent in the sense that any finite Boolean combination is nonempty). Then T is consistent as one can take for U_n the kernels of independent functionals. Let \mathfrak{C} be a monster model for T, and let H be the group $(\mathfrak{C}, +)$. Let G be the intersection $\bigcap \{U_n \mid n < \omega\}$. Then the index $[H : G] = 2^{\aleph_0}$. In fact, the cosets of G in H are exactly the types $X_\eta = \bigcap \{U_n^{\eta(n)} \mid n < \omega\}$ where $\eta : \omega \to 2$ and $U_n^0 = U_n, U_n^1 = \mathfrak{C} \setminus U_n$. Pick a basis $\{v_i \mid i < 2^{\aleph_0}\}$ for the space H/G. Any map $\eta : 2^{\aleph_0} \to 2$ defines a subspace V_η by taking the kernel of the functional mapping v_i to $\eta(i)$. Obviously, if η is not trivial then $[H : \pi^{-1}(V_\eta)] = 2$ (where π is the projection $H \to H/G$). So for at least one $\eta, \pi^{-1}(V_\eta)$ is not definable. But all of them are invariant as they are union of cosets X_η .

Example 3.40. Corollary 3.37 (5) does not hold when G is pseudo G_{δ} . For instance, let T = RCF, and add to the language constant symbols for the rational numbers \mathbb{Q} . Then \mathbb{Q} itself is pseudo open (in every model), so also pseudo G_{δ} , but definitely not closed (every closed infinite subset must have unbounded cardinality).

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