# AN EMBEDDING THEOREM OF $\mathbb{E}_{0}$ WITH MODEL THEORETIC APPLICATIONS 

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#### Abstract

We provide a new criterion for embedding $\mathbb{E}_{0}$, and apply it to equivalence relations in model theory. This generalize the results of the authors and Pierre Simon on the Borel cardinality of Lascar strong types equality, and Newelski's results about pseudo $F_{\sigma}$ groups.


## 1. Introduction

Given two topological spaces $X$ and $X^{\prime}$ and two equivalence relations $E$ and $E^{\prime}$ respectively on $X$ and $X^{\prime}$, we say that $E$ is Borel reducible to $E^{\prime}$ if there is a Borel map $f$ from $X$ to $X^{\prime}$ such that $x E y \Longleftrightarrow f(x) E^{\prime} f(y)$ for all $x, y \in X$. The quasi-order of Borel reducibility of Borel equivalence relations on Polish spaces is a well-studied object in descriptive set theory, and enjoys a number of remarkable properties. One of them is given by the Harrington-Kechris-Louveau dichotomy, which asserts that a Borel equivalence relation is either smooth (Borel reducible to equality on $2^{\omega}$ ) or at least as complicated as $\mathbb{E}_{0}$ (eventual equality on $2^{\omega}$ ). In other words, $\mathbb{E}_{0}$ is the first non-smooth Borel equivalence relation.

In Section 2, we provide a new criterion for being non-smooth. We also translate this criterion to another context, that of strong Choquet spaces.

In the majority of Section 3, we apply this criterion to bounded invariant equivalence relation in model theory.

Suppose $T$ is a complete first order theory and $\mathfrak{C}$ a $\kappa$-saturated model for some large $\kappa$. If $E$ is an equivalence relation on $\mathfrak{C}^{\alpha}$ which is a countable union of $\emptyset$-type definable sets $U_{n}$ (i.e., $U_{n}$ is definable by intersection of parameter free formulas), we say that it is bounded when the number of classes is smaller than $\kappa$. We call $E$ a bounded invariant pseudo $F_{\sigma}$ equivalence relation. Such relations appear naturally in model theory, and include the finest bounded invariant equivalence relation: equality of Lascar strong types $-\equiv_{L}^{\alpha}$. It turns out that if the $T$ and $\alpha$ are countable, one can interpret $E$ as an (honest) $F_{\sigma}$ equivalence relation on a compact Polish space in a very natural way, which equips $E$ with a well defined Borel cardinality. This was done for Lascar strong types in KPS12, where many examples are computed, but in fact works for any $E$. It is explained in details in Subsections 3.2.2 and 3.2.3.

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If $E$ is an invariant bounded pseudo $F_{\sigma}$ equivalence relation, we can assume by compactness that there are $\emptyset$-type definable sets $U_{n}$ which are reflexive, symmetric and $U_{n} \circ U_{n} \subseteq U_{n+1}$ with $E=\bigcup_{n<\omega} U_{n}$. Such a sequence $\left\langle U_{n} \mid n<\omega\right\rangle$ is called a normal form of $E$.

In New03, Corollary 1.12], Newelski proved that if $E$ is an invariant pseudo $F_{\sigma}$ equivalence relation on $\mathfrak{C}^{\alpha}$ with normal form $\left\langle U_{n} \mid n<\omega\right\rangle$, and $X$ is a type definable set, all its elements have the same type over $\emptyset$, then either $E \upharpoonright X=U_{n}$ for some $n$ or $|X / E| \geq 2^{\aleph_{0}}$.
${ }^{*}$ ) Equivalently, if there is some $x \in X$ such that $E \upharpoonright[x]_{E}$ is not already $U_{n} \upharpoonright[x]_{E}$ for some $n$ then $|X / E| \geq 2^{\aleph_{0}}$.

He continued to prove [New03, Theorem 3.1] that if $H$ is an $\emptyset$-type definable group and $G \leq H$ is generated by countably many sets $V_{n}$, each $\emptyset$-type definable, then $G$ is type definable iff $G$ is generated by finitely many $V_{n}$ 's in finitely many steps and if $G$ is not type definable then [ $H: G] \geq 2^{\aleph_{0}}$. In that case, if moreover $T$ is small (has only countably many types over $\emptyset$ ) and $H$ consists of finite tuples, then $[H: G]$ is unbounded. Let $X=H$ and $E=E_{G}^{H}$ be the orbit equivalence relation of the action of $G$ on $H$ (so it is an invariant pseudo $F_{\sigma}$ equivalence relation).
$\left(^{* *}\right)$ In this language this is equivalent to: if for some $x \in X, E \upharpoonright[x]_{E}$ is not already generated by finitely many of the $V_{n}$ 's in finitely many steps, then $|X / E| \geq 2^{\aleph_{0}}$.

An important example of such a pair $(G, H)$ is $\left(G, G_{\emptyset}^{000}\right)$ where $G$ is $\emptyset$-type definable and $G_{\emptyset}^{000}$ is the minimal $\emptyset$-invariant bounded index subgroup. See Gis11 for more.

In [KMS13] the authors dealt with the case where $X$ was a $K P$-strong type and $E=\equiv_{L}^{\alpha}$ The main result there is that if $E$ is not trivial on $X$ then it is non-smooth. This went through a stronger theorem KMS13, Main Theorem A] that stated that:
$\left({ }^{* * *}\right)$ If $Y$ is a pseudo $G_{\delta}, \equiv_{L}^{\alpha}$-invariant subset of $\mathfrak{C}^{\alpha}$ and for some $x \in Y,[x]_{\equiv_{L}^{\alpha}}$ has unbounded Lascar diameter, then $\equiv_{L}^{\alpha} \upharpoonright Y$ is non-smooth. "Unbounded Lascar diameter" means exactly that it is not the case that $\equiv_{L}^{\alpha} \upharpoonright[x]_{\equiv_{L}^{\alpha}}=U_{n} \upharpoonright[x]_{\equiv_{L}^{\alpha}}$ for some $n$, where $U_{n}(a, b)$ is the type saying that the Lascar distance between $a$ and $b$ is at most $n$. (This is a normal form for $\equiv \equiv_{L}^{\alpha}$.)

Here we try to generalize $\left({ }^{*}\right),\left({ }^{* *}\right)$ and $\left({ }^{* * *}\right)$ in a uniform way using the results from Section 2. So the idea is to prove, in each case (when everything is countable), that if $Y$ is a pseudo $G_{\delta}$, $E$-invariant and for some $x \in Y, E \upharpoonright[x]_{E}$ is not already $U_{n} \upharpoonright[x]_{E}$ for some $n$, then $E \upharpoonright X$ is not smooth.

While we do not successfully generalize $\left(^{*}\right)$, we do prove it if there is a subgroup of Aut $(\mathfrak{C})$ which acts nicely on $[x]_{E}$, for instance when it is transitive on this class and preserves all classes. This is done in Subsection 3.2.5, and includes also $\left({ }^{* * *)}\right.$ (the subgroup in that case is Aut $f_{L}(\mathfrak{C})$ ). $\left({ }^{* *}\right)$ is successfully generalized and moreover stated for group actions (with an extra technical assumption called "shiftiness" which holds in the case where the action is free).

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## 2. A sufficient condition for embedding $\mathbb{E}_{0}$

### 2.1. Preliminaries.

Definition 2.1. Suppose $X$ and $Y$ are topological spaces, and $E$ and $F$ are Borel equivalence relations on $X$ and $Y$. We say that a function $f: X \rightarrow Y$ is a reduction of $E$ to $F$ if for all $x_{0}, x_{1} \in X,\left(x_{0}, x_{1}\right) \in E \operatorname{iff}\left(f\left(x_{0}\right), f\left(x_{1}\right)\right) \in F$.
(1) We say that $E$ is Borel reducible to $F$, denoted by $E \leq_{B} F$, when there is a Borel reduction $f: X \rightarrow Y$ of $E$ to $F$.
(2) We write $E \sqsubseteq_{c} F$ when there is a continuous injective reduction $f: X \rightarrow Y$ of $E$ to $F$.
(3) We say that $E$ and $F$ are Borel bi-reducible, denoted by $E \sim_{B} F$, when $E \leq_{B} F$ and $F \leq_{B} E$.
(4) We write $E<_{B} F$ to mean that $E \leq_{B} F$ but $E \not \chi_{B} F$.

Example 2.2. For a Polish space $X$, the relations $\Delta(X)$ denotes equality on $X$. Then $\Delta(1)<_{B}$ $\Delta(2)<_{B} \ldots<_{B} \Delta(\omega)<_{B} \Delta\left(2^{\omega}\right)$.

Definition 2.3. We say that $E$ is smooth iff $E \leq_{B} \Delta\left(2^{\omega}\right)$.
Fact 2.4. Sil80] (Silver dichotomy) For all Borel equivalence relations $E$ on Polish spaces, $E \leq_{B}$ $\Delta(\omega)$ or $\Delta\left(2^{\omega}\right) \sqsubseteq_{c} E$. It follows that $\Delta\left(2^{\omega}\right)$ is the successor of $\Delta(\omega)$.

Fact 2.5. Closed equivalence relations are smooth.

Example 2.6. Let $\mathbb{E}_{0}$ be the following equivalence relation on the Cantor space $2^{\omega}:(\eta, \nu) \in \mathbb{E}_{0}$ iff there exists some $n<\omega$ such that for all $m>n, \eta(m)=\nu(m)$.

Fact 2.7. The relation $\mathbb{E}_{0}$ is non-smooth.
In addition, we have the following dichotomy:

Fact 2.8. HKL90 (Harrington-Kechris-Louveau dichotomy) For every Borel equivalence relation $E$ on a Polish space either $E \leq_{B} \Delta\left(2^{\omega}\right)$ (i.e., $E$ is smooth) or $\mathbb{E}_{0} \sqsubseteq_{c} E$. It follows that $\mathbb{E}_{0}$ is the successor of $\Delta\left(2^{\omega}\right)$.
2.2. The ideal embedding theorem. Suppose that $X$ is a topological space. Associated with each family $\mathcal{U}$ of open subsets of $X$ is the corresponding family $\mathcal{I}_{\mathcal{U}}$ of subsets of $X$ given by $F \in \mathcal{I}_{\mathcal{U}} \Longleftrightarrow \forall U \in \mathcal{U} \exists$ open $V \supseteq F V \cup U \in \mathcal{U}$.

Equivalently,

Remark 2.9. $F \in \mathcal{I}_{\mathcal{U}} \Longleftrightarrow \forall U \in \mathcal{U} \exists V \in \mathcal{U} F \cup U \subseteq V$.

Proposition 2.10. Suppose that $X$ is a topological space and $\mathcal{U}$ is a family of open subsets of $X$. Then $\mathcal{I}_{\mathcal{U}}$ is an ideal.

Proof. To see that $\mathcal{I}_{\mathcal{U}}$ is downward closed, note that if $F \in \mathcal{I}_{\mathcal{U}}$ and $F^{\prime} \subseteq F$, then for each $U \in \mathcal{U}$, there exists an open set $V \supseteq F \supseteq F^{\prime}$ with the property that $V \cup U \in \mathcal{U}$, thus $F^{\prime} \in \mathcal{I}_{\mathcal{U}}$.

To see that $\mathcal{I}_{\mathcal{U}}$ is closed under finite unions, note that if $F, F^{\prime} \in \mathcal{I}_{\mathcal{U}}$, then for each $U \in \mathcal{U}$, there exists an open set $V \supseteq F$ with $V \cup U \in \mathcal{U}$, so there exists an open set $V^{\prime} \supseteq F^{\prime}$ with $\left(V \cup V^{\prime}\right) \cup U \in \mathcal{U}$, thus $F \cup F^{\prime} \in \mathcal{I}_{\mathcal{U}}$.

Proposition 2.11. Suppose that $X$ is a topological space, $\Gamma$ is a group of homeomorphisms of $X$, $Y \subseteq X$ is $\Gamma$-invariant, and $\mathcal{U}$ is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ with $Y \subseteq \Delta \cdot U$. Then for all $\mathcal{I}_{\mathcal{U}}$-positive sets $F \subseteq X$ and all open sets $W \supseteq F$, there is a finite set $\Delta \subseteq \Gamma$ such that whenever $I$ is a finite set, $\left\langle F_{i} \mid i \in I\right\rangle$ is a finite sequence of subsets of $X$ whose union contains $F$, and $\left\langle\lambda_{i} \mid i \in I\right\rangle$ is a sequence of elements of $\Gamma$, there exists $\delta \in \Delta$ and $i \in I$ for which $\overline{\delta \cdot W} \cap \lambda_{i} \cdot F_{i}$ is $\mathcal{I}_{\mathcal{U}}$-positive.

Proof. We will use Remark 2.9]. Fix $U \in \mathcal{U}$ such that for no $V \in \mathcal{U}$ is $F \cup U \subseteq V$. Then there is a finite set $\Delta \subseteq \Gamma$ with $Y \subseteq \Delta \cdot(U \cup W)$. In light of Proposition 2.10, it is sufficient to show that there is no finite set $I$, sequence $\left\langle F_{i} \mid i \in I\right\rangle$ of subsets of $X$ whose union contains $F$, and sequence $\left\langle\lambda_{i} \mid i \in I\right\rangle$ such that $\overline{\Delta \cdot W} \cap \bigcup_{i \in I} \lambda_{i} \cdot F_{i} \in \mathcal{I}_{\mathcal{U}}$.

Suppose, towards a contradiction, that there is such a triple. Then $Y \backslash \overline{\Delta \cdot W} \subseteq \Delta \cdot U$, so $\sim \overline{\Delta \cdot W} \cup \bigcup_{i \in I} \lambda_{i} \cdot U \in \mathcal{U}$. Fix $V \in \mathcal{U}$ with $\bigcup_{i \in I} \lambda_{i} \cdot\left(U \cup F_{i}\right) \subseteq\left(\overline{\Delta \cdot W} \cap \bigcup_{i \in I} \lambda_{i} \cdot F_{i}\right) \cup(\sim \overline{\Delta \cdot W} \cup$ $\left.\bigcup_{i \in I} \lambda_{i} \cdot U\right) \subseteq V$. Then $\bigcup_{i \in I} \lambda_{i}^{-1} \cdot V \in \mathcal{U}$ and $F \cup U \subseteq \bigcup_{i \in I} \lambda_{i}^{-1} \cdot V$, a contradiction.

Theorem 2.12. Suppose that $X$ is a complete metric space, $\Gamma$ is a group of homeomorphisms of $X, Y \subseteq X$ is $\Gamma$-invariant, $\mathcal{U}$ is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ such that $Y \subseteq \Delta \cdot U,\left\langle R_{n} \mid n \in \mathbb{N}\right\rangle$ is an increasing sequence of reflexive symmetric closed subsets of $X \times X$, and there is a compact $\mathcal{I}_{\mathcal{U}}$-positive set $K \subseteq X$ with the following properties:
(a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_{n}^{(4)} \gamma \cdot x$.
(b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in \Gamma \cdot K x R_{n} \gamma \cdot x$.

Then for some $x \in K$ there is a continuous injective homomorphism $\phi: 2^{\omega} \rightarrow \overline{\Gamma \cdot x}$ from $\left(\mathbb{E}_{0}, \sim \mathbb{E}_{0}\right)$ into $\left(E_{\Gamma}^{X}, \sim \bigcup_{n \in \mathbb{N}} R_{n}\right)$.

Proof. Let $\mathcal{V}$ denote the family of open sets $V \subseteq X$ containing compact $\mathcal{I}_{\mathcal{U}}$-positive subsets of $K$. We recursively construct $V_{n} \in \mathcal{V}$ and $\gamma_{n} \in \Gamma$, from which we define $\gamma_{s}=\prod_{i<n} \gamma_{i}^{s(i)}$ for $s \in 2^{<\omega}$, so as to ensure that at stage $n$ of the construction, the following conditions are satisfied:
(1) $\forall m<n \overline{V_{m+1}} \cup \gamma_{m} \cdot \overline{V_{m+1}} \subseteq V_{m}$.
(2) $\forall m<n \forall s \in 2^{m+1} \operatorname{diam}\left(\gamma_{s} \cdot V_{m+1}\right) \leq 1 / m$.
(3) $\forall m<n \forall s, t \in 2^{m}\left(\left(\gamma_{s} \cdot V_{m+1}\right) \times\left(\gamma_{t} \gamma_{m} \cdot V_{m+1}\right)\right) \cap R_{m}=\emptyset$.

We begin by setting $V_{0}=X$.

Suppose now that $n \in \mathbb{N}$ and we have found $V_{n}$ and $\left\langle\gamma_{i} \mid i<n\right\rangle$. Fix an $\mathcal{I}_{\mathcal{U}}$-positive compact set $L \subseteq K$ contained in $V_{n}$, as well as an open set $W \supseteq L$ for which $\bar{W} \subseteq V_{n}$. By Proposition 2.11, there is a finite set $\Delta \subseteq \Gamma$ such that whenever $I$ is a finite set, $\left\langle L_{i} \mid i \in I\right\rangle$ is a sequence of compact sets whose union is $L$, and $\left\langle\lambda_{i} \mid i \in I\right\rangle$ is a sequence of elements of $\Gamma$, there are $\delta \in \Delta$ and $i \in I$ for which $\overline{\delta \cdot W} \cap \lambda_{i} \cdot L_{i}$ is $\mathcal{I}_{\mathcal{U}}$-positive. Condition (b) yields $m \geq n$ such that $\forall x \in \Gamma \cdot K \forall \gamma \in$ $\left\{\gamma_{s} \mid s \in 2^{n}\right\} \cup \Delta^{-1} x R_{m} \gamma \cdot x$. In particular, it follows that:
$\left(^{*}\right)$ If $x \in K, \lambda \in \Gamma$ and $\neg x R_{m}^{(4)} \lambda \cdot x$, then for no $\delta \in \Delta$ and $s, t \in 2^{n}$ is it the case that $\gamma_{s} \cdot x R_{m} \gamma_{t} \delta^{-1} \lambda \cdot x$.

Thus condition (a) yields a finite set $I$, a sequence $\left\langle L_{i} \mid i \in I\right\rangle$ of compact subsets of $X$ whose union is $L$, and a sequence $\left\langle\lambda_{i} \mid i \in I\right\rangle$ of elements of $\Gamma$ with

$$
\forall \delta \in \Delta \forall i \in I \forall s, t \in 2^{n}\left(\gamma_{s} \cdot L_{i} \times \gamma_{t} \delta^{-1} \lambda_{i} \cdot L_{i}\right) \cap R_{m}=\emptyset
$$

Fix $\delta \in \Delta$ and $i \in I$ such that $\overline{\lambda_{i}^{-1} \delta \cdot W} \cap L_{i}$ is $\mathcal{I}_{\mathcal{U}}$-positive, and define $\gamma_{n}=\delta^{-1} \lambda_{i}$. Then $\forall s, t \in 2^{n}\left(\left(\gamma_{s} \cdot L_{i}\right) \times\left(\gamma_{t} \gamma_{n} \cdot L_{i}\right)\right) \cap R_{m}=\emptyset$. Proposition 2.10 ensures that by replacing $L_{i}$ with a compact $\mathcal{I}_{\mathcal{U}}$-positive subset of $\overline{\lambda_{i}^{-1} \delta \cdot W} \cap L_{i}$, we can assume that $\forall s \in 2^{n+1} \operatorname{diam}\left(\gamma_{s} \cdot L_{i}\right)<1 / n$. It follows that there is an open set $V_{n+1} \subseteq X$ containing $L_{i}$ such that $\overline{V_{n+1}} \cup\left(\gamma_{n} \cdot \overline{V_{n+1}}\right) \subseteq V_{n}$, $\forall s \in 2^{n+1} \operatorname{diam}\left(\gamma_{s} \cdot V_{n+1}\right) \leq 1 / n$, and $\forall s, t \in 2^{n}\left(\left(\gamma_{s} \cdot V_{n+1}\right) \times\left(\gamma_{t} \gamma_{n} \cdot V_{n+1}\right)\right) \cap R_{n}=\emptyset$. This completes the recursive construction.

Conditions (1) and (2) ensure that we obtain a continuous function $\phi: 2^{\omega} \rightarrow X$ by insisting that $\{\phi(c)\}=\bigcap_{n \in \mathbb{N}} \gamma_{c \uparrow n} \cdot V_{n}$ for all $c \in 2^{\omega}$. To see that $\phi$ is a homomorphism from $\mathbb{E}_{0}$ to $E_{\Gamma}^{X}$, suppose that $c \in 2^{\omega}, k \in \mathbb{N}$, and $s \in 2^{k}$, and observe that

$$
\left\{\gamma_{s} \cdot \phi\left((0)^{k} \frown c\right)\right\}=\bigcap_{n \in \mathbb{N}} \gamma_{s} \gamma_{\left((0)^{k} \frown c\right) \upharpoonright n} \cdot V_{n}=\{\phi(s \frown c)\}
$$

To see that $\phi$ is an injective homomorphism from $\sim \mathbb{E}_{0}$ to $\sim \bigcup_{n \in \mathbb{N}} R_{n}$, suppose that $c, d \in 2^{\omega}$, $n \in \mathbb{N}, c(n)=0$, and $d(n)=1$, and observe that $\phi(c) \in \gamma_{c \upharpoonright n} \cdot V_{n+1}$ and $\phi(d) \in \gamma_{d \upharpoonright n} \gamma_{n} \cdot V_{n+1}$, in which case condition (3) ensures that $\neg \phi(c) R_{n} \phi(d)$. Finally, set $x=\phi\left((0)^{\infty}\right)$ and note that $\phi\left[2^{\omega}\right] \subseteq \overline{\Gamma \cdot x}$ and $x \in K$.

We give a slight variant of Theorem [2.12, adding an extra assumption.

Theorem 2.13. Suppose that $X, \Gamma, Y, \mathcal{U},\left\langle R_{n} \mid n \in \mathbb{N}\right\rangle$ are as in Theorem 2.12. Suppose that there is a compact $\mathcal{I}_{\mathcal{U}}$-positive set $K \subseteq X$ with the following properties:
(a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_{n}^{(2)} \gamma \cdot x$.
(b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in K x R_{n} \gamma \cdot x$. Note that this condition is weaker than (b) in Theorem 2.12.
(c) $\forall \gamma \in \Gamma \forall x, y \in \Gamma \cdot K \forall n \in \mathbb{N} x R_{n} y \Rightarrow \gamma \cdot x R_{n} \gamma \cdot y$.

Then for some $x \in K$ there is a continuous injective homomorphism $\phi: 2^{\omega} \rightarrow \overline{\Gamma \cdot x}$ from $\left(\mathbb{E}_{0}, \sim \mathbb{E}_{0}\right)$ into $\left(E_{\Gamma}^{X}, \sim \bigcup_{n \in \mathbb{N}} R_{n}\right)$.

Proof. The proof is parallel to the proof of Theorem 2.12, reading the same up to $\left(^{*}\right)$, but we choose $m$ so that $\forall x \in K \forall \gamma_{1}, \gamma_{2} \in\left\{\gamma_{s} \mid s \in 2^{n}\right\} \forall \delta \in \Delta x R_{m} \delta \gamma_{1}^{-1} \gamma_{2} \cdot x$. By (c), we get:
$\left(^{* *}\right)$ If $x \in K, \lambda \in \Gamma$ and $\neg x R_{m}^{(2)} \lambda \cdot x$, then for no $\delta \in \Delta$ and $s, t \in 2^{n}$ is it the case that $\gamma_{s} \cdot x R_{m} \gamma_{t} \delta^{-1} \lambda \cdot x$.
The rest of the proof is exactly the same.
2.3. Choquet spaces. The proof of Theorem 2.12 easily goes through in the context of strong Choquet spaces.

Definition 2.14. The Choquet game on a topological space $X$ is a two player game in $\omega$-many rounds. In round $n$, player A chooses a nonempty open set $U_{n} \subseteq V_{n-1}$ (where $V_{-1}=X$ ), and player B responds by choosing a nonempty open subset $V_{n} \subseteq U_{n}$. Player B wins if the intersection $\bigcap\left\{V_{n} \mid n<\omega\right\}$ is not empty.

The strong Choquet game is similar: in round $n$ player A chooses an open set $U_{n} \subseteq V_{n-1}$ and $x_{n} \in U_{n}$, and player B responds by choosing an open set $V_{n} \subseteq U_{n}$ containing $x_{n}$. Again, player B wins when the intersection $\bigcap\left\{V_{n} \mid n<\omega\right\}$ is not empty.

A topological space $X$ is a (strong) Choquet space if player B has a winning strategy in every (strong) Choquet game.

It is easy to see that:

Example 2.15. Every Polish space is strong Choquet.
But for our purposes, we shall need the following example:

Example 2.16. If $X$ is compact (not necessarily Hausdorff) and has a basis consisting of clopen sets then it is strong Choquet.

Proof. In round $n$, player B will choose a clopen set $x_{n} \in V_{n} \subseteq U_{n}$. By compactness, the intersection $\bigcap\left\{V_{n} \mid n<\omega\right\}$ is not empty.

Fact 2.17. (see e.g., KMS13) If $X$ is strong Choquet and $\emptyset \neq U \subseteq X$ is $G_{\delta}$, then $U$ is also strong Choquet.

Theorem 2.18. Suppose that $X$ is a regular strong Choquet space, $\Gamma$ is a group of homeomorphisms of $X, Y \subseteq X$ is $\Gamma$-invariant, $\mathcal{U}$ is the family of open sets $U \subseteq X$ for which there is no finite set $\Delta \subseteq \Gamma$ such that $Y \subseteq \Delta \cdot U,\left\langle R_{n} \mid n \in \mathbb{N}\right\rangle$ is an increasing sequence of reflexive symmetric closed subsets of $X \times X$, and there is a compact $\mathcal{I}_{\mathcal{U}}$-positive set $K \subseteq X$ with the following properties:
(a) $\forall n \in \mathbb{N} \forall x \in K \exists \gamma \in \Gamma \neg x R_{n}^{(4)} \gamma \cdot x$.
(b) $\forall \gamma \in \Gamma \exists n \in \mathbb{N} \forall x \in \Gamma \cdot K x R_{n} \gamma \cdot x$.

Then there is a map $\phi: 2^{\omega} \rightarrow \mathcal{P}(X)$ such that for every $y, z \in 2^{\omega}$ :

- $\phi(y)$ is a nonempty closed $G_{\delta}$ subset of $X$.
- If $z \mathbb{E}_{0} y$ then there is some $\gamma \in \Gamma$ such that $\gamma \cdot \phi(z)=\phi(y)$.
- If $\sim z \mathbb{E}_{0} y$ then $(\phi(y) \times \phi(z)) \subseteq \sim \bigcup_{n \in \mathbb{N}} R_{n}$.

In particular, there is homomorphism $\phi: 2^{\omega} \rightarrow X$ from $\left(\mathbb{E}_{0}, \sim \mathbb{E}_{0}\right)$ into $\left(E_{\Gamma}^{X}, \sim \bigcup_{n \in \mathbb{N}} R_{n}\right)$.
Moreover, if $X$ is compact, then we can choose $\phi$ so that its range is $\overline{\Gamma \cdot x}$ for some $x \in K$.

Proof. Fix a winning strategy for Player B in the strong Choquet game. The main point is that in the construction done in the proof of Theorem 2.12, instead of decreasing the diameter of the open sets, we choose them according to the strategy. So in addition to choosing $V_{n}$ and $\gamma_{n}$, we also choose points $x_{n} \in X$ and open sets $U_{n} \in \mathcal{V}$ (the family of open sets containing compact $\mathcal{I}_{\mathcal{U}}$-positive subsets of $\left.K\right)$ such that $x_{n} \in U_{n} \subseteq V_{n}$, and the new construction will satisfy:
(1) $\forall m<n \overline{U_{m}} \cup \gamma_{m} \cdot \overline{U_{m}} \subseteq V_{m}$.
(2) $\forall m<n \forall s, t \in 2^{m}\left(\left(\gamma_{s} \cdot U_{m}\right) \times\left(\gamma_{t} \gamma_{m} \cdot U_{m}\right)\right) \cap R_{m}=\emptyset$.
(3) $\forall m<n$ and $\forall s \in 2^{m+1}, \gamma_{s} \cdot V_{m+1}$ is contained in an open set which is played according to Player B's strategy in the Strong Choquet game in which Player A plays $\left\langle\left(\gamma_{s\lceil i+1} \cdot U_{i}, \gamma_{s\lceil i+1} \cdot x_{i}\right) \mid i \leq m\right\rangle$ and Player B plays according to his strategy.

For the construction, we follow the proof of Theorem 2.12, and note that:

Claim. Suppose $L$ is an $I_{\mathcal{U}}$-positive compact set contained in some open set $U$, and suppose $\Delta$ is a finite subset of $\Gamma$. Furthermore, suppose that for any $\gamma \in \Delta, \gamma \cdot U$ is contained in an open set which is chosen by Player B in some finite strong Choquet play according to his strategy. Then, there is some $x \in L$ such that if Player A plays $(\gamma \cdot U, \gamma \cdot x)$ then there is a set $V \in \mathcal{V}$ contained in $U$ such that $\gamma \cdot V$ is contained in Player B's response for all $\gamma \in \Delta$.

Proof. Indeed, for each point $x \in L$, let Player A play $(\gamma \cdot U, \gamma \cdot x)$, and let $\gamma \cdot U_{\gamma, x}$ be Player B's response. Let $U_{x}=\bigcap_{\gamma \in \Delta} U_{x, \gamma}$, and let $U_{x}^{\prime}$ be such that $x \in U_{x}^{\prime}$ and $\overline{U_{x}^{\prime}} \subseteq U_{x}$. By compactness and by Proposition 2.10, for some $x \in L, \overline{U_{x}^{\prime}} \cap L$ is $I_{\mathcal{U}}$-positive. Let $V=U_{x}$.

Now we let $U_{n}$ be the set denoted $V_{n+1}$ in the proof of Theorem 2.12 (without the condition on the diameter), and proceed using the claim. Finally, we let $\phi(c)=\bigcap_{n \in \mathbb{N}} \gamma_{c \upharpoonright n} \cdot V_{n}$.

For the moreover part, choose any $x \in K \cap \phi\left((0)^{\omega}\right)$, and note that by compactness $\overline{\Gamma \cdot x} \cap \phi(c) \neq$ $\emptyset$.

We also have an analog to Theorem 2.13, which we state briefly.

Theorem 2.19. Suppose that $X, \Gamma, Y, \mathcal{U},\left\langle R_{n} \mid n \in \mathbb{N}\right\rangle$ are as in Theorem 2.18. Suppose that there is a compact $\mathcal{I}_{\mathcal{U}}$-positive set $K \subseteq X$ satisfying the assumptions of Theorem 2.13.

Then the conclusion of Theorem 2.18 hold.
Problem 2.20. All the applications we found use a weak version of Theorem [2.12, i.e., we apply it with (a) replaced by: $\forall n \in \mathbb{N} \exists \gamma \in \Gamma \forall x \in K \neg x R_{n}^{(4)} \gamma \cdot x$. Is there an interesting application that uses the full strength of the theorem?

## 3. Applications

3.1. Application to compact group actions. Most of our applications will be model theoretic, but we start with a simple topological one.

Corollary 3.1. Suppose that $G$ is a compact topological group and that $\cdot: G \times X \rightarrow X$ is a continuous action on a complete metric space $X$. Let $H$ a subgroup of $G$, and suppose $H=$ $\bigcup_{n<\omega} V_{n}$ where $V_{n}$ are closed subsets of $G, e \in V_{n}^{-1}=V_{n}, V_{n}^{2} \subseteq V_{n+1}$. Then, if there is some $x \in X$ such that $H \cdot x \neq V_{n} \cdot x$ for all $n<\omega$, then $\mathbb{E}_{0} \sqsubseteq_{c} E_{H}^{X}$. If not and $X$ is Polish then $E_{H}^{X}$ is smooth.

Proof. First assume that there is such an $x \in X$. Let $Y=H \cdot x, R_{n}=\left\{(x, y) \in X^{2} \mid \exists h \in V_{n}(y=h \cdot x)\right\}$, $\Gamma=H$ and $K=\{x\}$. All the conditions of Theorem 2.12 but the condition that $K$ is $I_{\mathcal{U}}$-positive hold trivially (note that $R_{n}$ is closed by the compactness of $V_{n}$ ). To show that $K$ is $I_{\mathcal{U}}$-positive it is enough to see that for any open $x \in V \subseteq X$, there is some finite $\Delta \subseteq H$ such that $\Delta \cdot V \supseteq Y$. Suppose not. Recursively choose $h_{n} \in H$ for $n<\omega$ such that $h_{n} \cdot x \notin \bigcup_{i<n} h_{i} \cdot V$.

Let $\kappa=|G|^{+}$, and let $L=G^{\kappa}$ equipped with the product topology (so it is compact). For a finite $s \subseteq \kappa$, let $F_{s}=\left\{\eta \in L \mid \forall i \in s\left(\eta(i) \cdot x \notin \bigcup_{j<i, j \in s} \eta(j) \cdot V\right)\right\}$. By the construction above, this is a closed nonempty set. By compactness, there is some $\eta \in \bigcap_{s \subseteq \kappa,|s|<\omega} F_{s}$. In particular $\eta: \kappa \rightarrow G$ is injective - contradiction.

Now assume that $X$ is Polish and that there is no such $x$ but $E_{H}^{X}$ is not smooth. By assumption, for all $x \in X, H \cdot x=V_{n} \cdot x$ for some $n<\omega$ and as $G$ is compact it follows that all classes are compact, so also $G_{\delta}$. But a Borel equivalence relation $E$ with $G_{\delta}$ classes on a Polish space $X$ must be smooth. Otherwise Fact 2.8 gives us a continuous embedding of $\mathbb{E}_{0}$ into $E$, and it follows that every $\mathbb{E}_{0}$ class is $G_{\delta}$. But $\mathbb{E}_{0}$ classes are also dense - contradiction.

Corollary 3.2. If $G$ is a compact complete metric group, and $H$ is an $F_{\sigma}$ subgroup, then either $H$ is closed (in which case, if $G$ is Polish, $E_{H}^{G}$ is smooth), or $\mathbb{E}_{0} \sqsubseteq_{c} E_{H}^{G}$.
3.2. Applications to model theory. In applying Theorem 2.12 or any of its variations, we need to find the space $X$, the set $Y$, the group $\Gamma$, the closed sets $R_{n}$ and the compact $I_{\mathcal{U}}$-positive set $K$. In all our applications, $X$ will be some subspace of $S_{\alpha}(M)$ for model $M$, invariant under $E$,
$Y$ will be the projection of some $E$-class $C, R_{n}$ will be the projections of $U_{n}$ (from $E$ 's normal form), $\Gamma$ will be some group of homeomorphisms of $X$, which is either induced by automorphisms of the model $M$ or by a type definable model theoretic group and $K$ will be the projection of some type of the form $U_{n}(x, a)$. The main point is to show that $K$ is $I_{\mathcal{U}}$-positive, which we will call here "proper".
3.2.1. Preliminaries. We briefly introduce our notation, which is fully explained in KMS13.

- $T$ is a complete (perhaps many sorted) first order theory.
- $\alpha$ is some ordinal.
- $S_{\alpha}(A)$ is the Stone space of complete $\alpha$-types over $A$, which comes equipped with a compact Hausdorff topology, and $L_{\alpha}(A)$ is the set of formulas in the first $\alpha$ variables.
- $\mathfrak{C}$ is a monster model of $T$ - a $\kappa$-saturated, $\kappa$-homogeneous model where $\kappa$ is a big cardinal.
- All parameter sets and models considered will be small (i.e., of cardinality less than $\kappa$ ) subsets and elementary substructures of $\mathfrak{C}$.
- $\equiv$ is equality of types, $\equiv_{L}^{\alpha}$ is equality of Lascar strong types of $\alpha$-tuples (if $A$ is a small set, then $\equiv_{A}$ denotes types equality over $A$, etc.).
- Aut $(\mathfrak{C} / A)$ is the group of automorphisms of $\mathfrak{C}$ that fix $A$ pointwise, and an $A$-invariant subset of $\mathfrak{C}^{\alpha}$ is one invariant under the action of this group.
- A subset $X \subseteq \mathfrak{C}^{\alpha}$ is pseudo closed if $X$ is type definable over some small set. A pseudo open set is a complement of a pseudo closed set. Pseudo $G_{\delta}$ sets and pseudo $F_{\sigma}$ sets are defined in the obvious way.
- If $Y \subseteq \mathfrak{C}^{\alpha}$ is some set, and $M$ some model then $Y_{M}=\left\{p \in S_{\alpha}(M) \mid \exists a \in Y(p=\operatorname{tp}(a / M))\right\}$. This is also denoted by $S_{M}(Y)$.

We also recall the notion of an indiscernible sequence:

Definition 3.3. Let $A$ be a small set. Let $(I,<)$ be some linearly ordered set. A sequence $\bar{a}=\left\langle a_{i} \mid i \in I\right\rangle \in\left(\mathfrak{C}^{\alpha}\right)^{I}$ is called $A$-indiscernible (or indiscernible over $A$ ) if for all $n<\omega$, every increasing $n$-tuple from $\bar{a}$ realizes the same type over $A$. When $A$ is omitted, it is understood that $A=\emptyset$.

Also recall:

## Fact 3.4.

(1) [TZ12, Lemma 5.1.3] Let $\left(I,<_{I}\right),\left(J,<_{J}\right)$ be small linearly ordered sets, and let $A$ be some small set. Suppose $\bar{b}=\left\langle b_{j} \mid j \in J\right\rangle$ is some sequence of elements from $\mathfrak{C}^{\alpha}$. Then there exists an indiscernible sequence $\bar{a}=\left\langle a_{i} \mid i \in I\right\rangle \in\left(\mathfrak{C}^{\alpha}\right)^{I}$ such that:

- For any $n<\omega$ and $\varphi \in L_{\alpha \cdot n}$, if $\mathfrak{C} \models \varphi\left(b_{j_{0}}, \ldots, b_{j_{n-1}}\right)$ for every $j_{0}<_{J} \ldots<_{J} j_{n-1}$ from $J$ then $\mathfrak{C} \models \varphi\left(a_{i_{0}}, \ldots, a_{i_{n-1}}\right)$ for every $i_{0}<_{I} \ldots<_{I} i_{n-1}$ from $I$.
(2) Ker07, proof of Proposition 3.1.4] If $M$ is a small model and $a \equiv_{M} b$, then there is an indiscernible sequence $\bar{c}=\left\langle c_{i} \mid i<\omega\right\rangle$ such that both $a \frown \bar{c}$ and $b \frown \bar{c}$ are indiscernible.
3.2.2. Equivalence relations on $\mathfrak{C}^{\alpha}$.

Definition 3.5. An equivalence relation $E$ on a set $X$ is called bounded if $|X / E|<\kappa$.

See KMS13, Remark 1.12] for a discussion of bounded invariant equivalence relations.
Suppose that $A$ is some small set, $X \subseteq \mathfrak{C}^{\alpha}$ is type definable over $A$, and that $E$ is some $\emptyset$-invariant relation on $\mathfrak{C}^{\alpha \cdot 2}$ such that $E \upharpoonright X$ is a bounded equivalence relation on $X$.

Definition 3.6. Let $M \supseteq A$ be any model. For $p, q \in S_{X}(M)$, we write $p E^{M} q$ iff $\exists a \models p, b \models$ $q(a E b)$.

Note that this does not depend on the choice of representatives, i.e.,:
Proposition 3.7. For $p, q \in S_{X}(M), p E^{M} q$ iff $\forall a \models p, \forall b \models q(a E b)$.
Proof. Since $E$ is bounded, $\equiv_{L, A}^{\alpha}$ refines it on $X$, so if $a \equiv_{M} b$ for $a, b \in X$ then $a E b$.
Remark 3.8. Suppose $Y \subseteq X$ is pseudo $G_{\delta}$. For a model $M, Y_{M}$ is not necessarily $G_{\delta}$. But in case $Y$ is $\equiv_{L, A}^{\alpha}$-invariant and $A \subseteq M$, it is. Indeed, $\mathfrak{C}^{\alpha} \backslash Y$ is pseudo $F_{\sigma}$, and so $\left(\mathfrak{C}^{\alpha} \backslash Y\right)_{M}$ is $F_{\sigma}$. But since $\equiv_{M}$ refines $\equiv_{L, A}^{\alpha}$, $\left(\mathfrak{C}^{\alpha} \backslash Y\right)_{M} \cap Y_{M}=\emptyset$. In addition, if $A, T$ and $\alpha$ are countable, $Y$ is pseudo closed and $\equiv_{L, A}^{\alpha}$-invariant, then $Y_{M}$ is $G_{\delta}$, so $Y$ is pseudo $G_{\delta}$. In fact, in that case $Y$ is type definable over $M$.

Assume that $E$ is pseudo $F_{\sigma}$. This is equivalent to saying that there are $\emptyset$-type definable sets $U_{n} \subseteq \mathfrak{C}^{\alpha \cdot 2}$ for $n<\omega$ such that $E=\bigcup\left\{U_{n} \mid n<\omega\right\}$ (this follows by compactness, as $E$ is Ø-invariant). In this case the set $U_{n}^{M}=\pi\left(U_{n, M}\right) \subseteq S_{\alpha}(M)^{2}$ is closed (where $\pi: S_{\alpha \cdot 2}(M) \rightarrow$ $S_{\alpha}(M)^{2}$ is the projection) and hence $E_{M}=\bigcup\left\{U_{n}^{M} \mid n<\omega\right\}$ is $F_{\sigma}$. We assume that the sequence $\left\langle U_{n} \mid n<\omega\right\rangle$ is in normal form, i.e., $U_{0}$ contains the diagonal $\Delta_{X}, U_{n}$ is symmetric and:

$$
U_{n} \circ U_{n} \upharpoonright X=\left\{(a, b) \in X^{2} \mid \exists c \in X(a, c) \in U_{n} \wedge(c, b) \in U_{n}\right\} \subseteq U_{n+1} .
$$

So the $U_{n}$ are increasing on $X$.

Definition 3.9. Suppose $Y \subseteq X$ is $E$ invariant. We say $E$ is strongly closed on $Y$ if there exists some $n<\omega$ such that $E \upharpoonright Y=Y^{2} \cap U_{n}$. Note that this may depend on the choice of the $U_{n}$ 's.
3.2.3. Countable language. Suppose $T$ and $\alpha$ are countable. In this setting we will translate our relation $E$ into an $F_{\sigma}$ relation on $X_{M}$, as was done in KPS12].

For a countable model $A \subseteq M, S_{\alpha}(M)$ is Polish and if $Y$ is as in Remark 3.8 then $Y_{M}$ is a Polish space (every $G_{\delta}$ set is), and similarly to KMS13, Proposition 1.41] (with the same proof as there) we have:

Proposition 3.10. Fix a pseudo $G_{\delta}$ set $Y \subseteq X$, such that $Y$ is $E$-invariant. Then for any two models $A \subseteq M, N$ we have:

$$
E^{M} \upharpoonright Y_{M} \sim_{B} E^{N} \upharpoonright Y_{N}
$$

So with this assumption and Proposition 3.10, we can refer to the Borel cardinality of the $F_{\sigma}$ equivalence relation $E \upharpoonright Y$ without specifying the model.
3.2.4. Countable or uncountable language. Let $T$ be any complete first order theory and $\alpha$ any ordinal.

Definition 3.11. We say that a set $Y \subseteq \mathfrak{C}^{\alpha}$ for some small $\alpha$ is pseudo strong Choquet if $Y_{M}$ is strong Choquet for all $M$.

Example 3.12. If $Y \subseteq \mathfrak{C}^{\alpha}$ is pseudo closed or pseudo $G_{\delta}$ and $\equiv_{L}^{\alpha}$-invariant, then by Remark 3.8 and Proposition 2.17 it is pseudo strong Choquet.

Remark 3.13. For countable $T$ and $\alpha$, "pseudo strong Choquet" is the correct analog of pseudo $G_{\delta}$ for $\equiv_{L}^{\alpha}$-invariant sets. This follows from [Kec95, Theorem 8.17].
3.2.5. Invariant equivalence relations with a nice automorphism group. Let $C$ be some subset of $X$. Suppose that $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$.

Definition 3.14. (1) A formula $\varphi \in L_{\alpha}(\mathfrak{C})$ is said to be $C$-generic if finitely many translates of $\varphi$ under the action of $\Gamma$ cover $C$.
(2) The formula $\varphi$ is said to be $C$-weakly generic if there is a non- $C$-generic formula $\psi \in L_{\alpha}(\mathfrak{C})$ such that $\varphi \vee \psi$ is $C$-generic.
(3) A partial type $p \subseteq L_{\alpha}(\mathfrak{C})$ is said to be $C$-generic ( $C$-weakly generic) if all its formulas are.
(4) A partial type $p \subseteq L_{\alpha}(\mathfrak{C})$ which is is closed under conjunctions is said to be $C$-proper if there is a non- $C$-generic formula $\psi$ such that for all $\varphi \in p, \varphi \vee \psi$ is $C$-generic. In general, $p$ is $C$-proper when its closure under finite conjunctions is.

For the most part we will omit $C$ from the notation.
For $n<\omega$, let $p_{n}(x, y)$ be the type defining $U_{n}$.

Proposition 3.15. Suppose that $\Gamma$ is $C$-transitive: for all $a, b \in C$ there is some $\sigma \in \Gamma$ such that $\sigma(a)=b$. Then, for some $n<\omega$ and for all $a \in C, p_{n}(x, a)$ is proper. Moreover, there is a formula $\psi(x, y)$ such that $\psi(x, a)$ is the non-generic formula that witnesses this.

Proof. First observe that if $p_{n}(x, a)$ is proper for some $a \in C, \psi(x, a)$ witnesses this and $b \in C$, then $p_{n}(x, b)$ is proper with $\psi(x, b)$ witnessing it. So fix some $a \in C$.

Note that if $\psi(x, a)$ is not generic, then we can construct inductively a sequence $a_{i} \in C$ for $i<\omega$ such that $\neg \psi\left(a_{i}, a_{j}\right)$ for $j<i$ : let $a_{0}=a$, and for $n+1$, let $\sigma_{0}, \ldots, \sigma_{n} \in \Gamma$ be such that $\sigma_{i}(a)=a_{i}$ (so $\sigma_{0}=\mathrm{id}$ ) and let $a_{n+1} \not \vDash \bigvee_{i \leq n} \sigma(\psi(x, a))=\bigvee_{i \leq n} \psi\left(x, a_{i}\right)$. By Ramsey and compactness (Fact 3.4), there is an $A$-indiscernible sequence $\left\langle b_{i} \mid i<\omega\right\rangle$ in $X$ with the property that $\neg \psi\left(b_{i}, b_{j}\right)$ for $j<i$. Here we used the fact that $X$ is type definable.

Now suppose that for no $n<\omega$ is $p_{n}(x, a)$ proper. This allows us to inductively construct formulas $\varphi_{n}(x, a) \in p_{n}(x, a)$ such that $\bigvee_{i<n} \varphi_{n}$ is not generic. By the remark above and compactness, there is an $A$-indiscernible sequence $\left\langle b_{i} \mid i<\omega\right\rangle$ in $X$ such that for all $n<\omega$ and $j<i<\omega$, $\neg \varphi_{n}\left(b_{i}, b_{j}\right)$. But this means that $\left(b_{i}, b_{j}\right) \notin U_{n}$ for all $j<i<\omega$ and $n<\omega$, so $\neg E\left(b_{i}, b_{j}\right)$. By compactness, we may increasing the length of the sequence to any length, contradicting the fact that $E$ is bounded on $X$.

Now assume that $C$ is $\Gamma$ invariant, and fix some $a \in C$. By taking a countable union of models $M_{i}$ and a countable union of subsets $\Gamma_{i}$ of $\Gamma$, we can find a model $M$ of size $|A|+|L|+|\alpha|$ containing $A$ and a subgroup $\Gamma^{*} \leq \Gamma$ of the size $|\alpha|+|L|$ such that:
(1) $\{a\} \cup A \subseteq M$.
(2) For all $\sigma \in \Gamma^{*}, \sigma(M)=M$ setwise.
(3) If $\varphi$ is a formula over $a$ which is generic, then there are finitely many elements from $\Gamma^{*}$ which witness this.

Recall that the Stone space $S_{\alpha}(M)$ has a natural topology in which basic open sets are of the form $[\varphi]=\left\{p \in S_{\alpha}(M) \mid \varphi \in p\right\}$. When $r$ is a partial type, i.e., a consistent set of formulas over $M$, we denote by $[r]$ the set $\left\{p \in S_{\alpha}(M) \mid r \subseteq p\right\}$. This set is compact.

By (2) above, $\Gamma^{*}$ is a group of homeomorphisms of $S_{\alpha}(M)$.

Lemma 3.16. Suppose $[a]_{E} \subseteq C \subseteq Y \subseteq X$ is $\Gamma$ invariant and that $\Gamma$ is $C$-transitive. Let $\mathcal{U}$ be the family of open sets $U \subseteq Y_{M}$ for which there is no finite set $\Delta \subseteq \Gamma^{*}$ with $C_{M} \subseteq \Delta \cdot U$ (all in the induced Stone space topology). Then for some $n<\omega$, the compact set $\left[p_{n}(x, a)\right] \subseteq Y_{M}$ is $I_{\mathcal{U}}$-positive.

Proof. By Proposition 3.15, for some $n<\omega, p_{n}(x, a)$ is proper. By (3) above, if a formula $\varphi$ over $M$ is generic then $[\varphi] \cap Y_{M} \notin \mathcal{U}$ and the converse also holds. Unwinding the definitions, the proposition is clear.

Assume now that $E \upharpoonright C$ is not strongly closed, that $\Gamma$ is $C$-transitive and that $C=[a]_{E}$. Since $U_{n}$ is $\emptyset$-invariant and $\Gamma$ is $C$-transitive, this means that for any $n<\omega$, there is some $b \in C$ such that $(a, b) \notin U_{n}$. By enlarging $\Gamma^{*}$ and $M$, we may assume:
(4) For all $n<\omega$ there is $\sigma \in \Gamma^{*}$ such that $(a, \sigma(a)) \notin U_{n}$.

We are now ready to state our result:

Theorem 3.17. Assume that $T, A \subseteq \mathfrak{C}$ and $\alpha$ are countable. Suppose that:
(1) $X \subseteq \mathfrak{C}^{\alpha}$ is some type definable set over $A$.
(2) $E$ is a pseudo $F_{\sigma} \emptyset$-invariant equivalence relation on $X$ with normal form $\left\langle U_{n} \mid n<\omega\right\rangle$ and $E$ is bounded on $X$.
(3) $C \subseteq X$ is an $E$ class, and $E \upharpoonright C$ is not strongly closed (with respect to $\left\langle U_{n} \mid n<\omega\right\rangle$ ).
(4) $C \subseteq Y \subseteq X$ is pseudo $G_{\delta}$ and $E$ invariant.
(5) $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ is $C$-transitive, and preserves all $E$-classes (in particular, it preserves $X$ ).

Then $E \upharpoonright Y$ is not smooth (see Proposition 3.10).

Proof. Keeping the notation from above, this follows directly from Theorem [2.13 with $X$ there being $Y_{M}$ (note that it is $\Gamma^{*}$ invariant by assumptions (4) and (5) and that it is Polish by (4) and Remark (3.8), $\Gamma$ there being $\Gamma^{*}$ here, $Y$ there being $C_{M}$ here, $R_{n}$ there being $U_{n}^{M} \upharpoonright Y_{M}$ here and $K$ there being $\left[p_{k}(x, a)\right]$ for some $k<\omega$, chosen by Proposition 3.15 (note that as $Y$ contains $C$, $Y_{M}$ contains $\left[p_{k}(x, a)\right]$, so it is compact). By assumption (5), Theorem 2.13s $E_{\Gamma}^{X}$ is contained in $E_{M} \upharpoonright Y_{M}$, so checking that the conditions of this theorem hold will suffice:

By Lemma 3.16, $K$ is $I_{\mathcal{U}}$-positive.
Condition (a) there follows from assumption (3) here. Note that if $p \in\left[p_{m}(x, a)\right], q R_{n} p$ and $b \models q$ then $(a, b) \in U_{\max \{n, m\}+1}$ (because there is some $b^{\prime} \models q, c \vDash p$ such that $\left(b^{\prime}, c\right) \in U_{n}$, but $(c, a) \in U_{m}$ so $\left(b^{\prime}, a\right) \in U_{\max \{n, m\}+1}$ but $\left.b \equiv_{a} b^{\prime}\right)$. So $q \in\left[p_{\max \{n, m\}+1}(x, a)\right]$. From this computation it follows that if $p \in\left[p_{k}(x, a)\right]$ and $q R_{n}^{(2)} p$ for some $n \geq k$ then for all $b \models q$, $(a, b) \in U_{n+2}$. So if $\sigma \in \Gamma^{*}$ is such that $(a, \sigma(a)) \notin U_{n+3}$ for $n \geq k$, then for all $p \in\left[p_{k}(x, a)\right]$, $(p, \sigma(p)) \notin R_{n}^{(2)}$ (because for $\left.b \models p,(\sigma(a), \sigma(b)) \in U_{k}\right)$.

Condition (b) there follows similarly. As $\Gamma$ preserves $E$ classes, there is some $n<\omega$ such that $(a, \sigma(a)) \in U_{n}$. So if $p \in\left[p_{k}(x, a)\right]$, then for all $b \models p,(\sigma(b), b) \in U_{\max \{n, k\}+3}$.

Condition (c) there follows from the fact that $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ and that for all $n<\omega, U_{n}$ is $\emptyset$-invariant.

Theorem 3.18. Let T, A and $\alpha$ be of any (small) size. Then under the same conditions as Theorem 3.17 replacing (4) with:
(4) $C \subseteq Y \subseteq X$ is pseudo strong Choquet and $E$ invariant.
$E \upharpoonright Y$ has at least $2^{\aleph_{0}}$ classes.

Proof. Follows similarly from Theorem 2.19 as in the proof of Theorem 3.17.

Theorem 3.19. Suppose T, $A$ and $\alpha$ are countable, and the same assumptions as in Theorem 3.17 hold, except (4) and (5) which we replace by:
(4) $C \subseteq Y \subseteq X$ is pseudo $G_{\delta}$ and $\Gamma$ invariant.
(5) $\Gamma \leq \operatorname{Aut}(\mathfrak{C})$ is $C$-transitive, and for all $\sigma \in \Gamma$ there is some $n<\omega$ such that for all $c \in C$, $(c, \sigma(c)) \in U_{n}$.

Then $E \upharpoonright Y$ is not smooth.

Proof. To prove this theorem we could use either Theorem 2.12 or Theorem 2.13 similarly to the proof of Theorem 3.17. The conditions there hold, but since $\Gamma$ may not preserve $E$ classes, it is not clear that $E_{\Gamma}^{X}$ is contained in $E^{M} \upharpoonright Y_{M}$. To solve this problem, we note that for any $x \in K$ (which is just $\left[p_{k}(x, a)\right]$ for some $\left.a \in C, k<\omega\right), E_{\Gamma}^{X} \upharpoonright \overline{\Gamma \cdot x}$ is contained in $E^{M} \upharpoonright Y_{M}$, and recall that the the image of the embedding $\phi$ of either Theorem 2.12 or Theorem 2.13 is into $\overline{\Gamma \cdot x}$ for some $x \in K$.

Indeed, fix some $p \in\left[p_{k}(x, a)\right]$ and $\sigma \in \Gamma^{*}$, and let $n<\omega$ correspond to (5). Then for any $q \in \overline{\Gamma^{*} \cdot p},(\sigma(q), q) \in U_{n}^{M}$ as this is a closed condition.

As above we give a general analog (using Theorem [2.18 or Theorem 2.19). Unfortunately, in this case, being pseudo strong Choquet is not enough in order to prove the theorem since we do not know that the range of $\phi$ can be chosen to be $\overline{\Gamma \cdot x}$.

Theorem 3.20. Let T, A and $\alpha$ be of any (small) size. Then under the same conditions as Theorem 3.19 replacing (4) with:
(4) $C \subseteq Y \subseteq X$ is pseudo closed and and $\Gamma$ invariant.
$E \upharpoonright Y$ has at least $2^{\aleph_{0}}$ classes.

Corollary 3.21. For $E=\equiv_{L}^{\alpha}$, the group Aut $f_{L}(\mathfrak{C})$ satisfies both the condition of Theorem 3.17 and Theorem 3.19, and so [KMS13, Main Theorems A and B] both follow directly.

In addition, [KMS13, Fact 1.1] has an obvious analog (at least in the countable case) for the cases described in Theorem 3.17 and Theorem 3.19. In particular, in these cases, an Elass is closed iff it is $G_{\delta}$ iff $E$ is strongly closed on it, and if $T$ is small and $\alpha$ is finite then all classes are closed.

We can also deduce that [New03, Corollary 1.12] hold for the cases described above (both for countable and uncountable languages), which begs the question:

Problem 3.22. Do our result extend to any $\emptyset$-invariant $F_{\sigma}$ relation?

Remark 3.23. One of the properties of $\equiv_{L}$ is that if $a \equiv_{M} b$ for some model $M$, then $d_{L}(a, b) \leq 2$ where $d_{L}$ is the Lascar metric. An analog for $E$ and its normal form would be that for some $n<\omega$, if $M \supseteq A$ and $a \equiv_{M} b$ then $(a, b) \in U_{n}$. This has no reason to hold in general. However, if $\Gamma$ is $C$-transitive then for some $n<\omega$ and all $M$ and $\Gamma^{*}$ as in (1)-(3) above, there is a nonempty $\Gamma^{*}-$ invariant closed subset $S \subseteq S_{X}(M)$ such that for any $p \in S \cap C_{M}$, if $b, c \models p$ then $(b, c) \in U_{n+1}$. Moreover, it is dense in the following sense: for every $b \in C$, there is some $c \in C$ such that $\operatorname{tp}(c / M) \in S$ and $(b, c) \in U_{n}$.

Indeed, let $n<\omega$ be such that $p_{n}(x, a)$ is proper for all $a \in C$, and let $\psi(x, y)$ be the formula that witnesses this (see Proposition 3.15). Let $a \in C, M$ and $\Gamma^{*}$ be as in (1)-(3). Let $S$ be the set of types $\left[\left\{\neg \psi(x, \sigma(a)) \mid \sigma \in \Gamma^{*}\right\}\right]$. This is obviously closed and $\Gamma^{*}$-invariant. Suppose $p \in S \cap C_{M}$ and $b, c \models p$. We will show that $(b, c) \in U_{n+1}$, i.e., $(b, c) \models p_{n+1}$. Let $\xi(x, y) \in p_{n+1}$, and let $\chi(x, y) \in p_{n}$ be such that $\chi(x, y) \wedge \chi(z, y) \rightarrow \xi(x, z)$. Since $\chi(x, a) \vee \psi(x, a)$ is generic, for some $\sigma \in \Gamma^{*}, b, c \models \chi(x, \sigma(a)) \vee \psi(x, \sigma(a))$, but by the definition of $S, b, c \models \chi(x, \sigma(a))$. It follows that $\xi(b, c)$ holds.

We also need to show the denseness property. Fix some $b \in C$. It is enough to show that the set $\left\{\neg \psi(x, \sigma(a)) \mid \sigma \in \Gamma^{*}\right\} \cup p_{n}(x, b)$ is consistent. Suppose not, so for some $\xi(x, y) \in p_{n}(x, y)$ and some finite $\Delta \subseteq \Gamma^{*}, \xi(x, b) \rightarrow \bigvee_{\sigma \in \Delta} \psi(x, \sigma(a))$. Since $\Gamma$ is $C$-transitive, for some $\tau \in \Gamma$, $\tau(b)=a$, so $\xi(x, a)$ implies $\bigvee_{\sigma \in \Delta} \psi(x, \tau \circ \sigma(a))$. But then $\bigvee_{\sigma \in \Delta} \psi(x, \tau \circ \sigma(a)) \vee \psi(x, a)$ is generic - contradiction.

This observation could have been used in the proof of e.g., Theorem 3.17, using $S \cap Y_{M}$ as our Polish space.

### 3.2.6. Definable and type definable group action.

Definition 3.24. For an ordinal $\beta,(H, \cdot)$ is a type definable group contained in $\mathfrak{C}^{\beta}$ when $H$ is type definable and the multiplication is type definable.

Suppose $(H, \cdot)$ is a type definable group over $\emptyset$. Let $G$ be an $\emptyset$-invariant pseudo $F_{\sigma}$ subgroup. In this case $G$ has a normal form: $G=\bigcup\left\{V_{n} \mid m<\omega\right\}$ where $V_{n}$ is $\emptyset$-type definable, $\{e\} \in V_{n}, V_{n}$ is symmetric $\left(V_{n}=V_{n}^{-1}\right)$, and $V_{n}^{\cdot 2} \subseteq V_{n+1}$.

Suppose that $X \subseteq \mathfrak{C}^{\alpha}$ is $\emptyset$-type definable and that $*$ is an $\emptyset$-type definable group action of $H$ on $X$. In particular, the orbit equivalence relation of the action $E_{H}^{X}$ is a closed invariant equivalence relation on $X$ and $E_{G}^{X}$ is an $\emptyset$-invariant pseudo $F_{\sigma}$ equivalence relation in the sense discussed in the previous subsection, with normal form defined by:

$$
U_{n}=\left\{(a, b) \in X \times X \mid \exists g \in V_{n}(g * a=b)\right\}
$$

for $n<\omega$.

Definition 3.25. To simplify notation, we call such a tuple $\bar{D}=\left(\alpha, \beta, G, H,\left\langle V_{n}, U_{n} \mid n<\omega\right\rangle, \cdot, X, *\right)$ an $F_{\sigma}$ action. If $E_{G}^{X}$ is bounded, we call $\bar{D}$ a bounded $F_{\sigma}$ action.

Example 3.26. For an $\emptyset$-type definable group $G \subseteq \mathfrak{C}^{\alpha}, G_{\emptyset}^{000}$ is defined as the smallest bounded index invariant subgroup of $G$ and it is generated by the set $\left\{a^{-1} \cdot b \mid a \equiv_{L}^{\alpha} b, a, b \in G\right\}$. So, letting $W_{n}=\left\{a^{-1} \cdot b \mid d_{L}(a, b) \leq n\right\}$ where $d_{L}$ is the Lascar distance, we see that $G=\bigcup_{n<\omega} V_{n}$ where $V_{n}=\left\{\prod_{i<n} c_{i}^{ \pm 1} \mid c_{i} \in W_{n}\right\}$. See Gis11] for more. So $\left(\alpha, \alpha, G_{\emptyset}^{000}, G,\left\langle V_{n}, U_{n} \mid n<\omega\right\rangle, \cdot, G, \cdot\right)$ is a bounded $F_{\sigma}$ action.

We shall need a technical assumption that seems necessary for this approach to work.

Definition 3.27. We say that $a \in X$ is shifty if one of the following holds:
(1) (Right shifty) For every $k<\omega$ there exists $n=n_{k}<\omega$ such that for any $g_{1}, g_{2} \in H$ if $\left(g_{1} * a, g_{2} * a\right) \in U_{k}$ then $\left(\left(g_{1} \cdot g_{2}^{-1}\right) * a, a\right) \in U_{n}$ or:
(2) (Left shifty) For every $k<\omega$ there exists $n=n_{k}<\omega$ such that for any $g_{1}, g_{2} \in H$ if $\left(g_{1} * a, g_{2} * a\right) \in U_{k}$ then $\left(\left(g_{1}^{-1} \cdot g_{2}\right) * a, a\right) \in U_{n}$ and if $\left(\left(g_{1}^{-1} \cdot g_{2}\right) * a, a\right) \in U_{k}$ then $\left(g_{1} * a, g_{2} * a\right) \in U_{n}$.

Remark 3.28. In both cases, we may safely assume that $n_{k} \geq k$.
Example 3.29. Suppose $a \in X$ and $\operatorname{stab}_{H}(a) \unlhd H$. Then $a$ is right shifty.
Proof. Let $k<\omega$ be given and let $n=k$. If $g_{2} * a=\left(h \cdot g_{1}\right) * a$ for $h \in V_{n}$ then $\left(g_{2}^{-1} \cdot h \cdot g_{1}\right) * a=a$ and since $\operatorname{stab}_{H}(a)$ is normal, $\left(h \cdot g_{1} \cdot g_{2}^{-1}\right) * a=a$ so $\left(g_{1} \cdot g_{2}^{-1}\right) * a=h^{-1} * a$. As $V_{n}$ is symmetric, we are done.

Example 3.30. Suppose that for every $k<\omega$ there exists $n<\omega$ such that for any $c, d \in G * a$ and $g \in H$, if $(c, d) \in U_{k}$ then $(g * c, g * d) \in U_{n}$. Then $a$ is left shifty. This happens for instance when $V_{n}$ is definable for all $n<\omega$ and $G$ is a normal subgroup of $H$.

Proof. If $V_{n}$ is definable, then by compactness for every $k<\omega$ there is some $n<\omega$ such that for all $g \in H, g V_{k} g^{-1} \subseteq V_{n}$. So if there is some $h \in V_{k}$ such that $c=h * d$, then $g * c=(g \cdot h) * d$, but $g \cdot h=h^{\prime} \cdot g$ for $h^{\prime} \in V_{n}$ so $(g * c, g * d) \in U_{n}$.

Lemma 3.31. Suppose $\varphi(x, y)$ is some formula where $x$ comes from the first $\alpha$ variables. Then there is a formula $\psi\left(x^{\prime}, y, z\right)$ with $x^{\prime}$ coming from the first $\alpha$ variables such that for every $g \in H$, and any $a \in \mathfrak{C}^{\lg (y)}, g *\left(\varphi\left(\mathfrak{C}^{\alpha}, a\right) \cap X\right)=\left(\psi\left(\mathfrak{C}^{\alpha}, a, g\right) \cap X\right)$.

Proof. If $\alpha$, and $\beta$ were finite, so that $*$ and $\cdot$ were definable, then we could just define $\psi(x, y, z)=$ $\varphi\left(z^{-1} \cdot x, y\right)$ (so $x=x^{\prime}$ ). Otherwise, it is a standard compactness argument. Note that we need that both $X$ and $H$ are closed.

Lemma 3.31 defines an action of $H$ on sets of the form $X \cap \varphi\left(\mathfrak{C}^{\alpha}\right)$. In order to ease notation, we will write $g * \varphi$ instead of $g *\left(\varphi\left(\mathfrak{C}^{\alpha}\right) \cap X\right)$. This induces a natural action of $H$ on the set of types in $X$. If $\operatorname{dcl}(A)=A$, then $H \cap A$ (and also $G \cap A$ ) is a subgroup of $H$, and so it acts naturally by homeomorphisms on $S_{X}(A)$ (with the usual Stone topology). In that case, for any $g \in H \cap A$, $c \models p$ iff $g * c \models g * p$.

Fix an $E_{G}^{X}$-class $C \subseteq X$. Similarly to Definition 3.14, we define $C$-generic and $C$-weakly generic formulas and $C$-proper types, replacing the action of an automorphism group $\Gamma$ by the action of $G$ on $L_{\alpha}(\mathfrak{C})$ (note: $G$ and not $H$ ). We omit the details, since it is exactly as above.

For $n<\omega$, let $p_{n} \subseteq L_{\beta}(\emptyset)$ be the partial types defining $V_{n}$ and let $q_{n}(x, a)$ be the partial type saying $x \in X \wedge \exists g \in V_{n}(g * a=x)$.

Lemma 3.32. Suppose $a \in X$ is shifty. Then, for some $n<\omega, q_{n}(x, a)$ is $a G * a$-proper type.

Proof. The proof uses the same basic idea as in Lemma 3.15, but one has to be a bit careful.
Assume first that $a$ is right shifty. Suppose $\pi_{*}$ is the partial type defining $*$ and that $\pi_{X}$ is the type defining $X$. We may assume that these types, as well as $p_{n}$ and $q_{n}$ are closed under conjunctions. First we need to establish the following:

Claim. For each $k<\omega$ there is some $n<\omega$ such that for all formulas $\varphi \in p_{n}, \theta \in \pi_{*}$ there are formulas $\psi \in p_{k}$ and $\theta^{\prime} \in \pi_{*}$ such that for every $g_{1}, g_{2} \in H$, if

$$
\exists z\left(\psi(z) \wedge \theta^{\prime}\left(z, g_{1} * a, g_{2} * a\right)\right)
$$

then

$$
\exists z\left(\varphi(z) \wedge \theta\left(z, a,\left(g_{1} \cdot g_{2}^{-1}\right) * a\right)\right)
$$

Proof of claim. Let $k<\omega$ be given, and let $n<\omega$ be the corresponding number from Definition 3.27. Then the following is inconsistent: there are $g_{1}, g_{2} \in H$ such that $g_{2} * a \in V_{k} *\left(g_{1} * a\right)$ but $\left(g_{1} \cdot g_{2}^{-1}\right) * a \notin V_{n} * a$. Applying compactness, we are done. Note that these formulas may depend on $a$ (but not on $g_{1}, g_{2}$ ).

Assume that for all $n<\omega, q_{n}$ is not proper. For each $k<\omega$, let $n_{k}<\omega$ be the corresponding number from the claim.

Since $q_{n_{k}}$ is not proper for all $k<\omega$, we can find formulas $\varphi_{k} \in p_{n_{k}}$ and $\theta_{k} \in \pi_{*}$ such that $\bigvee_{k<m} \psi_{k}^{\prime}$ is not generic for all $m<\omega$ where $\psi_{k}^{\prime}(x)=\exists y\left(\varphi_{k}(y) \wedge \theta_{k}(y, a, x)\right)$. (note: a formula in $q_{n_{k}}$ generally looks like $\psi_{k}^{\prime} \wedge \tau$ for $\tau \in \pi_{X}$, but this does not matter for genericity.)

For each $k<\omega$, the claim provides formulas $\psi_{k} \in p_{k}$ and $\theta_{k}^{\prime} \in \pi_{*}$ such that:
If $g_{1}, g_{2} \in H$ and $g_{2}^{-1} * a \notin g_{1}^{-1} * \psi_{k}^{\prime}$ then $\neg \exists z\left(\psi_{k}(z) \wedge \theta_{k}^{\prime}\left(z, g_{1} * a, g_{2} * a\right)\right)$. Note that this latter condition implies that $\left(g_{1} * a, g_{2} * a\right) \notin U_{k}$.

Fix some $n<\omega$ and let $\psi^{\prime}=\bigvee_{k<n} \psi_{k}^{\prime}$. Since $\psi^{\prime}$ is not generic, there is a sequence $\left\langle g_{i} \in G \mid i<\omega\right\rangle$ such that $g_{i}^{-1} * a \notin g_{j}^{-1} * \psi^{\prime}$ for $j<i$. This means that $\left(g_{j} * a, g_{i} * a\right) \notin U_{k}$ for $k<n$, and for each $k$, this is because of $\psi_{k}$ and $\theta_{k}^{\prime}$. Note that although $V_{k} \subseteq V_{k+1}$, we do not get that $\psi_{k}$ implies $\psi_{k+1}$, so we really need to keep all the formulas.

Now, by compactness we can find a sequence $\left\langle a_{i} \in X \mid i<\omega\right\rangle$ such that for all $j<i<\omega$, $\left(a_{j}, a_{i}\right) \notin U_{k}$ for all $k<\omega$ (and each time because of the same formulas). By Ramsey and compactness (Fact 3.4) we may assume that this sequence is indiscernible. But this is a contradiction to our assumption that the action is bounded.

If $a$ is left shifty, the proof is exactly the same, replacing $g_{i}^{-1}$ by $g_{i}$.
Recall that if $E_{G}^{X} \upharpoonright G * a$ is not strongly closed for some $a \in X$ (see Definition 3.9), then for all $n<\omega$ there are $g_{1}, g_{2} \in G$ such that $\left(g_{1} * a, g_{2} * a\right) \notin U_{n+1}$. But then either $\left(a, g_{1} * a\right) \notin U_{n}$ or $\left(a, g_{2} * a\right) \notin U_{n}$. So we may always assume that $g_{1}=e$.

Theorem 3.33. Suppose $T$ is a complete countable first-order theory, $\alpha, \beta$ countable ordinals. Suppose that $\left(\alpha, \beta, G, H,\left\langle V_{n}, U_{n} \mid n<\omega\right\rangle, \cdot, X, *\right)$ is a bounded $F_{\sigma}$ action and suppose $Y \subseteq \mathfrak{C}^{\alpha}$ is a pseudo $G_{\delta}$ set contained in $X$ which is $E_{G}^{X}$ invariant. If for some shifty $a \in Y, E_{G}^{X} \upharpoonright G * a$ is not strongly closed, then $E_{G}^{X} \upharpoonright Y$ is non-smooth.

Proof. This follows from Theorem 2.12, just like the proof of Theorem 3.17 By Lemma 3.32, for some $k<\omega, q_{k}(x, a)$ is $G * a$-proper, and this is witnessed by some non-generic formula $\psi$. Construct recursively a countable model $M$ such that:
(1) $a \in M$ and $\psi$ is over $M$.
(2) If $\varphi \in L_{\alpha}(M)$ is $G * a$-generic, then for some $\Delta \subseteq G \cap M, \Delta * \varphi$ contains $G * a$.
(3) For all $n<\omega$, there is some $g \in G \cap M$ such that $(a, g * a) \notin U_{n}$.

In the language of Theorem 2.12, $X$ is $Y_{M}, \Gamma$ is $G \cap M, Y$ is $(G * a)_{M}, R_{n}$ is $U_{n}^{M} \cap Y_{M}^{2}$ and $K$ is the compact set $\left[q_{k}(x, a)\right]$. The fact that $\left[q_{k}(x, a)\right]$ is $I_{\mathcal{U}}$-positive follows from (2) and the fact that $q_{k}(x, a)$ is proper (see the proof of Lemma 3.16).

Condition (a) follows from (3) above: if $(a, g * a) \notin U_{N}$ for $N$ big enough, then for all $p \in Y_{M}$ containing $q_{k}(x, a),(g * p, p) \notin U_{n}^{M,(4)}$. We illustrate this: if $(p, g * p) \in U_{n}^{M,(2)}$, then for some $q \in X_{M},(p, q) \in U_{n}^{M}$ and $(q, g * p) \in U_{n}^{M}$. So for some $b_{1}, b_{2} \models p$ and $c_{1}, c_{2} \models q,\left(b_{1}, c_{1}\right) \in$ $U_{n},\left(c_{2}, g * b_{2}\right) \in U_{n}$. Since $\left(b_{1}, a\right) \in U_{k}$, and $b_{1} \equiv_{a} b_{2},\left(b_{1}, b_{2}\right) \in U_{k+1}$. Similarly, $\left(c_{1}, c_{2}\right) \in$ $U_{\max \{n, k\}+2}$. It follows that $\left(b_{2}, g * b_{2}\right) \in U_{\max \{k, n\}+4}$. Suppose $a$ is right shifty. As $b_{2} \in G * a$ and since $a$ is right shifty, we get that $(a, g * a) \in U_{N}$ for some $N$. If $a$ is left shifty, then as $\left(a, b_{2}\right) \in U_{k},\left(g * a, g * b_{2}\right) \in U_{n_{k}}$ for some large $n_{k}$, so $(a, g * a) \in U_{n_{k}+2}$.

Condition (b) is trivial, since any $g \in G \cap M$ belongs to some $V_{n}$.
Problem 3.34. Is shiftiness of $a$ necessary?

Corollary 3.35. With the same assumptions of Theorem 3.33, if the action of $G$ is free (if $g * x=h * x$ then $h=g$ ) then either $G=V_{n}$ for some $n<\omega$, in which case $E_{G}^{X}$ is strongly closed so smooth or $E_{G}^{X} \upharpoonright Y$ is non-smooth.

Proof. Note that by assumption, every $a \in X$ is right shifty (since $\operatorname{stab}_{H}(a)=e$ is a normal subgroup). Now, if $E_{G}^{X} \upharpoonright Y$ is smooth, then by Theorem 3.33, for every $a \in Y, E \upharpoonright G * a$ is strongly closed. So for some $a \in Y$ and $n<\omega$ for all $b \in Y$, a $E_{G}^{X} b$ iff $(a, b) \in U_{n}$. Since the action is free, it follows that then $G=V_{n}$.

Theorem 3.36. Suppose $T$ is a complete first-order theory, $\alpha, \beta$ small ordinals. Suppose that $\left(\alpha, \beta, G, H,\left\langle V_{n}, U_{n} \mid n<\omega\right\rangle, \cdot, X, *\right)$ is a bounded $F_{\sigma}$ action and suppose $Y \subseteq \mathfrak{C}^{\alpha}$ is a pseudo a strong Choquet set contained in $X$ which is $E_{G}^{X}$ invariant. Suppose also that for some shifty $a \in Y, E_{G}^{X} \upharpoonright G * a$ is not strongly closed. Then $\left|Y / E_{G}^{X}\right| \geq 2^{\aleph_{0}}$.

Proof. Follows similarly from Theorem 2.18,

We can also recover Newelski's results [New03, Theorem 3.1] about groups generated by countably many type definable sets over $\emptyset$.

Corollary 3.37. Let $T$ be any first order theory and $\alpha$ any small ordinal. Suppose that $(H, \cdot)$ is a $\emptyset$-type definable group such that $H \subseteq \mathfrak{C}^{\alpha}$. Suppose that $G \leq H$ is a subgroup which is generated by countably many sets $V_{n}$ for $n<\omega$ which are $\emptyset$-type definable. Suppose that $G \leq H_{0} \leq H$ is a subgroup which is pseudo $G_{\delta}$ or pseudo closed (or even pseudo strong Choquet). Assume also that [ $H: G]$ is bounded. Then:
(1) If $G$ is pseudo $G_{\delta}$ or pseudo closed or even pseudo strong Choquet then $G$ is pseudo closed and in fact generated by finitely many of the sets $V_{n}$ in finitely many steps.
(2) If $G$ is not pseudo closed then $\left[H_{0}: G\right] \geq 2^{\aleph_{0}}$.
(3) If $T$ and $\alpha$ are countable then either $G$ is pseudo closed or the equivalence relation $E_{G}^{H_{0}}$ on $H_{0}$ of being in the same coset modulo $G$ is not smooth.
(4) If $T$ is small and $\alpha$ is finite then $G$ is pseudo closed.
(5) If we remove the assumption that $[H: G]$ is bounded, we still get (1) for pseudo closed $G$.

Proof. We may assume that $V_{n}$ is symmetric $\left(V_{n}=V_{n}^{-1}\right), V_{0}=\left\{1_{H}\right\}$, and $V_{n}^{\cdot 2} \subseteq V_{n+1}$. Consider the action of $G$ on $H$ by left multiplication and the orbit equivalence relation $E_{G}^{H}$ on $H$. Then by Theorem 3.36 with $X=H, Y=G, \alpha=\beta, a=e_{G}$ we get (1). Applying it again with $X=H, Y=H_{0}$ we get (2). (3) follows from Corollary 3.35,
(4) Suppose not. Since $T$ is small, the set $S_{\alpha}(\emptyset)$ is countable. Thus every subset of it is $G_{\delta}$, in particular the set

$$
Q=\left\{q \in S_{\alpha}(\emptyset) \mid \forall b \models q(b \in G)\right\}
$$

But then $G$ is pseudo $G_{\delta}$ so it is pseudo closed by (1).
(5) Note that in that case $G$ is type definable over $\emptyset$, so we can replace $H$ by $G$.

Corollary 3.38. Suppose $H$ is a definable group, and $G$ an $\emptyset$-invariant subgroup which is a union of countably many type definable sets. Then if $[H: G]<\infty$ then $G$ is definable.

Proof. By Corollary 3.37 (2), $G$ must be type definable over $\emptyset$. But then its complement is also type definable since it is a finite union of type definable sets, so it is definable by compactness.

Example 3.39. (with Pierre Simon) Theorem 3.33 does not hold in a very strong sense, if the group $G$ is only $\emptyset$-invariant and not pseudo $F_{\sigma}$. More precisely there is a countable theory $T$ where Corollary 3.38 fails.

Let $T$ be the theory of an infinite dimensional vector space over $\mathbb{F}_{2}$ in the language $\{+, 0\}$. Add predicates $U_{n}$ to the language and add axioms saying that $U_{n}$ are independent subspaces of co-dimension 1 (independent in the sense that any finite Boolean combination is nonempty). Then $T$ is consistent as one can take for $U_{n}$ the kernels of independent functionals. Let $\mathfrak{C}$ be a monster model for $T$, and let $H$ be the group $(\mathfrak{C},+)$. Let $G$ be the intersection $\bigcap\left\{U_{n} \mid n<\omega\right\}$. Then the index $[H: G]=2^{\aleph_{0}}$. In fact, the cosets of $G$ in $H$ are exactly the types $X_{\eta}=\bigcap\left\{U_{n}^{\eta(n)} \mid n<\omega\right\}$ where $\eta: \omega \rightarrow 2$ and $U_{n}^{0}=U_{n}, U_{n}^{1}=\mathfrak{C} \backslash U_{n}$. Pick a basis $\left\{v_{i} \mid i<2^{\aleph_{0}}\right\}$ for the space $H / G$. Any map $\eta: 2^{\aleph_{0}} \rightarrow 2$ defines a subspace $V_{\eta}$ by taking the kernel of the functional mapping $v_{i}$ to $\eta(i)$. Obviously, if $\eta$ is not trivial then $\left[H: \pi^{-1}\left(V_{\eta}\right)\right]=2$ (where $\pi$ is the projection $H \rightarrow H / G$ ). So for at least one $\eta, \pi^{-1}\left(V_{\eta}\right)$ is not definable. But all of them are invariant as they are union of cosets $X_{\eta}$.

Example 3.40. Corollary 3.37(5) does not hold when $G$ is pseudo $G_{\delta}$. For instance, let $T=R C F$, and add to the language constant symbols for the rational numbers $\mathbb{Q}$. Then $\mathbb{Q}$ itself is pseudo open (in every model), so also pseudo $G_{\delta}$, but definitely not closed (every closed infinite subset must have unbounded cardinality).

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