The existence of measures of a given cocycle, II: Probability measures

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Abstract. Given a Polish space X, a countable Borel equivalence relation E on X, and a Borel cocycle $\rho : E \to (0, \infty)$, we characterize the circumstances under which there is a probability measure μ on X such that $\rho(\phi^{-1}(x), x) = [d(\phi_* \mu)/d\mu](x)$ μ-almost everywhere, for every Borel injection ϕ whose graph is contained in E.

1. *Introduction*

A topological space is *Polish* if it is separable and admits a complete metric. An equivalence relation is *finite* if all of its equivalence classes are finite, and *countable* if all of its equivalence classes are countable. By a *measure* on a Polish space, we shall always mean a measure defined on its Borel subsets which is not identically zero. A measure is *atomless* if every Borel set of positive measure contains a Borel set of strictly smaller positive measure. Measures μ and ν are *equivalent*, or $\mu \sim \nu$, if they have the same null sets. Given a measure μ on X and a Borel function $\phi : X \to Y$, let $\phi_* \mu$ denote the measure on Y given by $\phi_* \mu(B) = \mu(\phi^{-1}(B))$.

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is a measure on X, and $\rho : E \to (0, \infty)$ is Borel. Let $[\![E]\!]$ denote the set of all Borel injections $\phi : A \to B$, where $A, B \subseteq X$ are Borel and $\mathrm{graph}(\phi) \subseteq E$. We say that μ is *E-quasi-invariant* if $\phi_* \mu \sim \mu$, for all $\phi \in [\![E]\!]$. We say that ρ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$, for all $xEyEz$. We say that μ is *ρ-invariant* if

$$\phi_* \mu(B) = \int_B \rho(\phi^{-1}(x), x) \, d\mu(x),$$

for all $\phi \in [\![E]\!]$ and Borel sets $B \subseteq \mathrm{rng}(\phi)$. When $\rho \equiv 1$, we say that μ is *E-invariant*.

These notions typically arise in a slightly different guise in the context of group actions. The *orbit equivalence relation* associated with an action of a countable group Γ by Borel automorphisms of X is given by $x E_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y)$. It is easy to see that if

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$\gamma_*\mu \sim \mu$, for all $\gamma \in \Gamma$, then μ is E^X_Γ-quasi-invariant, and similarly, if $\rho : E^X_\Gamma \to (0,\infty)$ is a Borel cocycle such that

$$\gamma_*\mu(B) = \int_B \rho(\gamma^{-1} \cdot x, x)\, d\mu(x),$$

for all $\gamma \in \Gamma$ and Borel sets $B \subseteq X$, then μ is ρ-invariant.

Our goal here is to characterize the circumstances under which there is a ρ-invariant probability measure on X. Before getting to our main results, we will review the well known answer to the special case of our question for E-invariant measures. First, however, we need to lay out some terminology. The *E-class* of x is given by $[x]_E = \{y \in X : xEy\}$. A set $B \subseteq X$ is a *partial transversal* of E if it intersects every E-class in at most one point. We say that E is *smooth* if X is the union of countably many Borel partial transversals. The *E-saturation* of B is given by $[B]_E = \{x \in X : \exists y \in B\ (xEy)\}$, and we say that B is *E-invariant* if $B = [B]_E$. We say that μ is *E-ergodic* if every E-invariant Borel set is μ-null or μ-conull.

A *compression* of E is a function $\phi \in [\![E]\!]$ such that $\mathrm{dom}(\phi) = X$ and $\mathrm{rng}(\phi)$ misses a point of every E-class. We say that E is *compressible* if there is a compression of E. Although the main result of [**6**] is stated only for Borel automorphisms, the argument can be easily modified so as to obtain the following:

THEOREM 1 (NADKARNI) *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*
1. *E is compressible;*
2. *There is an E-invariant probability measure on X.*

In order to characterize the existence of probability measures beyond the E-invariant case, we must first generalize the notion of compressibility. Given a function $\phi : X \to \mathbb{R}$, an E-class C, a set $S \subseteq C$, and a point $x \in C$, define

$$I_S(\phi) = \frac{\sum_{y \in S} \phi(y)\rho(y,x)}{\sum_{y \in S} \rho(y,x)}.$$

We leave $I_S(\phi)$ undefined in case this ratio is of the form $0/0$ or $\pm\infty/\infty$. The fact that ρ is a cocycle ensures that $I_S(\phi)$ does not depend on the choice of $x \in C$. Intuitively, the quantity $I_S(\phi)$ represents the best guess at the integral of ϕ with respect to a ρ-invariant probability measure on X, given only $\phi|S$. For each set $B \subseteq X$, let $\mu_S(B) = I_S(\chi_B)$. Given an increasing sequence $\langle F_k \rangle_{k \in \mathbb{N}}$ of finite Borel equivalence relations on X, let $\mu_x(B) = \lim_{k \to \infty} \mu_{[x]_{F_k}}(B)$. We leave $\mu_x(B)$ undefined if this limit does not exist.

We say that an E-invariant Borel set $B \subseteq X$ is *ρ-compressible of type I* if there is an increasing sequence $\langle F_k \rangle_{k \in \mathbb{N}}$ of finite Borel subequivalence relations of E and a partition $\langle B_n \rangle_{n \in \mathbb{N}}$ of B into Borel sets such that (1) $\mu_{[x]_{F_k}}(B_n)$ converges uniformly to $\mu_x(B_n)$, for all $n \in \mathbb{N}$, and (2) $\sum_{n \in \mathbb{N}} \mu_x(B_n) < 1$, for all $x \in B$. Let $[E]$ denote the group of all Borel automorphisms of X in $[\![E]\!]$. We say that an E-invariant Borel set $B \subseteq X$ is *ρ-compressible of type II* if there is a smooth Borel subequivalence relation F of E, a Borel set $A \subseteq B$, and $T \in [E]$ such that $\sum_{y \in T(A) \cap [x]_F} \rho(y,x) < \sum_{y \in T(A \cap [x]_F)} \rho(y,x)$, for all $x \in B$. We say that a set is *ρ-compressible* if it is contained in the union of countably many

Borel sets which are ρ-compressible of types I or II, and we say that ρ is *compressible* if X is ρ-compressible. It is not difficult to see that if $\rho \equiv 1$, then E is compressible if and only if ρ is compressible (see the remark following the proof of Proposition 6.3), thus the following fact generalizes Theorem 1:

THEOREM 2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is compressible;*
2. *There is a ρ-invariant probability measure on X.*

Theorem 2 still leaves something to be desired, however, as it is natural to look for a characterization that is closer to the usual notion of compressibility. We say that a function $\phi \in [\![E]\!]$ is *ρ-invariant* if $\rho(\phi(x), x) = 1$, for all $x \in \mathrm{dom}(\phi)$. Perhaps the most natural attempt at generalizing the notion of compressibility is to replace it with ρ-invariant compressibility. Unfortunately, this is far too restrictive, as there are Borel cocycles $\rho : E \to (0, \infty)$ for which there are neither ρ-invariant probability measures on X nor non-trivial ρ-invariant elements of $[\![E]\!]$. In order to alleviate this problem, we consider an enlarged version of $[\![E]\!]$ which necessarily contains a plethora of functions which satisfy a natural analog of ρ-invariance.

The *fuzzy domain and range* of a function $\phi = (\phi_d, \phi_r) : X \times X \to [0, 1] \times [0, 1]$ are the functions $\mathrm{fdom}(\phi), \mathrm{frng}(\phi) : X \to [0, \infty]$ given by

$$[\mathrm{fdom}(\phi)](x) = \sum_{y \in X} \phi_d(x, y) \text{ and } [\mathrm{frng}(\phi)](y) = \sum_{x \in X} \phi_r(x, y).$$

We say that ϕ is a *fuzzy partial injection* if $\mathrm{fdom}(\phi), \mathrm{frng}(\phi) \leq 1$. Intuitively, we think of ϕ as sending a fraction of x of size $\phi_d(x, y)$ to a fraction of y of size $\phi_r(x, y)$. The *fuzzy analog* of $[\![E]\!]$ is the set of all Borel fuzzy partial injections $\phi = (\phi_d, \phi_r)$ with the property that $\mathrm{supp}(\phi_d), \mathrm{supp}(\phi_r) \subseteq E$. We say that ϕ is *ρ-invariant* if $\phi_r(x, y) = \phi_d(x, y)\rho(x, y)$, for all xEy, and we use $[\![\rho]\!]$ to denote the set of all ρ-invariant fuzzy partial injections in the fuzzy analog of $[\![E]\!]$. A *fuzzy compression* of ρ is a fuzzy partial injection $\phi \in [\![\rho]\!]$ such that $\mathrm{fdom}(\phi) \equiv 1$ and $\mathrm{frng}(\phi)$ is not identically 1 on any E-class. We say that ρ is *fuzzily compressible* if there is a fuzzy compression of ρ.

THEOREM 3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is fuzzily compressible;*
2. *There is a ρ-invariant probability measure on X.*

The organization of the paper is as follows. In §2, we discuss some basic facts concerning equivalence relations, cocycles, and measures. In §3, we review the construction of measures from finitely additive measures. In §4, we prove Theorem 2. In §5, we obtain a version of Theorem 2 which characterizes the existence of suitably non-trivial, ρ-invariant probability measures, as well as a new proof of Ditzen's quasi-invariant ergodic decomposition theorem (see [**1**]). In §6, we prove Theorem 3.

2. *Preliminaries*

Associated with each Borel cocycle $\rho : E \to (0, \infty)$ is a way of thinking of each E-class as a single mass which has been divided into countably many pieces. When xEy, we think of $\rho(x, y)$ as the ratio of the mass of x to that of y. For each set $S \subseteq [x]_E$, we use

$$|S|_x = \sum_{y \in S} \rho(y, x)$$

to denote the quantity which intuitively represents the mass of S relative to that of x.

Although $|S|_x$ depends on x, whether $|S|_x$ is finite does not. We say that S is *ρ-finite* if $|S|_x$ is finite, for all $x \in S$, and we say that S is *ρ-infinite* otherwise. We say that ρ is *finite* if every E-class is ρ-finite, and we say that ρ is *aperiodic* if every E-class is ρ-infinite. The *aperiodic part* of ρ is given by $\mathrm{Aper}(\rho) = \{x \in X : |[x]_E|_x = \infty\}$. We say that a set is *$\rho$-negligible* if it is null with respect to every ρ-invariant probability measure on X.

PROPOSITION 2.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle.*

1. *If ρ is finite, then E is smooth.*
2. *If E is smooth, then the aperiodic part of ρ is ρ-negligible.*

Proof. To see (1), note that if ρ is finite, then for each $n \in \mathbb{N}$, the set

$$B_n = \{x \in X : \forall y \in [x]_E \ (\rho(x, y) \geq 1/n)\}$$

intersects each E-class in a finite set. Then $X = \bigcup_{n \in \mathbb{N}} B_n$ and the Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of [**4**]) implies that each B_n is Borel, thus Proposition 2.4 of [**5**] (and the remark thereafter) ensures that E is smooth.

To see (2), it is enough to show that if $B \subseteq \mathrm{Aper}(\rho)$ is a Borel partial transversal of E and μ is a ρ-invariant probability measure on X, then $\mu(B) = 0$. By Theorem 1 of [**2**] (see also Proposition 2.1 of [**5**]), there is a group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms of X such that $E = E_\Gamma^X$. Fix an enumeration $\langle S_k \rangle_{k \in \mathbb{N}}$ of the family $[\mathbb{N}]^{<\mathbb{N}}$ of finite subsets of $\mathbb{N}$, and recursively define $k_n : B \to \mathbb{N}$ by letting $k_n(x) \in \mathbb{N}$ be least such that:

1. $|\{\gamma_i \cdot x\}_{i \in S_{k_n(x)}}|_x \geq 1$;
2. $\forall i, j \in S_{k_n(x)} \ (\gamma_i \cdot x = \gamma_j \cdot x \Rightarrow i = j)$;
3. $\forall m < n \forall i \in S_{k_m(x)} \forall j \in S_{k_n(x)} \ (\gamma_i \cdot x \neq \gamma_j \cdot x)$.

Let $B_n = \bigcup_{k \in \mathbb{N}} \bigcup_{i \in S_k} \gamma_i(k_n^{-1}(k))$, and observe that

$$\begin{aligned}
\mu(B_n) &= \sum_{k \in \mathbb{N}} \sum_{i \in S_k} \mu(\gamma_i(k_n^{-1}(k))) \\
&= \sum_{k \in \mathbb{N}} \sum_{i \in S_k} \int_{k_n^{-1}(k)} \rho(\gamma_i \cdot x, x) \, d\mu(x) \\
&= \sum_{k \in \mathbb{N}} \int_{k_n^{-1}(k)} |\{\gamma_i \cdot x\}_{i \in S_k}|_x \, d\mu(x) \\
&\geq \sum_{k \in \mathbb{N}} \mu(k_n^{-1}(k)) \\
&= \mu(B).
\end{aligned}$$

As $\langle B_n \rangle_{n \in \mathbb{N}}$ is a pairwise disjoint sequence of Borel sets, it follows that $\mu(B) = 0$. □

Recall from [**5**] that a set $B \subseteq X$ is *E-complete* if it intersects every E-class, and a *transversal* is an E-complete partial transversal.

PROPOSITION 2.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant probability measure on X, $B \subseteq X$ is a Borel transversal of E, and $\phi : X \to [0, \infty]$ is Borel. Then*

$$\int \phi(x)\ d\mu(x) = \int_B I_{[x]_E}(\phi)|[x]_E|_x\ d\mu(x).$$

Proof. Fix a group $\Gamma = \{\gamma_n\}_{n \in \mathbb{N}}$ of Borel automorphisms of X such that $E = E_\Gamma^X$, set $B_n = \gamma_n(B) \setminus \bigcup_{m<n} \gamma_m(B)$, and observe that

$$\begin{aligned}
\int \phi(x)\ d\mu(x) &= \sum_{n \in \mathbb{N}} \int_{B_n} \phi(x)\ d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int_{\gamma_n^{-1}(B_n)} \phi(\gamma_n \cdot x)\rho(\gamma_n \cdot x, x)\ d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int \chi_{\gamma_n^{-1}(B_n)}(x)\phi(\gamma_n \cdot x)\rho(\gamma_n \cdot x, x)\ d\mu(x) \\
&= \int \sum_{n \in \mathbb{N}} \chi_{B_n}(\gamma_n \cdot x)\phi(\gamma_n \cdot x)\rho(\gamma_n \cdot x, x)\ d\mu(x) \\
&= \int_B \sum_{y \in [x]_E} \phi(y)\rho(y, x)\ d\mu(x) \\
&= \int_B I_{[x]_E}(\phi)|[x]_E|_x\ d\mu(x),
\end{aligned}$$

which completes the proof of the proposition. □

PROPOSITION 2.3. *Suppose that X is a Polish space, E is a smooth countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant probability measure on X, and $\phi : X \to [0, \infty]$ is Borel. Then*

$$\int \phi(x)\ d\mu(x) = \int I_{[x]_E}(\phi)\ d\mu(x).$$

Proof. By Proposition 2.6 of [**5**] (and the remark thereafter), there is a Borel transversal $B \subseteq X$ of E. Proposition 2.1 ensures that after throwing out an E-invariant, μ-null Borel set, we can assume that ρ is finite. Define $\psi : X \to [0, \infty]$ by $\psi(x) = I_{[x]_E}(\phi)$, noting that $I_{[x]_E}(\phi) = I_{[x]_E}(\psi)$, for all $x \in X$. Two applications of Proposition 2.2 ensure that

$$\begin{aligned}
\int \phi(x)\ d\mu(x) &= \int_B I_{[x]_E}(\phi)|[x]_E|_x\ d\mu(x) \\
&= \int_B I_{[x]_E}(\psi)|[x]_E|_x\ d\mu(x) \\
&= \int \psi(x)\ d\mu(x) \\
&= \int I_{[x]_E}(\phi)\ d\mu(x),
\end{aligned}$$

which completes the proof of the proposition. □

While the following fact can also be obtained as a corollary of the Hurewicz ergodic theorem (see, for example, Exercise 3.8.3 of [**7**]), we are now in position to give an elementary proof:

PROPOSITION 2.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant probability measure on X, $\langle F_k \rangle_{k \in \mathbb{N}}$ is an increasing sequence of finite Borel subequivalence relations of E, and $B \subseteq X$ is Borel. Then $\mu_x(B)$ exists μ-almost everywhere and $\mu(B) = \int \mu_x(B) \; d\mu(x)$.*

Proof. First, we will show that $\mu(B) \geq \int \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x)$. Given $\epsilon > 0$, choose $n \in \mathbb{N}$ sufficiently large that the set

$$A = \{x \in X : \exists m \leq n \; (\mu_{[x]_{F_m}}(B) \geq \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) - \epsilon)\}$$

is of μ-measure at least $1 - \epsilon$. For each $x \in A$, fix $n(x) \leq n$ largest such that

$$\mu_{[x]_{F_{n(x)}}}(B) \geq \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) - \epsilon,$$

and define an equivalence relation $F \subseteq F_n$ on A by setting

$$xFy \Leftrightarrow xF_{n(x)}y.$$

Proposition 2.3 ensures that

$$\begin{aligned}\mu(B) &\geq \int_A \mu_{[x]_F}(B) \; d\mu(x) \\ &\geq \int_A \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) - \epsilon \; d\mu(x) \\ &\geq \int \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x) - 2\epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\mu(B) \geq \int \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x)$.

A similar argument shows that $\mu(B) \leq \int \liminf_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x)$, thus

$$\mu(B) = \int \liminf_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x) = \int \limsup_{k \to \infty} \mu_{[x]_{F_k}}(B) \; d\mu(x),$$

and the proposition follows. □

We next check that compressible cocycles do not admit invariant probability measures:

PROPOSITION 2.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, and $B \subseteq X$ is an E-invariant Borel set which is ρ-compressible of type I. Then B is ρ-negligible.*

Proof. Fix an increasing sequence $\langle F_k \rangle_{k \in \mathbb{N}}$ of finite Borel subequivalence relations of E and a partition $\langle B_n \rangle_{n \in \mathbb{N}}$ of B into Borel sets such that $\sum_{n \in \mathbb{N}} \mu_x(B_n) < 1$, for all $x \in B$.

If μ is a ρ-invariant probability measure on X, then Proposition 2.4 ensures that

$$\begin{aligned}\mu(B) &= \sum_{n\in\mathbb{N}} \mu(B_n) \\ &= \sum_{n\in\mathbb{N}} \int \mu_x(B_n)\, d\mu(x) \\ &= \int_B \sum_{n\in\mathbb{N}} \mu_x(B_n)\, d\mu(x),\end{aligned}$$

thus $\mu(B) = 0$. □

Let $[E]^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}}[E]^n$, where $[E]^n$ denotes the family of sets $S \subseteq X$ of cardinality n such that $\forall x, y \in S\ (xEy)$. It is not difficult to see that $[E]^n$ carries a Polish topology with respect to which a set $B \subseteq [E]^n$ is Borel if and only if $\{(x_1, \ldots, x_n) \in X^n : \{x_1, \ldots, x_n\} \in B\}$ is a Borel subset of X^n. Similarly, the set $[E]^{<\mathbb{N}}$ carries a Polish topology with respect to which a subset of $[E]^n$ is Borel if and only if it is Borel when viewed as a subset of $[E]^n$. Let $\tilde{E}$ denote the equivalence relation on $[E]^{<\mathbb{N}}$ given by

$$S\tilde{E}T \Leftrightarrow [S]_E = [T]_E.$$

Note that if $S\tilde{E}T$ and $C = [S]_E = [T]_E$, then $|S|_x/|T|_x$ is independent of the choice of $x \in C$. We therefore obtain a cocycle $\tilde{\rho} : \tilde{E} \to (0, \infty)$ by setting

$$\tilde{\rho}(S, T) = |S|_x/|T|_x,$$

for $x \in C$. It should be noted that an E-invariant Borel set $B \subseteq X$ is ρ-compressible of type II if and only if there is a smooth Borel subequivalence relation F of E, a Borel set $A \subseteq B$, and $T \in [E]$ such that $\tilde{\rho}(T(A) \cap [x]_F, T(A \cap [x]_F)) < 1$, for all $x \in B$.

PROPOSITION 2.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, and $B \subseteq X$ is an E-invariant Borel set which is ρ-compressible of type II. Then B is ρ-negligible.*

Proof. Fix a smooth Borel subequivalence relation F of E, a Borel set $A \subseteq B$, and $T \in [E]$ such that $\tilde{\rho}(T(A) \cap [x]_F, T(A \cap [x]_F)) < 1$, for all $x \in B$. Define $\phi : X \to [0, \infty]$ by $\phi(x) = \chi_A(x)\rho(T(x), x)$, and observe that if μ is a ρ-invariant probability measure, then Proposition 2.3 implies that

$$\begin{aligned}\mu(T(A)) &= \int_A \rho(T(x), x)\, d\mu(x) \\ &= \int_B I_{[x]_F}(\phi)\, d\mu(x) \\ &= \int_B \frac{\sum_{y\in[x]_F} \chi_A(y)\rho(T(y), y)\rho(y, x)}{\sum_{y\in[x]_F} \rho(y, x)}\, d\mu(x) \\ &= \int_B \frac{\sum_{y\in A\cap[x]_F} \rho(T(y), x)}{\sum_{y\in[x]_F} \rho(y, x)}\, d\mu(x) \\ &= \int_B \tilde{\rho}(T(A \cap [x]_F), [x]_F)\, d\mu(x),\end{aligned}$$

and one more application of Proposition 2.3 ensures that

$$\begin{aligned}\mu(T(A)) &= \int_B \mu_{[x]_F}(T(A))\, d\mu(x)\\ &= \int_B \frac{\sum_{y\in T(A)\cap[x]_F}\rho(y,x)}{\sum_{y\in[x]_F}\rho(y,x)}\, d\mu(x)\\ &= \int_B \tilde\rho(T(A)\cap[x]_F,[x]_F)\, d\mu(x)\\ &= \int_B \tilde\rho(T(A)\cap[x]_F, T(A\cap[x]_F))\tilde\rho(T(A\cap[x]_F),[x]_F)\, d\mu(x).\end{aligned}$$

As $\tilde\rho(T(A)\cap[x]_F, T(A\cap[x]_F)) < 1$, for all $x\in B$, it follows that $\mu(B)=0$. □

We close this section with two cases in which ρ-compressibility can be easily inferred:

PROPOSITION 2.7. *Suppose that X is a Polish space, E is a smooth countable Borel equivalence relation on X, and $\rho : E \to (0,\infty)$ is an aperiodic Borel cocycle. Then X is ρ-compressible of type I.*

Proof. Fix an enumeration $\langle S_k\rangle_{k\in\mathbb{N}}$ of $[\mathbb{N}]^{<\mathbb{N}}\setminus\{\emptyset\}$, a Borel transversal $B\subseteq X$ of E, and a group $\Gamma=\{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms of X such that $E=E_\Gamma^X$, and recursively define $k_n : B\to\mathbb{N}$ by letting $k_n(x)\in\mathbb{N}$ be least such that:

1. $\gamma_n\cdot x\in\bigcup_{m\le n}\{\gamma_i\cdot x\}_{i\in S_{k_m(x)}}$;
2. $\forall m<n\ (\tilde\rho(\{\gamma_i\cdot x\}_{i\in S_{k_n(x)}},\{\gamma_i\cdot x\}_{i\in S_{k_m(x)}})\ge 1)$;
3. $\forall i,j\in S_{k_n(x)}\ (\gamma_i\cdot x=\gamma_j\cdot x\Rightarrow i=j)$;
4. $\forall m<n\forall i\in S_{k_m(x)}\forall j\in S_{k_n(x)}\ (\gamma_i\cdot x\ne\gamma_j\cdot x)$.

Let $B_n=\bigcup_{k\in\mathbb{N}}\bigcup_{i\in S_k}\gamma_i(k_n^{-1}(k))$, and define F_k on X by setting

$$xF_ky \Leftrightarrow x=y \text{ or } \Big(xEy \text{ and } x,y\in\bigcup_{n\le k}B_n\Big).$$

Then $\mu_{[x]_{F_k}}(B_n)\le 1/(k-n+1)$, for all $k\ge n$, thus X is ρ-compressible of type I. □

Recall from [**5**] that a set $B\subseteq X$ is *ρ-discrete* if there is an open neighborhood U of 1 such that $\rho(x,y)\in U\Rightarrow x=y$, for all $(x,y)\in E|B$, and ρ is *σ-discrete* if X is the union of countably many ρ-discrete Borel sets.

PROPOSITION 2.8. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E\to(0,\infty)$ is an aperiodic, σ-discrete Borel cocycle. Then ρ is compressible.*

Proof. Fix a cover $\langle A_n\rangle_{n\in\mathbb{N}}$ of X by ρ-discrete Borel sets. Define $\phi_n\in[\![E|A_n]\!]$ by

$$\phi_n(x)=y\Leftrightarrow \rho(x,y)<1 \text{ and } \forall z\in[x]_{E|A_n}\ (\rho(x,z)<1\Rightarrow\rho(y,z)\le 1).$$

By throwing out an E-invariant Borel set on which E is smooth (as Proposition 2.7 allows us to do), we can assume that ϕ_n is a Borel automorphism of A_n. Set $B_n=[A_n]_E$.

By the Lusin-Novikov uniformization theorem, there is a Borel function $\psi_n : B_n\to A_n$ which fixes the points of A_n and whose graph is contained in E. Define F_n on B_n by setting $xF_ny\Leftrightarrow\psi_n(x)=\psi_n(y)$, and let T_n be the Borel automorphism of X which agrees with ϕ_n on A_n and fixes the points of $X\setminus A_n$. Then $\tilde\rho(T_n(A_n)\cap[x]_{F_n}, T_n(A_n\cap[x]_{F_n}))<1$, for all $x\in B_n$, so B_n is ρ-compressible of type II. □

3. *The construction of measures from finitely additive measures*

Let $\mathcal{P}(X)$ denote the family of all subsets of X. We say that a set $\mathcal{U} \subseteq \mathcal{P}(X)$ is an *algebra* if it is closed under complements, intersections, and unions. We say that a function $\mu : \mathcal{U} \to [0, \infty]$ is a *finitely additive measure* if (1) $\mu(\emptyset) = 0$, and (2) $\mu(A \cup B) = \mu(A) + \mu(B)$, for all disjoint sets $A, B \in \mathcal{U}$. When $\mu(X) = 1$, we say that μ is a *finitely additive probability measure*. We say that a function $\mu : \mathcal{P}(X) \to [0, \infty]$ is an *outer measure* if (1) $\mu(\emptyset) = 0$, (2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$, for all $A, B \subseteq X$, and (3) $\mu(\bigcup_{n \in \mathbb{N}} B_n) \leq \sum_{n \in \mathbb{N}} \mu(B_n)$, for all $B_0, B_1, \ldots \subseteq X$.

Given an algebra $\mathcal{U} \subseteq \mathcal{P}(X)$ and a finitely additive measure $\mu : \mathcal{U} \to [0, \infty]$, define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(B) = \inf_{\mathcal{V} \subseteq \mathcal{U} \text{ covers } B} \sum_{V \in \mathcal{V}} \mu(V).$$

PROPOSITION 3.1. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is an algebra and $\mu : \mathcal{U} \to [0, \infty]$ is a finitely additive measure. Then μ^* is an outer measure.*

Proof. It is clear that $\mu^*(\emptyset) \leq \mu(\emptyset) = 0$, and if $A \subseteq B \subseteq X$, then every cover of B is a cover of A, thus $\mu^*(A) \leq \mu^*(B)$. Given $B_0, B_1, \ldots \subseteq X$, set $B = \bigcup_{n \in \mathbb{N}} B_n$, and for $\epsilon > 0$, fix a cover $\mathcal{U}_n \subseteq \mathcal{U}$ of B_n such that $\sum_{U \in \mathcal{U}_n} \mu(U) \leq \mu^*(B_n) + \epsilon/2^{n+1}$, for all $n \in \mathbb{N}$. Then $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ covers B, thus

$$\begin{aligned} \mu^*(B) &\leq \sum_{n \in \mathbb{N}} \sum_{U \in \mathcal{U}_n} \mu(U) \\ &\leq \sum_{n \in \mathbb{N}} \mu^*(B_n) + \epsilon/2^{n+1} \\ &= \epsilon + \sum_{n \in \mathbb{N}} \mu^*(B_n). \end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\mu^*(B) \leq \sum_{n \in \mathbb{N}} \mu^*(B_n)$. □

PROPOSITION 3.2. *Suppose that $\mathcal{U} \subseteq \mathcal{P}(X)$ is an algebra, $\mu : \mathcal{U} \to [0, \infty]$ is a finitely additive measure, and $\mathcal{B}$ is the σ-algebra generated by $\mathcal{U}$. Then $\mu^* | \mathcal{B}$ is a measure.*

Proof. By Proposition 3.1 and results of Carathéodory (see, for example, Theorems 11.B and 11.C of [**3**]), it is enough to show that $\mu^*(B \cap U) + \mu^*(B \setminus U) \leq \mu^*(B)$, for all $U \in \mathcal{U}$ and $B \subseteq X$. Towards this end, suppose that $U \in \mathcal{U}$ and $B \subseteq X$, and given $\epsilon > 0$, fix a cover $\mathcal{V} \subseteq \mathcal{U}$ of B such that $\sum_{V \in \mathcal{V}} \mu(V) \leq \mu^*(B) + \epsilon$. Then

$$\mu^*(B \cap U) + \mu^*(B \setminus U) \leq \sum_{V \in \mathcal{V}} \mu(V \cap U) + \sum_{V \in \mathcal{V}} \mu(V \setminus U) \leq \mu^*(B) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, it follows that $\mu^*(B \cap U) + \mu^*(B \setminus U) \leq \mu^*(B)$. □

A metric space is *Polish* if it is complete and separable. Given a Polish metric space X and an algebra $\mathcal{U} \subseteq \mathcal{P}(X)$, we say that a finitely additive measure $\mu : \mathcal{U} \to [0, \infty]$ is *decomposable* if for every $U \in \mathcal{U}$ and $\epsilon > 0$ there is a sequence $\langle U_n \rangle \in \mathcal{U}^{\mathbb{N}}$ of subsets of U of diameter at most ϵ with the property that $\mu(U) = \lim_{n \to \infty} \mu(\bigcup_{m \leq n} U_m)$.

PROPOSITION 3.3. *Suppose that X is a Polish metric space, $\mathcal{U}$ is an algebra of clopen subsets of X, and $\mu : \mathcal{U} \to [0, \infty]$ is a decomposable finitely additive probability measure. Then $\mu = \mu^*|\mathcal{U}$.*

Proof. Suppose, towards a contradiction, that there exists $U \in \mathcal{U}$ such that $\mu^*(U) < \mu(U)$, and fix $\epsilon > 0$ such that $\mu^*(U) < \mu(U) - \epsilon$. Decomposability ensures that for each $n \in \mathbb{N}$, there exist $k_n \in \mathbb{N}$ and a sequence $\langle U_{nk} \rangle \in \mathcal{U}^{k_n}$ of subsets of U of diameter at most $1/(n+1)$ such that $\mu(U) \leq \mu(\bigcup_{k<k_n} U_{nk}) + \epsilon/2^{n+1}$. Then the set $K = \bigcap_{n\in\mathbb{N}} \bigcup_{k<k_n} U_{nk}$ is compact (see, for example, Proposition 4.2 of [**4**]). As $K \subseteq U$, there is a finite cover $\mathcal{V} \subseteq \mathcal{U}$ of K such that $\sum_{V\in\mathcal{V}} \mu(V) \leq \mu(U) - \epsilon$.

LEMMA 3.4. *There exists $N \in \mathbb{N}$ such that $\bigcap_{n<N} \bigcup_{k<k_n} U_{nk} \subseteq \bigcup \mathcal{V}$.*

Proof. Simply note that if $\bigcap_{n<N} \bigcup_{k<k_N} U_{nk} \setminus \bigcup \mathcal{V} \neq \emptyset$, for each $N \in \mathbb{N}$, then there are natural numbers $l_n < k_n$ such that $\bigcap_{n<N} U_{nl_n} \setminus \bigcup \mathcal{V} \neq \emptyset$, in which case the unique point of $\bigcap_{n\in\mathbb{N}} U_{nl_n}$ is in $K \setminus \bigcup \mathcal{V}$, which contradicts the fact that $\mathcal{V}$ covers K. □

It now follows that $\mu(U) - \epsilon < \mu(\bigcap_{n<N} \bigcup_{k<k_n} U_{nk}) \leq \sum_{V\in\mathcal{V}} \mu(V) \leq \mu(U) - \epsilon$, which is the desired contradiction. □

Let $C_b(X)$ denote the space of bounded continuous functions $\phi : X \to \mathbb{R}$. We say that a linear space $\Phi \subseteq C_b(X)$ *contains* a set $\mathcal{U} \subseteq \mathcal{P}(X)$ if $\mathbf{1}_U \in \Phi$, for all $U \in \mathcal{U}$. A *mean* on Φ is a positive linear functional $I : \Phi \to \mathbb{R}$ such that $I(\mathbf{1}) = 1$. We say that I is *decomposable* if Φ contains an algebra of sets $\mathcal{U}$ which is a basis for X, and the finitely additive probability measure $\mu : \mathcal{U} \to [0, 1]$ given by $\mu(U) = I(\mathbf{1}_U)$ is decomposable. Associated with each decomposable mean I on Φ is the mean I^* on $C_b(X)$ given by

$$I^*(\phi) = \int \phi \, d\mu^*.$$

Proposition 3.3 ensures that I^* does not depend on the choice of $\mathcal{U}$.

PROPOSITION 3.5. *Suppose that X is a Polish metric space, Φ is a linear subspace of $C_b(X)$, and I is a decomposable mean on Φ. Then $I = I^*|\Phi$.*

Proof. Fix an algebra $\mathcal{U}$ contained in Φ such that the finitely additive measure $\mu : \mathcal{U} \to [0, 1]$ given by $\mu(U) = I(\mathbf{1}_U)$ is decomposable. Given $\phi \in \Phi$ and $\epsilon > 0$, fix a partition $\mathcal{V} \subseteq \mathcal{U}$ of X and a function $\psi : \mathcal{V} \to \mathbb{R}$ such that $\psi(V) < \phi(x) < \psi(V) + \epsilon$, for all $V \in \mathcal{V}$ and $x \in V$, as well as a finite set $\mathcal{W} \subseteq \mathcal{V}$ such that $\sum_{W\in\mathcal{W}} \mu(W) \geq 1 - \epsilon$. Set $W' = X \setminus \bigcup \mathcal{W}$ and $b = \sup_{x\in X} |\phi(x)|$. Proposition 3.3 ensures that

$$\begin{aligned}
I(\phi) &\leq I\Big(b\mathbf{1}_{W'} + \sum_{W\in\mathcal{W}} (\psi(W) + \epsilon)\mathbf{1}_W\Big) \\
&= I^*\Big(b\mathbf{1}_{W'} + \sum_{W\in\mathcal{W}} (\psi(W) + \epsilon)\mathbf{1}_W\Big) \\
&\leq b\epsilon + \epsilon + I^*\Big(\sum_{W\in\mathcal{W}} \psi(W)\mathbf{1}_W\Big) \\
&\leq b\epsilon + \epsilon + b\epsilon + I^*(\phi).
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $I(\phi) \leq I^*(\phi)$. A similar argument shows that $I(\phi) \geq I^*(\phi)$, and the proposition follows. □

4. *A characterization of the existence of ρ-invariant probability measures*

A *graph* on X is an irreflexive, symmetric set $\mathcal{G} \subseteq X \times X$. A *coloring* of $\mathcal{G}$ is a function $c : X \to Y$ such that $c(x_1) \neq c(x_2)$, for all $(x_1, x_2) \in \mathcal{G}$. When Y is Polish and c is Borel, we say that c is a *Borel coloring* of $\mathcal{G}$. The *Borel chromatic number* of $\mathcal{G}$ is given by $\chi_B(\mathcal{G}) = \min\{|c(X)| : c \text{ is a Borel coloring of } \mathcal{G}\}$.

Let $\mathcal{G}_E = \{(S,T) \in [E]^{<\mathbb{N}} \times [E]^{<\mathbb{N}} : S \neq T \text{ and } S \cap T \neq \emptyset\}$. It is not hard to see that $\chi_B(\mathcal{G}_E|[E]^2) \leq \aleph_0$ means exactly that E is the union of the graphs of countably many Borel involutions. The latter fact is a simple corollary (and consequence of the proof of) Theorem 1 of [**2**]. Strengthening this, we have the following:

PROPOSITION 4.1. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then $\chi_B(\mathcal{G}_E) \leq \aleph_0$.*

Proof. Fix a Borel linear ordering $\leq$ of X, as well as Borel involutions $I_n : X \to X$ such that $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(I_n)$. For each $S \in [E]^{<\mathbb{N}}$, let $\langle x_i^S \rangle_{i<|S|}$ denote the $\leq$-increasing enumeration of S, and let $c(S)$ denote the lexicographically least sequence $\langle k_{ij} \rangle_{i,j<|S|}$ of natural numbers such that $I_{k_{ij}}(x_i^S) = x_j^S$, for all $i, j < |S|$. Suppose, towards a contradiction, that c is not a coloring. Fix $(S,T) \in \mathcal{G}_E$ such that $c(S) = c(T) = \langle k_{ij} \rangle_{i,j<|S|}$, put $n = |S| = |T|$, and fix $i, j < n$ such that $x_i^S = x_j^T$. Then

$$\begin{aligned} i < j &\Leftrightarrow x_i^S < x_j^S \\ &\Leftrightarrow x_i^S < I_{k_{ij}}(x_i^S) \\ &\Leftrightarrow x_j^T < I_{k_{ij}}(x_j^T) \\ &\Leftrightarrow x_j^T < x_i^T \\ &\Leftrightarrow j < i, \end{aligned}$$

so $i = j$, thus $x_i^S = x_i^T$. It follows that $x_m^S = I_{k_{im}}(x_i^S) = I_{k_{im}}(x_i^T) = x_m^T$, for all $m < n$, thus $S = T$, which contradicts our assumption that $(S,T) \in \mathcal{G}_E$. □

A set $\Phi \subseteq [E]^{<\mathbb{N}}$ is *pairwise disjoint* if $S \neq T \Rightarrow S \cap T = \emptyset$, for all $S, T \in \Phi$.

PROPOSITION 4.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\Phi \subseteq [E]^{<\mathbb{N}}$ is Borel. Then there is a maximal pairwise disjoint Borel subset of Φ.*

Proof. Fix a Borel coloring $c : [E]^{<\mathbb{N}} \to \mathbb{N}$ of $\mathcal{G}_E$, set $\Psi_0 = \emptyset$, and recursively define

$$\Psi_{n+1} = \Psi_n \cup \{S \in \Phi : c(S) = n \text{ and } \forall T \in \Psi_n \ (S \cap T = \emptyset)\}.$$

A straightforward induction shows that each of the sets Ψ_n is pairwise disjoint and Borel (by the Lusin-Novikov uniformization theorem), thus so too is the set $\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n$. To see that Ψ is a maximal pairwise disjoint subset of Φ, simply observe that if $S \in \Phi \setminus \Psi$, then $S \in \Phi \setminus \Psi_{c(S)+1}$, so there exists $T \in \Psi_{c(S)}$ such that $S \cap T \neq \emptyset$. □

Given a Borel function $\phi : X \to \mathbb{R}$ and $\epsilon > 0$, we say that a finite Borel subequivalence relation F of E is *(ϕ, ϵ)-approximating* if $|I_{[x]_F}(\phi) - I_{[y]_F}(\phi)| \leq \epsilon$, for all xEy. It is important to note that if F is (ϕ, ϵ)-approximating, then so too is every finite Borel subequivalence relation of E which contains F.

PROPOSITION 4.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, $\phi : X \to \mathbb{R}$ is Borel, $\epsilon > 0$, and $F \subseteq E$ is a (ϕ, ϵ)-approximating finite Borel equivalence relation. Then there is a finite Borel subequivalence relation F' of E containing F and an E-invariant Borel set $B \subseteq X$ such that $F'|(X \setminus B)$ is $(\phi|(X \setminus B), 3\epsilon/4)$-approximating and $\rho|(E|B)$ is σ-discrete.*

Proof. For each E-class C, set $I_C(\phi) = (\inf_{x \in C} I_{[x]_F}(\phi) + \sup_{x \in C} I_{[x]_F}(\phi))/2$ and let Φ denote the family of all F-invariant sets $S \in [E]^{<\mathbb{N}}$ such that $|I_S(\phi) - I_{[S]_E}(\phi)| \leq \epsilon/4$. By Proposition 4.2, there is a maximal pairwise disjoint Borel set $\Psi \subseteq \Phi$. Set

$$xF'y \Leftrightarrow xFy \text{ or } \exists S \in \Psi \ (x, y \in S),$$

and define $B = \{x \in X : \exists y, z \in [x]_E \ (|I_{[y]_{F'}}(\phi) - I_{[z]_{F'}}(\phi)| > 3\epsilon/4)\}$. Then $F'|(X \setminus B)$ is $(\phi|(X \setminus B), 3\epsilon/4)$-approximating and B is E-invariant and Borel (by the Lusin-Novikov uniformization theorem), so it only remains to prove that $\rho|(E|B)$ is σ-discrete.

Suppose, towards a contradiction, that $\rho|(E|B)$ is not σ-discrete. Fix a Borel transversal A of $F|B$, and define $\rho' : E|A \to (0, \infty)$ by $\rho'(x, y) = \tilde{\rho}([x]_F, [y]_F)$.

LEMMA 4.4. *ρ' is not σ-discrete.*

Proof. Suppose, towards a contradiction, that there is a cover $\langle A_n \rangle_{n \in \mathbb{N}}$ of A by ρ'-discrete Borel sets. For each $n \in \mathbb{N}$, define $B_n = \{x \in B : \forall y \in [x]_F \ (\tilde{\rho}([x]_F, \{y\}) \leq n)\}$. Recall from [**5**] that a set $C \subseteq X$ is *almost ρ-discrete* if there is an open neighborhood U of 1 such that for each $x \in C$, there are only finitely many $y \in [x]_{E|C}$ with $\rho(x, y) \in U$.

SUBLEMMA 4.5. *Each set of the form $[A_m]_F \cap B_n$ is almost ρ-discrete.*

Proof. Suppose that $x, y \in [A_m]_F \cap B_n$ are E-related and fix $x' \in A_m \cap [x]_F$ and $y' \in A_m \cap [y]_F$. As $\rho(x, y) = \tilde{\rho}(\{x\}, [x]_F)\rho'(x', y')\tilde{\rho}([y]_F, \{y\})$, it follows that

$$(1/n)\rho'(x', y') \leq \rho(x, y) \leq n\rho'(x', y'),$$

so the ρ'-discreteness of A_m implies that $[A_m]_F \cap B_n$ is almost ρ-discrete. □

As $B = \bigcup_{m,n \in \mathbb{N}} [A_m]_F \cap B_n$, Proposition 2.4 of [**5**] ensures that $\rho|(E|B)$ is σ-discrete, the desired contradiction. □

Now define $(E|B)$-complete Borel sets $Y = \{y \in A : I_{[y]_{F'}}(\phi) < I_{[y]_E}(\phi) - \epsilon/4\}$ and $Z = \{z \in A : I_{[z]_{F'}}(\phi) > I_{[z]_E}(\phi) + \epsilon/4\}$, noting that Y and Z are disjoint from $\bigcup \Psi$, thus $F|(Y \cup Z) = F'|(Y \cup Z)$.

LEMMA 4.6. *There exist $x \in A$, $y \in Y \cap [x]_E$, and $z \in Z \cap [x]_E$ with the property that for every open neighborhood U of 1, there are infinitely many $y' \in Y \cap [x]_E$ and $z' \in Z \cap [x]_E$ such that $\rho'(y', y), \rho'(z', z) \in U$.*

Proof. For each Borel set $C \subseteq A$ and open neighborhood U of 1, define $C_U \subseteq C$ by

$$C_U = \{x \in C : |\{x' \in [x]_{E|C} : \rho'(x', x) \in U\}| < \infty\}.$$

The Lusin-Novikov uniformization theorem implies that C_U is Borel, and Propositions 2.4 and 2.5 of [**5**] ensure that C_U is the union of countably many $(\rho', (1/2, 2))$-discrete Borel sets. Letting $C_n = C_{(1-1/n, 1+1/n)}$, it follows from Proposition 2.6 of [**5**] that the set $D = \bigcup_{n>0} [Y_n]_E \cup [Z_n]_E$ is the union of countably many ρ'-discrete Borel sets, thus there exists $x \in A \setminus D$, and it is clear that any $y \in Y \cap [x]_E$ and $z \in Z \cap [x]_E$ are as desired. □

Choose $m, n \in \mathbb{N}$ such that $1/2 < (m/n)\,\rho'(y,z) < 2$, as well as $\delta > 0$ such that

$$\frac{1}{2} < \frac{m(1-\delta)|[y]_F|_x}{n(1+\delta)|[z]_F|_x}, \frac{m(1+\delta)|[y]_F|_x}{n(1-\delta)|[z]_F|_x} < 2,$$

and fix pairwise distinct points $y_i \in Y \cap [x]_E$ and $z_j \in Z \cap [x]_E$ such that $1-\delta < \rho'(y_i, y), \rho'(z_j, z) < 1+\delta$, for all $i < m$ and $j < n$. Set $Y' = \bigcup_{i<m}[y_i]_F$ and $Z' = \bigcup_{j<n}[z_j]_F$, and note that

$$m(1-\delta)|[y]_F|_x < |Y'|_x < m(1+\delta)|[y]_F|_x$$

and

$$n(1-\delta)|[z]_F|_x < |Z'|_x < n(1+\delta)|[z]_F|_x,$$

thus

$$\frac{m(1-\delta)|[y]_F|_x}{n(1+\delta)|[z]_F|_x} < \frac{|Y'|_x}{|Z'|_x} < \frac{m(1+\delta)|[y]_F|_x}{n(1-\delta)|[z]_F|_x}.$$

As the middle quantity is by definition $\tilde{\rho}(Y', Z')$, it follows that $\tilde{\rho}(Y', Z'), \tilde{\rho}(Z', Y') < 2$, so $\tilde{\rho}(Y' \cup Z', Y'), \tilde{\rho}(Y' \cup Z', Z') < 3$. Observe now that

$$\begin{aligned} I_{Y' \cup Z'}(\phi) &= \frac{\sum_{y' \in Y'} \phi(y')\rho(y', x) + \sum_{z' \in Z'} \phi(z')\rho(z', x)}{\sum_{w' \in Y' \cup Z'} \rho(w', x)} \\ &= \left(\frac{\sum_{y' \in Y'} \phi(y')\rho(y', x)}{\sum_{y' \in Y'} \rho(y', x)}\right)\left(\frac{\sum_{y' \in Y'} \rho(y', x)}{\sum_{w' \in Y' \cup Z'} \rho(w', x)}\right) + \\ &\quad \left(\frac{\sum_{z' \in Z'} \phi(z')\rho(z', x)}{\sum_{z' \in Z'} \rho(z', x)}\right)\left(\frac{\sum_{z' \in Z'} \rho(z', x)}{\sum_{w' \in Y' \cup Z'} \rho(w', x)}\right) \\ &= I_{Y'}(\phi)\tilde{\rho}(Y', Y' \cup Z') + I_{Z'}(\phi)\tilde{\rho}(Z', Y' \cup Z'). \end{aligned}$$

It follows that

$$\begin{aligned} I_{Y' \cup Z'}(\phi) &= \tilde{\rho}(Y', Y' \cup Z')I_{Y'}(\phi) + \tilde{\rho}(Z', Y' \cup Z')I_{Z'}(\phi) \\ &< (1/3)I_{Y'}(\phi) + (2/3)I_{Z'}(\phi) \\ &< (1/3)(I_{[x]_E}(\phi) - \epsilon/4) + (2/3)(I_{[x]_E}(\phi) + \epsilon/2) \\ &= I_{[x]_E}(\phi) + \epsilon/4, \end{aligned}$$

and similarly $I_{Y' \cup Z'}(\phi) > I_{[x]_E}(\phi) - \epsilon/4$, thus $|I_{Y' \cup Z'}(\phi) - I_{[x]_E}(\phi)| < \epsilon/4$, which contradicts the maximality of Ψ. □

We are now ready to prove our main theorem:

THEOREM 4.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is compressible;*
2. *There is a ρ-invariant probability measure on X.*

Proof. Propositions 2.5 and 2.6 yield $(1) \Rightarrow \neg(2)$, so it is enough to show $\neg(1) \Rightarrow (2)$. Towards this end, suppose that ρ is not compressible. If there is a ρ-finite E-class C, then there is a unique ρ-invariant probability measure which concentrates on C, so we can assume that ρ is aperiodic.

Fix a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$. For each $\gamma \in \Gamma$, define $\rho_\gamma : X \to (0, \infty)$ by $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$. By standard change of topology results (see, for example, §13 of [**4**]), we can assume that there is a countable, Γ-invariant algebra $\mathcal{U}$ of subsets of X which is a basis and contains every set of the form $\rho_\gamma^{-1}(I)$, where $\gamma \in \Gamma$ and $I \subseteq (0, \infty)$ is an open interval with rational endpoints. From this point forward, we work only with this topology and a fixed compatible, complete metric. Fix an enumeration $\langle \phi_n \rangle_{n \in \mathbb{N}}$ of the bounded functions of the form $\rho_\gamma \mathbf{1}_U$, where $\gamma \in \Gamma$ and $U \in \mathcal{U}$, and let Φ denote the linear subspace of $C_b(X)$ spanned by $\langle \phi_n \rangle_{n \in \mathbb{N}}$.

We will now construct an increasing sequence $\langle F_k \rangle_{k \in \mathbb{N}}$ of finite Borel subequivalence relations of E. We begin by setting $F_0 = \Delta(X)$. Given F_k, by applying Proposition 4.3 finitely many times and throwing out the corresponding E-invariant, ρ-discrete Borel sets (as Proposition 2.8 allows us to do), we obtain a finite Borel subequivalence relation F_{k+1} of E containing F_k which is $(\phi_n, 1/k)$-approximating, for all $n \leq k$.

For each $x \in X$, define $I_x : \Phi \to \mathbb{R}$ by $I_x(\phi) = \lim_{k \to \infty} I_{[x]_{F_k}}(\phi)$. Then I_x is a mean on Φ, and the function $\mu_x : \mathcal{U} \to [0,1]$ given by $\mu_x(U) = I_x(\mathbf{1}_U)$ is a finitely additive probability measure. Propositions 3.1 and 3.2 ensure that μ_x^* is a measure.

For each $U \in \mathcal{U}$ and $n \in \mathbb{N}$, fix a partition $\langle U_n \rangle \in \mathcal{U}^\mathbb{N}$ of U into sets of diameter less than $1/(n+1)$. Then the E-invariant Borel set

$$A_{U,n} = \Big\{ x \in X : \mu_x(U) \neq \lim_{n \to \infty} \mu_x\Big(\bigcup_{m<n} U_m \Big) \Big\}$$

is ρ-compressible of type I. By throwing out every set of this form, we can assume that each μ_x is decomposable. Proposition 3.5 then implies that $I_x = I_x^* | \Phi$, for all $x \in X$. In particular, it follows that each μ_x^* is a probability measure.

For $n \in \mathbb{N}$, $\gamma \in \Gamma$, and $U \in \mathcal{U}$ such that $\rho_\gamma | U$ is bounded, the E-invariant Borel set

$$B_{\gamma,U,n} = \{ x \in X : \forall y \in [x]_E \ (\tilde{\rho}(\gamma(U) \cap [y]_{F_n}, \gamma(U \cap [y]_{F_n})) > 1) \}$$

is ρ-compressible of type II, as is the E-invariant Borel set

$$C_{\gamma,U,n} = \{ x \in X : \forall y \in [x]_E \ (\tilde{\rho}(\gamma(U) \cap [y]_{F_n}, \gamma(U \cap [y]_{F_n})) < 1) \}.$$

We will complete the proof of the theorem by showing that if x is not in the union of the sets of this form, then μ_x^* is ρ-invariant. Suppose, towards a contradiction, that there exist $\gamma \in \Gamma$ and a Borel set $B \subseteq X$ such that

$$\mu_x^*(\gamma(B)) \neq \int_B \rho(\gamma \cdot y, y) \, d\mu_x^*(y).$$

We can clearly assume that $B = U$, for some $U \in \mathcal{U}$ (see, for example, Theorem 17.10 of [**4**]), and we can also assume that $\rho_\gamma | U$ is bounded, thus $\mu_x(\gamma(U)) \neq I_x(\rho_\gamma \mathbf{1}_U)$.

If $\mu_x(\gamma(U)) > I_x(\rho_\gamma \mathbf{1}_U)$, then there exists $n \in \mathbb{N}$ such that $\mu_{[y]_{F_n}}(\gamma(U)) > I_{[y]_{F_n}}(\rho_\gamma \mathbf{1}_U)$, for all $y \in [x]_E$, and it follows that $x \in B_{\gamma,U,n}$, a contradiction. Similarly, if $\mu_x(\gamma(U)) < I_x(\rho_\gamma \mathbf{1}_U)$, then there exists $n \in \mathbb{N}$ such that $\mu_{[y]_{F_n}}(\gamma(U)) < I_{[y]_{F_n}}(\rho_\gamma \mathbf{1}_U)$, for all $y \in [x]_E$, and it follows that $x \in C_{\gamma,U,n}$, a contradiction. □

5. *A characterization of the existence of non-trivial, ρ-invariant probability measures*
In the spirit of [**5**], we now characterize the circumstances under which there is a suitably non-trivial, ρ-invariant probability measure on X:

THEOREM 5.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is an aperiodic Borel cocycle. Then the following are equivalent:*
1. *There is a ρ-invariant probability measure on X;*
2. *There is an atomless, ρ-invariant probability measure on X;*
3. *There is an E-ergodic, ρ-invariant probability measure on X;*
4. *There is an atomless, E-ergodic, ρ-invariant probability measure on X;*
5. *There is a ρ-invariant probability measure on X which concentrates off of Borel partial transversals of E;*
6. *There is a ρ-invariant probability measure on X which concentrates off of ρ-discrete Borel sets.*

Proof. The aperiodicity of ρ ensures that every ρ-invariant probability measure is atomless, thus $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$. Proposition 2.15 of [**5**] gives $(4) \Rightarrow (5)$, Proposition 2.16 of [**5**] gives $(5) \Rightarrow (6)$, and $(6) \Rightarrow (1)$ is trivial, so it only remains to prove $(1) \Rightarrow (3)$. By Theorem 4.7, it is sufficient to show that if ρ is not compressible, then there is an E-ergodic, ρ-invariant probability measure on X. Fix Γ, $\langle \rho_\gamma \rangle_{\gamma \in \Gamma}$, $\mathcal{U}$, $\langle \phi_n \rangle_{n \in \mathbb{N}}$, and Φ as in the proof of Theorem 4.7, as well as an enumeration $\langle I_n \rangle_{n \in \mathbb{N}}$ of the set of subintervals of $(0, \infty)$ with rational endpoints.

We will again construct an increasing sequence $\langle F_k \rangle_{k \in \mathbb{N}}$ of finite Borel subequivalence relations of E. This time, we will simultaneously construct Polish topologies τ_k on X, bases $\mathcal{U}_k$ for (X, τ_k), and sequences $\langle \phi_{kn} \rangle_{n \in \mathbb{N}}$ which span a linear subspace of $C_b(X, \tau_k)$ containing $\mathcal{U}_k$. We begin by setting $F_0 = \Delta(X), \mathcal{U}_0 = \mathcal{U}$, and $\phi_{0n} = \phi_n$. We also let τ_0 denote the topology discussed in the proof of Theorem 4.7. Given $(F_k, \tau_k, \mathcal{U}_k, \langle \phi_{kn} \rangle_{n \in \mathbb{N}})$, we can again apply Proposition 4.3 finitely many times so as to obtain a finite Borel subequivalence relation F'_k of E containing F_k which is $(\phi_{ij}, 1/k)$-approximating, for all $i, j \leq k$ (of course, we must again remove finitely many E-invariant, ρ-discrete Borel sets, as Proposition 2.8 allows us to do). This time, however, we shall approximate more sets. For $i, j, m, n \leq k$, define

$$X_{ijmn} = \{x \in X : \forall y \in [x]_E \ (I_{[y]_{F_m}}(\phi_{ij}) \in I_n)\}.$$

By applying Proposition 4.3 finitely many times and again throwing out the corresponding E-invariant, ρ-discrete Borel sets, we obtain a finite Borel subequivalence relation F_{k+1} of E containing F'_k which is $(\mathbf{1}_{X_{ijmn}}, 1/k)$-approximating, for all $i, j, m, n \leq k$. Fix a Polish topology τ_{k+1} on X containing τ_k for which there is a countable, Γ-invariant algebra $\mathcal{U}_{k+1}$ of sets which is a basis for (X, τ_{k+1}) and contains each of the sets X_{ijmn}, for $i, j, m, n \leq k$. Fix an enumeration $\langle \phi_{(k+1)n} \rangle_{n \in \mathbb{N}}$ of the bounded functions of the form $\rho_\gamma \mathbf{1}_U$, where $\gamma \in \Gamma$ and $U \in \mathcal{U}_{k+1}$.

As in the proof of Theorem 4.7, by throwing out countably many E-invariant, ρ-compressible Borel sets, we can assume that each of the corresponding means I_x is decomposable, and each of the maps μ_x^* is a ρ-invariant probability measure on X. Define an equivalence relation F on X containing E by setting

$$\begin{aligned} xFy &\Leftrightarrow \mu_x^* = \mu_y^* \\ &\Leftrightarrow \forall U \in \mathcal{U}\ (\mu_x(U) = \mu_y(U)) \\ &\Leftrightarrow \neg\exists i, j, m, n_x, n_y \in \mathbb{N}\ (I_{n_x} \cap I_{n_y} = \emptyset \text{ and } x \in X_{ijmn_x} \text{ and } y \in X_{ijmn_y}). \end{aligned}$$

As X_{ijmn_y} is E-invariant, Proposition 3.3 ensures that if $I_{n_x} \cap I_{n_y} = \emptyset$ and $x \in X_{ijmn_x}$, then $\mu_x^*(X_{ijmn_y}) = \mu_x(X_{ijmn_y}) = 0$. Letting

$$S_x = \{(i, j, m, n_y) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \exists n_x \in \mathbb{N}\ (I_{n_x} \cap I_{n_y} = \emptyset \text{ and } x \in X_{ijmn_x})\},$$

it follows that $\mu_x^*([x]_F) \geq 1 - \sum_{(i,j,m,n_y) \in S_x} \mu_x^*(X_{ijmn_y}) = 1$.

It remains to check that μ_x^* is E-ergodic. Towards this end, suppose that $C \subseteq X$ is an E-invariant Borel set of positive μ_x^*-measure. Given $0 < \epsilon < \mu_x^*(C)$, fix a set $U \in \mathcal{U}$ such that $\mu_x^*(U) > \epsilon$ and $\mu_x^*(U \setminus C) \leq \epsilon^2$ (see, for example, Theorem 17.10 of [**4**]), and put

$$D = \{y \in [x]_F : \mu_y(C), \mu_y(U \setminus C) \text{ exist and } \mu_y(U \setminus C) \leq \epsilon\}.$$

Proposition 2.4 ensures that $\mu_y(C), \mu_y(U \setminus C)$ exist μ_x^*-almost everywhere and

$$\epsilon^2 \geq \mu_x^*(U \setminus C) \geq \int_{[x]_F \setminus D} \mu_y(U \setminus C)\ d\mu_x^*(y) \geq \epsilon(1 - \mu_x^*(D)),$$

thus $\mu_x^*(D) \geq 1 - \epsilon$. Observe now that if $y \in D$, then

$$\mu_y(C) = \mu_y(U) - \mu_y(U \setminus C) = \mu_x^*(U) - \mu_y(U \setminus C) > 0,$$

so $C \cap [y]_E \neq \emptyset$, thus $y \in C$. As $y \in D$ was arbitrary, it follows that $D \subseteq C$, hence $\mu_x^*(C) \geq \mu_x^*(D) \geq 1 - \epsilon$. As $0 < \epsilon < \mu_x^*(C)$ was arbitrary, it follows that $\mu_x^*(C) = 1$. □

Let $P(X)$ denote the standard Borel space of all probability measures on X (see, for example, §17 of [**4**]). The idea behind the above proof can be used to give a new proof of:

THEOREM 5.2 (DITZEN) *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is an incompressible Borel cocycle. Then there is a Borel function $\pi : X \to P(X)$ such that:*

1. *Each of the measures $\pi(x)$ is E-ergodic and ρ-invariant;*
2. *$\mu(\{x \in X : \pi(x) = \mu\}) = 1$, for every E-ergodic, ρ-invariant $\mu \in P(X)$;*
3. *$\mu = \int \pi\ d\mu$, for every ρ-invariant $\mu \in P(X)$.*

Proof. We will assume that ρ is aperiodic, as it is clear how to proceed when ρ is finite. Let $\pi(x) = \mu_x^*$, where μ_x is defined as in the proof of Theorem 5.1. Clearly we can ignore the ρ-negligible set on which there are no ρ-invariant probability measures, so that (1) holds. Note that if μ is ρ-invariant, then for each $U \in \mathcal{U}$, Proposition 2.4 implies that $\mu(U) = \int \mu_x(U)\ d\mu(x) = \int \mu_x^*(U)\ d\mu(x)$, and (3) follows. To see (2), note that if μ is E-ergodic and ρ-invariant, then for each $U \in \mathcal{U}$, the function $\mu_x(U)$ is constant μ-almost everywhere. Proposition 2.4 then implies that $\mu_x^*(U) = \mu_x(U) = \mu(U)$ μ-almost everywhere. As $\mathcal{U}$ is countable, it follows that $\mu(\{x \in X : \pi(x) = \mu\}) = 1$. □

6. *A fuzzy characterization of the existence of ρ-invariant probability measures*

A *fuzzy Borel set* is a Borel function $b : X \to [0,1]$. A *fuzzy ρ-injection* of a into b is a fuzzy partial injection $\phi \in [\![\rho]\!]$ such that $\mathrm{fdom}(\phi) = a$ and $\mathrm{frng}(\phi) \leq b$.

PROPOSITION 6.1. *Suppose that X is a Polish space, E is a smooth countable Borel equivalence relation on X, $\rho : E \to (0,\infty)$ is a Borel cocycle, and a, b are fuzzy Borel sets with $I_{[x]_E}(a) \leq I_{[x]_E}(b)$, for all $x \in X$. Then there is a fuzzy ρ-injection of a into b.*

Proof. This is a straightforward consequence of the smoothness of E. □

The following two facts imply that compressible cocycles are fuzzily compressible:

PROPOSITION 6.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0,\infty)$ is a Borel cocycle, and $B \subseteq X$ is an E-invariant Borel set which is ρ-compressible of type I. Then $\rho|(E|B)$ is fuzzily compressible.*

Proof. Fix an increasing sequence $\langle F_k \rangle_{k\in\mathbb{N}}$ of finite Borel subequivalence relations of E and a partition $\langle B_n \rangle_{n\in\mathbb{N}}$ of B into Borel sets such that (1) $\mu_{[x]_{F_k}}(B_n)$ converges uniformly to $\mu_x(B_n)$, for each $n \in \mathbb{N}$, and (2) $\sum_{n\in\mathbb{N}} \mu_x(B_n) < 1$, for all $x \in B$. For each $x \in B$, fix $n(x) \in \mathbb{N}$ least such that $\sum_{n\geq n(x)} \mu_x(B_n) \leq \lim_{n\to\infty} \mu_x(\bigcup_{m>n} B_m)$, set $B'_n = \{x \in X : x \in B_{n+n(x)}\}$, and define $B' = \bigcup_{n\in\mathbb{N}} B'_n$. For each $n \in \mathbb{N}$, fix $k_n(x) \in \mathbb{N}$ least such that

$$\sum_{m\leq n} \mu_{[y]_{F_{k_n(x)}}}(B'_m) \leq \mu_{[y]_{F_{k_n(x)}}}\Big(\bigcup_{m>n} B'_m\Big),$$

for all $y \in [x]_E$, noting that $\langle k_n(x) \rangle_{n\geq n(x)}$ is non-decreasing. Define equivalence relations F'_n on B by setting $xF'_ny \Leftrightarrow xF_{k_n(x)}y$, noting that $\langle F'_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence of finite Borel subequivalence relation of E and $\langle B'_n \rangle_{n\in\mathbb{N}}$ is a partition of B' such that

$$\sum_{m\leq n} \mu_{[y]_{F'_n}}(B'_m) \leq \mu_{[y]_{F'_n}}\Big(\bigcup_{m>n} B'_m\Big),$$

for all $n \in \mathbb{N}$, $x \in B$, and $y \in [x]_E$. Set $\rho_n = \rho|F'_n$.

We will now recursively define fuzzy ρ_n-injections ϕ_n of $\mathbf{1}_{B'_n}$ into $\sum_{m>n} \mathbf{1}_{B'_m}$. Suppose that we have already defined $\langle \phi_m \rangle_{m<n}$. Then for all $x \in B$,

$$I_{[x]_{F'_n}}(\mathbf{1}_{B'_n}) \leq I_{[x]_{F'_n}}\Big(\sum_{m>n} \mathbf{1}_{B'_m} - \sum_{m<n} \mathbf{1}_{B'_m}\Big) = I_{[x]_{F'_n}}\Big(\sum_{m>n} \mathbf{1}_{B'_m} - \sum_{m<n} \mathrm{frng}(\phi_m)\Big),$$

thus Proposition 6.1 ensures that there is a fuzzy ρ_n-injection ϕ_n of $\mathbf{1}_{B'_n}$ into $\sum_{m>n} \mathbf{1}_{B'_m} - \sum_{m<n} \mathrm{frng}(\phi_m)$. This completes the recursive construction. Clearly $\phi = \sum_{n\in\mathbb{N}} \phi_n$ is a fuzzy compression of $\rho|(E|B')$, thus $\rho|(E|B)$ is fuzzily compressible. □

PROPOSITION 6.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0,\infty)$ is a Borel cocycle, and $B \subseteq X$ is an E-invariant Borel set which is ρ-compressible of type II. Then $\rho|(E|B)$ is fuzzily compressible.*

Proof. Fix a smooth Borel subequivalence relation F of E, a Borel set $A \subseteq B$, and $T \in [E]$ such that $\sum_{y\in T(A)\cap[x]_F} \rho(y,x) < \sum_{y\in T(A\cap[x]_F)} \rho(y,x)$, for all $x \in B$. The smoothness of F easily implies that there is a fuzzy compression ϕ of $\rho|(E|B)$ such that $\mathrm{supp}(\phi_d(x,\cdot)) \subseteq T([x]_F)$, for all $x \in X$. □

Remark. In the special case that $\rho \equiv 1$, it is not difficult to see that if E is compressible, then X is ρ-compressible of types I and II, and the idea behind the proofs of Propositions 6.2 and 6.3 can be used to show that if ρ is compressible, then E is compressible. Together with Theorem 4.7, this gives a new proof of Nadkarni's Theorem [**6**].

Next, we show that fuzzy compressibility rules out ρ-invariant probability measures:

PROPOSITION 6.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a fuzzily compressible Borel cocycle. Then there is no ρ-invariant probability measure on X.*

Proof. Suppose, towards a contradiction, that μ is a ρ-invariant probability measure on X, and fix a fuzzy compression ϕ of ρ and Borel involutions $I_n : X \to X$ such that $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(I_n)$. Set $B_n = \{x \in X : \forall m < n \ (I_n(x) \neq I_m(x))\}$, and observe that

$$\begin{aligned}
\int [\operatorname{frng}(\phi)](y)\, d\mu(y) &= \int \sum_{x \in [y]_E} \phi_d(x, y) \rho(x, y)\, d\mu(y) \\
&= \int \sum_{n \in \mathbb{N}} \mathbf{1}_{B_n}(y) \phi_d(I_n(y), y) \rho(I_n(y), y)\, d\mu(y) \\
&= \sum_{n \in \mathbb{N}} \int_{B_n} \phi_d(I_n(y), y) \rho(I_n(y), y)\, d\mu(y) \\
&= \sum_{n \in \mathbb{N}} \int_{B_n} \phi_d(x, I_n(x))\, d\mu(x) \\
&= \int \sum_{n \in \mathbb{N}} \mathbf{1}_{B_n}(x) \phi_d(x, I_n(x))\, d\mu(x) \\
&= \int \sum_{y \in [x]_E} \phi_d(x, y)\, d\mu(x) \\
&= \int [\operatorname{fdom}(\rho)](x)\, d\mu(x).
\end{aligned}$$

As $\operatorname{fdom}(\phi) \equiv 1$, it follows that $[\operatorname{frng}(\phi)](x) = 1$, for μ-almost every $x \in X$. As $\operatorname{frng}(\phi)$ is not identically 1 on any E-class, this contradicts the fact that μ is E-quasi-invariant. □

We are now ready for our final theorem:

THEOREM 6.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is fuzzily compressible;*
2. *There is a ρ-invariant probability measure on X.*

Proof. This follows from Theorem 4.7 and Propositions 6.2, 6.3, and 6.4. □

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References

[1] A. Ditzen. Definable equivalence relations on Polish spaces. Ph.D. thesis, *California Institute of Technology* (1992)

[2] J. Feldman and C.C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Amer. Math. Soc.*, **234 (2)**, (1977), 289–324

[3] P. Halmos. *Measure Theory*. D. Van Nostrand Company, Inc., New York (1950)

[4] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (1995)

[5] B.D. Miller. The existence of measures of a given cocycle, I: Atomless, ergodic σ-finite measures. *Ergodic Theory Dynam. Systems*, this issue

[6] M. Nadkarni. On the existence of a finite invariant measure. *Proc. Indian Acad. Sci. Math. Sci.*, **100 (3)**, (1990), 203–220

[7] K. Petersen. *Ergodic theory*, volume 2 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (1983)