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The existence of measures of a given cocycle, I: Atomless, ergodic σ-finite measures

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Abstract. Given a Polish space X, a countable Borel equivalence relation E on X, and a Borel cocycle $\rho : E \to (0, \infty)$, we characterize the circumstances under which there is a suitably non-trivial σ-finite measure μ on X such that $\rho(\phi^{-1}(x), x) = [d(\phi_*\mu)/d\mu](x)$ μ-almost everywhere, for every Borel injection ϕ whose graph is contained in E.

1. *Introduction*

A topological space is *Polish* if it is separable and admits a complete metric. A topological group is *Polish* if its topology is Polish. An equivalence relation is *finite* if all of its equivalence classes are finite, and *countable* if all of its equivalence classes are countable. By a *measure* on a Polish space, we shall always mean a measure defined on its Borel subsets which is not identically zero. A measure is *atomless* if every Borel set of positive measure contains a Borel set of strictly smaller positive measure. Measures μ and ν are *equivalent*, or $\mu \sim \nu$, if they have the same null sets. Given a measure μ on X and a Borel function $\phi : X \to Y$, let $\phi_*\mu$ denote the measure on Y given by $\phi_*\mu(B) = \mu(\phi^{-1}(B))$.

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, μ is a measure on X, and $\rho : E \to G$ is Borel. Let $[\![E]\!]$ denote the set of all Borel injections $\phi : A \to B$, where $A, B \subseteq X$ are Borel and $\mathrm{graph}(\phi) \subseteq E$. We say that μ is *E-quasi-invariant* if $\phi_*\mu \sim \mu$, for all $\phi \in [\![E]\!]$. We say that ρ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$, for all $xEyEz$. In the special case that G is the group $(0, \infty)$ of positive real numbers under multiplication, we say that μ is *ρ-invariant* if

$$\phi_*\mu(B) = \int_B \rho(\phi^{-1}(x), x)\, d\mu(x),$$

for all $\phi \in [\![E]\!]$ and Borel sets $B \subseteq \mathrm{rng}(\phi)$. When $\rho \equiv 1$, we say that μ is *E-invariant*.

These notions typically arise in a slightly different guise in the context of group actions. The *orbit equivalence relation* associated with an action of a countable group Γ by Borel automorphisms of X is given by $xE^X_\Gamma y \Leftrightarrow \exists \gamma \in \Gamma\ (\gamma \cdot x = y)$. It is easy to see that if

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$\gamma_* \mu \sim \mu$, for all $\gamma \in \Gamma$, then μ is E_Γ^X-quasi-invariant, and similarly, if $\rho : E_\Gamma^X \to (0, \infty)$ is a Borel cocycle such that

$$\gamma_* \mu(B) = \int_B \rho(\gamma^{-1} \cdot x, x) \, d\mu(x),$$

for all $\gamma \in \Gamma$ and Borel sets $B \subseteq X$, then μ is ρ-invariant.

Theorem 1 of [**1**] and the Radon-Nikodym Theorem (see, for example, Theorem 6.10 of [**5**]) easily imply that if μ is E-quasi-invariant and σ-finite, then there is a Borel cocycle $\rho : E \to (0, \infty)$ such that μ is ρ-invariant, and moreover, this cocycle is unique modulo μ-null sets. Here we investigate the conditions under which we can go in the other direction. That is, given a Borel cocycle $\rho : E \to (0, \infty)$, we characterize the circumstances under which there is a suitably non-trivial, ρ-invariant σ-finite measure on X. The problem of finding such a characterization was posed originally in [**6**].

Before getting to our main results, we will review the well known answer to the special case of our question for E-invariant measures. First, however, we need to lay out some terminology. The *E-class* of x is given by $[x]_E = \{y \in X : xEy\}$. A set $B \subseteq X$ is a *partial transversal* of E if it intersects every E-class in at most one point. We say that E is *smooth* if X is the union of countably many Borel partial transversals. The *E-saturation* of B is given by $[B]_E = \{x \in X : \exists y \in B \; (xEy)\}$, and we say that B is *E-invariant* if $B = [B]_E$. We say that μ is *E-ergodic* if every E-invariant Borel set is μ-null or μ-conull.

It is not difficult to show that there is always an E-ergodic, ρ-invariant σ-finite measure on X, and if X is uncountable, then there is always an atomless, ρ-invariant σ-finite measure on X. Although the main result of [**9**] is stated in terms of quasi-invariant measures for Borel automorphisms, a straightforward modification of the argument gives:

THEOREM 1 (SHELAH-WEISS) *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*
1. *E is smooth;*
2. *There is an atomless, E-ergodic, E-invariant σ-finite measure on X.*

In order to characterize the existence of measures beyond the E-invariant case, we must first generalize the notion of smoothness. Given a set $U \subseteq G$ and a Borel cocycle $\rho : E \to G$, we say that a set $B \subseteq X$ is *(ρ, U)-discrete* if $\rho(x, y) \in U \Rightarrow x = y$, for all $(x, y) \in E|B$. We say that B is *ρ-discrete* if there is an open neighborhood U of 1_G such that B is (ρ, U)-discrete, and we say that ρ is *σ-discrete* if X is the union of countably many ρ-discrete Borel sets. It is not difficult to see that if $\rho \equiv 1_G$, then a set is ρ-discrete if and only if it is a partial transversal of E, so ρ is σ-discrete if and only if E is smooth, thus the following fact generalizes Theorem 1:

THEOREM 2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*
1. *ρ is σ-discrete;*
2. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X.*

Much as in [**9**], we obtain Theorem 2 as a corollary of a descriptive set-theoretic Glimm-Effros style dichotomy theorem. Using this theorem, we also obtain:

THEOREM 3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0,\infty)$ is a Borel cocycle. Then the following are equivalent:*
1. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X;*
2. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X which is equivalent to an atomless, E-ergodic, E-invariant σ-finite measure on X;*
3. *There is a ρ-invariant σ-finite measure on X which concentrates off of Borel partial transversals of E;*
4. *There is a ρ-invariant σ-finite measure on X which concentrates off of ρ-discrete Borel sets;*
5. *There is a family of continuum-many atomless, E-ergodic, ρ-invariant σ-finite measures on X with pairwise disjoint supports;*
6. *There is a finer Polish topology τ on X such that for every τ-comeager set $C \subseteq X$, there is an atomless, E-ergodic, ρ-invariant σ-finite measure concentrating on C.*

It is worth noting that while the analogs of conditions (1), (3), and (4) for probability measures are equivalent, the analogs of conditions (2) and (5) are strictly stronger, and the analog of condition (6) never holds (see Theorem 13.1 of [**3**]).

We say that a set $B \subseteq X$ is *globally Baire* if for every Polish space Y and Borel function $\pi : Y \to X$, the set $\pi^{-1}(B)$ has the property of Baire. It is well known that every $\sigma(\boldsymbol{\Sigma}^1_1)$ set is globally Baire, and under strong set-theoretic hypotheses, the class of globally Baire sets contains much more (see, for example, Theorem 38.17 of [**2**]). In fact, it is consistent with $\mathtt{ZF}+\mathtt{DC}$ that every subset of a Polish space is globally Baire (see [**7**]). Again using our descriptive set-theoretic Glimm-Effros style dichotomy theorem, we obtain the following alternative characterization of σ-discrete Borel cocycles:

THEOREM 4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, and $\rho : E \to G$ is a Borel cocycle. Then the following are equivalent:*
1. *X is the union of countably many ρ-discrete Borel sets;*
2. *X is the union of countably many ρ-discrete globally Baire sets.*

We say that an equivalence relation E is *hyperfinite* if there are finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$. As a corollary of Theorem 2, we also obtain the following characterization of hyperfiniteness:

THEOREM 5. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*
1. *E is hyperfinite;*
2. *For every Borel cocycle $\rho : E \to (0,\infty)$, there is an atomless, E-ergodic, ρ-invariant σ-finite measure on X.*

The organization of the paper is as follows. In §2, we discuss some basic facts concerning equivalence relations, cocycles, and measures. In §3, we prove our descriptive set-theoretic Glimm-Effros style dichotomy theorem, Theorem 4, and a descriptive set-theoretic analog of Theorem 5. In §4, we establish Theorems 2, 3, and 5.

2. *Preliminaries*

The following fact appeared originally as Theorem 1 of [**1**]:

PROPOSITION 2.1 (FELDMAN-MOORE) *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a countable group Γ of Borel automorphisms of X such that $E = E^X_\Gamma$.*

Proof. Suppose that ϕ is an injection of a subset of X into X. The *orbit equivalence relation* associated with ϕ is given by

$$xE^X_\phi y \Leftrightarrow \exists n \in \mathbb{Z}\ (\phi^n(x) = y),$$

and the *orbit* of a point x under ϕ is given by $[x]_\phi = \{\phi^n(x) : n \in \mathbb{Z} \text{ and } x \in \mathrm{dom}(\phi^n)\}$. Let $[E]$ denote the group of all Borel automorphisms of X in $[\![E]\!]$.

LEMMA 2.2. *For all $\phi \in [\![E]\!]$, there exists $T \in [E]$ such that $E^X_T = E^X_\phi$.*

Proof. We define $T|[x]_\phi$ by examining the sets $D_x = [x]_\phi \setminus \mathrm{dom}(\phi)$ and $R_x = [x]_\phi \setminus \mathrm{rng}(\phi)$. If both D_x and R_x are empty, then we set $T|[x]_\phi = \phi|[x]_\phi$. If only D_x is empty, then there is a unique point $y \in R_x$, and we define $T|[x]_\phi$ by

$$T(w) = \begin{cases} \phi^{2(n+1)}(y) & \text{if } w = \phi^{2n}(y), \\ \phi^{2n+1}(y) & \text{if } w = \phi^{2(n+1)+1}(y), \\ y & \text{if } w = \phi(y). \end{cases}$$

If only R_x is empty, then there is a unique point $z \in D_x$, and we define $T|[x]_\phi$ by

$$T(w) = \begin{cases} \phi^{-2(n+1)}(z) & \text{if } w = \phi^{-2n}(z), \\ \phi^{-(2n+1)}(z) & \text{if } w = \phi^{-(2(n+1)+1)}(z), \\ z & \text{if } w = \phi^{-1}(z). \end{cases}$$

If neither D_x nor R_x is empty, then there are unique points $y \in R_x$ and $z \in D_x$, and we define $T|[x]_\phi$ by

$$T(w) = \begin{cases} y & \text{if } w = z, \\ \phi(w) & \text{otherwise.} \end{cases}$$

It is straightforward to check that T is as desired. □

By the Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of [**2**]), there are Borel functions $\phi_m : X \to X$ such that $E = \bigcup_{m \in \mathbb{N}} \mathrm{graph}(\phi_m)$. As each of these functions is necessarily countable-to-one, there are Borel sets $B_{mn} \subseteq X$ such that $\phi_m|B_{mn}$ is injective and $X = \bigcup_{n \in \mathbb{N}} B_{mn}$ (see, for example, Exercise 18.15 of [**2**]). By Lemma 2.2, there are Borel automorphisms $T_{mn} \in [E]$ such that $E^X_{T_{mn}} = E^X_{\phi_m|B_{mn}}$, and the group generated by these automorphisms is clearly as desired. □

A *directed graph* on X is an irreflexive set $\mathcal{G} \subseteq X \times X$. We say that $\mathcal{G}$ has *finite out-degree* if the set $\mathcal{G}_x = \{y \in X : (x, y) \in \mathcal{G}\}$ is finite, for each $x \in X$. A *coloring* of $\mathcal{G}$ is a function $c : X \to Y$ such that $c(x_1) \neq c(x_2)$, for all $(x_1, x_2) \in \mathcal{G}$. When Y is Polish and c is Borel, we say that c is a *Borel coloring* of $\mathcal{G}$. The *Borel chromatic number* of $\mathcal{G}$ is given by $\chi_B(\mathcal{G}) = \min\{|c(X)| : c \text{ is a Borel coloring of } \mathcal{G}\}$. The following fact is a straightforward consequence of the directed analogs of the arguments of §4 of [**4**]:

PROPOSITION 2.3 (KECHRIS-SOLECKI-TODORČEVIĆ) *Suppose that X is a Polish space and $\mathcal{G}$ is a Borel directed graph on X with finite out-degree. Then $\chi_B(\mathcal{G}) \leq \aleph_0$.*

Proof. Fix a countable sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of Borel subsets of X which is closed under finite intersection and *separates points*, in the sense that for all distinct $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x \in U_n$ and $y \notin U_n$. For each $n \in \mathbb{N}$, define $B_n \subseteq X$ by

$$B_n = \{x \in U_n : \mathcal{G}_x \cap U_n = \emptyset\}.$$

Then $\mathcal{G} \cap (B_n \times B_n) = \emptyset$, the Lusin-Novikov uniformization theorem implies that B_n is Borel, and our assumption that $\mathcal{G}$ has finite out-degree ensures that $X = \bigcup_{n \in \mathbb{N}} B_n$, thus the function $c(x) = \min\{n \in \mathbb{N} : x \in B_n\}$ is a Borel coloring of $\mathcal{G}$. □

We say that a set $B \subseteq X$ is *almost (ρ, U)-discrete* if for each $x \in B$, there are only finitely many $y \in [x]_{E|B}$ such that $\rho(x, y) \in U$.

PROPOSITION 2.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, $\rho : E \to G$ is a Borel cocycle, $U \subseteq G$ is Borel, and $B \subseteq X$ is an almost (ρ, U)-discrete Borel set. Then B is the union of countably many (ρ, U)-discrete Borel sets.*

Proof. Let $\mathcal{G}$ denote the directed graph on X given by

$$\mathcal{G} = \{(x, y) \in E|B : x \neq y \text{ and } \rho(x, y) \in U\}.$$

By Proposition 2.3, there is a Borel coloring $c : X \to \mathbb{N}$ of $\mathcal{G}$. Then each of the sets $B_n = B \cap c^{-1}(n)$ is (ρ, U)-discrete, and $B = \bigcup_{n \in \mathbb{N}} B_n$. □

Remark. In the special case that $\rho \equiv 1_G$, Proposition 2.4 implies that if $B \subseteq X$ is a Borel set which intersects every E-class in a finite set, then $E|B$ is smooth.

We say that a set $B \subseteq X$ is *almost ρ-discrete* if there is an open neighborhood U of 1_G such that B is almost (ρ, U)-discrete.

PROPOSITION 2.5. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, $\rho : E \to G$ is a Borel cocycle, $K \subseteq G$ is compact, and $B \subseteq X$ is almost ρ-discrete. Then B is almost (ρ, K)-discrete.*

Proof. Suppose, towards a contradiction, that there exists $x \in B$ for which there are infinitely many $y \in [x]_{E|B}$ such that $\rho(x, y) \in K$. Fix an open neighborhood U of 1_G such that B is almost (ρ, U)-discrete. By the continuity of inversion and multiplication, there is a non-empty open set $V \subseteq G$ such that $V^{-1}V \subseteq U$. The compactness of K ensures that it can be covered by finitely many left translates of V, thus there exist $g \in G$ and an infinite set $S \subseteq [x]_{E|B}$ such that $\rho(x, y) \in gV$, for all $y \in S$. Then $\rho(y, z) = \rho(y, x)\rho(x, z) \in (gV)^{-1}gV = V^{-1}V \subseteq U$, for all $y, z \in S$, which contradicts our assumption that B is almost (ρ, U)-discrete. □

We say that a set $B \subseteq X$ is *E-complete* if it intersects every E-class.

PROPOSITION 2.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, $\rho : E \to G$ is a Borel cocycle, and U is an open neighborhood of 1_G with compact closure. Then the following are equivalent:*

1. *ρ is σ-discrete;*
2. *There is an E-complete, (ρ, U)-discrete Borel set.*

Proof. To see $(1) \Rightarrow (2)$, note that if ρ is σ-discrete, then Proposition 2.5 implies that there is a cover of X by countably many almost $(\rho, \overline{U})$-discrete Borel sets, and Proposition 2.4 then ensures that there is a cover $\langle A_n \rangle_{n \in \mathbb{N}}$ of X by $(\rho, \overline{U})$-discrete Borel sets. Put $B_n = A_n \setminus \bigcup_{m<n} [A_m]_E$, and observe that the set $B = \bigcup_{n \in \mathbb{N}} B_n$ is as desired.

To see $(2) \Rightarrow (1)$, it is enough to show that if $\phi \in [\![E]\!]$ and $B \subseteq \mathrm{dom}(\phi)$ is a (ρ, U)-discrete Borel set, then $\phi(B)$ can be covered with countably many ρ-discrete Borel sets (by Proposition 2.1). Towards this end, fix a basis $\langle U_n \rangle_{n \in \mathbb{N}}$ for G, let $S = \{(m, n) \in \mathbb{N} \times \mathbb{N} : U_n^{-1} U_m U_n \subseteq U\}$, and set $B_n = \{\phi(x) : x \in B \text{ and } \rho(\phi(x), x) \in U_n\}$, for each $n \in \mathbb{N}$.

LEMMA 2.7. *For each $(m, n) \in S$, the set B_n is (ρ, U_m)-discrete.*

Proof. Simply observe that if $\phi(x), \phi(y) \in B_n$ and $\rho(\phi(x), \phi(y)) \in U_m$, then

$$\rho(x, y) = \rho(x, \phi(x))\rho(\phi(x), \phi(y))\rho(\phi(y), y) \in U_n^{-1} U_m U_n \subseteq U,$$

so $x = y$, thus $\phi(x) = \phi(y)$. □

Observe now that for each $x \in B$, the continuity of inversion and multiplication ensures that there are open neighborhoods U_m of 1_G and U_n of $\rho(\phi(x), x)$ such that $U_n^{-1} U_m U_n \subseteq U$, so $(m, n) \in S$ and $\phi(x) \in B_n$, thus $\phi(B) \subseteq \bigcup_{(m,n) \in S} B_n$. □

Remark. A *transversal* is an E-complete partial transversal. In the special case that $\rho \equiv 1_G$, Proposition 2.6 implies that E is smooth if and only if E has a Borel transversal.

We say that a set $B \subseteq X$ is *ρ-bounded* if $\overline{\rho(E|B)}$ is compact.

PROPOSITION 2.8. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, $\rho : E \to G$ is a Borel cocycle, and $B \subseteq X$ is ρ-bounded and ρ-discrete. Then B intersects every E-class in a finite set.*

Proof. Proposition 2.5 ensures that B is almost $(\rho, \overline{\rho(E|B)})$-discrete, which immediately implies that B intersects every E-class in a finite set. □

The standard example of a non-smooth equivalence relation is E_0 on $2^{\mathbb{N}}$, given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N} \forall m \geq n \; (\alpha(m) = \beta(m)).$$

PROPOSITION 2.9. *Suppose that $B \subseteq 2^{\mathbb{N}}$ has the property of Baire and intersects each E_0-class in a finite set. Then B is meager.*

Proof. Let $2^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} 2^n$, where 2^n denotes the set of sequences of zeros and ones of length n. For each $s \in 2^{<\mathbb{N}}$, let $\mathcal{N}_s = \{\alpha \in 2^{\mathbb{N}} : s \subseteq \alpha\}$. Suppose, towards a contradiction, that B is non-meager, and fix $n \in \mathbb{N}$ such that the set $B_n = \{\alpha \in B : |[\alpha]_{E_0|B}| < 2^n\}$ is non-meager. As B_n has the property of Baire, localization (see, for example, Proposition 8.26 of [**2**]) yields $s \in 2^{<\mathbb{N}}$ such that $B_n \cap \mathcal{N}_s$ is comeager in $\mathcal{N}_s$. Fix a transitive permutation τ of 2^n, and define $\pi : \mathcal{N}_s \to \mathcal{N}_s$ by $\pi(st\alpha) = s\tau(t)\alpha$. Then $\pi^i(B_n \cap \mathcal{N}_s)$ is comeager in $\mathcal{N}_s$, for all $i < 2^n$. Fix $\alpha \in \bigcap_{i<2^n} \pi^i(B_n \cap \mathcal{N}_s)$, and observe that $|[\alpha]_{E_0|B}| = |[\alpha]_{E_0|B_n}| \geq 2^n$, a contradiction. □

A metric space X is *Polish* if it is complete and separable. A metric d on X is an *ultrametric* if $d(x,z) \leq \max(d(x,y), d(y,z))$, for all $x, y, z \in X$. An example is the usual metric d on $2^{\mathbb{N}}$, given by $d(\alpha, \beta) = 1/(n+1)$, where $\alpha, \beta \in 2^{\mathbb{N}}$ are distinct and $n \in \mathbb{N}$ is least such that $\alpha(n) \neq \beta(n)$. Let $\mathcal{B}(x, \epsilon) = \{y \in X : d(x,y) < \epsilon\}$. We will need the following analog of the Lebesgue density theorem for Polish ultrametric spaces:

PROPOSITION 2.10. *Suppose that X is a Polish ultrametric space, μ is a probability measure on X, and $B \subseteq X$ is Borel. Then*

$$\lim_{\epsilon \to 0} \frac{\mu(B \cap \mathcal{B}(x, \epsilon))}{\mu(\mathcal{B}(x, \epsilon))} = 1,$$

for μ-almost every $x \in B$.

Proof. By subtracting a μ-null open set from B, we can assume that no point of B is contained in a μ-null open set. It is easily verified that for $0 < \delta < 1$, the set

$$B_\delta = \{x \in B : \liminf_{\epsilon \to 0} \mu(B \cap \mathcal{B}(x, \epsilon))/\mu(\mathcal{B}(x, \epsilon)) < 1 - \delta\}$$

is Borel. Suppose, towards a contradiction, that there exists $0 < \delta < 1$ such that $\mu(B_\delta) > 0$. Fix a compact set $K \subseteq B_\delta$ of positive μ-measure and an open set $U \supseteq K$ such that $\mu(K)/\mu(U) > 1 - \delta$ (see, for example, Theorem 17.11 of [**2**]). For each $x \in K$, fix $\epsilon_x > 0$ such that $\mathcal{B}(x, \epsilon_x) \subseteq U$ and $\mu(B \cap \mathcal{B}(x, \epsilon_x))/\mu(\mathcal{B}(x, \epsilon_x)) < 1 - \delta$. Since K is compact, there exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{1 \leq i \leq n} \mathcal{B}(x_i, \epsilon_{x_i})$. As X is an ultrametric space, by thinning out $x_1, \ldots, x_n$ we can ensure that the sets $\mathcal{B}(x_i, \epsilon_{x_i})$ partition their union $V = \bigcup_{1 \leq i \leq n} \mathcal{B}(x_i, \epsilon_{x_i})$. Then

$$\begin{aligned} \mu(B \cap V) &= \sum_{1 \leq i \leq n} \mu(B \cap \mathcal{B}(x_i, \epsilon_{x_i})) \\ &< (1 - \delta) \sum_{1 \leq i \leq n} \mu(\mathcal{B}(x_i, \epsilon_{x_i})) \\ &= (1 - \delta)\mu(V), \end{aligned}$$

thus $1 - \delta < \mu(K)/\mu(U) \leq \mu(B \cap V)/\mu(V) < 1 - \delta$, the desired contradiction. □

As a corollary, we obtain the following well-known fact:

PROPOSITION 2.11. *The usual $(1/2, 1/2)$ product measure μ on $2^{\mathbb{N}}$ is E_0-ergodic.*

Proof. Suppose that $B \subseteq 2^{\mathbb{N}}$ is an E_0-invariant Borel set of positive μ-measure. Given $\epsilon > 0$, Proposition 2.10 ensures that there exist $n \in \mathbb{N}$ and $s \in 2^n$ such that $\mu(B \cap \mathcal{N}_s)/\mu(\mathcal{N}_s) \geq 1 - \epsilon$. Then $\mu(B \cap \mathcal{N}_t)/\mu(\mathcal{N}_t) \geq 1 - \epsilon$, for all $t \in 2^n$, thus $\mu(B) \geq 1 - \epsilon$. As $\epsilon > 0$ was arbitrary, it follows that $\mu(B) = 1$. □

The facts we have mentioned thus far will be used in §3 to transform the usual $(1/2, 1/2)$ product measure on $2^{\mathbb{N}}$ into an atomless, $(E|B)$-ergodic, $(E|B)$-invariant σ-finite measure on a ρ-bounded Borel set $B \subseteq X$. We next discuss some facts which will be used in §4 to turn this into an atomless, E-ergodic, ρ-invariant σ-finite measure on X.

PROPOSITION 2.12. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $B \subseteq X$ is an E-complete Borel set. Then every $(E|B)$-invariant σ-finite measure μ on B has a unique extension to an E-invariant σ-finite measure on X.*

Proof. By Proposition 2.1, there is a group $\Gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ of Borel automorphisms of X such that $E = E_\Gamma^X$. For each $n \in \mathbb{N}$, define $B_n = \gamma_n(B) \setminus \bigcup_{m<n} \gamma_m(B)$, and let ν denote the σ-finite extension of μ given by

$$\nu(A) = \sum_{n\in\mathbb{N}} (\gamma_n)_* \mu(A \cap B_n).$$

If $\phi \in [\![E]\!]$ and $A \subseteq \operatorname{rng}(\phi)$, then $(\gamma_m^{-1} \circ \phi^{-1} \circ \gamma_n)|\gamma_n^{-1}(\phi(B_m) \cap B_n) \in [\![E|B]\!]$, so

$$\begin{aligned}
\nu(A) &= \sum_{n\in\mathbb{N}} \mu(\gamma_n^{-1}(A \cap B_n)) \\
&= \sum_{m,n\in\mathbb{N}} \mu(\gamma_n^{-1}(A \cap \phi(B_m) \cap B_n)) \\
&= \sum_{m,n\in\mathbb{N}} \mu(\gamma_m^{-1} \circ \phi^{-1}(A \cap \phi(B_m) \cap B_n)) \\
&= \sum_{m\in\mathbb{N}} \mu(\gamma_m^{-1}(\phi^{-1}(A) \cap B_m)) \\
&= \nu(\phi^{-1}(A)),
\end{aligned}$$

thus ν is E-invariant, and it is clear that ν is the only E-invariant extension of μ. □

We say that $\rho : E \to G$ is a *Borel coboundary* if there is a Borel function $w : X \to G$ such that $\rho(x, y) = w(x)w(y)^{-1}$, for all xEy.

PROPOSITION 2.13. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel coboundary, and μ is an E-invariant σ-finite measure on X. Then there is a ρ-invariant σ-finite measure $\nu \sim \mu$.*

Proof. Fix a Borel function $w : X \to (0, \infty)$ such that $\rho(x, y) = w(x)/w(y)$, for all xEy, define a σ-finite measure $\nu \sim \mu$ by setting

$$\nu(B) = \int_B w(x)\, d\mu(x),$$

and observe that if $\phi \in [\![E]\!]$ and $B \subseteq \operatorname{rng}(\phi)$, then

$$\begin{aligned}
\nu(\phi^{-1}(B)) &= \int_{\phi^{-1}(B)} w(x)\, d\mu(x) \\
&= \int_B w(\phi^{-1}(x))\, d\mu(x) \\
&= \int_B \rho(\phi^{-1}(x), x) w(x)\, d\mu(x) \\
&= \int_B \rho(\phi^{-1}(x), x)\, d\nu(x),
\end{aligned}$$

thus ν is ρ-invariant. □

PROPOSITION 2.14. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*
1. *There is an E-complete, ρ-bounded Borel set $B \subseteq X$;*
2. *ρ is a Borel coboundary.*

Proof. To see $(1) \Rightarrow (2)$, suppose that $B \subseteq X$ is an E-complete, ρ-bounded Borel set, and define $w : X \to (0, \infty)$ by

$$w(x) = \sup\{\rho(x, y) : y \in B \cap [x]_E\}.$$

Given xEy and $0 < \epsilon < \min(w(x), w(y))$, fix $z \in B \cap [x]_E$ such that $\rho(x, z) \geq w(x) - \epsilon$ and $\rho(y, z) \geq w(y) - \epsilon$, and observe that

$$\frac{w(x) - \epsilon}{w(y)} \leq \frac{\rho(x, z)}{\rho(y, z)} \leq \frac{w(x)}{w(y) - \epsilon},$$

so $\rho(x, y) = \rho(x, z)/\rho(y, z) = w(x)/w(y)$, thus ρ is a Borel coboundary.

To see $(2) \Rightarrow (1)$, suppose that $w : X \to (0, \infty)$ is a Borel function such that $\rho(x, y) = w(x)/w(y)$, for all xEy, fix an enumeration $\langle k_n \rangle_{n \in \mathbb{N}}$ of $\mathbb{Z}$, define

$$B_n = w^{-1}([2^{k_n}, 2^{k_n+1})) \setminus \bigcup_{m<n} [w^{-1}([2^{k_m}, 2^{k_m+1}))]_E,$$

and observe that $B = \bigcup_{n \in \mathbb{N}} B_n$ is an E-complete, ρ-bounded Borel set. □

We close this section with circumstances under which certain sets are necessarily null:

PROPOSITION 2.15. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, μ is an atomless, E-ergodic measure on X, and $B \subseteq X$ is a Borel partial transversal of E. Then B is μ-null.*

Proof. Simply observe that if $\mu(B) > 0$, then there is a Borel set $A \subseteq B$ such that $0 < \mu(A) < \mu(B)$, and it follows that $[A]_E$ and $[B \setminus A]_E$ are disjoint Borel sets of positive μ-measure, which contradicts the E-ergodicity of μ. □

PROPOSITION 2.16. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho : E \to (0, \infty)$ is a Borel cocycle, μ is a ρ-invariant σ-finite measure on X which concentrates off of Borel partial transversals of E, and $B \subseteq X$ is a ρ-discrete Borel set. Then B is μ-null.*

Proof. Suppose, towards a contradiction, that $\mu(B) > 0$. By thinning out B, we can assume that $\mu(B) < \infty$. Define $\phi \in [\![E|B]\!]$ by

$$\phi(x) = y \Leftrightarrow \rho(x, y) < 1 \text{ and } \forall z \in [x]_{E|B} \ (\rho(x, z) < 1 \Rightarrow \rho(y, z) \leq 1).$$

By throwing out a Borel set on which E is smooth, we can assume that $\phi \in [E|B]$, so

$$\mu(B) = \mu(\phi(B)) = \int_B \rho(\phi(x), x) \, d\mu(x) > \mu(B),$$

the desired contradiction. □

3. *A descriptive set-theoretic characterization of σ-discrete cocycles*
An *embedding* of E_0 into E is an injection $\pi : 2^{\mathbb{N}} \to X$ such that $\alpha E_0 \beta \Leftrightarrow \pi(\alpha) E \pi(\beta)$, for all $\alpha, \beta \in 2^{\mathbb{N}}$. We say that π is (ρ, U)*-bounded* if $\rho(E|\pi(2^{\mathbb{N}})) \subseteq U$.

THEOREM 3.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a Polish group, U is an open neighborhood of 1_G with compact closure, and $\rho : E \to G$ is a Borel cocycle. Then exactly one of the following holds:*
1. ρ is σ-discrete;
2. There is a (ρ, U)-bounded continuous embedding of E_0 into E.

Proof. To see that conditions (1) and (2) are mutually exclusive, simply observe that if $\langle B_n \rangle_{n \in \mathbb{N}}$ is a sequence of ρ-discrete Borel sets which cover X and $\pi : 2^{\mathbb{N}} \to X$ is a (ρ, U)-bounded Borel embedding of E_0 into E, then $\pi(2^{\mathbb{N}})$ is a ρ-bounded Borel set (see, for example, Theorem 15.1 of [**2**]), so Proposition 2.8 implies that for each $n \in \mathbb{N}$, the set $B_n \cap \pi(2^{\mathbb{N}})$ intersects each E-class in a finite set. Then $A_n = \pi^{-1}(B_n \cap \pi(2^{\mathbb{N}}))$ is a Borel set which intersects each E_0-class in a finite set, and since $\langle A_n \rangle_{n \in \mathbb{N}}$ covers $2^{\mathbb{N}}$, this contradicts Proposition 2.9.

It remains to show $\neg(1) \Rightarrow (2)$. Suppose that (1) fails, or equivalently, that X is not in the σ-ideal $\mathcal{I}$ generated by the ρ-discrete Borel subsets of X. By Proposition 2.1, there is a countable group Γ of Borel automorphisms of X such that $E = E_\Gamma^X$. Fix an increasing sequence of finite, symmetric sets $\Gamma_n \subseteq \Gamma$ containing 1_Γ such that $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$.

LEMMA 3.2. *There is a sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of open neighborhoods of 1_G such that $(U_0 \cdots U_n)(U_0 \cdots U_n)^{-1} \subseteq U$, for all $n \in \mathbb{N}$.*

Proof. Set $U_{-1} = U$ and recursively appeal to the continuity of inversion and multiplication to obtain a sequence of open, symmetric neighborhoods U_n of 1_G such that $(U_n)^3 \subseteq U_{n-1}$. A straightforward induction shows that $\langle U_n \rangle_{n \in \mathbb{N}}$ is as desired. □

By standard change of topology results (see, for example, §13 of [**2**]), there is a zero-dimensional Polish topology on X, finer than the given one but generating the same Borel sets, with respect to which Γ acts by homeomorphisms and each map of the form $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$ is continuous. If $\pi : 2^{\mathbb{N}} \to X$ is continuous with respect to this new topology, then it is continuous with respect to the original topology, so from this point forward we work only with the new topology and a fixed compatible, complete metric.

We will recursively find clopen sets $X_n \subseteq X$ and group elements $\gamma_n \in \Gamma$. From these, we define group elements γ_s, for $s \in 2^{<\mathbb{N}}$, by setting $\gamma_\emptyset = 1_\Gamma$ and $\gamma_s = \gamma_0^{s(0)} \cdots \gamma_n^{s(n)}$, for $s \in 2^{n+1}$. We will ensure that for all $n \in \mathbb{N}$, the following conditions are satisfied:
(a) $X_n \notin \mathcal{I}$;
(b) $\rho_{\gamma_n}(X_{n+1}) \subseteq U_n$;
(c) $X_{n+1} \cup \gamma_n(X_{n+1}) \subseteq X_n$;
(d) $\forall s, t \in 2^n \forall \gamma \in \Gamma_n \ (\gamma_n(X_{n+1}) \cap \gamma_t^{-1} \gamma \gamma_s(X_{n+1}) = \emptyset)$;
(e) $\forall u \in 2^{n+1} \ (\operatorname{diam}(\gamma_u(X_{n+1})) \leq 1/(n+1))$.
We begin by setting $X_0 = X$. Now suppose that we have found $\langle X_i \rangle_{i \leq n}$ and $\langle \gamma_i \rangle_{i < n}$ which satisfy conditions (a) – (e). For each $\delta \in \Gamma$, define $V_\delta \subseteq X$ by

$$V_\delta = \{x \in X_n \cap \delta^{-1}(X_n) \cap \rho_\delta^{-1}(U_n) : \forall s, t \in 2^n \forall \gamma \in \Gamma_n \ (\delta \cdot x \neq \gamma_t^{-1} \gamma \gamma_s \cdot x)\}.$$

LEMMA 3.3. *There exists $\delta \in \Gamma$ such that $V_\delta \notin \mathcal{I}$.*

Proof. Set $C = X_n \setminus \bigcup_{\delta \in \Gamma} V_\delta$, and observe that if $x, y \in C$ and $\rho(x, y) \in U_n$, then there exists $\delta \in \Gamma$ such that $x = \delta \cdot y$, and the fact that $y \notin V_\delta$ ensures that there exist $s, t \in 2^n$ and $\gamma \in \Gamma_n$ such that $\delta \cdot y = \gamma_t^{-1} \gamma \gamma_s \cdot y$, so $y = \gamma_s^{-1} \gamma^{-1} \gamma_t \cdot x$, thus C is almost (ρ, U_n)-discrete. Proposition 2.4 then implies that $C \in \mathcal{I}$, thus the set $X_n \setminus C = \bigcup_{\delta \in \Gamma} V_\delta$ is not in $\mathcal{I}$, and the lemma follows. □

By Lemma 3.3, there exists $\gamma_n \in \Gamma$ such that $V_{\gamma_n} \notin \mathcal{I}$. As V_{γ_n} is open, it is the union of countably many clopen sets $W_k \subseteq V_{\gamma_n}$ such that $\gamma_n(W_k) \cap \gamma_t^{-1} \gamma \gamma_s(W_k) = \emptyset$ and $\operatorname{diam}(\gamma_u(W_k)) \leq 1/(n+1)$, for all $s, t \in 2^n$, $\gamma \in \Gamma_n$, and $u \in 2^{n+1}$. Fix k such that $W_k \notin \mathcal{I}$, and put $X_{n+1} = W_k$. This completes the recursive construction.

For each $\alpha \in 2^{\mathbb{N}}$, condition (c) implies that the sequence $\langle \gamma_{\alpha|n}(X_n) \rangle_{n \in \mathbb{N}}$ is decreasing, and condition (e) ensures that the diameter of the sets along this sequence is vanishing. As a consequence, we obtain a function $\pi : 2^{\mathbb{N}} \to X$ by setting

$$\pi(\alpha) = \text{the unique element of } \bigcap_{n \in \mathbb{N}} \gamma_{\alpha|n}(X_n).$$

Condition (d) implies that π is injective, and condition (e) ensures that π is continuous.

To see that $\alpha E_0 \beta \Rightarrow \pi(\alpha) E \pi(\beta)$, it is enough to check the following:

LEMMA 3.4. *Suppose that $k \in \mathbb{N}$, $s \in 2^k$, and $\alpha \in 2^{\mathbb{N}}$. Then $\pi(s\alpha) = \gamma_s \cdot \pi(0^k \alpha)$.*

Proof. Simply observe that

$$\begin{aligned} \{\pi(s\alpha)\} &= \bigcap_{n \in \mathbb{N}} \gamma_{(s\alpha)|n}(X_n) \\ &= \bigcap_{n \in \mathbb{N}} \gamma_s \gamma_{0^k(\alpha|n)}(X_{k+n}) \\ &= \gamma_s \Big(\bigcap_{n \in \mathbb{N}} \gamma_{0^k(\alpha|n)}(X_{k+n}) \Big) \\ &= \gamma_s \Big(\bigcap_{n \in \mathbb{N}} \gamma_{(0^k\alpha)|n}(X_n) \Big) \\ &= \gamma_s(\{\pi(0^k\alpha)\}), \end{aligned}$$

thus $\pi(s\alpha) = \gamma_s \cdot \pi(0^k \alpha)$. □

To see that $(\alpha, \beta) \notin E_0 \Rightarrow (\pi(\alpha), \pi(\beta)) \notin E$, it is enough to check the following:

LEMMA 3.5. *Suppose that $\alpha(n) \neq \beta(n)$. Then $\forall \gamma \in \Gamma_n \ (\gamma \cdot \pi(\alpha) \neq \pi(\beta))$.*

Proof. By reversing the roles of α and β if necessary, we can assume that $\alpha(n) = 0$. Suppose, towards a contradiction, that there exists $\gamma \in \Gamma_n$ such that $\gamma \cdot \pi(\alpha) = \pi(\beta)$. Set $s = \alpha|n$ and $t = \beta|n$, and put $x = \gamma_s^{-1} \cdot \pi(\alpha)$ and $y = \gamma_n^{-1} \gamma_t^{-1} \cdot \pi(\beta)$. Then $x, y \in X_{n+1}$ and $\gamma_n \cdot y = \gamma_t^{-1} \gamma \gamma_s \cdot x$, which contradicts condition (d). □

It only remains to check that π is (ρ, U)-bounded. Towards this end, suppose that $\alpha E_0 \beta$ and fix $n \in \mathbb{N}$ such that $\forall m > n \ (\alpha(m) = \beta(m))$. Set $s = \alpha(0) \ldots \alpha(n)$ and

$t = \beta(0) \dots \beta(n)$, noting that $\gamma_s^{-1} \cdot \pi(\alpha) = \gamma_t^{-1} \cdot \pi(\beta)$, by Lemma 3.4. Then

$$\rho(\pi(\alpha), \gamma_s^{-1} \cdot \pi(\alpha)) = \prod_{i<n} \rho(\gamma_{s|i}^{-1} \cdot \pi(\alpha), \gamma_{s|(i+1)}^{-1} \cdot \pi(\alpha)) \in \prod_{i<n} U_i$$

and

$$\rho(\pi(\beta), \gamma_t^{-1} \cdot \pi(\beta)) = \prod_{i<n} \rho(\gamma_{t|i}^{-1} \cdot \pi(\beta), \gamma_{t|(i+1)}^{-1} \cdot \pi(\beta)) \in \prod_{i<n} U_i,$$

by condition (b), thus $\rho(\pi(\alpha), \pi(\beta)) = \rho(\pi(\alpha), \gamma_s^{-1} \cdot \pi(\alpha))\rho(\gamma_t^{-1} \cdot \pi(\beta), \pi(\beta)) \in U$. □

THEOREM 3.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a locally compact Polish group, and $\rho : E \to G$ is a Borel cocycle. Then the following are equivalent:*

1. *X is the union of countably many ρ-discrete Borel sets;*
2. *X is the union of countably many ρ-discrete globally Baire sets.*

Proof. It is clear that $(1) \Rightarrow (2)$. To see $(2) \Rightarrow (1)$ suppose, towards a contradiction, that there is a sequence $\langle B_n \rangle_{n \in \mathbb{N}}$ of ρ-discrete globally Baire sets which cover X, but ρ is not σ-discrete. Fix an open neighborhood U of 1_G with compact closure. By Theorem 3.1, there is a (ρ, U)-bounded continuous embedding $\pi : 2^{\mathbb{N}} \to X$ of E_0 into E. Then $\pi(2^{\mathbb{N}})$ is a ρ-bounded Borel set, so Proposition 2.8 implies that for each $n \in \mathbb{N}$, the set $B_n \cap \pi(2^{\mathbb{N}})$ intersects each E-class in a finite set. As $B_n \cap \pi(2^{\mathbb{N}})$ is globally Baire, it follows that the set $A_n = \pi^{-1}(B_n \cap \pi(2^{\mathbb{N}}))$ has the property of Baire and intersects each E_0-class in a finite set. Since $\langle A_n \rangle_{n \in \mathbb{N}}$ covers $2^{\mathbb{N}}$, this contradicts Proposition 2.9. □

Remark. A similar argument gives the universally measurable analog of Theorem 3.6.

THEOREM 3.7. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:*

1. *E is hyperfinite;*
2. *There is a σ-discrete Borel cocycle $\rho : E \to (0, \infty)$.*

Proof. It is easily verified that if E is smooth, then E is hyperfinite and every Borel cocycle from E to a Polish group is σ-discrete.

To see $(1) \Rightarrow (2)$, suppose that E is hyperfinite. By throwing out an E-invariant Borel set on which E is smooth, we can assume that every E-class is infinite. By a result of [**8**] and [**10**] (see also Theorem 6.6 of [**3**]), there exists $T \in [E]$ such that $E = E_T^X$. Let $\rho : E \to (0, \infty)$ be the Borel cocycle given by $\rho(T^n(x), x) = 2^n$, and observe that X is $(\rho, (1/2, 2))$-discrete, thus ρ is σ-discrete.

To see $(2) \Rightarrow (1)$, suppose that $\rho : E \to (0, \infty)$ is a σ-discrete Borel cocycle, and fix a cover $\langle B_n \rangle_{n \in \mathbb{N}}$ of X by ρ-discrete Borel sets. It is enough to show that $E|B_n$ is hyperfinite, for each $n \in \mathbb{N}$. Towards this end, define $\phi_n \in [\![E|B_n]\!]$ by

$$\phi_n(x) = y \Leftrightarrow \rho(x, y) < 1 \text{ and } \forall z \in [x]_{E|B_n} \ (\rho(x, z) < 1 \Rightarrow \rho(y, z) \leq 1).$$

By throwing out an $(E|B_n)$-invariant Borel set on which $E|B_n$ is smooth, we can assume that ϕ_n is a Borel automorphism of B_n such that $E_\phi^{B_n} = E|B_n$, thus the previously mentioned result of [**8**] and [**10**] implies that $E|B_n$ is hyperfinite. □

4. *Characterizations of the existence of non-trivial σ-finite measures*

We begin this section with the proof of our main theorem:

THEOREM 4.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *ρ is σ-discrete;*
2. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X.*

Proof. Propositions 2.15 and 2.16 immediately imply that (1) and (2) are mutually exclusive. To see $\neg(1) \Rightarrow (2)$, suppose that ρ is not σ-discrete, and appeal to Theorem 3.1 to obtain a $(\rho, (1/2, 2))$-bounded embedding π of E_0 into E. Set $B = \pi(2^{\mathbb{N}})$, and note that we can push the usual $(1/2, 1/2)$ product measure on $2^{\mathbb{N}}$ through π to obtain an $(E|B)$-invariant probability measure μ on B. Proposition 2.12 implies that μ extends to an E-invariant σ-finite measure ν which concentrates on $[B]_E$, Proposition 2.14 implies that $\rho|(E|[B]_E)$ is a Borel coboundary, and Proposition 2.13 then ensures that there is a ρ-invariant σ-finite measure $\xi \sim \nu$. As μ is atomless and $(E|B)$-ergodic (by Proposition 2.11), it follows that ξ is atomless and E-ergodic. □

Next we establish various equivalents of the existence of non-trivial measures:

THEOREM 4.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*

1. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X;*
2. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X which is equivalent to an atomless, E-ergodic, E-invariant σ-finite measure on X.*

Proof. It is enough to establish $(1) \Rightarrow (2)$. By Theorem 4.1, it is sufficient to show that if ρ is not σ-discrete, then (2) holds, and this follows from the proof of Theorem 4.1. □

THEOREM 4.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*

1. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X;*
2. *There is a ρ-invariant σ-finite measure on X which concentrates off of Borel partial transversals of E;*
3. *There is a ρ-invariant σ-finite measure on X which concentrates off of ρ-discrete Borel sets.*

Proof. Proposition 2.15 gives $(1) \Rightarrow (2)$, and Proposition 2.16 gives $(2) \Rightarrow (3)$. To see $(3) \Rightarrow (1)$, observe that if there is a measure on X which concentrates off of ρ-discrete Borel sets, then ρ is not σ-discrete, thus Theorem 4.1 ensures that (1) holds. □

THEOREM 4.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*

1. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X;*
2. *There is a family of continuum-many atomless, E-ergodic, ρ-invariant σ-finite measures on X with pairwise disjoint supports.*

Proof. It is enough to show $(1) \Rightarrow (2)$. Towards this end, suppose that (1) holds. Theorem 4.1 implies that ρ is not σ-discrete, and Theorem 3.1 ensures that there is a $(\rho, (1/2, 2))$-bounded continuous embedding π of E_0 into E.

LEMMA 4.5. *There is a sequence $\langle \pi_\alpha \rangle_{\alpha \in 2^\mathbb{N}}$ of embeddings of E_0 into E_0 such that*

$$\forall \alpha, \beta \in 2^\mathbb{N} \ (\alpha \neq \beta \Rightarrow [\pi_\alpha(2^\mathbb{N})]_{E_0} \cap [\pi_\beta(2^\mathbb{N})]_{E_0} = \emptyset).$$

Proof. The functions $\pi_\alpha(\gamma) = (\alpha|0)\gamma(0)(\alpha|1)\gamma(1)\ldots$ are clearly as desired. □

For each $\alpha \in 2^\mathbb{N}$, the proof of Theorem 4.1 yields an atomless, E-ergodic, ρ-invariant σ-finite measure μ_α which concentrates on $[\pi \circ \pi_\alpha(2^\mathbb{N})]_E$. □

THEOREM 4.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\rho : E \to (0, \infty)$ is a Borel cocycle. Then the following are equivalent:*
1. *There is an atomless, E-ergodic, ρ-invariant σ-finite measure on X;*
2. *There is a finer Polish topology τ on X such that for every τ-comeager set $C \subseteq X$, there is an atomless, E-ergodic, ρ-invariant σ-finite measure concentrating on C.*

Proof. It is enough to show $(1) \Rightarrow (2)$. Towards this end, suppose that (1) holds. Theorem 4.1 then implies that X is not ρ-discrete, and Theorem 3.1 ensures that there is a $(\rho, (1/2, 2))$-bounded continuous embedding π of E_0 into E. Let τ_1 denote the push-forward of the usual topology on $2^\mathbb{N}$ through π. By standard change of topology results, there is a Polish topology τ_2 on $X \setminus \pi(2^\mathbb{N})$ such that the topology τ generated by τ_1 and τ_2 is Polish and finer than the given Polish topology on X (and therefore generates the same Borel sets). It remains to check that if $C \subseteq X$ is τ-comeager, then there is an atomless, E-ergodic, ρ-invariant σ-finite measure concentrating on C. Clearly we can assume that C is Borel, and by Theorem 4.1, it is enough to show that $\rho|(E|C)$ is not σ-discrete. Suppose, towards a contradiction, that $\langle B_n \rangle_{n \in \mathbb{N}}$ is a sequence of ρ-discrete Borel sets which cover C. As $\pi(2^\mathbb{N})$ is ρ-bounded, Proposition 2.8 implies that for each $n \in \mathbb{N}$, the set $B_n \cap \pi(2^\mathbb{N})$ intersects each E-class in a finite set. Then $A_n = \pi^{-1}(B_n \cap \pi(2^\mathbb{N}))$ is a Borel set which intersects each E_0-class in a finite set, and since $\pi^{-1}(C)$ is comeager and $\langle A_n \rangle_{n \in \mathbb{N}}$ covers $\pi^{-1}(C)$, this contradicts Proposition 2.9. □

We close with our promised characterization of hyperfiniteness:

THEOREM 4.7. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then exactly one of the following holds:*
1. *E is hyperfinite;*
2. *For every Borel cocycle $\rho : E \to (0, \infty)$, there is an atomless, E-ergodic, ρ-invariant σ-finite measure on X.*

Proof. This is a straightforward consequence of Theorems 3.7 and 4.1. □

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