DICHOTOMY THEOREMS FOR COUNTABLY INFINITE DIMENSIONAL ANALYTIC HYPERGRAPHS

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Abstract. We give classical proofs, strengthenings, and generalizations of Lecomte's characterizations of analytic ω-dimensional hypergraphs with countable Borel chromatic number.

1. Introduction

An *ω-dimensional* (*directed*) *hypergraph* on a set X is a family $G \subseteq {}^\omega X$ of non-constant sequences. A (Y-)*coloring* of such a hypergraph is a function $c\colon X \to Y$ which sends sequences in G to non-constant sequences in ${}^\omega Y$. More generally, a *homomorphism* from an ω-dimensional hypergraph G on X to an ω-dimensional hypergraph H on Y is a function $\varphi\colon X \to Y$ which sends sequences in G to sequences in H.

In [4], Kechris-Solecki-Todorcevic isolated an acyclic $D_2(\mathbf{\Sigma}^0_1)$ graph on ${}^\omega 2$ that is minimal among all analytic graphs which do not have Borel ω-colorings. In [5], Lecomte proved that an analogous ω-dimensional hypergraph is minimal among all analytic ω-dimensional hypergraphs which do not have Borel ω-colorings.

Here we give a classical proof of a strengthening of Lecomte's result. This allows us to provide new insight into the curious fact that the notion of minimality appearing in the ω-dimensional case is weaker than that appearing in the Kechris-Solecki-Todorcevic theorem. We also give generalizations of Lecomte's result to κ-Souslin graphs.

We work in `ZF` except where stated otherwise.

2. Preliminaries

A topological space is *analytic* if it is the continuous image of a closed subset of ${}^\omega\omega$. Given a set $R \subseteq \prod_{i\in I} X_i$, we say that a sequence $(A_i)_{i\in I}$ is *R-discrete* if $A_i \subseteq X_i$ for all $i \in I$ and $\prod_{i\in I} A_i$ is disjoint from R.

Proposition 1. *Suppose that $(X_i)_{i\in I}$ is a countable sequence of Hausdorff spaces, $R \subseteq \prod_{i\in I} X_i$ is analytic, and $(A_i)_{i\in I}$ is an R-discrete sequence of analytic sets. Then there exist a Borel set $S \subseteq \prod_{i\in I} X_i$ and an S-discrete sequence $(B_i)_{i\in I}$ of Borel sets such that $R \subseteq S$ and $A_i \subseteq B_i$ for all $i \in I$.*

Proof. This is a straightforward generalization of the Novikov separation theorem (see, for example, Theorem 28.5 of [3]). ⊠

The *restriction* of G to a set $A \subseteq X$ is given by $G \restriction A = G \cap {}^{\omega}A$. We say that a set $A \subseteq X$ is *G-discrete* if $G \restriction A = \emptyset$.

Proposition 2. *Suppose that X is a Hausdorff space, G is an analytic ω-dimensional hypergraph on X, and $A \subseteq X$ is a G-discrete analytic set. Then there is a G-discrete Borel set $B \subseteq X$ such that $A \subseteq B$.*

Proof. By Proposition 1, there is a G-discrete sequence $(B_n)_{n\in\omega}$ of Borel sets such that $A \subseteq B_n$ for all $n \in \omega$. Clearly the set $B = \bigcap_{n\in\omega} B_n$ is as desired. ⊠

For each set $I \subseteq {}^{<\omega}\omega$, let G_I denote the ω-dimensional hypergraph on ${}^{\omega}\omega$ given by $G_I = \{(s^\frown i^\frown x)_{i\in\omega} \mid s \in I \text{ and } x \in {}^{\omega}\omega\}$. We say that a set $I \subseteq {}^{<\omega}\omega$ is *dense* if $\forall s \in {}^{<\omega}\omega \exists t \in I\ (s \sqsubseteq t)$.

Proposition 3. *Suppose that $A \subseteq {}^{\omega}\omega$ is a non-meager set with the Baire property and $I \subseteq {}^{<\omega}\omega$ is dense. Then A is not G_I-discrete.*

Proof. Fix $s \in {}^{<\omega}\omega$ such that A is comeager in $\mathcal{N}_s$, fix $t \in I$ such that $s \sqsubseteq t$, and fix $x \in {}^{\omega}\omega$ such that $t^\frown i^\frown x \in A$ for all $i \in \omega$. As $(t^\frown i^\frown x)_{i\in\omega} \in G_I$, it follows that A is not G_I-discrete. ⊠

Fix sequences $s_n \in {}^{n}\omega$ for which the set $I = \{s_n \mid n \in \omega\}$ is dense, and define $G_0(\omega) = G_I$.

3. Dichotomy theorems

The primary dichotomy in [5] concerns the existence of continuous homomorphisms from $G_0(\omega) \restriction X_0$ to G, where X_0 denotes the dense G_δ set of sequences $x \in {}^{\omega}\omega$ such that $s_n{}^\frown 0 \sqsubseteq x$ for infinitely many $n \in \omega$. We will establish the analogous result concerning the existence of continuous homomorphisms from $G_0(\omega) \restriction X_z$ to G, where $z \in {}^{\omega}\omega$ is strictly increasing and X_z denotes the dense G_δ set of sequences $x \in {}^{\omega}\omega$ such that $x \restriction n \in {}^{n}z(n)$ for infinitely many $n \in \omega$.

Note that if $z(n+1) > \max_{i\in n} s_n(i)$ for all $n \in \omega$, then $X_0 \subseteq X_z$, so the inclusion map is a continuous homomorphism from $G_0(\omega) \restriction X_0$ to $G_0(\omega) \restriction X_z$. The following fact therefore yields the original result:

Theorem 4. *Suppose that X is a Hausdorff space and G is an analytic ω-dimensional hypergraph on X. Then for all strictly increasing sequences $z \in {}^{\omega}\omega$, exactly one of the following holds:*

1. *There is a Borel ω-coloring of G.*
2. *There is a continuous homomorphism from $G_0(\omega) \restriction X_z$ to G.*

Proof. To see that (1) and (2) are mutually exclusive suppose, towards a contradiction, that $c\colon X \to \omega$ is a Borel coloring of G, $C \subseteq {}^\omega\omega$ is a dense G_δ set, and $\varphi\colon C \to X$ is a Baire measurable homomorphism from $G_0(\omega) \restriction C$ to G. Then the function $c_0 = c \circ \varphi$ is Baire measurable, so there exists $n \in \omega$ such that $c_0^{-1}(\{n\})$ is a non-meager set with the Baire property. As c_0 is a coloring of $G_0(\omega) \restriction C$, it follows that $c_0^{-1}(\{n\})$ is also $G_0(\omega)$-discrete, which contradicts Proposition 3.

It remains to show that at least one of (1) and (2) holds. We can clearly assume that G is non-empty, in which case there are continuous surjections $\varphi_G\colon {}^\omega\omega \to G$ and $\varphi_X\colon {}^\omega\omega \to \mathrm{dom}(G)$, where

$$\mathrm{dom}(G) = \{x \in X \mid \exists n \in \omega \exists y \in G\ (x = y(n))\}.$$

Suppose that $n \in \omega$. A *global* (n-)*approximation* is a pair of the form $p = ((u_m^p)_{m \in n+1}, (v_m^p)_{m \in n+1})$, where $u_m^p\colon {}^m z(m) \to {}^m\omega$ and $v_m^p\colon {}^{<m} z(m) \to {}^m\omega$ for all $m \in n+1$, which satisfies the following conditions:

(a) $\forall l \in m \in n+1 \forall s \in {}^l z(l) \forall t \in {}^m z(m)\ (s \sqsubseteq t \Longrightarrow u_l^p(s) \sqsubseteq u_m^p(t))$.
(b) $\forall l \in m \in n+1 \forall s \in {}^{<l} z(l) \forall t \in {}^{<m} z(m)$
$((s \sqsubseteq t \text{ and } m - l = |t| - |s|) \Longrightarrow v_l^p(s) \sqsubseteq v_m^p(t))$.

Fix an enumeration $(p_k)_{k \in \omega}$ of the set of all global approximations.

An *extension* of a global m-approximation p is a global n-approximation q such that $u_l^p = u_l^q$ and $v_l^p = v_l^q$ for all $l \in m+1$. In the special case that $n = m+1$, we say that q is a *one-step extension* of p.

A *local* (n-)*approximation* is a pair of the form $l = (f^l, g^l)$, where $f^l\colon {}^n\omega \to {}^\omega\omega$ and $g^l\colon {}^{<n}\omega \to {}^\omega\omega$, with the property that

$$\forall k \in n \forall t \in {}^{n-(k+1)}\omega\ (\varphi_G \circ g^l(t) = (\varphi_X \circ f^l(s_k \frown i \frown t))_{i \in \omega}).$$

We say that l is *compatible* with a global n-approximation p if the following conditions are satisfied:

(i) $\forall m \in n+1 \forall s \in {}^m z(m) \forall t \in {}^n\omega\ (s \sqsubseteq t \Longrightarrow u_m^p(s) \sqsubseteq f^l(t))$.
(ii) $\forall m \in n+1 \forall s \in {}^{<m} z(m) \forall t \in {}^{<n}\omega$
$((s \sqsubseteq t \text{ and } n - m = |t| - |s|) \Longrightarrow v_m^p(s) \sqsubseteq g^l(t))$.

We say that l is *compatible* with a set $Y \subseteq X$ if $\varphi_X \circ f^l[{}^n\omega] \subseteq Y$.

Suppose now that $Y \subseteq X$ is a Borel set, α is a countable ordinal, and $c\colon X \setminus Y \to \omega \cdot \alpha$ is a Borel coloring of $G \restriction (X \setminus Y)$. Associated with each global n-approximation p is the set $L(p, Y)$ of local n-approximations which are compatible with both p and Y, as well as the set

$$A(p, Y) = \{\varphi_X \circ f^l(s_n) \mid l \in L(p, Y)\}.$$

We say that p is Y-*terminal* if $L(q, Y) = \emptyset$ for all one-step extensions q of p. Let $T(Y)$ denote the set of Y-terminal global approximations.

Lemma 5. *Suppose that p is a global approximation and $A(p,Y)$ is not G-discrete. Then p is not Y-terminal.*

Proof of lemma. Fix $n \in \omega$ such that p is a global n-approximation, as well as local n-approximations $l_i \in L_n(p,Y)$ with $(\varphi_X \circ f^{l_i}(s_n))_{i\in\omega} \in G$. Then there exists $x \in {}^\omega\omega$ for which $\varphi_G(x) = (\varphi_X \circ f^{l_i}(s_n))_{i\in\omega}$. Let l denote the local $(n+1)$-approximation given by $f^l(s^\frown i) = f^{l_i}(s)$ for $i \in \omega$ and $s \in {}^n\omega$, $g^l(\emptyset) = x$, and $g^l(t^\frown i) = g^{l_i}(t)$ for $i \in \omega$ and $t \in {}^{<n}\omega$. As l is compatible with a one-step extension of p, it follows that p is not Y-terminal. ⊠

Proposition 2 and Lemma 5 ensure that for each Y-terminal global approximation p, there is a G-discrete Borel set $B(p,Y) \subseteq X$ such that $A(p,Y) \subseteq B(p,Y)$. Define $Y' \subseteq Y$ by

$$Y' = Y \setminus \bigcup \{B(p,Y) \mid p \in T(Y)\}.$$

For each $y \in Y \setminus Y'$, set

$$k(y) = \min\{k \in \omega \mid p_k \in T(Y) \text{ and } y \in B(p_k,Y)\},$$

and define an extension $c' \colon X \setminus Y' \to \omega \cdot (\alpha+1)$ of $c \colon X \setminus Y \to \omega \cdot \alpha$ by setting $c'(y) = \omega \cdot \alpha + k(y)$ for $y \in Y \setminus Y'$.

Lemma 6. *The function c' is a coloring of $G \upharpoonright (X \setminus Y')$.*

Proof of lemma. Suppose, towards a contradiction, that there exist $\beta \in \omega \cdot (\alpha+1)$ and $(x_i)_{i\in\omega} \in G \upharpoonright (X \setminus Y')$ such that $c'(x_i) = \beta$ for all $i \in \omega$. Then there exists $k \in \omega$ with $\beta = \omega \cdot \alpha + k$, in which case p_k is Y-terminal and $(x_i)_{i\in\omega} \in G \upharpoonright B(p_k,Y)$, the desired contradiction. ⊠

Lemma 7. *Suppose that p is a global approximation whose one-step extensions are all Y-terminal. Then p is Y'-terminal.*

Proof of lemma. For each one-step extension q of p, the sets $A(q,Y)$ and Y' are disjoint, so $L(q,Y') = \emptyset$, thus p is Y'-terminal. ⊠

Recursively define Borel sets $Y_\alpha \subseteq X$ and Borel colorings $c_\alpha \colon X \setminus Y_\alpha \to \omega \cdot \alpha$ of $G \upharpoonright (X \setminus Y_\alpha)$ for all countable ordinals α by setting

$$(Y_\alpha, c_\alpha) = \begin{cases} (X, \emptyset) & \text{if } \alpha = 0, \\ (Y'_\beta, c'_\beta) & \text{if } \alpha = \beta+1, \text{ and} \\ (\bigcap_{\beta\in\alpha} Y_\beta, \lim_{\beta\to\alpha} c_\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

As there are only countably many global approximations and the sequence $(T(Y_\alpha))_{\alpha\in\omega_1}$ is increasing, there is a countable ordinal α with the property that $T(Y_\alpha) = T(Y_{\alpha+1})$.

If the unique global 0-approximation p^0 is Y_α-terminal, then the fact that $A(p^0, Y_\alpha) = \mathrm{dom}(G) \cap Y_\alpha$ ensures that c_α extends to a Borel $(\omega \cdot \alpha + 1)$-coloring of G, thus there is a Borel ω-coloring of G.

Otherwise, by repeatedly applying Lemma 7 we obtain one-step extensions p^{n+1} of p^n for all $n \in \omega$, none of which are Y_α-terminal. For each $k \in \omega$, let $X_{z,k}$ denote the dense G_δ set of sequences $x \in {}^\omega\omega$ with $x \restriction n \in {}^n z(k + n + 1)$ for infinitely many $n \in \omega$. Define continuous functions $\psi_X : X_z \to {}^\omega\omega$ and $\psi_k : X_{z,k} \to {}^\omega\omega$ for $k \in \omega$ by

$$\psi_X(x) = \lim_{n \to \omega} u_n^{p^n}(x \restriction n) \text{ and } \psi_k(x) = \lim_{n \to \omega} v_{k+n+1}^{p^{k+n+1}}(x \restriction n),$$

where the limits are taken over all $n \in \omega$ for which the maps are defined.

To see that $\varphi_X \circ \psi_X$ is a homomorphism from $G_0(\omega) \restriction X_z$ to G, it is enough to show that $\varphi_G \circ \psi_k(x) = (\varphi_X \circ \psi_X(s_k {}^\frown i {}^\frown x))_{i \in \omega}$ for all $k \in \omega$ and $x \in X_{z,k}$. By the continuity of φ_G and φ_X, it is enough to show that for every open neighborhood U of $\psi_k(x)$ and every open neighborhood V of $(\psi_X(s_k {}^\frown i {}^\frown x))_{i \in \omega}$, there exists $(y, (y_i)_{i \in \omega}) \in U \times V$ with $\varphi_G(y) = (\varphi_X(y_i))_{i \in \omega}$. Towards this end, fix $m \in \omega$ and an open set $W \subseteq {}^m({}^\omega\omega)$ such that $(\psi_X(s_k {}^\frown i {}^\frown x))_{i \in m} \in W$ and $W \times {}^\omega({}^\omega\omega) \subseteq V$. Then there exists $n \in \omega$ such that $s_k {}^\frown i {}^\frown (x \restriction n) \in {}^{k+n+1}z(k + n + 1)$ for all $i \in m$, $\mathcal{N}_{\psi_k(x) \restriction (k+n+1)} \subseteq U$, and $\prod_{i \in m} \mathcal{N}_{\psi_X(s_k {}^\frown i {}^\frown x) \restriction (k+n+1)} \subseteq W$. Fix a local approximation $l \in L(p^{k+n+1}, Y_\alpha)$. Then the points $y = g^l(x \restriction n)$ and $y_i = f^l(s_k {}^\frown i {}^\frown (x \restriction n))$ for $i \in \omega$ are as desired. ⊠

The following fact implies Lecomte's result that $G_0(\omega) \restriction X_z$ cannot be replaced with $G_0(\omega)$ in the statement of Theorem 4:

Proposition 8. *Suppose that $z \in {}^\omega\omega$ is strictly increasing. Then there is no continuous homomorphism from $G_0(\omega)$ to $G_0(\omega) \restriction X_z$.*

Proof. We say that a set $P \subseteq {}^\omega\omega$ is a *prism* if there is a co-infinite set $I \subseteq \omega$ and a sequence $y \in {}^I\omega$ such that $P = \{x \in {}^\omega\omega \mid y = x \restriction I\}$.

Lemma 9 (Lecomte)**.** *Suppose that $\varphi : {}^\omega\omega \to {}^\omega\omega$ is a continuous homomorphism from $G_0(\omega)$ to $G_0(\omega)$. Then $\varphi[{}^\omega\omega]$ contains a prism.*

Proof of lemma. This follows from the proof of Theorem 3 of [5]. ⊠

By Lemma 9, it is enough to show that no prism P is contained in X_z. Towards this end, fix $I \subseteq \omega$ and $y \in {}^I\omega$ with $P = \{x \in {}^\omega\omega \mid y = x \restriction I\}$, let $(i_k)_{k \in \omega}$ denote the strictly increasing enumeration of $\omega \setminus I$, and define $x \in P$ by setting $x(i_k) = z(i_{k+1})$ for $k \in \omega$. Then $x \restriction n \notin {}^n z(n)$ for all $n \in \omega \setminus (i_0 + 1)$, so $x \notin X_z$, thus $P \not\subseteq X_z$. ⊠

As originally noted by Lecomte, there is nevertheless a weak version of Theorem 4 in which $G_0(\omega) \restriction X_z$ can be replaced with $G_0(\omega)$:

Theorem 10 (Lecomte)**.** *Work in* ZFC*. Suppose that X is a Hausdorff space and G is an analytic ω-dimensional hypergraph on X. Then exactly one of the following holds:*

1. *There is a Borel ω-coloring of G.*
2. *There is a Baire measurable homomorphism from $G_0(\omega)$ to G.*

Proof. The proof that (1) and (2) of Theorem 4 are mutually exclusive works just as well here. To see that $\neg(1) \implies (2)$, let $E_0(\omega)$ denote the equivalence relation on ${}^\omega\omega$ given by

$$xE_0(\omega)y \iff \exists m \in \omega \forall n \in \omega \setminus m \ (x(n) = y(n)).$$

Recall that a set is *invariant* with respect to an equivalence relation if it can be expressed as the union of equivalence classes. Note that the range of every sequence in $G_0(\omega)$ is contained in a single equivalence class of $E_0(\omega)$. In particular, it follows that we can construct partial homomorphisms from $G_0(\omega)$ to G by pasting together partial homomorphisms defined on disjoint $E_0(\omega)$-invariant subsets of ${}^\omega\omega$.

Fix a strictly increasing sequence $z \in {}^\omega\omega$, and appeal to Theorem 4 to obtain a continuous homomorphism $\varphi_z \colon X_z \to X$ from $G_0(\omega) \restriction X_z$ to G. For each equivalence class C of $E_0(\omega)$, fix a strictly increasing sequence $z_C \in {}^\omega\omega$ with $C \subseteq X_{z_C}$, and appeal to Theorem 4 to obtain a continuous homomorphism $\varphi_C \colon X_{z_C} \to X$ from $G_0(\omega) \restriction X_{z_C}$ to G.

As X_z is $E_0(\omega)$-invariant, we obtain a homomorphism $\varphi \colon {}^\omega\omega \to X$ from $G_0(\omega)$ to G by setting $\varphi(x) = \varphi_z(x)$ for $x \in X_z$ and $\varphi(x) = \varphi_C(x)$ for $x \notin X_z$, where C denotes the $E_0(\omega)$-class of x.

To see that φ is Baire measurable, observe that if $U \subseteq X$ is open, then $\varphi_z^{-1}(U) = \varphi^{-1}(U) \cap X_z$. As the former set is Borel and X_z is comeager, it follows that $\varphi^{-1}(U)$ has the Baire property. ⊠

Theorem 4, Proposition 8, and Theorem 10 lead to the following:

Question 11 (Lecomte)**.** Can the homomorphism in part (2) of Theorem 10 be taken to be Borel? Equivalently, is there a Borel homomorphism from $G_0(\omega)$ to $G_0(\omega) \restriction X_z$ for every (equivalently, some) strictly increasing sequence $z \in {}^\omega\omega$?

In light of Theorem 10, perhaps the most natural attempt at producing a negative answer to Question 11 is to find a Polish topology τ on ${}^\omega\omega$, compatible with the underlying Borel structure of ${}^\omega\omega$, for which there is no τ-Baire measurable homomorphism from $G_0(\omega)$ to G. Similarly, one could look for a σ-finite measure μ on ${}^\omega\omega$ for which there is no μ-measurable homomorphism from $G_0(\omega)$ to G.

Theorem 4 immediately implies that neither strategy can succeed. Simply choose a strictly increasing sequence $z \in {}^{\omega}\omega$ such that X_z is μ-conull and τ-comeager, and proceed as in the proof of Theorem 10.

We can consistently obtain an even stronger result. Recall that a subset of a Polish space X is *universally measurable* if it is μ-measurable for every Borel probability measure μ on X, and a function $\varphi \colon X \to Y$ is *universally measurable* if $\varphi^{-1}(U)$ is universally measurable for every open set $U \subseteq Y$. Similarly, a subset of a Polish space X is *ω-universally Baire* if its pre-image under every continuous function $\psi \colon {}^{\omega}\omega \to X$ has the Baire property, and a function $\varphi \colon X \to Y$ is *ω-universally Baire measurable* if $\varphi^{-1}(U)$ is ω-universally Baire for every open set $U \subseteq Y$.

Theorem 12. *Work in* $\mathtt{ZFC} + \mathtt{add}(\mathtt{null}) = \mathfrak{c}$. *Suppose that X is a Hausdorff space and G is an analytic ω-dimensional hypergraph on X. Then exactly one of the following holds:*

1. *There is a Borel ω-coloring of G.*
2. *There is a homomorphism from $G_0(\omega)$ to G which is universally measurable and ω-universally Baire measurable.*

Proof. By Theorem 10, it is enough to show that $\neg(1) \implies (2)$. Towards this end, fix enumerations $(\mu_\alpha)_{\alpha \in \mathfrak{c}}$ of the set of all Borel probability measures on ${}^{\omega}\omega$, $(\psi_\alpha)_{\alpha \in \mathfrak{c}}$ of the set of all continuous functions from ${}^{\omega}\omega$ to ${}^{\omega}\omega$, and $(x_\alpha)_{\alpha \in \mathfrak{c}}$ of ${}^{\omega}\omega$.

For each $\alpha \in \mathfrak{c}$, fix a strictly increasing sequence $z_\alpha \in {}^{\omega}\omega$ such that X_{z_α} is μ_α-conull, $\psi_\alpha^{-1}(X_{z_\alpha})$ is comeager, and $x_\alpha \in X_{z_\alpha}$, and appeal to Theorem 4 to obtain a continuous homomorphism $\varphi_\alpha \colon X_{z_\alpha} \to X$ from $G_0(\omega) \restriction X_{z_\alpha}$ to G.

As each of the sets X_{z_α} is $E_0(\omega)$-invariant, we obtain a homomorphism $\varphi \colon {}^{\omega}\omega \to X$ from $G_0(\omega)$ to G by setting $\varphi(x) = \varphi_\alpha(x)$ for all $\alpha \in \mathfrak{c}$ and $x \in X_{z_\alpha} \setminus \bigcup_{\beta \in \alpha} X_{z_\beta}$.

To see that φ is universally measurable, suppose that μ is a Borel probability measure on ${}^{\omega}\omega$, fix $\alpha \in \mathfrak{c}$ with $\mu = \mu_\alpha$, and observe that if $U \subseteq X$, then $\varphi^{-1}(U) \cap X_{z_\alpha} = \bigcup_{\beta \in \alpha+1} (\varphi_{z_\beta}^{-1}(U) \setminus \bigcup_{\gamma \in \beta} X_{z_\gamma}) \cap X_{z_\alpha}$. In particular, if U is open, then our assumption that $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$ ensures that the latter set is μ-measurable. As X_{z_α} is μ-conull, it follows that $\varphi^{-1}(U)$ is μ-measurable.

Similarly, to see that φ is ω-universally Baire measurable, suppose that $\psi \colon {}^{\omega}\omega \to {}^{\omega}\omega$ is continuous, fix $\alpha \in \mathfrak{c}$ with $\psi = \psi_\alpha$, and observe that if $U \subseteq X$, then $(\varphi \circ \psi)^{-1}(U) \cap \psi^{-1}(X_{z_\alpha})$ can be expressed as

$$\bigcup_{\beta \in \alpha+1} \left((\varphi_{z_\beta} \circ \psi)^{-1}(U) \setminus \bigcup_{\gamma \in \beta} \psi^{-1}(X_{z_\gamma}) \right) \cap \psi^{-1}(X_{z_\alpha}).$$

By a result of Bartoszynski [1], our assumption that $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$ ensures that $\mathtt{add}(\mathtt{meager}) = \mathfrak{c}$. In particular, if U is open, then the latter set has the Baire property. As $\psi^{-1}(X_{z_\alpha})$ is comeager, it follows that $(\varphi \circ \psi)^{-1}(U)$ has the Baire property. ⊠

We close by noting generalizations of Lecomte's results to broader classes of definable sets. Suppose that κ is an aleph. A topological space is *κ-Souslin* if it is the continuous image of a closed subset of ${}^\omega\kappa$. By removing our use of Proposition 1 from the proof of Theorem 4 and replacing ω with κ as appropriate, we obtain the following:

Theorem 13. *Suppose that κ is an aleph, X is a Hausdorff space, and G is a κ-Souslin ω-dimensional hypergraph on X. Then for all strictly increasing sequences $z \in {}^\omega\omega$, at least one of the following holds:*

1. *There is a κ-coloring of G.*
2. *There is a continuous homomorphism from $G_0(\omega) \upharpoonright X_z$ to G.*

By employing techniques of Kanovei [2], we can do even better:

Theorem 14. *Suppose that κ is an aleph, X is a Hausdorff space, and G is a κ-Souslin ω-dimensional hypergraph on X. Then for all strictly increasing sequences $z \in {}^\omega\omega$, at least one of the following holds:*

1. *There is a κ^+-Borel κ-coloring of G.*
2. *There is a continuous homomorphism from $G_0(\omega) \upharpoonright X_z$ to G.*

Question 15. Is there a classical proof of Theorem 14?

References

[1] T. Bartoszynski. Additivity of measure implies additivity of category. *Trans. Amer. Math. Soc.*, 209–213, **281 (1)**, 1984.

[2] V. Kanovei. Two dichotomy theorems on colourability of non-analytic graphs. *Fund. Math.*, 183–201, **154 (2)**, 1997.

[3] A.S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[4] A.S. Kechris, S. Solecki, and S. Todorcevic. Borel chromatic numbers. *Adv. Math.*, 1–44, **141 (1)**, 1999.

[5] D. Lecomte. A dichotomy characterizing analytic digraphs of uncountable Borel chromatic number in any dimension. *Trans. Amer. Math. Soc.*, 4181–4193, **361 (8)**, 2009.