

# AN ANTIBASIS RESULT FOR GRAPHS OF INFINITE BOREL CHROMATIC NUMBER

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ABSTRACT. We answer in the negative a question posed by Kechris-Solecki-Todorćević as to whether the shift graph on Baire space is minimal among graphs of indecomposably infinite Borel chromatic number. To do so, we use ergodic-theoretic techniques to construct a new graph amalgamating various properties of the shift actions of free groups. The resulting graph is incomparable with any graph induced by a function. We then generalize this construction and collect some of its useful properties.

## 1. INTRODUCTION

Recall that the *chromatic number*, written  $\chi(G)$ , of a (possibly infinite) graph  $G = (V, E)$  with vertex set  $V$  and edge set  $E$  is defined to be the smallest cardinal  $\kappa$  for which there is a *coloring function*  $c : V \rightarrow \kappa$  such that  $c(x) \neq c(y)$  whenever  $x E y$ . In the traditional set-theoretic framework of ZFC this notion is quite well studied, and invoking the Axiom of Choice in the form of a compactness argument establishes that such a graph has finite chromatic number if and only if there is a finite bound on the chromatic numbers of its finite induced subgraphs. In the absence of relatively strong choice principles, however, this equivalence fails drastically. For example, it is consistent with ZF+DC (assuming as usual the consistency of ZFC) that there is a graph  $G$  all of whose finite induced subgraphs have chromatic number at most two while  $\chi(G) = 2^{\aleph_0}$ , the cardinality of the continuum (an example,  $G_0$ , is discussed below).

One can clarify the dependency of these arguments on the Axiom of Choice, and shift the question into the descriptive set-theoretic milieu, by placing definability restrictions on the coloring functions. More precisely, one can fix a standard Borel space  $X$  and examine graphs  $G$  whose edge relation is realized as a symmetric, ir-reflexive subset of  $X^2$ . One may then consider the *Borel chromatic number*, defined as the least cardinality of a standard Borel space  $Y$  for which there is a Borel coloring function  $c : X \rightarrow Y$  of  $G$ , with the usual requirement that  $G$ -adjacent vertices get different colors. This number was investigated by Kechris-Solecki-Todorćević, who noticed that there are acyclic Borel graphs which have no Borel  $\aleph_0$ -coloring. Indeed, in [6, §6] such an example  $G_0$  on  $2^{\mathbb{N}}$  is produced with the following remarkable minimality property: an analytic graph  $G$  on a Hausdorff space  $X$  has uncountable Borel chromatic number iff there is a continuous homomorphism from  $G_0$  to  $G$ . Moreover, in a model of ZF+DC in which all subsets of the reals have the property of Baire (see, e.g., [9, Theorem 7.16]), we have  $\chi(G_0) = 2^{\aleph_0}$ .

There are many consequences of the existence of such a minimal object. It reduces the inherent complexity of the statement “ $G$  cannot be colored with countably many colors,” since it is equivalent to the existential statement “there is a continuous homomorphism from  $G_0$  to  $G$ .” Moreover, recent work by the second author has isolated several classical descriptive set-theoretic dichotomies as consequences of precisely this form of graph minimality, suggesting that these graph coloring concerns are sufficiently flexible to encode seemingly unrelated set-theoretic notions.

Kechris-Solecki-Todorćevic have asked whether there is an analogous minimal graph among the collection of graphs with infinite Borel chromatic number. More precisely, they posed the question of whether the graph  $G_s$  on infinite sets of natural numbers, where two such sets are adjacent if one can be obtained from the other by removing its least element, is such a minimal graph [6, Question 8.1]. In addition to the benefits listed above, the existence of such a minimal graph would provide a nice analog to the ease of testing in ZFC whether a graph has infinite chromatic number.

In §2, we exhibit an acyclic, locally finite, Borel graph  $G_{\mathbb{F}}$  of infinite Borel chromatic number with the property that there is no Borel homomorphism from any graph associated with a Borel function to  $G_{\mathbb{F}}$ . In particular, this gives a negative answer to [6, Question 8.1]. The graph  $G_{\mathbb{F}}$  can be viewed as an amalgamation of graphs associated with shift actions of free groups. In §3, we discuss a more general way of amalgamating acyclic graphs into larger graphs, and catalog some useful properties of this amalgamation.

## 2. A GRAPH INCOMPARABLE WITH THE SHIFT

A *graph* on a set  $X$  is a symmetric, irreflexive set  $G \subseteq X \times X$ . The *degree* of a point  $x$  with respect to  $G$  is given by  $\deg_G(x) = |\{y \in X : (x, y) \in G\}|$ . The graph  $G$  is *locally countable* if every point has countable degree, and the graph  $G$  is *locally finite* if every point has finite degree. The *restriction* of  $G$  to a set  $A \subseteq X$  is given by  $G \upharpoonright A = G \cap (A \times A)$ . We say that  $A$  is *(G)-independent* if  $G \upharpoonright A = \emptyset$ .

A  $\kappa$ -*coloring* of  $G$  is a function  $c : X \rightarrow \kappa$  such that  $c^{-1}(\{\alpha\})$  is independent for each  $\alpha \in \kappa$ . The *chromatic number* of  $G$ ,  $\chi(G)$ , is the least cardinal  $\kappa$  for which there exists a  $\kappa$ -coloring of  $G$ . Analogously, the *Borel chromatic number*,  $\chi_B(G)$ , of a graph on a standard Borel space  $X$  is the least cardinality of a standard Borel space  $Y$  for which there is a Borel function  $c : X \rightarrow Y$  with  $c^{-1}(\{y\})$  independent for each  $y \in Y$ .

For a graph  $G$ , let  $E_G$  denote the equivalence relation generated by  $G$ . The classes of  $E_G$  are called the *connected components* of  $G$ , and  $G$  is *connected* if  $E_G$  has only one class. We say that  $G$  has *indecomposably infinite Borel chromatic number* if  $X$  cannot be partitioned into countably many Borel  $E_G$ -invariant sets on which  $G$  has finite Borel chromatic number.

We identify the space  $[\mathbb{N}]^{\mathbb{N}}$  of infinite subsets of the natural numbers with the collection of strictly increasing sequences of natural numbers. The *unilateral shift* on  $[\mathbb{N}]^{\mathbb{N}}$  is the function  $s : [\mathbb{N}]^{\mathbb{N}} \rightarrow [\mathbb{N}]^{\mathbb{N}}$  given by

$$s(x)(i) = x(i + 1),$$

for all  $x \in [\mathbb{N}]^{\mathbb{N}}$  and  $i \in \mathbb{N}$ . Let  $G_s$  denote the graph on  $[\mathbb{N}]^{\mathbb{N}}$  given by

$$x G_s y \Leftrightarrow x = s(y) \text{ or } y = s(x).$$

Thus,  $G_s$  is an acyclic, locally finite, Borel graph on  $[\mathbb{N}]^{\mathbb{N}}$ . By a straightforward application of the Galvin-Prikry theorem, it is shown in [6, Example 3.2] that  $\chi_B(G_s) = \aleph_0$ , and it is therein conjectured that  $G_s$  is in some sense minimal among the collection of graphs of infinite Borel chromatic number.

If  $\Gamma$  is a countable group with generating set  $S$  (assumed not to contain the identity) and  $a$  is a free, Borel action of  $\Gamma$  on  $X$ , we define the graph  $G(S, a)$  on  $X$  by

$$x G(S, a) y \Leftrightarrow \exists s \in S (x = s \cdot^a y \text{ or } y = s \cdot^a x).$$

To analyze such graphs, it will be useful to work with a measure. If  $G$  is a Borel graph on a standard Borel space  $X$  and  $\mu$  is a Borel probability measure on  $X$ , the  $(\mu)$ -measurable chromatic number of  $G$ ,  $\chi_\mu(G)$ , is the least cardinality of a Polish space  $Y$  for which there is a  $(\mu)$ -measurable function  $c : X \rightarrow Y$  with  $c^{-1}(y)$  a  $G$ -independent set for each  $y \in Y$ . To avoid trivial degeneracies, we assume from now on that  $(X, \mu)$  is a standard probability space. We see immediately that  $\chi_\mu(G) \leq \chi_B(G)$ .

We denote by  $\mathbb{F}_n$  ( $n \geq 2$ ) the free group on  $n$  generators, and by  $\mathbb{F}_\infty$  the free group on  $\aleph_0$ -many generators. Fixing a set of free generators  $\{\gamma_0, \gamma_1, \dots\}$  for  $\mathbb{F}_\infty$ , we may canonically identify  $\mathbb{F}_n$  with the subgroup of  $\mathbb{F}_\infty$  generated by  $\{\gamma_1, \dots, \gamma_n\}$  (note that  $\gamma_0$  is unused). Equip  $2^{\mathbb{F}_\infty}$  with the  $(1/2, 1/2)$  product measure  $\mu_0$ , and denote by  $a_\infty$  the shift action of  $\mathbb{F}_\infty$  on  $(X_0, \mu_0)$ , where  $X_0 \subseteq 2^{\mathbb{F}_\infty}$  is the conull set on which this shift action is free. For each  $n \geq 2$ , let  $a_n$  denote the (free) action of  $\mathbb{F}_n$  on  $(X_0, \mu_0)$  obtained by restricting  $a_\infty$  to  $\langle F_n \rangle$ . Finally, let  $G_n = G(F_n, a_n) \subseteq G(F_\infty, a_\infty)$ , so that each  $G_n$  is an acyclic, Borel graph with respect to which every point has degree  $2n$ . Moreover, each  $E_{G_n}$  is ergodic since the shift action of  $\mathbb{F}_\infty$  is mixing [5, Example 3.1].

From [6, Proposition 4.6] it follows that  $\chi_B(G_n) \leq 2n+1$ , and we will next sketch the argument that  $\chi_{\mu_0}(G_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Let  $b_n$  denote the free part of the shift of  $\mathbb{F}_n$  on  $2^{\mathbb{F}_n}$ , equipped with the usual product measure. Each  $a_n$  is isomorphic to the product  $(b_n)^{\mathbb{N}}$  acting on  $(2^{\mathbb{F}_n})^{\mathbb{N}}$ . In turn,  $(b_n)^{\mathbb{N}}$  is isomorphic to the shift action of  $\mathbb{F}_n$  on  $(2^{\mathbb{N}})^{\mathbb{F}_n}$ . Bowen [1] has shown that this last shift action is weakly equivalent to  $b_n$  itself (for a definition and discussion of weak equivalence, see [4, Chapter 10]). In [2] and independently [7] it is shown that the graph associated with the action  $b_n$  (and a free set of generators) has an upper bound on the measure of an independent set tending towards zero as  $n \rightarrow \infty$ . Since this bound respects isomorphism and weak equivalence, we see by the above observations that  $\chi_{\mu_0}(G_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . More details of this argument may be found in [2, Corollary 4.2, Theorem 4.17, Remark 4.18].

We amalgamate these graphs into a single graph  $G_{\mathbb{F}}$  on  $X_{\mathbb{F}} = X_0 \times 2^{\mathbb{N}}$  (with the product measure  $\mu_{\mathbb{F}} = \mu_0 \times (1/2, 1/2)^{\mathbb{N}}$ ) with countably infinite Borel and measurable chromatic numbers. Partition  $2^{\mathbb{N}}$  into countably many Borel sets  $A_2, A_3, \dots$  of positive measure. Fix an aperiodic, ergodic, measure-preserving, Borel automorphism  $\sigma : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ , and define  $T : X_{\mathbb{F}} \rightarrow X_{\mathbb{F}}$  by  $T(x, y) = (\gamma_0 \cdot x, \sigma(y))$ . Finally, define  $G_{\mathbb{F}}$  on  $X_{\mathbb{F}}$  by

$$(x, y) G_{\mathbb{F}} (x', y') \Leftrightarrow \exists n \geq 2 (x G_n x' \text{ and } y = y' \in A_n) \text{ or } (x, y) = T^{\pm 1}(x', y').$$

Here we are pasting continuum-many copies of  $G_n$  into each  $A_n$ , and using  $T$  to tie them together. We see that  $G_{\mathbb{F}}$  is an acyclic, locally finite, Borel graph with  $\chi_B(G_{\mathbb{F}}) = \chi_\mu(G_{\mathbb{F}}) = \aleph_0$ . Moreover,  $E_{G_{\mathbb{F}}}$  is an ergodic equivalence relation, so there

is no way to partition  $X_{\mathbb{F}}$  into countably many  $E_{G_{\mathbb{F}}}$ -invariant pieces on which  $G_{\mathbb{F}}$  has finite  $\mu_{\mathbb{F}}$ -measurable chromatic number.

Of course, the same idea works in general to sew countably many acyclic graphs together using an ergodic automorphism, and acyclicity is preserved provided this automorphism is sufficiently “free” from the original graphs. We examine this idea in more detail in §3.

Recall that a *homomorphism* from a graph  $G$  on  $X$  to a graph  $H$  on  $Y$  is a function  $\varphi : X \rightarrow Y$  such that

$$x G x' \Rightarrow \varphi(x) H \varphi(x').$$

With any function  $f : X \rightarrow X$  we may associate the graph  $G_f$  on  $X$  given by

$$x G_f y \Leftrightarrow x \neq y \text{ and } (x = f(y) \text{ or } y = f(x)).$$

We abbreviate  $E_{G_f}$  by  $E_f$ .

**Proposition 1.** *Suppose that  $f : X \rightarrow X$  is a Borel function. Then there is no Borel homomorphism from  $G_{\mathbb{F}}$  to  $G_f$ .*

*Proof.* Suppose, towards a contradiction, that  $\varphi : X_{\mathbb{F}} \rightarrow X$  is a Borel homomorphism from  $G_{\mathbb{F}}$  to  $G_f$ . Denote by  $(\mu_{\mathbb{F}})_*$  the push-forward of  $\mu_{\mathbb{F}}$  by  $\varphi$ , i.e., the measure given by

$$(\mu_{\mathbb{F}})_*(A) = \mu_{\mathbb{F}}(\varphi^{-1}(A)),$$

for all Borel  $A \subseteq X$ . By [8, Theorem A] there is a  $(\mu_{\mathbb{F}})_*$ -measurable 3-coloring  $c : X \rightarrow 3$ . But then  $c \circ \varphi$  is a  $\mu_{\mathbb{F}}$ -measurable 3-coloring of  $G_{\mathbb{F}}$ , the desired contradiction.  $\square$

**Remark 2.** The above argument in fact also rules out a Borel homomorphism from  $G_{\mathbb{F}}$  to any acyclic Borel graph  $G$  whose associated equivalence relation  $E_G$  is measure hyperfinite.

**Proposition 3.** *Suppose that  $f : X \rightarrow X$  is a Borel function. If there is a Borel homomorphism from  $G_f$  to  $G_{\mathbb{F}}$ , then  $\chi_B(G_f) \leq 3$ .*

*Proof.* Suppose that  $\varphi : X \rightarrow X_{\mathbb{F}}$  is a Borel homomorphism from  $G_f$  to  $G_{\mathbb{F}}$ . We use  $\varphi$  to pull a particular partition of the edges of  $G_{\mathbb{F}}$  back to  $G_f$ . Define subgraphs  $H_T, H$  of  $G_f$  by

$$\begin{aligned} x H_T y &\Leftrightarrow x G_f y \text{ and } \varphi(x) = T^{\pm 1}(\varphi(y)), \text{ and} \\ x H y &\Leftrightarrow x G_f y \text{ and } \text{proj}_{2^{\mathbb{N}}}(\varphi(x)) = \text{proj}_{2^{\mathbb{N}}}(\varphi(y)), \end{aligned}$$

where  $\text{proj}_{2^{\mathbb{N}}}$  denotes projection onto  $2^{\mathbb{N}}$ . Observe that  $G_f = H_T \sqcup H$ . Moreover, since  $T$  is a Borel automorphism, it follows from [6, Proposition 4.7] that  $H_T$  has Borel chromatic number at most 3. It therefore suffices to argue that  $\chi_B(H)$  is finite, since  $\chi_B(G_f) \leq 3\chi_B(H)$ , thus [6, Theorem 5.1] would imply that  $\chi_B(G_f) \leq 3$ . We will in fact show that  $\chi_B(H) \leq 3$  as well.

**Lemma 4.** *Suppose that  $G$  is a Borel graph on  $X$  with  $G \subseteq G_f$ . Then there is a Borel function  $g : X \rightarrow X$  with  $G = G_g$ . In particular, if  $\chi_B(G)$  is finite, then  $\chi_B(G) \leq 3$ .*

*Proof.* Note that each connected component of  $G$  contains at most one point  $x$  such that  $x$  and  $f(x)$  are not  $G$ -related. Simply define  $g : X \rightarrow X$  by

$$g(x) = \begin{cases} f(x) & \text{if } x G f(x), \text{ and} \\ x & \text{otherwise,} \end{cases}$$

so that  $G = G_g$ . If also  $\chi_B(G_g)$  is finite we have  $\chi_B(G_g) \leq 3$  (see [6, Theorem 5.1]).  $\square$

For  $n \geq 2$ , put  $B_n = \varphi^{-1}(X_0 \times A_n)$ , where  $A_n \subseteq 2^{\mathbb{N}}$  is as in the definition of  $G_{\mathbb{F}}$ . Each  $B_n$  is a union of connected components of  $H$  (since the only edges in  $G_{\mathbb{F}}$  connecting distinct  $X_0 \times A_n, X_0 \times A_m$  are those induced by  $T$ ). Moreover, for each  $n \geq 2$ , the Borel chromatic number of  $H|_{B_n}$  is finite, since  $(\varphi \times \varphi)[H]|_{(X_0 \times A_n)}$  has degree bounded by  $2n$  (again applying [6, Proposition 4.6]). By then applying Lemma 4 to  $H$  on each  $B_n$ , we see that  $\chi_B(H) \leq 3$  as promised.  $\square$

**Corollary 5.** *There is neither a Borel homomorphism from  $G_s$  to  $G_{\mathbb{F}}$  nor a Borel homomorphism from  $G_{\mathbb{F}}$  to  $G_s$ .*

We now shift our attention to the existence of graphs below both  $G_s$  and  $G_{\mathbb{F}}$ . While it remains unknown whether there exists a graph of indecomposably infinite Borel chromatic number which homomorphs into both, we can rule out such graphs under additional injectivity assumptions on one of these homomorphisms. For graphs  $G$  on  $X$  and  $H$  on  $Y$ , we say that  $\varphi : X \rightarrow Y$  is *locally injective* if for all  $w \in X$  and all  $G$ -neighbors  $x_0, x_1$  of  $w$ ,  $\varphi(x_0) = \varphi(x_1) \Rightarrow x_0 = x_1$ .

**Corollary 6.** *Suppose that  $f : X \rightarrow X$  is a Borel function. Then there is no Borel graph  $G$  of infinite Borel chromatic number which admits a locally injective Borel homomorphism to  $G_f$  and a Borel homomorphism to  $G_{\mathbb{F}}$ .*

*Proof.* Suppose, towards a contradiction, that there is such a graph  $G$  on  $X$ . Fix a locally injective Borel homomorphism  $\varphi$  from  $G$  to  $G_f$ , and define a function  $g : X \rightarrow X$  by

$$g(x) = \begin{cases} y & \text{if } x G y \text{ and } \varphi(y) = f \circ \varphi(x), \text{ and} \\ x & \text{otherwise.} \end{cases}$$

It is easy to see that  $G = G_g$ , and the result follows from Proposition 3.  $\square$

We close this section by considering the finer partial order of Borel reducibility among graphs. Given graphs  $G$  on  $X$  and  $H$  on  $Y$ , we say that  $\varphi : X \rightarrow Y$  is a *reduction* of  $G$  to  $H$  if  $x_0 G x_1 \Leftrightarrow \varphi(x_0) H \varphi(x_1)$ . When we restrict our attention to reductions in the locally countable context, we can discard the assumption of local injectivity in the previous result.

**Proposition 7.** *There is no locally countable Borel graph  $G$  of infinite Borel chromatic number which admits both a Borel reduction to  $G_s$  and a Borel homomorphism to  $G_{\mathbb{F}}$ .*

*Proof.* Suppose, towards a contradiction, that there is such a graph  $G$  on  $X$ . Since  $G$  is locally countable, we may assume without loss of generality that every vertex has at least one  $G$ -neighbor. Fix a Borel reduction  $\varphi$  of  $G$  to  $G_s$ . We consider the Borel equivalence relation  $E$  on  $X$  defined by  $x E y \Leftrightarrow \varphi(x) = \varphi(y)$ . Every equivalence class of  $E$  is countable, since  $[x]_E \subseteq \{y \in X : x \text{ and } y \text{ have the same set}$

of neighbors}. As  $\varphi$  witnesses the smoothness of  $F$ , there is a Borel transversal  $A$  of  $F$  [5, Proposition 6.4]. Then  $\varphi \upharpoonright A$  is an injective Borel reduction of  $G \upharpoonright A$  to  $G_s$ . As  $\varphi \upharpoonright A$  is a Borel homomorphism from  $G \upharpoonright A$  to  $G_{\mathbb{F}}$ , Corollary 6 implies that  $\chi_B(G \upharpoonright A) < \aleph_0$ .

Fix  $k \in \omega$  and a Borel coloring  $c : A \rightarrow k$  of  $G \upharpoonright A$ . Define  $c' : X \rightarrow k$  by  $c'(x) = c(y)$ , where  $y$  is the unique element of  $A \cap [x]_F$ . Then  $c'$  is a coloring of  $G$ , since the  $F$ -saturation of any  $G$ -independent set remains  $G$ -independent. Since the range of  $c'$  is finite, this contradicts our assumption that  $G$  has infinite Borel chromatic number.  $\square$

### 3. A MORE GENERAL CONSTRUCTION

Suppose that  $E$  and  $F$  are equivalence relations on a measure space  $(X, \mu)$ . We say that  $E$  and  $F$  are *independent*, in symbols  $E \perp F$ , if there is no sequence  $x_0, x_1, \dots, x_n = x_0$  with  $n > 1$ ,  $x_i \neq x_j$  ( $0 \leq i < j < n$ ), and  $x_0 E x_1 F x_2 E x_3 \dots$  (i.e., there are no nontrivial cycles whose “edges” alternate between  $E$  and  $F$ ). We say that  $E$  and  $F$  are  $\mu$ -*independent* if they are independent after removing a  $\mu$ -null set. We denote by  $\text{Aut}(X, \mu)$  the set of  $\mu$ -preserving automorphisms of  $X$ , and say that the *support* of  $T \in \text{Aut}(X, \mu)$  is the set  $\{x \in X : T(x) \neq x\}$ .

Recall that  $\text{Aut}(X, \mu)$  is a Polish space once equipped with the weak topology (see [4, Chapter 1]). The basic open sets in the weak topology are determined by finite partitions  $\{A_0, \dots, A_n\}$ ,  $\{B_0, \dots, B_n\}$  of  $X$  into Borel sets and  $\varepsilon > 0$ , and have the form  $\{T \in \text{Aut}(X, \mu) : \mu(T(A_i) \triangle B_i) < \varepsilon\}$ . Equivalently, each nonempty basic open set is determined by a partition  $\mathcal{P}$  of  $X$  into countably many Borel sets,  $T_0 \in \text{Aut}(X, \mu)$ , and  $\varepsilon > 0$ , having the form  $\{T : \forall P \in \mathcal{P} \mu(T(P) \triangle T_0(P)) < \varepsilon\}$ .

The following is a generalization of the main lemma (and the subsequent remark) in [10, III]. As usual, a measure  $\mu$  is  $E$ -quasi-invariant if the  $E$ -saturation of a  $\mu$ -null set remains  $\mu$ -null.

**Theorem 8.** *Suppose that  $E$  is a countable Borel equivalence on a standard Borel space  $X$  and  $\mu$  is an  $E$ -quasi-invariant standard Borel probability measure on  $X$ . Then the set of automorphisms  $T$  such that  $E$  and  $E_T$  are  $\mu$ -independent is comeager in  $\text{Aut}(X, \mu)$ .*

Before jumping into the proof of Theorem 8, we establish some lemmas in the more restrictive Borel context.

**Lemma 9.** *Suppose that  $T_1, \dots, T_k : X \rightarrow X$  are Borel automorphisms of a standard Borel space  $X$  and  $Y \subseteq X$  is such that for all  $x \in Y$  and  $i, j \leq k$ ,  $T_i(x) \neq T_j(x)$ . Then there is a partition of  $Y$  into  $N = k^2 + 1$  Borel parts  $Y_1, \dots, Y_N$  such that for each  $n \leq N$  the collection  $\{T_i(Y_n) : i \leq k\}$  forms a pairwise disjoint family.*

*Proof.* Consider the Borel graph  $G$  on  $Y$  given by

$$x G y \Leftrightarrow x \neq y \text{ and } \exists i, j (T_i(x) = T_j(y)).$$

The graph  $G$  has degree bounded by  $k^2$  and thus admits a Borel coloring by  $k^2 + 1 = N$  colors [6, Theorem 4.6]. We choose these colors as our sets  $Y_n$ . Suppose, towards a contradiction, that  $T_i(Y_n) \cap T_j(Y_n) \neq \emptyset$ , and fix  $x, y \in Y_n$  with  $T_i(x) = T_j(y)$ . Since  $Y_n$  is  $G$ -independent, we must have  $x = y$ , contradicting the hypothesis of the lemma.  $\square$

Given a partition  $\mathcal{P}$  of  $X$  and an automorphism  $T : X \rightarrow X$ , we say that  $T$  *factors over*  $\mathcal{P}$  if  $T(P) = P$  for all  $P \in \mathcal{P}$ .

**Lemma 10.** *Suppose that  $\mathcal{P}$  is a partition of  $X$  into countably many Borel parts, and  $B$  is a Borel subset of  $X$ . Then there is a Borel involution  $I : X \rightarrow X$  factoring over  $\mathcal{P}$  such that  $\text{supp}(I)$  is a cocountable subset of  $B$ . Moreover, if  $\mu$  is a nonatomic Borel probability measure on  $X$  then  $I$  may be chosen to preserve  $\mu$ .*

*Proof.* Simply note that for each  $P \in \mathcal{P}$  there is a Borel involution  $I_P$  with support contained in  $B \cap P$  fixing at most one element of  $B \cap P$  (such a fixed point is only necessary if  $|B \cap P|$  is finite and odd). Set  $I = \bigcup_{P \in \mathcal{P}} I_P$ . To ensure that this is  $\mu$ -preserving, for each  $P \in \mathcal{P}$  with  $\mu(B \cap P) > 0$ , choose  $I_P$  to flip two subsets of  $B \cap P$  of equal measure.  $\square$

For each  $i \in \mathbb{N}$ , let  $\iota_i$  be a Borel involution of full support. We say that a formal word  $w$  in the alphabet  $\{\iota_i : i \in \mathbb{N}\} \sqcup \{\tau, \tau^{-1}\}$  is *reduced* if no two elements of  $\{\iota_i : i \in \mathbb{N}\}$  appear consecutively, nor do  $\tau$  and  $\tau^{-1}$  appear consecutively. With each reduced word  $w$  and Borel automorphism  $T : X \rightarrow X$ , we may associate a Borel automorphism  $T_w$  inductively by

$$\begin{aligned} T_\emptyset &= \text{id}, \\ T_{\iota_i \frown w} &= \iota_i \circ T_w, \text{ for } i \in \mathbb{N}, \\ T_{\tau \frown w} &= T \circ T_w, \\ T_{\tau^{-1} \frown w} &= T^{-1} \circ T_w. \end{aligned}$$

Intuitively, we simply plug in  $T$  for the symbol  $\tau$ . We say that  $v$  is a *subword* of  $w$ , written  $v \sqsubseteq w$ , if there exist possibly empty words  $w_0, w_1$  such that  $w = w_0 \frown v \frown w_1$ ; if  $v \neq w$ , we say that  $v$  is a *proper subword* of  $w$ , written  $v \sqsubset w$ . We say that  $v$  is an *initial subword* (resp., *terminal subword*) of  $w$  if there exists  $w_0$  such that  $w = v \frown w_0$  (resp.,  $w = w_0 \frown v$ ). With each reduced word  $w$  we may associate its *inverse*  $w^{-1}$  defined inductively so that  $\iota_i$  is its own inverse,  $\tau$  and  $\tau^{-1}$  are inverses, and  $(w_0 \frown w_1)^{-1} = w_1^{-1} \frown w_0^{-1}$ .

If  $T$  is a bijection on  $X$  and  $\mathcal{P}$  is a partition of  $X$ , let  $T[\mathcal{P}]$  denote the partition  $\{T(P) : P \in \mathcal{P}\}$ . If  $\mathcal{P}, \mathcal{P}'$  are two partitions of  $X$ , denote by  $\mathcal{P} \vee \mathcal{P}'$  the partition  $\{P \cap P' : P \in \mathcal{P}, P' \in \mathcal{P}'\}$ . If  $\mathcal{A}$  is a finite collection of subsets of  $X$ , we write  $\mathcal{P}(\mathcal{A})$  to denote the finite partition generated by  $\mathcal{A}$  (or, equivalently, the atoms of the algebra generated by  $\mathcal{A}$ ).

**Lemma 11.** *Suppose that  $\mathcal{P}$  is a partition of  $X$  into countably many Borel sets,  $w$  is a nonempty reduced word, and  $T$  is a Borel automorphism of  $X$  such that  $T_v$  has cocountable support for all nonempty proper subwords  $v \sqsubset w$ . Then there is a Borel automorphism  $S$  factoring over  $\mathcal{P}$  such that  $(T \circ S)_w$  has cocountable support. Moreover, if  $\mu$  is a nonatomic Borel probability measure on  $X$  then  $S$  may be chosen to preserve  $\mu$ .*

*Proof.* The conclusion is immediate if  $w = \iota_i$  for some  $i$ . Replacing  $w$  by  $w^{-1}$  if necessary, we may assume  $\tau \sqsubseteq w$ , and write  $w = w_0 \frown \tau \frown w_1$ . Discarding a countable set, we may assume that  $T_v$  has full support for all nonempty proper subwords  $v \sqsubset w$ . By Lemma 9 there is a partition  $\{Y_1, \dots, Y_N\}$  of  $X$  into Borel sets such that for all distinct proper terminal subwords  $v, v' \sqsubset w$  and all  $n \leq N$ ,  $T_v(Y_n) \cap T_{v'}(Y_n) = \emptyset$ . We handle the sets  $Y_n$  one at a time, via the following fact.

**Sublemma 12.** *Suppose that  $T'$  is a Borel automorphism of  $X$ ,  $\mathcal{P}'$  is a partition of  $X$  into countably many Borel sets, and  $Y$  is a Borel subset of  $X$  such that  $T'_v(Y) \cap T'_{v'}(Y) = \emptyset$  for all distinct proper terminal subwords  $v, v' \sqsubset w$ . Then there is a Borel involution  $I$  of  $X$  factoring over  $\mathcal{P}'$  such that  $(T' \circ I)_w$  has cocountable support. Moreover, if  $\mu$  is a nonatomic Borel probability measure on  $X$  then  $I$  may be chosen to preserve  $\mu$ .*

*Proof.* Set  $Y^{\text{fix}} = Y \setminus \text{supp}(T'_w)$  and choose (by Lemma 10) a Borel involution  $I$  which factors over  $\mathcal{P}'$  such that  $\text{supp}(I)$  is a cocountable subset of  $T'_{w_1}(Y^{\text{fix}})$ . Then for all  $y \in Y$ ,  $(T' \circ I)_{w_1}(y) = T'_{w_1}(y)$ , since  $w_1$  does not begin with  $\tau^{-1}$  and  $\text{supp}(I)$  is disjoint from  $T'_v(Y)$  for all proper subwords  $v \sqsubset w_1$ . Analogously, for all  $y \in T'_{\tau \sim w_1}(Y)$ , we have  $(T' \circ I)_{w_0}(y) = T'_{w_0}(y)$ . Now if  $y \in Y \cap \text{supp}(T'_w)$ , we see  $(T' \circ I)_w(y) = T'_{w_0} \circ T' \circ I \circ T'_{w_1}(y) = T'_{w_0} \circ T' \circ T'_{w_1}(y) = T'_w(y) \neq y$ . On the other hand, for cocountably many  $y \in Y^{\text{fix}}$  we have  $(T' \circ I)_w(y) = T'_{w_0} \circ T' \circ I \circ T'_{w_1}(y) \neq T'_{w_0} \circ T' \circ T'_{w_1}(y) = T'_w(y) = y$ , since  $I \circ T'_{w_1}(y) = T'_{w_1}(y)$  for only countably many  $y$ . Thus,  $I$  is as required, and of course may be chosen to preserve  $\mu$  (again by Lemma 10).  $\square$

We now iteratively apply the sublemma to each  $Y_n$ . First, we apply the sublemma to the automorphism  $T$ , the partition  $\mathcal{P}_1 = \mathcal{P} \vee \mathcal{P}(\{T_v(Y_n) : v \sqsubset w \text{ and } n \leq N\})$ , and the set  $Y_1$  to obtain an involution  $I_1$  with  $Y_1 \setminus \text{supp}((T \circ I_1)_w)$  countable. Apply Lemma 9 to partition  $\text{supp}((T \circ I_1)_w) \cap Y_1$  into Borel sets  $Y_1^1, \dots, Y_1^M$  such that  $T_v(Y_1^M) \cap T_{v'}(Y_1^M) = \emptyset$  for all distinct terminal subwords  $v, v' \sqsubset w$ . Note that here we allow one of  $v, v'$  to equal  $w$ . Next, apply the sublemma to the automorphism  $T_1 = T \circ I_1$ , the partition  $\mathcal{P}_2 = \mathcal{P}_1 \vee \mathcal{P}(\{(T_1)_v(Y_1^m) : v \sqsubset w \text{ and } m \leq M\})$ , and the set  $Y_2$  to obtain a second involution  $I_2$ . Note that the hypotheses of the sublemma are met, since for all proper terminal subwords  $v \sqsubset w$  and  $n \leq N$  we have  $(T_1)_v(Y_n) = T_v(Y_n)$ , as the involution  $I_1$  factors over the partition  $\mathcal{P}(\{T_v(Y_n) : v \sqsubset w \text{ and } n \leq N\})$ .

In general, at stage  $n$  we set  $T_{n-1} = T \circ I_1 \circ \dots \circ I_{n-1}$  and first apply Lemma 9 to partition  $\text{supp}((T_{n-1})_w) \cap Y_{n-1}$  into Borel sets  $Y_{n-1}^1, \dots, Y_{n-1}^M$  such that  $T_v(Y_{n-1}^M) \cap T_{v'}(Y_{n-1}^M) = \emptyset$  for all distinct terminal subwords  $v, v' \sqsubset w$ . Subsequently, we apply the sublemma to the automorphism  $T_{n-1}$ , along with the partition  $\mathcal{P}_n = \mathcal{P}_{n-1} \vee \mathcal{P}(\{(T_{n-1})_v(Y_{n-1}^m) : v \sqsubset w \text{ and } m \leq M\})$ , and the set  $Y_n$  to obtain the next involution  $I_n$ . We claim that the automorphism  $S = I_1 \circ \dots \circ I_N$  satisfies the conclusion of the lemma.

That is, we want to show that the set of  $x$  such that  $(T \circ S)_w(x) = x$  is countable. It suffices to show that if  $x \in Y_n$  and  $(T_n)_w(x) \neq x$ , then  $(T \circ S)(x) \neq x$ , where as before  $T_n = T \circ I_1 \circ \dots \circ I_n$ . Fix  $m \leq M$  such that  $x \in Y_n^m$ . Since each of the involutions  $I_{n+1}, \dots, I_N$  factors over  $\mathcal{P}(\{(T_n)_v(Y_n^m) : v \sqsubset w\})$ , we have  $(T \circ S)_v(Y_n^m) = (T_n)_v(Y_n^m)$  for all  $v \sqsubset w$ . In particular,  $(T \circ S)_w(Y_n^m) \cap Y_n^m = \emptyset$ , and thus  $(T \circ S)_w(x) \neq x$ .  $\square$

*Proof of Theorem 8.* Suppose that  $E$  is a countable Borel equivalence on a standard Borel space  $X$  and  $\mu$  is an  $E$ -quasi-invariant Borel probability measure on  $X$ . We want to show that the set of automorphisms  $T$  such that  $E$  and  $E_T$  are  $\mu$ -independent is comeager in  $\text{Aut}(X, \mu)$  (with respect to the weak topology).

We may (see [3]) write  $E$  as the union of the graphs of countably many Borel involutions. A straightforward maximality argument then allows us to write  $E \setminus \{(x, x) : x \in X\}$  as the union of the graphs of countably many fixed-point-free



Borel involutions  $\iota_0, \iota_1, \dots$ . Using our earlier notation, it suffices to show that for all nonempty reduced words  $w$  in the alphabet  $\{\iota_0, \iota_1, \dots, \tau, \tau^{-1}\}$ , the set  $\{T : \mu(\text{supp}(T_w)) = 1\}$  is comeager in  $\text{Aut}(X, \mu)$ . We may assume by induction that for each proper subword  $v \sqsubset w$  the set  $\{T : \mu(\text{supp}(T_v)) = 1\}$  is comeager.

We first check that  $U_\varepsilon = \{T : \mu(\text{supp}(T_w)) > 1 - \varepsilon\}$  is open for each  $\varepsilon > 0$ . Fix  $T_0 \in U_\varepsilon$ . By Lemma 9 there is a partition  $Y_1, \dots, Y_N$  of  $\text{supp}((T_0)_w)$  into finitely many sets such that  $Y_n \cap (T_0)_w(Y_N) = \emptyset$  for each  $n \leq N$ . Set  $\mathcal{P} = \mathcal{P}(\{(T_0)_v(y_n) : v \sqsubset w \text{ and } n \leq N\})$ . Then, since each  $(T_0)_v$  is  $\mu$ -class preserving, for each  $\delta > 0$  there is some  $\varepsilon_\delta > 0$  such that whenever  $T$  satisfies  $\mu(T(P) \Delta T_0(P)) < \varepsilon_\delta$  for all  $P \in \mathcal{P}$  then  $\mu(Y_n \cap T_w(Y_n)) < \delta$ . That is, choosing  $\delta$  smaller than  $\mu(\text{supp}((T_0)_w)) - (1 - \varepsilon)$ , the set  $\{T : \forall P \in \mathcal{P} \mu(T(P) \Delta T_0(P)) < \varepsilon_\delta\}$  is a neighborhood of  $T_0$  contained in  $U_\varepsilon$ . Consequently, the set  $\{T : \mu(\text{supp}(T_w)) = 1\}$  is  $G_\delta$  in  $\text{Aut}(X, \mu)$ .

So all that remains is to check that the set  $\{T : \mu(\text{supp}(T_v)) = 1\}$  is dense. Fix an open set  $U$  and by the inductive assumption an automorphism  $T_0 \in U \cap \{T : \forall v \sqsubset w (\mu(\text{supp}(T_v)) = 1)\}$ . Without loss of generality, we may assume there is a finite partition  $\mathcal{P}$  and  $\varepsilon > 0$  such that  $U = \{T : \forall P \in \mathcal{P} (\mu(T(P) \Delta T_0(P)) < \varepsilon)\}$ . By Lemma 11, there is a  $\mu$ -preserving automorphism  $S$  factoring over  $\mathcal{P}$  such that  $(T \circ S)_v$  has conull support for all  $v \sqsubset w$ . Since  $T_0 \circ S(P) = T(P)$  for all  $P \in \mathcal{P}$ , we see  $T_0 \circ S \in U \cap \{T : \forall v \sqsubset w (\mu(\text{supp}(T_v)) = 1)\}$  as desired.  $\square$

Before generalizing the construction of  $G_{\mathbb{F}}$  in the previous section, we digress a bit to discuss costs. The relevant notions are defined in [5, III].

**Corollary 13.** *Suppose that  $E$  is a  $\mu$ -preserving, aperiodic, countable, Borel equivalence relation on a standard probability space  $(X, \mu)$ . Then there is a free,  $\mu$ -preserving action of  $F_\infty$  whose associated orbit equivalence relation is independent of  $E$ . Consequently, for all  $\alpha > 0$  there is a  $\mu$ -preserving, countable, Borel equivalence relation  $E_\alpha \supseteq E$  such that  $C_\mu(E_\alpha) = C_\mu(E) + \alpha$ . Moreover, if  $E$  is treeable then  $E_\alpha$  may be taken to be treeable.*

*Proof.* Apply Theorem 8 in succession to obtain a sequence of automorphisms  $T_0, T_1, \dots \in \text{Aut}(X, \mu)$  such that for each  $i$  we have  $E_{T_i} \perp E \vee \bigvee_{j < i} E_{T_j}$ . Then the set  $\{T_i : i \in \mathbb{N}\}$  generates a free  $F_\infty$  action whose associated equivalence relation is independent of  $E$ .

To obtain a superequivalence relation increasing the cost of  $E$  by  $\alpha$ , first write  $\alpha = n + \varepsilon$  for some  $n \in \mathbb{N}$  and  $0 \leq \varepsilon < 1$ . Fix a Borel set  $A_\varepsilon \subseteq X$  with  $\mu(A_\varepsilon) = \varepsilon$  and define  $T'_n$  by

$$T'_n(x) = \begin{cases} T_n(x) & \text{if } x \in A_\varepsilon, \\ x & \text{if } x \notin A_\varepsilon. \end{cases}$$

Then, by a fundamental result of Gaboriau's theory of costs (see, e.g., [5, Theorem 27.2]), the equivalence relation

$$E_\alpha = E \vee \bigvee_{i < n} E_{T_i} \vee E_{T'_n}$$

has cost  $C_\mu(E) + \alpha$ .

If  $E$  is treeable, it is evident that this  $E_\alpha$  is also treeable by simply adding to the treeing of  $E$  the (symmetrized) graphs of  $T_0, \dots, T_{n-1}, T'_n$ .  $\square$

Suppose now that we have a countable sequence of measure-preserving, Borel graphs  $(G_n)_{n \in \mathbb{N}}$  on standard probability spaces  $(X_n, \mu_n)$ , respectively (i.e., for each

$n \in \mathbb{N}$ ,  $\mu_n$  is  $E_{G_n}$ -invariant). We build another standard measure space  $(X, \mu)$  by setting  $X = \bigsqcup_{n \in \mathbb{N}} X_n$  and for Borel  $A \subseteq X$ ,

$$\mu(A) = \sum_{n \in \mathbb{N}} 2^{-(n+1)} \mu_n(A \cap X_n).$$

We then may define a Borel graph  $G_{\mathbb{N}}$  on  $X$  by

$$x G_{\mathbb{N}} y \Leftrightarrow \exists n \in \mathbb{N} (x, y \in X_n \text{ and } x G_n y).$$

Note then that  $\mu$  is  $E_{G_{\mathbb{N}}}$ -invariant.

By Theorem 8, the generic  $T \in \text{Aut}(X, \mu)$  satisfies  $E_T \perp E_{G_{\mathbb{N}}}$  (after, as usual, discarding a  $\mu$ -null set). We may fix such a  $T$  which is ergodic, since of course the generic element of  $\text{Aut}(X, \mu)$  is ergodic [4, Theorem 2.6]. The *ergodic amalgamation* of  $(G_n)$  by  $T$  is the Borel graph

$$G_{(G_n), T} = G_{\mathbb{N}} \cup G_T$$

on  $X$ . When the particular choice of  $T$  is unimportant, we say  $G$  is an *ergodic amalgamation* of  $(G_n)$  if  $G = G_{(G_n), T}$  for some  $T$  as above. Note of course that  $E_G$  is an ergodic equivalence relation. We collect here some useful facts about ergodic amalgamations.

**Proposition 14.** *Suppose that  $(G_n)_{n \in \mathbb{N}}$  is a sequence of Borel, measure-preserving graphs on  $(X_n, \mu_n)$ , respectively, and that  $G$  on  $(X, \mu)$  is an ergodic amalgamation of  $(G_n)$ .*

- (i) *If each  $G_n$  is acyclic, then  $G$  is acyclic. Moreover, the girth (the length of the shortest cycle) of  $G$  is equal to the least girth among the graphs  $G_n$ , and the clique number (the size of the largest set of vertices with all pairs adjacent) of  $G$  is equal to the largest clique number among the graphs  $G_n$ .*
- (ii) *If each  $G_n$  is locally finite, then  $G$  is locally finite. Moreover, if every  $G_n$  has degree bounded by a fixed  $d$ , then  $G$  has degree bounded by  $d + 2$ .*
- (iii) *The Borel chromatic number of  $G$ ,  $\chi_B(G)$ , is bounded below by  $\sup_n \chi_B(G_n)$  and above by  $3 \sup_n \chi_B(G_n)$ .*
- (iv) *The measurable chromatic number of  $G$ ,  $\chi_\mu(G)$ , is bounded below by  $\sup_n \chi_{\mu_n}(G_n)$  and above by  $3 \sup_n \chi_{\mu_n}(G_n)$ .*

In particular, if  $G$  is an ergodic amalgamation of some sequence  $(G_n)_{n \in \mathbb{N}}$  of graphs with measurable chromatic numbers tending towards infinity while each having bounded degree, and  $f$  is any Borel function on a Polish space such that  $G_f$  has infinite Borel chromatic number, the arguments in Proposition 1 and 3 ensure that there is no Borel homomorphism from  $G$  to  $G_f$  nor from  $G_f$  to  $G$ .

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