ON THE EXISTENCE OF LARGE ANTICHAINS FOR DEFINABLE QUASI-ORDERS

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Abstract. We simultaneously generalize Silver's perfect set theorem for co-analytic equivalence relations and Harrington-Marker-Shelah's Dilworth-style perfect set theorem for Borel quasi-orders, establish the analogous theorem at the next definable cardinal, and give further generalizations under weaker definability conditions.

A *quasi-order* is a reflexive transitive binary relation. Associated with every such relation R on a set X are the equivalence relation $x \equiv_R y \iff (x \mathrel{R} y \text{ and } y \mathrel{R} x)$ and the incomparability relation $x \perp_R y \iff (\neg x \mathrel{R} y \text{ and } \neg y \mathrel{R} x)$. We say that a set $Y \subseteq X$ is an *R-antichain* if $R \restriction Y$ is the diagonal on Y, and an *R-chain* if $\perp_R \restriction Y$ is empty. We say that a subset of a topological space X is *Borel* if it is in the σ-algebra generated by the open sets, *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, *co-analytic* if its complement is analytic, and $\aleph_0$-*universally Baire* if its pre-image under every continuous function $\phi \colon 2^{\mathbb{N}} \to X$ has the Baire property.

In §1, we simultanously generalize the main result of [Sil80] and [HMS88, Theorem 5.1]:

Theorem 1. *Suppose that X is a Hausdorff space and R is an $\aleph_0$-universally Baire quasi-order on X for which $\perp_R$ is analytic. Then exactly one of the following holds:*

1. *The space X is a union of countably-many Borel R-chains.*
2. *There is a continuous injection of $2^{\mathbb{N}}$ into an R-antichain.*

Our proof uses only Baire category arguments and the $\mathbb{G}_0$ dichotomy (see [KST99, Theorem 6.3]), which itself has a classical proof (see

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[Mil12]). An interesting wrinkle is that, while such arguments typically utilize just one application of the $\mathbb{G}_0$ dichotomy, ours requires infinitely many.

A *homomorphism* from a binary relation R on X to a binary relation S on Y is a function $\phi\colon X \to Y$ such that $(\phi \times \phi)(R) \subseteq S$, and a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$. A Borel equivalence relation E on an analytic Hausdorff space X is *smooth* if there is a Borel-measurable reduction of E to equality on $2^{\mathbb{N}}$, and an analytic set $A \subseteq X$ is *E-smooth* if $E \restriction A$ is smooth. In §2, we establish the analog of [HMS88, Theorem 5.1] at the next Borel cardinal:

Theorem 2. *Suppose that X is an analytic Hausdorff space and R is an $\aleph_0$-universally Baire quasi-order on X for which $\equiv_R$ is Borel. Then exactly one of the following holds:*

1. *There is a smooth Borel superequivalence relation of $\equiv_R$ whose equivalence classes are R-chains.*
2. *There is an $\equiv_R$-non-smooth compact set whose quotient by $\equiv_R$ is an $(R/{\equiv_R})$-antichain.*

Our proof uses only Baire category arguments and the $\mathbb{E}_0$ dichotomy (see [HKL90, Theorem 1.1]), which itself has a classical proof (see [Mil12]), and reveals that the theorem holds for the simple reason that its two alternatives are equivalent to those of the $\mathbb{E}_0$ dichotomy (for $\equiv_R$).

Although dichotomy results such as Theorems 1 and 2 are typically stated for Polish spaces, we instead work with Hausdorff spaces, as this reflects the natural generality in which our arguments go through. It is well known, however, that the special cases of such theorems for $\mathbb{N}^{\mathbb{N}}$ yield their generalizations to analytic Hausdorff spaces (and, in most cases, to Hausdorff spaces). This is because the structure involved can be pulled back to $\mathbb{N}^{\mathbb{N}}$ through a witness to analyticity, where the special case of the dichotomy can be applied, and the witness to the theorem pushed forward to the original space. In the simpler case of the dichotomy, the latter will not necessarily be Borel, but Lusin's separation theorem (see, for example, [Kec95, Theorem 14.7], noting that the same argument goes through in Hausdorff spaces) can be used to turn it into a Borel witness.

A subset of a topological space is *κ-Borel* if it is in the κ-complete algebra generated by the open sets, *κ-Souslin* if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$, *co-κ-Souslin* if its complement is

κ-Souslin, and *bi-κ-Souslin* if it is both κ-Souslin and co-κ-Souslin. An *embedding* is an injective reduction. We adopt the convention that $\kappa < \infty$ for every aleph κ, and we use $\mathtt{add}(\mathcal{M})$ to denote the least aleph κ for which there is a sequence $(M_\alpha)_{\alpha<\kappa}$ of meager subsets of $2^{\mathbb{N}}$ whose union is not meager, or ∞ if no such aleph exists. When $\kappa < \mathtt{add}(\mathcal{M})$, we say that a κ^+-Borel equivalence relation E on an analytic Hausdorff space X is *smooth* if there is a κ^+-Borel-measurable reduction of E to equality on 2^κ, and an analytic set $A \subseteq X$ is *E-smooth* if $E \upharpoonright A$ is κ-smooth. In §3, we note that our arguments can be combined with those of [Kan97] and [Mil12] to obtain the following generalizations of Theorems 1 and 2, in the spirit of [HSh82]:

THEOREM 3. *Suppose that $\kappa < \mathtt{add}(\mathcal{M})$ is an aleph, X is an analytic Hausdorff space, and R is an $\aleph_0$-universally-Baire quasi-order on X for which $\perp_R$ is κ-Souslin. Then exactly one of the following holds:*

1. *The space X is a union of κ-many κ^+-Borel R-chains.*
2. *There is a continuous injection of $2^{\mathbb{N}}$ into an R-antichain.*

THEOREM 4. *Suppose that $\kappa < \mathtt{add}(\mathcal{M})$ is an aleph, X is an analytic Hausdorff space, and R is an $\aleph_0$-universally-Baire quasi-order on X for which $\equiv_R$ is bi-κ-Souslin. Then exactly one of the following holds:*

1. *There is a smooth κ^+-Borel superequivalence relation of $\equiv_R$ whose equivalence classes are R-chains.*
2. *There is an $\equiv_R$-non-smooth compact set whose quotient by $\equiv_R$ is an $(R/{\equiv_R})$-antichain.*

A subset of an analytic Hausdorff space is $\utilde{\mathbf{\Sigma}}^1_1$ if it is analytic, $\utilde{\mathbf{\Pi}}^1_n$ if its complement is $\utilde{\mathbf{\Sigma}}^1_n$, $\utilde{\mathbf{\Sigma}}^1_{n+1}$ if it is a continuous image of a $\utilde{\mathbf{\Pi}}^1_n$ set, and $\utilde{\mathbf{\Delta}}^1_n$ if it is both $\utilde{\mathbf{\Pi}}^1_n$ and $\utilde{\mathbf{\Sigma}}^1_n$. Let $\utilde{\boldsymbol{\delta}}^1_n$ denote the supremum of the lengths of all $\utilde{\mathbf{\Delta}}^1_n$ pre-wellorderings of $\mathbb{N}^{\mathbb{N}}$. When $\mathtt{AD}$ holds, we say that a $\utilde{\mathbf{\Delta}}^1_{2n+1}$ equivalence relation E on an analytic Hausdorff space X is *smooth* if there exists $\kappa < \utilde{\boldsymbol{\delta}}^1_{2n+1}$ for which there is a $\utilde{\mathbf{\Delta}}^1_{2n+1}$-measurable reduction of E to equality on 2^κ, and an analytic set $A \subseteq X$ is *E-smooth* if $E \upharpoonright A$ is smooth. Taking the known structure theory of the projective sets as a black box, we note that our arguments also provide classical proofs of the relevant special cases of Theorems 3 and 4 necessary to obtain the following generalizations of Theorems 1 and 2, in the spirit of [HSa79]:

THEOREM 5 (AD). *Suppose that $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a quasi-order on X for which $\perp_R$ is $\utilde{\mathbf{\Sigma}}^1_{2n+1}$. Then exactly one of the following holds:*

1. *The space X is a union of $(< \utilde{\boldsymbol{\delta}}^1_{2n+1})$-many $\utilde{\mathbf{\Delta}}^1_{2n+1}$ R-chains.*
2. *There is a continuous injection of $2^{\mathbb{N}}$ into an R-antichain.*

THEOREM 6 (AD). *Suppose that $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a quasi-order on X for which $\equiv_R$ is $\utilde{\mathbf{\Delta}}^1_{2n+1}$. Then exactly one of the following holds:*

1. *There is a smooth $\utilde{\mathbf{\Delta}}^1_{2n+1}$ superequivalence relation of $\equiv_R$ whose equivalence classes are R-chains.*
2. *There is an $\equiv_R$-non-smooth compact set whose quotient by $\equiv_R$ is an $(R/\equiv_R)$-antichain.*

We work in $\mathtt{ZF} + \mathtt{DC}$ throughout the paper.

§1. Perfect antichains. For each discrete set D and sequence $s \in D^{<\mathbb{N}}$, we use $\mathcal{N}_s$ to denote the basic open set consisting of all extensions of s in $D^{\mathbb{N}}$. We use the notation $\forall^* x \in X\ P(x)$ to indicate that $\{x \in X \mid \neg P(x)\}$ is meager, and $\exists^* x \in X\ P(x)$ to indicate that $\{x \in X \mid P(x)\}$ is non-meager. Following standard convention, we use $\mathbb{E}_0$ to denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathrel{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n\ c(m) = d(m)$. Fix sequences $\mathbb{s}_n \in 2^n$ such that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq \mathbb{s}_n$, and define $\mathbb{G}_0 = \bigcup_{n \in \mathbb{N}} \{(\mathbb{s}_n \frown (i) \frown c, \mathbb{s}_n \frown (1-i) \frown c) \mid c \in 2^{\mathbb{N}} \text{ and } i < 2\}$. While our proof of the characterization of the existence of a continuous injection of $2^{\mathbb{N}}$ into an antichain requires infinitely-many applications of the $\mathbb{G}_0$ dichotomy, we need only one to establish the following:

THEOREM 7. *Suppose that X is a Hausdorff space, R is an $\aleph_0$-universally Baire quasi-order on X for which $\perp_R$ is analytic, and X is not a union of countably-many Borel R-chains. Then there are compact sets $K_i \subseteq X$ that are not unions of countably-many Borel R-chains such that $\prod_{i<2} K_i \subseteq \perp_R$.*

PROOF. As $\perp_R$ is analytic and X is not a union of countably-many Borel R-chains, the $\mathbb{G}_0$ dichotomy yields a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $\perp_R$. As the set $R_0 = (\phi \times \phi)^{-1}(R)$ has the Baire property, so too does $\perp_{R_0}$.

LEMMA 8. *The relation $\perp_{R_0}$ is non-meager.*

PROOF. Suppose, towards a contradiction, that $\perp_{R_0}$ is meager, and fix non-empty open sets $U_i \subseteq 2^{\mathbb{N}}$ for which R_0 is comeager in $\prod_{i<2} U_i$ (see, for example, [Kec95, Proposition 8.26]). The Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) ensures that the sets $C_0 = \{c \in 2^{\mathbb{N}} \mid \exists^* d \in U_0 \ c \ R_0 \ d\}$ and $C_1 = \{d \in 2^{\mathbb{N}} \mid \forall^* c \in U_0 \ c \ R_0 \ d\}$ have comeager union, and [Kec95, Theorem 16.1] and the Kuratowski-Ulam theorem imply that they have the Baire property. The Kuratowski-Ulam theorem also ensures that C_0 is non-meager, since otherwise $\forall^* c, d \in U_0$ ($\neg c \ R_0 \ d$ and $d \ R_0 \ c$), and C_1 is non-meager. As the $\mathbb{E}_0$-saturation of every non-meager set with the Baire property is comeager (see, for example, [Kec95, Theorem 8.47]), there are comeagerly-many $c \in 2^{\mathbb{N}}$ for which the sets $C_i \cap [c]_{\mathbb{E}_0}$ are each non-empty and together cover $[c]_{\mathbb{E}_0}$. As a straightforward induction reveals that $\mathbb{E}_0$ is the equivalence relation generated by $\mathbb{G}_0$, it follows that $(\prod_{i<2} C_i) \cap \mathbb{G}_0 \neq \emptyset$, contradicting the fact that $\prod_{i<2} C_i \subseteq R_0$. ⊠

LEMMA 9. *There are continuous homomorphisms* $\phi_i \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *from* $\mathbb{G}_0$ *to itself for which* $\prod_{i<2} \phi_i(2^{\mathbb{N}}) \subseteq \perp_{R_0}$.

PROOF. By Lemma 8, there are non-empty open sets $U_i \subseteq 2^{\mathbb{N}}$ and dense open sets $V_n \subseteq \prod_{i<2} U_i$ such that $\bigcap_{n \in \mathbb{N}} V_n \subseteq \perp_{R_0}$. Recursively construct $u_{i,n} \in 2^{<\mathbb{N}}$ and $k_{i,n} \in \mathbb{N}$ such that $\prod_{i<2} \mathcal{N}_{\phi_{i,n}(t_i)} \subseteq V_n$ for all $t_0, t_1 \in 2^n$ and $\phi_{i,n}(\mathbb{s}_n) = \mathbb{s}_{k_{i,n}}$ for all $i < 2$, where $\phi_{i,n} \colon 2^n \to 2^{<\mathbb{N}}$ is given by $\phi_{i,n}(t) = u_{i,0} \frown \bigoplus_{m<n} (t(m)) \frown u_{i,m+1}$. Then the functions $\phi_i \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\phi_i(c) = \bigcup_{n \in \mathbb{N}} \phi_{i,n}(c \upharpoonright n)$ are as desired. ⊠

It only remains to observe that if the functions ϕ_i are as in Lemma 9, then the sets $K_i = (\phi \circ \phi_i)(2^{\mathbb{N}})$ are as desired. ⊠

We now establish our characterization of the existence of a continuous injection of $2^{\mathbb{N}}$ into an antichain:

PROOF OF THEOREM 1. Conditions (1) and (2) are clearly mutually exclusive. To see $\neg(1) \Longrightarrow (2)$, note that if condition (1) fails, then $\mathrm{proj}_X(\perp_R)$ is not a union of countably-many Borel $(R \upharpoonright \mathrm{proj}_X(\perp_R))$-chains, fix a continuous surjection $\phi \colon \mathbb{N}^{\mathbb{N}} \to \mathrm{proj}_X(\perp_R)$, and recursively appeal to Theorem 7 to obtain functions $\psi_n \colon 2^n \to \mathbb{N}^n$ and sequences $(F_s)_{s \in 2^n}$ of closed subsets of $\mathrm{proj}_X(\perp_R)$ with the following properties:

1. $\forall s \in 2^n \ F_s$ is not a countable union of Borel $(R \upharpoonright F_s)$-chains.
2. $\forall s \in 2^n \ F_s \subseteq \phi(\mathcal{N}_{\psi_n(s)})$.
3. $\forall s \in 2^n \ F_{s \frown (0)} \cup F_{s \frown (1)} \subseteq F_s$.

4. $\forall s \in 2^n \ F_{s \frown (0)} \times F_{s \frown (1)} \subseteq \perp_R$.
5. $\forall i < 2 \forall s \in 2^n \ \psi_n(s) \sqsubseteq \psi_{n+1}(s \frown (i))$.

Define $\psi \colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $\psi(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \upharpoonright n)$, as well as $\pi = \phi \circ \psi$, noting that $\pi(c) \in \bigcap_{n \in \mathbb{N}} F_{c \upharpoonright n}$ for all $c \in 2^{\mathbb{N}}$. To see that π is the desired injection, observe that if $c, d \in 2^{\mathbb{N}}$ are distinct, then there is a maximal natural number $n \in \mathbb{N}$ for which $c \upharpoonright n = d \upharpoonright n$, so the fact that $\pi(c) \in F_{s \frown (c(n))}$ and $\pi(d) \in F_{s \frown (d(n))}$ ensures that $\pi(c) \perp_R \pi(d)$. ⊠

§2. Non-smooth antichains. We now establish our characterization of the existence of a non-smooth compact set whose quotient is an antichain:

Proof of Theorem 2. To see that conditions (1) and (2) are mutually exclusive, note that if E is a Borel superequivalence relation of $\equiv_R$ whose classes are R-chains, and $A \subseteq X$ is a set whose quotient by $\equiv_R$ is an $(R/\equiv_R)$-antichain, then $\equiv_R \upharpoonright A = E \upharpoonright A$. When A is analytic, it follows that if E is smooth, then so too is $\equiv_R \upharpoonright A$.

To see $\neg(1) \Longrightarrow (2)$, note that if (1) fails, then $\equiv_R$ is non-smooth, so the $\mathbb{E}_0$ dichotomy yields a continuous embedding $\phi \colon 2^{\mathbb{N}} \to X$ of $\mathbb{E}_0$ into $\equiv_R$. As the set $R_0 = (\phi \times \phi)^{-1}(R)$ has the Baire property, so too does $\perp_{R_0}$.

Lemma 10. *The relation* $\perp_{R_0}$ *is comeager.*

Proof. If there exist $n \in \mathbb{N}$ and $s, t \in 2^n$ for which R_0 is comeager in $\mathcal{N}_s \times \mathcal{N}_t$, then the fact that $\mathbb{E}_0 \subseteq R_0$ ensures that R_0 is comeager in $\mathcal{N}_{s'} \times \mathcal{N}_{t'}$ for all $s', t' \in 2^n$, and therefore comeager, thus so too is $\equiv_{R_0}$, contradicting the fact that the latter set is $\mathbb{E}_0$. ⊠

Lemma 11. *There is a continuous embedding* $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *of* $\mathbb{E}_0$ *into itself that is also a homomorphism from* $\sim\mathbb{E}_0$ *to* $\perp_{R_0}$.

Proof. By Lemma 10, there are dense open sets $U_n \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $\bigcap_{n \in \mathbb{N}} U_n \subseteq \perp_{R_0}$. We can clearly assume that these sets are decreasing and disjoint from the diagonal. Recursively construct $u_{i,n} \in 2^{<\mathbb{N}}$ such that $|u_{0,n}| = |u_{1,n}|$ and $\prod_{i<2} \mathcal{N}_{\psi_{n+1}(t_i \frown (i))} \subseteq U_n$ for all $t_0, t_1 \in 2^n$, where $\psi_{n+1} \colon 2^{n+1} \to 2^{<\mathbb{N}}$ is given by $\psi_{n+1}(t) = \bigoplus_{m \leq n}(t(m)) \frown u_{t(m),m}$. Then the map $\psi \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ given by $\psi(c) = \bigcup_{n \in \mathbb{N}} \psi_n(c \upharpoonright n)$ is as desired. ⊠

It only remains to observe that if the function ψ is as in Lemma 11, then the set $(\phi \circ \psi)(2^{\mathbb{N}})$ is as desired. ⊠

§3. Generalizations. Kanovei has generalized the $\mathbb{G}_0$ dichotomy to κ-Souslin graphs on analytic Hausdorff spaces (see [Kan97]), and the ideas underlying his arguments yield an analogous generalization of the $(\mathbb{G}_0, \mathbb{H}_0)$ dichotomy, which, in turn, can be used to obtain an analogous generalization of the $\mathbb{E}_0$ dichotomy to bi-κ-Souslin equivalence relations on analytic Hausdorff spaces (see [Mil12]). By using these facts in lieu of the usual dichotomies in our proofs of Theorems 1 and 2, we obtain proofs of Theorems 3 and 4.

If $\mathtt{AD}$ holds and $n \in \mathbb{N}$, then a subset of an analytic Hausdorff space is $\utilde{\boldsymbol{\Delta}}^1_{2n+1}$ if and only if it is $\utilde{\boldsymbol{\delta}}^1_{2n+1}$-Borel (see [Mar70, Mos71]), and there is a cardinal $\utilde{\boldsymbol{\kappa}}^1_{2n+1}$ for which $\utilde{\boldsymbol{\delta}}^1_{2n+1} = (\utilde{\boldsymbol{\kappa}}^1_{2n+1})^+$ (see [Kec74]) and a subset of an analytic Hausdorff space is $\utilde{\boldsymbol{\Sigma}}^1_{2n+1}$ if and only if it is $\utilde{\boldsymbol{\kappa}}^1_{2n+1}$-Souslin (see, for example, [Jac08, Theorem 2.21]). It follows that continuous images of $\utilde{\boldsymbol{\delta}}^1_{2n+1}$-Borel sets are $\utilde{\boldsymbol{\kappa}}^1_{2n+1}$-Souslin, a fact which alone ensures that the classical proofs of the $\mathbb{G}_0$ and $\mathbb{E}_0$ dichotomies yield the special cases of the Kanovei-style generalizations thereof at $\utilde{\boldsymbol{\kappa}}^1_{2n+1}$. By using these in lieu of the usual dichotomies in our proofs of Theorems 1 and 2, we obtain proofs of the Kanovei-style strengthenings of Theorems 3 and 4 at $\utilde{\boldsymbol{\kappa}}^1_{2n+1}$. As $\mathtt{AD}$ also ensures that every subset of a topological space is $\aleph_0$-universally Baire (see, for example, [Kec95, Theorem 38.17]), Theorems 5 and 6 easily follow.

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