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# INCOMPARABLE TREEABLE EQUIVALENCE RELATIONS

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We establish Hjorth's theorem that there is a family of continuum-many pairwise strongly incomparable free actions of free groups, and therefore a family of continuum-many pairwise incomparable treeable equivalence relations.

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## Prologue

The mathematical community was recently shocked to learn of the passing of Greg Hjorth, whose research over the last two decades has been an inspiration for so many. Much work was left unfinished, including his striking recent results on the existence of large families of pairwise incomparable treeable equivalence relations. Although he produced a preprint before his death, this left the quandary of how to publish these results, which we resolve by including his original preprint alongside this companion paper.

Our goal here is to explicitly isolate the essential ideas underlying Hjorth's arguments and establish them in their natural generality, while simultaneously discarding unnecessary machinery concerning amenability, almost invariant sets and vectors, and the theory of costs. We refer the reader to the original preprint for the history and motivation behind the results.

In §1, we establish a pair of basic facts connecting ergodicity with increasing unions and independence. In §2, we establish a connection between increasing unions and strong ergodicity. In §3, we introduce a notion of local rigidity for group actions, and establish the existence of locally rigid actions. In §4, we establish a strong separability property for orbit equivalence relations of such actions. In §5, we introduce a notion of stratification for aperiodic equivalence relations. In §6, we establish Hjorth's theorem. We close with a brief summary of related results.

## 1. Ergodicity

Suppose that $X$ is a standard Borel space. A *countable Borel equivalence relation* on $X$ is an equivalence relation on $X$ whose classes are all countable, and which is Borel when viewed as a subset of $X^2$. Suppose that $E$ is such an equivalence relation. A set is *$E$-invariant* if it is the union of a set of $E$-classes. The *$E$-saturation* of a set $Y \subseteq X$ is the smallest $E$-invariant set $[Y]_E$ containing $Y$. The uniformization theorem for Borel subsets of the plane with countable vertical sections (see, for example, Theorem 18.10 of [1]) ensures that $E$-saturations of Borel sets are Borel.

A Borel measure $\mu$ on $X$ is *$E$-ergodic* if every $E$-invariant Borel set is $\mu$-null or $\mu$-conull. Note that if $C \subseteq X$ is a $\mu$-conull Borel set, then $\mu$ is $E$-ergodic if and only if $\mu \restriction C$ is $(E \restriction C)$-ergodic. A Borel measure $\mu$ on $X$ is *$E$-quasi-invariant* if $E$-saturations of $\mu$-null sets are $\mu$-null. By Remark 10.3 of [2], there is always a $\mu$-conull Borel set $C \subseteq X$ for which $\mu \restriction C$ is $(E \restriction C)$-quasi-invariant.

**Proposition 1.1.** *Suppose that $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of countable Borel equivalence relations on $X$ whose union is $E$, $\mu$ is an $E$-ergodic Borel probability measure on $X$, and for some $k \in \mathbb{N}$ there is a cover $(B_i)_{i<k}$ of $X$ by $\mu$-positive Borel sets with the property that $\mu \restriction B_i$ is $(E_0 \restriction B_i)$-ergodic for all $i < k$. Then $\mu$ is $E_n$-ergodic for all sufficiently large $n \in \mathbb{N}$.*

**Proof.** By the above remarks, we can assume that $\mu$ is $E$-quasi-invariant. As $\mu$ is $E$-ergodic, there exists $n \in \mathbb{N}$ such that $[B_i]_{E_n} \cap B_j$ is $\mu$-positive for all $i, j < k$. To see that $\mu$ is $E_n$-ergodic, suppose that $B \subseteq X$ is an $E_n$-invariant $\mu$-positive Borel set, and fix $i < k$ such that $B \cap B_i$ is $\mu$-positive. Then $\mu(B \cap B_i) = \mu(B_i)$ by the $(E_0 \restriction B_i)$-ergodicity of $\mu \restriction B_i$. Given $j < k$, our choice of $n$ ensures that $[B_i]_{E_n} \cap B_j$ is $\mu$-positive, so the $E$-quasi-invariance of $\mu$ implies that $[B]_{E_n} \cap B_j$ is $\mu$-positive. As $B$ is $E_n$-invariant, it follows that $B \cap B_j$ is $\mu$-positive, so $\mu(B \cap B_j) = \mu(B_j)$ by the $(E_0 \restriction B_j)$-ergodicity of $\mu \restriction B_j$, thus $B$ is $\mu$-conull. □

The *join* of countable Borel equivalence relations $E$ and $F$ on $X$ is the smallest equivalence relation containing both $E$ and $F$. The uniformization theorem for Borel sets with countable vertical sections ensures that the join of $E$ and $F$ is also a countable Borel equivalence relation. We say that $E$ and $F$ are *independent* if for all positive integers $n$ and all sequences $(x_i)_{i\leq 2n}$ with $x_0 = x_{2n}$ and $x_{2i} \; E \; x_{2i+1} \; F \; x_{2i+2}$ for all $i < n$, there exists $i < 2n$ such that $x_i = x_{i+1}$. When this is the case, we use $E * F$ to denote the join of $E$ and $F$.

Let $D(E, F)$ denote the set of all $x \in X$ for which $[x]_E \neq [x]_F$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that $D(E, F)$ is Borel.

**Proposition 1.2.** *Suppose that $X$ is a standard Borel space, $E$ and $F$ are independent countable Borel equivalence relations on $X$, $F' \subseteq F$ is a countable Borel equivalence relation on $X$, and $\mu$ is an $E$-ergodic $(E * F)$-quasi-invariant Borel probability* *measure on $X$ for which $D(F, F')$ is $\mu$-positive. Then $D((E * F) \restriction B, (E * F') \restriction B)$ is $(\mu \restriction B)$-conull for all $\mu$-positive Borel sets $B \subseteq X$.*

**Proof.** Suppose, towards a contradiction, that there is a $\mu$-positive Borel set $B \subseteq X$ for which the set $A = B \setminus D((E * F) \restriction B, (E * F') \restriction B)$ is not $\mu$-null. Then $(E * F) \restriction [A]_E = (E * F') \restriction [A]_E$. As the $E$-ergodicity of $\mu$ ensures that the set $N = X \setminus [A]_E$ is $\mu$-null, the $(E * F)$-quasi-invariance of $\mu$ implies that the set $C = X \setminus [N]_{E*F}$ is $\mu$-conull. As $C$ is an $(E * F)$-invariant subset of $[A]_E$ and $E$ and $F$ are independent, it follows that $C \cap D(F, F') = \emptyset$, contradicting our assumption that $D(F, F')$ is $\mu$-positive. □

## 2. Strong ergodicity

Suppose that $X$ and $Y$ are standard Borel spaces and $E$ and $F$ are countable Borel equivalence relations on $X$ and $Y$. A *homomorphism* from $E$ to $F$ is a function $\phi\colon X \to Y$ sending $E$-equivalent points to $F$-equivalent points. A Borel measure $\mu$ on $X$ is $(E, F)$-*anti-ergodic* if there is a Borel homomorphism $\phi\colon X \to Y$ from $E$ to $F$ with the property that $\phi^{-1}(y)$ is $\mu$-null for all $y \in Y$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that if $C \subseteq X$ is a $\mu$-conull Borel set, then $\mu$ is $(E, F)$-anti-ergodic if and only if $\mu \restriction C$ is $(E \restriction C, F)$-anti-ergodic. We use $\mathbb{E}_0$ to denote the equivalence relation on $2^{\mathbb{N}}$ given by $x \mathbb{E}_0 y \iff \exists n \in \mathbb{N} \forall m \geq n\ x(m) = y(m)$.

**Proposition 2.1.** *Suppose that $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of countable Borel equivalence relations on $X$ whose union is $E$, and $\mu$ is a Borel probability measure on $X$ which is $(E_n, \mathbb{E}_0)$-anti-ergodic for all $n \in \mathbb{N}$. Then $\mu$ is $(E, \mathbb{E}_0)$-anti-ergodic.*

**Proof.** For each $n \in \mathbb{N}$, fix a Borel homomorphism $\phi_n\colon X \to 2^{\mathbb{N}}$ from $E_n$ to $\mathbb{E}_0$ with the property that $\phi_n^{-1}(z)$ is $\mu$-null for all $z \in 2^{\mathbb{N}}$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $f_{k,n}\colon X \to X$ such that $E_n = \bigcup_{k \in \mathbb{N}} \operatorname{graph}(f_{k,n})$.

Fix $\epsilon_n > 0$ with $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. For each $n \in \mathbb{N}$, fix $i_n \in \mathbb{N}$ such that the set

$$A_n = \{x \in X \mid \forall k, \ell \leq n\ \phi_n(x) \restriction [i_n, \infty) = \phi_n(f_{k,\ell}(x)) \restriction [i_n, \infty)\}$$

has $\mu$-measure strictly greater than $1 - \epsilon_n$, and fix $j_n \in \mathbb{N}$ such that the set

$$B_n = \{x \in A_n \mid \mu(\{y \in X \mid \phi_n(x) \restriction [i_n, j_n] = \phi_n(y) \restriction [i_n, j_n]\}) \leq 1/n\}$$

has $\mu$-measure at least $1 - \epsilon_n$. Then the set $B = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} B_m$ is $\mu$-conull.

Put $s_n(x) = \phi_n(x) \restriction [i_n, j_n]$ and define $\phi\colon B \to 2^{\mathbb{N}}$ by $\phi(x) = \bigoplus_{n \in \mathbb{N}} s_n(x)$. Then $\phi$ is a homomorphism from $E \restriction B$ to $\mathbb{E}_0$, since if $x\ (E \restriction B)\ y$, then there exist $n \in \mathbb{N}$ and $k, \ell \leq n$ with $x \in \bigcap_{m \geq n} B_m$ and $y = f_{k,\ell}(x)$, so $s_m(x) = s_m(f_{k,\ell}(x)) = s_m(y)$ for all $m \geq n$, thus $\phi(x)\ \mathbb{E}_0\ \phi(y)$. Moreover, if $x \in B$ and $\phi(x) = z$, then for all $\epsilon > 0$ there exists $n \geq 1/\epsilon$ such that $x \in B_n$. Setting $z_n = s_n(x)$, it follows that

$\mu(\phi^{-1}(z)) \leq \mu(s_n^{-1}(z_n)) \leq 1/n \leq \epsilon$, so $\phi^{-1}(z)$ is $\mu$-null, thus $\mu \restriction B$ is $(E \restriction B, \mathbb{E}_0)$-anti-ergodic, hence $\mu$ is $(E, \mathbb{E}_0)$-anti-ergodic. □

A Borel measure $\mu$ on $X$ is *$(E, F)$-ergodic* if for every Borel homomorphism $\phi\colon X \to Y$ from $E$ to $F$, there exists $y \in Y$ such that $\phi^{-1}([y]_F)$ is $\mu$-conull. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that if $\mu$ is $(E, F)$-ergodic, then $\mu \restriction B$ is $(E \restriction B, F)$-ergodic for all $\mu$-positive Borel sets $B \subseteq X$.

**Proposition 2.2.** *Suppose that $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of countable Borel equivalence relations on $X$ whose union is $E$, and $\mu$ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure on $X$. Then for all $\epsilon > 0$, there is a Borel set $B \subseteq X$ such that $\mu(B) > 1 - \epsilon$ and $\mu \restriction B$ is $(E_n \restriction B, \mathbb{E}_0)$-ergodic for all sufficiently large $n \in \mathbb{N}$.*

**Proof.** We establish first the weaker conclusion that there is a $\mu$-positive Borel set such that $\mu \restriction B$ is $(E_n \restriction B)$-ergodic for some $n \in \mathbb{N}$. Suppose, towards a contradiction, that this is not the case. Then a standard recursive construction yields $E_n$-invariant Borel sets $B_n \subseteq X$ of $\mu$-measure $1/2$ which are *$\mu$-independent* in the sense that for all $n \in \mathbb{N}$, the Boolean algebra generated by $\{B_i \mid i < n\}$ has exactly $2^n$ atoms, each having $\mu$-measure $1/2^n$. Then the function $\phi\colon X \to 2^{\mathbb{N}}$ given by $\phi(x)(n) = \chi_{B_n}(x)$ is a Borel homomorphism from $E$ to $\mathbb{E}_0$ with the property that $\phi^{-1}(z)$ is $\mu$-null for all $z \in 2^{\mathbb{N}}$, contradicting the $(E, \mathbb{E}_0)$-ergodicity of $\mu$.

By repeated application of this weaker conclusion, we obtain a partition of $X$ into a family $\mathscr{A}$ of $\mu$-positive Borel sets such that for all $A \in \mathscr{A}$, there exists $n \in \mathbb{N}$ for which $\mu \restriction A$ is $(E_n \restriction A)$-ergodic. Then for all $\epsilon > 0$, there is a finite set $\mathscr{B} \subseteq \mathscr{A}$ such that $\mu(\bigcup \mathscr{B}) > 1 - \epsilon$. Set $B = \bigcup \mathscr{B}$, and observe that Proposition 1.1 ensures the existence of $n \in \mathbb{N}$ for which $\mu \restriction B$ is $(E_n \restriction B)$-ergodic.

Suppose, towards a contradiction, that for no $m \geq n$ is it the case that $\mu \restriction B$ is $(E_m \restriction B, \mathbb{E}_0)$-ergodic. As $\mu \restriction B$ is $(E_n \restriction B)$-ergodic, it is $(E_m \restriction B, \mathbb{E}_0)$-anti-ergodic for all $m \geq n$. Proposition 2.1 therefore implies that $\mu \restriction B$ is $(E \restriction B, \mathbb{E}_0)$-anti-ergodic, contradicting the fact that $\mu$ is $(E, \mathbb{E}_0)$-ergodic. □

## 3. Local rigidity

We use $\Gamma \curvearrowright X$ to denote an action of a group $\Gamma$ on a set $X$, and $\mathrm{Stab}_\Gamma(x)$ to denote the stabilizer of $x$ under $\Gamma \curvearrowright X$. Let $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ denote the group of transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $T(x) = Ax + b$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$, and let $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ denote the corresponding action. Let $\mathbb{T}$ denote the space of all infinite rays through $\mathbb{R}^2$ emanating from the origin, let $\mathbb{T}^2$ denote $\mathbb{R}^2/\mathbb{Z}^2$, and let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ denote the actions induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

**Proposition 3.1.** *Suppose that $\theta \in \mathbb{T}$. Then $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$ is cyclic.*

**Proof.** We consider first the case that $\theta$ goes through a point of $\mathbb{Z}^2$. Let $v$ denote the unique such point of minimal magnitude. If $A \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$, then $A^{\pm 1}v \in \mathbb{Z}^2$, so minimality ensures $Av = v$, thus $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(v) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$. Minimality also implies that the coordinates of $v$ are relatively prime, so there exists $a \in \mathbb{Z}^2$ with $a \cdot v = 1$. Set $A = \left(\begin{smallmatrix} a_1 & a_2 \\ -v_2 & v_1 \end{smallmatrix}\right)$. Then $A \in \mathrm{SL}_2(\mathbb{Z})$ and $Av = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, so conjugation by $A$ yields an isomorphism of $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(v)$ with $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. But the latter group consists exactly of the matrices $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for $n \in \mathbb{Z}$, and is therefore isomorphic to $\mathbb{Z}$.

We consider now the case that $\theta$ does not go through a point of $\mathbb{Z}^2$. Fix $v \in \theta$, noting that $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(v) = \{I\}$. Observe that if $A \in \mathrm{SL}_2(\mathbb{Z})$, $\lambda > 0$, and $Av = \lambda v$, then the fact that $\det(A) = 1$ ensures that $1/\lambda$ is the other eigenvalue of $A$, so $\mathrm{trace}(A) = \lambda + 1/\lambda$. As $\mathrm{trace}(A) \in \mathbb{Z}$ and $\{\mu \geq 1 \mid \mu + 1/\mu \in \mathbb{Z}\}$ consists exactly of a single point in each interval of the form $[n, n+1)$ for $n \geq 1$, it follows that $\Lambda = \{\lambda > 0 \mid \exists A \in \mathrm{SL}_2(\mathbb{Z})\ Av = \lambda v\}$ is a discrete subgroup of $\mathbb{R}^+$, and is therefore cyclic. Fix $A \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$ such that the eigenvalue $\lambda$ of $v$ generates $\Lambda$, and note that if $B \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$, then there exists $n \in \mathbb{Z}$ for which $Bv = \lambda^n v$, in which case $A^n B^{-1} v = v$, so $B = A^n$, thus $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta) = \langle A \rangle$. □

Suppose that $X$ is a standard Borel space and $E$ is a countable Borel equivalence relation on $X$. We say that $E$ is *finite* if all of its classes are finite. We say that $E$ is *hyperfinite* if there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on $X$ whose union is $E$.

Suppose that $Y$ is a standard Borel space and $F$ is a countable Borel equivalence relation on $Y$. A *reduction* of $E$ to $F$ is a homomorphism from $E$ to $F$ sending $E$-inequivalent points to $F$-inequivalent points. As every hyperfinite Borel equivalence relation is Borel reducible to $\mathbb{E}_0$ (see Theorem 1 of [3]), it follows that a Borel measure is $(E, \mathbb{E}_0)$-ergodic if and only if it is $(E, F)$-ergodic for every hyperfinite Borel equivalence relation $F$ on a standard Borel space.

Suppose that $\Delta$ is a group and $\Delta \curvearrowright Y$. The *orbit* of $y$ under $\Delta \curvearrowright Y$ is the set $[y]_\Delta$ consisting of all points of the form $\delta \cdot y$ for $\delta \in \Delta$. The *orbit equivalence relation* of $\Delta \curvearrowright Y$ is the relation $E^Y_\Delta$ on $Y$ given by $y \ E^Y_\Delta \ z \iff \exists \delta \in \Delta \ y = \delta \cdot z$. Given $f, g \colon X \to Y$, let $D(f, g)$ denote the set of $x \in X$ for which $f(x) \neq g(x)$. Given a relation $R \subseteq X \times X$ and a function $\rho \colon R \to \Delta$, we say that a function $\phi \colon X \to Y$ is *$\rho$-invariant* if $\phi(w) = \rho(w, x) \cdot \phi(x)$ whenever $w \ R \ x$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that if $R$ is a Borel set with countable sections and $E$ is the smallest equivalence relation containing $R$, then every Borel function $\rho \colon R \to \Delta$ can be extended to a Borel function $\sigma \colon E \to \Delta$ so as to ensure that every $\rho$-invariant function is $\sigma$-invariant.

A Borel measure $\mu$ on $X$ is *$(\rho, \Delta \curvearrowright Y)$-ergodic* if for every $\rho$-invariant Borel function $\phi \colon X \to Y$, there exists $y \in Y$ such that $\phi^{-1}([y]_\Delta)$ is $\mu$-conull. A Borel measure $\mu$ on $X$ is *$(\rho, \Delta \curvearrowright Y)$-rigid* if $D(\phi, \psi)$ is $\mu$-null for all $\rho$-invariant Borel functions $\phi, \psi \colon X \to Y$. The action $\Delta \curvearrowright Y$ is *locally rigid* if whenever $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, and $\rho \colon E \to \Delta$ is Borel, every $(E, \mathbb{E}_0)$-ergodic Borel probability measure on $X$ is

$(\rho, \Delta \curvearrowright Y)$-ergodic or $(\rho, \Delta \curvearrowright Y)$-rigid.

**Theorem 3.2.** *The action* $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{R}^2$ *is locally rigid.*

**Proof.** We will show that if $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, $\rho \colon E \to \Delta$ is Borel, $\phi, \psi \colon X \to \mathbb{R}^2$ are $\rho$-invariant Borel functions, and $\mu$ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure on $X$ for which $D(\phi, \psi)$ is $\mu$-positive, then there exists $v \in \mathbb{R}^2$ such that $\phi^{-1}([v]_{\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2})$ is $\mu$-conull. Towards this end, define $\pi \colon D(\phi, \psi) \to \mathbb{T}$ and $\sigma \colon E \restriction D(\phi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\phi(x) - \psi(x))$ and $\sigma(x, y) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}(\rho(x, y))$. Observe that if $x \ (E \restriction D(\phi, \psi)) \ y$, then

$$\begin{aligned}
\pi(x) &= \mathrm{proj}_{\mathbb{T}}(\phi(x) - \psi(x)) \\
&= \mathrm{proj}_{\mathbb{T}}(\rho(x, y) \cdot \phi(y) - \rho(x, y) \cdot \psi(y)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x, y) \cdot \phi(y) - \sigma(x, y) \cdot \psi(y)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x, y) \cdot (\phi(y) - \psi(y))) \\
&= \sigma(x, y) \cdot \mathrm{proj}_{\mathbb{T}}(\phi(y) - \psi(y)) \\
&= \sigma(x, y) \cdot \pi(y),
\end{aligned}$$

so $\pi$ is $\sigma$-invariant, and thus a homomorphism from $E \restriction D(\phi, \psi)$ to the orbit equivalence relation of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$. As the latter is hyperfinite by the remark following the proof of Proposition 3.5 of [4], there exists $\theta \in \mathbb{T}$ for which $\pi^{-1}(\theta)$ is $\mu$-positive.

Proposition 3.1 ensures that the group $\Gamma = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\theta)$ is cyclic, so the orbit equivalence relation $F$ of $\Gamma \curvearrowright \mathbb{T}^2$ is hyperfinite by Lemma 1 of [5]. As the fact that $\sigma(E \restriction \pi^{-1}(\theta)) \subseteq \Gamma$ ensures $(\mathrm{proj}_{\mathbb{T}^2} \circ \phi) \restriction \pi^{-1}(\theta)$ is a homomorphism from $E \restriction \pi^{-1}(\theta)$ to $F$, there exists $t \in \mathbb{T}^2$ for which $(\mathrm{proj}_{\mathbb{T}^2} \circ \phi)^{-1}(t)$ is $\mu$-positive, so there exists $v \in \mathbb{R}^2$ such that $\phi^{-1}(v)$ is $\mu$-positive, thus $\phi^{-1}([v]_{\mathbb{Z}^2 \ltimes \Gamma})$ is $\mu$-conull. □

## 4. Homomorphisms and separability

Suppose that $X$ and $Y$ are standard Borel spaces, $E$ and $F$ are countable Borel equivalence relations on $X$ and $Y$, and $\mu$ is a finite Borel measure on $X$. Let $L(X, \mu, Y)$ denote the space of all $\mu$-measurable functions $\phi \colon X \to Y$, equipped with the pseudo-metric $d_\mu$ given by $d_\mu(f, g) = \mu(D(f, g))$. Let $\mathrm{Hom}(E, \mu, F)$ denote the space of all homomorphisms from $E$ to $F$ in $L(X, \mu, Y)$, and let $\mathrm{Hom}_0(E, \mu, F)$ denote the subspace of those $\phi$ with the property that $\phi^{-1}(y)$ is $\mu$-null for all $y \in Y$. We say that $F$ has *separable homomorphisms* if whenever $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, and $\mu$ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure on $X$, the space $\mathrm{Hom}_0(E, \mu, F)$ is separable.

**Theorem 4.1.** *Suppose that* $Y$ *is a standard Borel space and* $F$ *is the orbit equivalence relation of a locally rigid Borel action* $\Delta \curvearrowright Y$ *of a countable group. Then* $F$ *has separable homomorphisms.*

**Proof.** Suppose that $X$ is a standard Borel space, $E$ is a countable Borel equivalence relation on $X$, and $\mu$ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure on $X$. We can assume that $\mu$ is $E$-quasi-invariant. The uniformization theorem for Borel sets with countable vertical sections yields Borel functions $f_n \colon X \to X$ with $E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(f_n)$. For each $n \in \mathbb{N}$, set $R_n = \bigcup_{m<n} \operatorname{graph}(f_m)$ and let $E_n$ denote the smallest equivalence relation containing $R_n$.

Let $\mu_c$ denote the counting measure on $X$, and set $\mu_n = (\mu \times \mu_c) \restriction R_n$. The countability of $\Delta$ ensures that $L(R_n, \mu_n, \Delta)$ is separable. Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \mu_n, \Delta)$, and associate with each $n \in \mathbb{N}$, $\sigma \in \mathscr{D}_n$, and rational $\epsilon > 0$ for which it is possible a Borel function $\rho \colon R_n \to \Delta$, with $d_{\mu_n}(\sigma, \rho) < \epsilon$, and a $\rho$-invariant function $\psi \in \operatorname{Hom}_0(E, \mu, F)$. We will show that the set of $\psi$ is dense.

Suppose that $\phi \in \operatorname{Hom}_0(E, \mu, F)$ and $\epsilon > 0$ is rational. By Proposition 2.2, there exists $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq X$ such that $\mu(B) > 1 - \epsilon$ and $\mu \restriction B$ is $(E_n \restriction B, \mathbb{E}_0)$-ergodic. By the uniformization theorem for Borel sets with countable vertical sections, there is a Borel function $\rho \colon R_n \to \Delta$ for which $\phi$ is $\rho$-invariant. Fix $\epsilon_m > 0$ such that $\sum_{m \in \mathbb{N}} \epsilon_m < \infty$, and fix $\sigma_{m,n} \in \mathscr{D}_n$ with $d_{\mu_n}(\sigma_{m,n}, \rho) < \epsilon_m$. Let $\rho_{m,n}$ and $\psi_{m,n}$ denote the corresponding functions, and let $E_{m,n}$ denote the equivalence relation generated by $R_n \setminus \bigcup_{k \geq m} D(\rho, \rho_{k,n})$. The $E$-quasi-invariance of $\mu$ ensures that $\bigcup_{m \in \mathbb{N}} E_{m,n}$ agrees with $E_n$ on a $\mu$-conull set. By Proposition 2.2, there exists $m \in \mathbb{N}$ for which there is a Borel set $B_m \subseteq X$ with the property that $\mu(B_m) > 1 - \epsilon$ and $\mu \restriction B_m$ is $(E_{m,n} \restriction B_m, \mathbb{E}_0)$-ergodic. The local rigidity of $\Delta \curvearrowright Y$ therefore ensures that $\phi \restriction B_m = \psi_{m,n} \restriction B_m$, thus $d_\mu(\phi, \psi_{m,n}) < \epsilon$. □

## 5. Stratification

Suppose that $X$ is a standard Borel space and $E$ is a countable Borel equivalence relation on $X$. We say that $E$ is *aperiodic* if all of its classes are infinite. A *stratification* of $E$ is a sequence $(E_r)_{r \in \mathbb{R}}$ of equivalence relations on $X$ for which the $E$-class of each point of $X$ is the strictly increasing union of its $E_r$-classes. We say that such a stratification is *Borel* if $\{(r, (x, y)) \in \mathbb{R} \times (X \times X) \mid x \mathrel{E_r} y\}$ is Borel.

**Proposition 5.1.** *Suppose that $X$ is a standard Borel space and $E$ is an aperiodic countable Borel equivalence relation on $X$. Then $E$ has a Borel stratification.*

**Proof.** Recursively construct a sequence $(F_q)_{q \in \mathbb{Q}}$ of equivalence relations on $\mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $\mathbb{N}^2$ is the strictly increasing union of the $F_q$-classes of $n$. Then the relations $F_r = \bigcup_{q \leq r} F_q$ yield a Borel stratification of $\mathbb{N}^2$.

By the marker lemma (see, for example, Lemma 6.7 of [2]), there is a partition $(B_n)_{n \in \mathbb{N}}$ of $X$ into Borel sets intersecting every equivalence class of $E$. Let $n(x)$ denote the unique $n \in \mathbb{N}$ for which $x \in B_n$. Then the equivalence relations given by $x \mathrel{E_r} y \iff x \mathrel{E} y$ and $n(x) \mathrel{F_r} n(y)$ yield the desired Borel stratification of $E$. □

Suppose that $T$ is a permutation of $X$. The *orbit* of $x$ under $T$ is the set $[x]_T$ consisting of all points of the form $T^n(x)$ for $n \in \mathbb{Z}$. We say that $T$ is *aperiodic* if every such orbit is infinite. The *orbit equivalence relation* of $T$ is the relation $E_T^X$ on $X$ given by $x \, E_T^X \, y \iff \exists n \in \mathbb{Z} \; x = T^n(y)$. A *stratification* of $T$ is a sequence $(T_r)_{r \in \mathbb{R}}$ of permutations of $X$ for which $(E_{T_r}^X)_{r \in \mathbb{R}}$ is a stratification of $E_T^X$. We say that such a stratification is *Borel* if the function $(r, x) \mapsto T_r(x)$ is Borel.

**Proposition 5.2.** *Suppose that $X$ is a standard Borel space and $T \colon X \to X$ is an aperiodic Borel automorphism. Then $T$ has a Borel stratification.*

**Proof.** By Proposition 5.1, there is a Borel stratification $(E_r)_{r \in \mathbb{R}}$ of $E_T^X$. Define

$$Y_r = \{x \in X \mid [x]_{E_r} \cap \{T^n(x) \mid n < 0\} \text{ is finite}\}$$

and

$$Z_r = \{x \in X \mid [x]_{E_r} \cap \{T^n(x) \mid n > 0\} \text{ is finite}\},$$

as well as $X_r = X \setminus (Y_r \cup Z_r)$. Define $n_r^-(x) = |\{n < 0 \mid x \; E_r \; T^n(x)\}|$ and $n_r^+(x) = |\{n > 0 \mid x \; E_r \; T^n(x)\}|$, and fix a transitive permutation $\tau$ of $\mathbb{N}$. Then the permutations given by

$$T_r(x) = \begin{cases} T^n(x) & \text{if } x \in X_r \text{ and } n > 0 \text{ is least for which } x \; E_r \; T^n(x), \\ y & \text{if } x \in Y_r, \; x \; E_r \; y, \text{ and } \tau(n_r^-(x)) = n_r^-(y), \text{ and} \\ z & \text{if } x \in Z_r, \; x \; E_r \; z, \text{ and } \tau(n_r^+(x)) = n_r^+(z) \end{cases}$$

yield the desired Borel stratification of $T$. □

## 6. Denouement

Suppose that $X$ is a standard Borel space and $E$ is a countable Borel equivalence relation on $X$. A Borel measure $\mu$ on $X$ is *$E$-invariant* if every Borel automorphism $T \colon X \to X$ whose graph is contained in $E$ is $\mu$-measure preserving.

**Theorem 6.1 (Hjorth).** *Suppose that $n \in \{2, 3, \ldots, \aleph_0\}$. Then there is a standard Borel space $X$, a Borel probability measure $\mu$ on $X$, and countable Borel equivalence relations $E_r$ on $X$ such that for all $r, s \in \mathbb{R}$, the following conditions hold:*

- *The equivalence relation $E_r$ is induced by a free Borel action of $\mathbb{F}_n$.*
- *The measure $\mu$ is $(E_r, \mathbb{E}_0)$-ergodic and $E_r$-invariant.*
- *If $r \leq s$, then $E_r \subseteq E_s$.*
- *If $r \neq s$, then there is no $\mu$-measurable reduction of $E_r$ to $E_s$.*

**Proof.** We will handle only the case $n = 2$, as the other cases follow from an essentially identical argument. Let $m$ denote Lebesgue measure on $\mathbb{R}$, and let $\mu$ denote the Borel probability measure induced by $m^2$ on $\mathbb{T}^2$. By the remarks at the end of §5C of [6], there are matrices $A, B \in \mathrm{SL}_2(\mathbb{Z})$, generating a free subgroup of $\mathrm{SL}_2(\mathbb{Z})$ whose action on $\mathbb{T}^2$ is free, with the property that if $S, T \colon \mathbb{T}^2 \to \mathbb{T}^2$ are given by $S(x) = Ax$ and $T(x) = Bx$, then $\mu$ is $E_S^{\mathbb{T}^2}$-ergodic and $(E_S^{\mathbb{T}^2} * E_T^{\mathbb{T}^2}, \mathbb{E}_0)$-ergodic. By Proposition 5.2, there is a Borel stratification $(T_r)_{r \in \mathbb{R}}$ of $T$. Proposition

2.2 then ensures that by replacing $(T_r)_{r \in \mathbb{R}}$ with a terminal segment if necessary, we can assume that $\mu$ is $(E_S^{\mathbb{T}^2} * F, \mathbb{E}_0)$-ergodic, where $F = \bigcap_{r \in \mathbb{R}} E_{T_r}^{\mathbb{T}^2}$.

For each $r \in \mathbb{R}$, set $F_r = E_S^{\mathbb{T}^2} * E_{T_r}^{\mathbb{T}^2}$. Proposition 1.2 implies that for distinct $r, s \in \mathbb{R}$, the equivalence relations $F_r$ and $F_s$ differ on every $\mu$-positive set. Note that for all $s \in \mathbb{R}$, Theorems 3.2 and 4.1 ensure that $\mathrm{Hom}_0(F, \mu, F_s)$ is separable, so there are only countably many $r \in \mathbb{R}$ for which there is a $\mu$-measurable reduction of $F_r$ to $F_s$. As the set of pairs $(r, s) \in \mathbb{R}^2$ for which there exists such a reduction is analytic (this follows, for example, from Theorem 29.26 of [1]), it is therefore meager, so there is a perfect set $P \subseteq \mathbb{R}$ with the property that for no distinct $p, q \in P$ is there a $\mu$-measurable reduction of $F_p$ to $F_q$ (see, for example, Theorem 19.1 of [1]). Fix an order-preserving injection $\phi \colon \mathbb{R} \to P$, and observe that the equivalence relations $E_r = F_{\phi(r)}$ are as desired. □

## Epilogue

Here we consider further results that will be addressed in future papers.

Part of Hjorth's motivation for studying treeable equivalence relations stems from the use of product group actions in [7], along with the well-known incompatibility between orbit equivalence relations associated with such actions and treeability. In his preprint, Hjorth sketched an idea for obtaining a particularly general form of this incompatibility. Although it is not difficult to fill in the details, his use of lacunary sections for Borel actions of locally compact Polish groups (see Corollary 1.2 of [8]) is both limiting and unnecessary, and still more general results follow from a significantly simpler approach.

By adapting the arguments of [7], one can obtain analogous results on the descriptive complexity of Borel reducibility within the countable treeable Borel equivalence relations. In addition, the proof of Theorem 6.1 can be modified so as to handle various weakenings of Borel reducibility. On the other hand, at the cost of ergodicity one can modify the argument so as to produce families of incomparable countable treeable Borel equivalence relations lying within a single countable-to-one Borel bihomomorphism class.

Equivalence relations with separable homomorphisms have other applications. For example, if $E$ is a countable Borel equivalence relation with separable homomorphisms and $\mu$ is an $(E, \mathbb{E}_0)$-ergodic Borel probability measure, then there is no $\mu$-measurable reduction of $E \times \Delta(\mathbb{R})$ to $E$. Moreover, there is a $\mu$-conull Borel set $C$ for which there is no $(\mu \restriction C)$-measurable reduction of $(E \restriction C) \times \Delta(2)$ to $E \restriction C$.

The property of having separable homomorphisms is quite robust. It is downward closed under countable-to-one Borel homomorphism, and therefore under Borel reducibility and passage to Borel subequivalence relations. It is also closed under passage to Borel superequivalence relations of measure-hyperfinite index.

A *basis* for a family $\mathscr{F}$ of Borel equivalence relations is a family $\mathscr{E} \subseteq \mathscr{F}$ with the property that for all $F \in \mathscr{F}$, there exists $E \in \mathscr{E}$ for which there is a Borel reduction of $E$ to $F$. At the cost of being forced to employ a somewhat different notion of having separable homomorphisms, one can eliminate the need for strong ergodicity. In addition to further simplifying Hjorth's original argument, this observation allows one to establish that every basis for the class of non-measure-hyperfinite countable treeable Borel equivalence relations has cardinality at least $\mathtt{add}(\mathtt{null})$.

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