BASES FOR FUNCTIONS BEYOND THE FIRST BAIRE CLASS

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Abstract. We provide a finite basis for the class of Borel functions that are not in the first Baire class, as well as the class of Borel functions that are not σ-continuous with closed witnesses.

Introduction

A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$. A subset of a topological space is *Borel* if it is in the σ-algebra generated by open sets, F_σ if it is a union of countably-many closed sets, and G_δ if it is an intersection of countably-many open sets.

Suppose that X and Y are topological spaces. Given a family Γ of subsets of X, a function $\phi\colon X \to Y$ is Γ*-measurable* if $\phi^{-1}(V) \in \Gamma$ for every open set $V \subseteq Y$. A function is *Borel* if it is Borel-measurable, *Baire class one* if it is F_σ-measurable, and σ*-continuous with closed witnesses* if its domain is the union of countably-many closed sets on which it is continuous. A result of Jayne-Rogers (see [JR82, Theorem 1]) ensures that a function from an analytic metric space to a separable metric space has this property if and only if it is G_δ-measurable.

A *quasi-order* on a set Z is a reflexive transitive binary relation $\leq$ on Z. A set $B \subseteq Z$ is a *basis* under $\leq$ for Z if $\forall z \in Z \exists b \in B\ b \leq z$.

A *closed continuous embedding* of $\phi\colon X \to Y$ into $\phi'\colon X' \to Y'$ consists of a pair of closed continuous embeddings $\pi_X\colon X \to X'$ and $\pi_Y\colon \overline{\phi(X)} \to \overline{\phi'(X')}$ such that $\phi' \circ \pi_X = \pi_Y \circ \phi$. Note that the existence of such a pair depends not only on the graphs of the functions ϕ and ϕ', but on Y as well, since different choices of $Y \supseteq \phi(X)$ can lead to different values of $\overline{\phi(X)}$. Here we establish the following results.

Theorem 1. *There is a twenty-four-element basis under closed continuous embeddability for the class of non-Baire-class-one Borel functions between analytic metric spaces.*

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Theorem 2. *There is a twenty-seven-element basis under closed continuous embeddability for the class of non-σ-continuous-with-closed-witnesses Borel functions between analytic metric spaces.*

In §1, we discuss the compactification $\mathbb{N}^{\leq\mathbb{N}}_*$ of $\mathbb{N}^{\leq\mathbb{N}}$ underlying our arguments, as well as the corresponding compactification $\mathbb{N}^{\mathbb{N}}_*$ of $\mathbb{N}^{\mathbb{N}}$. In §2, we discuss the endomorphisms of $\mathbb{N}^{<\mathbb{N}}$ underlying our arguments. In §3, we provide a three-element basis for the class of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to separable metric spaces. In §4, we provide a three-element basis for the class of non-σ-continuous-with-closed-witnesses Baire-class-one functions from analytic metric spaces to separable metric spaces. In §5, we provide an eight-element basis for the class of all functions from $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ to analytic metric spaces. And in §6, we establish Theorems 1 and 2.

1. A compactification of $\mathbb{N}^{\leq\mathbb{N}}$

We use $s \frown t$ to denote the *concatenation* of sequences s and t, and we say that s is an *initial segment* of t, or $s \sqsubseteq t$, if there exists s' for which $t = s \frown s'$. Endow the set $\mathbb{N}^{\leq\mathbb{N}}_* = \mathbb{N}^{\leq\mathbb{N}} \cup \{t \frown (\infty) \mid t \in \mathbb{N}^{<\mathbb{N}}\}$ with the smallest topology with respect to which the sets of the form $\{t\}$ and $\mathcal{N}_t = \{c \in \mathbb{N}^{\leq\mathbb{N}}_* \mid t \sqsubseteq c\}$, where $t \in \mathbb{N}^{<\mathbb{N}}$, are clopen.

Proposition 1.1. *The family $\mathcal{B}$ of sets of the form $\{t\}$ and $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t\frown(j)})$, where $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, is a clopen basis for $\mathbb{N}^{\leq\mathbb{N}}_*$.*

Proof. Let τ be the topology generated by $\mathcal{B}$. As every set in $\mathcal{B}$ is clearly clopen, it is sufficient to show that the sets $\{t\}$ and $\mathcal{N}_t$ are τ-clopen for all $t \in \mathbb{N}^{<\mathbb{N}}$. As these sets are clearly τ-open, we need only show that they are τ-closed. As $\mathcal{N}_{t\frown(i)}$ is τ-closed in $\mathcal{N}_t$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, a straightforward induction shows that $\mathcal{N}_t$ is τ-closed for all $t \in \mathbb{N}^{<\mathbb{N}}$. As $\{t\}$ is τ-closed in $\mathcal{N}_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that $\{t\}$ is τ-closed for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 1.2. *The space $\mathbb{N}^{\leq\mathbb{N}}_*$ is compact.*

Proof. Suppose, towards a contradiction, that there is an open cover $\mathcal{U}$ of $\mathbb{N}^{\leq\mathbb{N}}_*$ with no finite subcover.

Lemma 1.3. *Suppose that $t \in \mathbb{N}^{<\mathbb{N}}$ and no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_t$. Then there exists $j \in \mathbb{N}$ such that no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t\frown(j)}$.*

Proof. Fix $U \in \mathcal{U}$ containing $t \frown (\infty)$. Proposition 1.1 then yields $i \in \mathbb{N}$ with $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j<i} \mathcal{N}_{t\frown(j)}) \subseteq U$, in which case no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\bigcup_{j<i} \mathcal{N}_{t\frown(j)}$, and it follows that there exists $j < i$ for which no finite set $\mathcal{V} \subseteq \mathcal{U}$ covers $\mathcal{N}_{t\frown(j)}$. ⊠

By recursively applying Lemma 1.3, we obtain $b \in \mathbb{N}^{\mathbb{N}}$ such that for no $i \in \mathbb{N}$ is there a finite set $\mathcal{V} \subseteq \mathcal{U}$ covering $\mathcal{N}_{b \upharpoonright i}$. But Proposition 1.1 implies that every open neighborhood of b contains some $\mathcal{N}_{b \upharpoonright i}$. ⊠

Given a countable set I and a topological space X, we say that a sequence $(x_i)_{i \in I} \in X^I$ *converges* to a point $x \in X$, or $x_i \to x$, if for every open neighborhood U of x there are only finitely many $i \in I$ with $x_i \notin U$. We endow $\mathbb{N}^{<\mathbb{N}}$ with the partial order $\sqsubseteq$, and when I and X are equipped with partial orders $\leq_I$ and $\leq_X$, we say that $(x_i)_{i \in I}$ is *decreasing* if $i \leq_I j \implies x_j \leq_X x_i$ for all $i, j \in I$.

Proposition 1.4. *The space $\mathbb{N}^{\leq \mathbb{N}}_*$ has a compatible ultrametric.*

Proof. Fix a decreasing sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero. Set $d(a, a) = 0$ for all $a \in \mathbb{N}^{\leq \mathbb{N}}_*$, as well as $d(a, b) = \max\{\epsilon_t \mid t \in \{a \upharpoonright \min(|a|, i(a, b)), b \upharpoonright \min(|b|, i(a, b))\} \cap \mathbb{N}^{<\mathbb{N}}\}$ for all distinct $a, b \in \mathbb{N}^{\leq \mathbb{N}}_*$, where $i(a, b) = \min\{i \in \mathbb{N} \mid a \upharpoonright i \neq b \upharpoonright i\}$.

To see that d is an ultrametric, suppose that $a, b, c \in \mathbb{N}^{\leq \mathbb{N}}_*$ are pairwise distinct. Observe that if $i(a, c) < \max\{i(a, b), i(b, c)\}$, then $d(a, c) \in \{d(b, c), d(a, b)\}$, so $d(a, c) \leq \max\{d(a, b), d(b, c)\}$. And if $i(a, c) = \max\{i(a, b), i(b, c)\}$, then setting $i = i(a, b) = i(a, c) = i(b, c)$, it follows that

$$
\begin{aligned}
d(a, c) &= \max\{\epsilon_t \mid t \in \{a \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{<\mathbb{N}}\} \\
&\leq \max\{\epsilon_t \mid t \in \{a \upharpoonright i, b \upharpoonright i, c \upharpoonright i\} \cap \mathbb{N}^{<\mathbb{N}}\} \\
&= \max\{d(a, b), d(b, c)\}.
\end{aligned}
$$

And if $i(a, c) > \max\{i(a, b), i(b, c)\}$, then setting $\epsilon = d(a, b) = d(b, c)$ and $t = a \upharpoonright i(a, b) = c \upharpoonright i(b, c)$, it follows that $d(a, c) \leq \epsilon_t \leq \epsilon$, and therefore $d(a, c) \leq \max\{d(a, b), d(b, c)\}$.

As $\{t\} = \mathcal{B}(t, \epsilon_t)$ and $\mathcal{N}_t \setminus \{t\} = \mathcal{B}(\mathcal{N}_t \setminus \{t\}, \epsilon_t)$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, and $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j \leq i} \mathcal{N}_{t \frown (j)}) = \mathcal{B}(\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j \leq i} \mathcal{N}_{t \frown (j)}), \min(\{\epsilon_{t \frown (j)} \mid j \leq i\}))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, Proposition 1.1 ensures that every open subset of $\mathbb{N}^{\leq \mathbb{N}}_*$ is d-open.

Given $b \in \mathbb{N}^{\mathbb{N}}$ and $\epsilon > 0$, fix $i \in \mathbb{N}$ with $\epsilon_{b \upharpoonright i} < \epsilon$, set $t = b \upharpoonright i$, and note that $\mathcal{N}_t \subseteq \mathcal{B}(b, \epsilon)$. Given $t \in \mathbb{N}^{<\mathbb{N}}$ and $\epsilon > 0$, fix $i \in \mathbb{N}$ with $\epsilon_{t \frown (j)} < \epsilon$ for all $j \geq i$, and observe that $\mathcal{N}_t \setminus (\{t\} \cup \bigcup_{j < i} \mathcal{N}_{t \frown (j)}) \subseteq \mathcal{B}(t \frown (\infty), \epsilon)$. Thus every d-open subset of $\mathbb{N}^{\leq \mathbb{N}}_*$ is open. ⊠

It follows that $\mathbb{N}^{\leq \mathbb{N}}_*$ is Polish. As the space $\mathbb{N}^{\mathbb{N}}_* = \mathbb{N}^{\leq \mathbb{N}}_* \setminus \mathbb{N}^{<\mathbb{N}}$ is a perfect subset of $\mathbb{N}^{\leq \mathbb{N}}_*$, a result of Brouwer's ensures that it is homeomorphic to $2^{\mathbb{N}}$ (see, for example, [Kec95, Theorem 7.4]).

2. Meet embeddings

The *meet* of sequences $s, t \in \mathbb{N}^{<\mathbb{N}}$ is the sequence $r = s \wedge t$ of maximal length for which $r \sqsubseteq s$ and $r \sqsubseteq t$. A $\wedge$-*embedding* is an injection $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(s \wedge t) = \pi(s) \wedge \pi(t)$ for all $s, t \in \mathbb{N}^{<\mathbb{N}}$.

Proposition 2.1. *Suppose that $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$. Then π is a $\wedge$-embedding if and only if the following conditions hold:*

1. $\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}\ \pi(t) \sqsubset \pi(t \frown (i))$.
2. $\forall i, j \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}$
$$(i \neq j \implies \pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)).$$

Proof. Suppose first that π is a $\wedge$-embedding. To see that condition (1) holds, observe that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then $\pi(t) = \pi(t) \wedge \pi(t \frown (i))$, so $\pi(t) \sqsubseteq \pi(t \frown (i))$, thus $\pi(t) \sqsubset \pi(t \frown (i))$. And to see that condition (2) holds, observe that if $i, j \in \mathbb{N}$ are distinct and $t \in \mathbb{N}^{<\mathbb{N}}$, then $\pi(t) = \pi(t \frown (i)) \wedge \pi(t \frown (j))$, so $\pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)$.

Suppose now that π satisfies conditions (1) and (2). To see that π is a $\wedge$-embedding, suppose that $s, t \in \mathbb{N}^{<\mathbb{N}}$ are distinct, and define $r = s \wedge t$. By reversing the roles of s and t if necessary, we can assume that $|s| > |r|$, so $\pi(r) \sqsubset \pi(s)$, thus either $r = t$ or ($|t| > |r|$ and $\pi(s)(|\pi(r)|) \neq \pi(t)(|\pi(r)|)$). In both cases, it follows that $\pi(s) \neq \pi(t)$ and $\pi(r) = \pi(s) \wedge \pi(t)$. ⊠

Remark 2.2. In particular, it follows that if $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ has the property that $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then π is a $\wedge$-embedding.

The *composition* of a finite sequence $(\pi_i)_{i \leq n}$ of functions is given by $\circ_{i \leq n} \pi_i = \pi_0 \circ \cdots \circ \pi_n$.

Proposition 2.3. *Suppose that $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ is a sequence of $\wedge$-embeddings with the property that $\pi_t(\mathbb{N}^{<\mathbb{N}}) \subseteq \mathcal{N}_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Then the function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t)$ is also a $\wedge$-embedding.*

Proof. Note that if $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \sqsubseteq \pi_{t \frown (i)}(t \frown (i))$, so Proposition 2.1 ensures that $(\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$, thus $\pi(t) \sqsubset (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i)) \sqsubseteq \pi(t \frown (i))$. It also implies that if $i \neq j$, then $(\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (i))(|\pi(t)|) \neq (\circ_{n \leq |t|} \pi_{t \upharpoonright n})(t \frown (j))(|\pi(t)|)$, so $\pi(t \frown (i))(|\pi(t)|) \neq \pi(t \frown (j))(|\pi(t)|)$. One last application of Proposition 2.1 therefore ensures that π is a $\wedge$-embedding. ⊠

We next consider the connection between $\wedge$-embeddings and closed continuous embeddings.

Proposition 2.4. *Every $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ has a unique extension to a (necessarily injective) continuous map $\overline{\pi} \colon \mathbb{N}^{\leq \mathbb{N}}_* \to \mathbb{N}^{\leq \mathbb{N}}_*$,* *given by* $\overline{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$ *and* $\overline{\pi}(t \frown (\infty)) = \pi(t) \frown (\infty)$ *for all* $b \in \mathbb{N}^{\mathbb{N}}$ *and* $t \in \mathbb{N}^{<\mathbb{N}}$.

Proof. Suppose that $\overline{\pi}\colon \mathbb{N}_*^{\leq \mathbb{N}} \to \mathbb{N}_*^{\leq \mathbb{N}}$ is a continuous extension of π. If $b \in \mathbb{N}^{\mathbb{N}}$, then $b \upharpoonright i \to b$, and since $(\pi(b \upharpoonright i))_{i \in \mathbb{N}}$ is strictly increasing by Proposition 2.1, it follows that $\overline{\pi}(b) = \bigcup_{i \in \mathbb{N}} \pi(b \upharpoonright i)$. If $t \in \mathbb{N}^{<\mathbb{N}}$, then $t \frown (i) \to t \frown (\infty)$, and since $\pi(t) = \pi(t \frown (i)) \wedge \pi(t \frown (j))$ for all distinct $i, j \in \mathbb{N}$, it follows that $\overline{\pi}(t \frown (\infty)) = \pi(t) \frown (\infty)$.

To see that these constraints actually define a continuous function, note that if $t \in \mathbb{N}^{<\mathbb{N}}$, then either $\overline{\pi}^{-1}(\mathcal{N}_t) = \emptyset$ or there exists $s \in \mathbb{N}^{<\mathbb{N}}$ of minimal length with $t \sqsubseteq \pi(s)$, in which case $\overline{\pi}^{-1}(\mathcal{N}_t) = \mathcal{N}_s$.

To see that $\overline{\pi}$ is injective, it is enough to check that its restriction to $\mathbb{N}^{\mathbb{N}}$ is injective. Towards this end, suppose that $a, b \in \mathbb{N}^{\mathbb{N}}$ are distinct, fix $i \in \mathbb{N}$ least for which $a(i) \neq b(i)$, set $t = a \upharpoonright i = b \upharpoonright i$, and observe that $\pi(t \frown (a(i)))(|\pi(t)|) \neq \pi(t \frown (b(i)))(|\pi(t)|)$ by Proposition 2.1, thus $\overline{\pi}(a)$ and $\overline{\pi}(b)$ are distinct. ⊠

Remark 2.5. It follows that the extension associated with the composition of two $\wedge$-embeddings is the composition of their extensions.

Given a function $\phi\colon X \to Y$ and sets $X' \subseteq X$ and $Y' \supseteq \phi(X')$, let $\phi \upharpoonright X' \to Y'$ denote the function $\psi\colon X' \to Y'$ given by $\phi(x) = \psi(x)$ for all $x \in X'$. Compactness ensures that if π is a $\wedge$-embedding, then $\overline{\pi}$ and $\overline{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}}$ are closed continuous embeddings. The following observations show that so too are $\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $\overline{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$.

Proposition 2.6. *Suppose that* $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ *is a* $\wedge$*-embedding. Then* $\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ *is closed.*

Proof. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\overline{\pi}(b_n))_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^{\mathbb{N}}$ is itself convergent to an element of $\mathbb{N}^{\mathbb{N}}$. As $(\overline{\pi}(b_n) \upharpoonright i)_{n \in \mathbb{N}}$ is eventually constant for all $i \in \mathbb{N}$, a simple induction shows that $(b_n \upharpoonright i)_{n \in \mathbb{N}}$ is also eventually constant for all $i \in \mathbb{N}$, so $(b_n)_{n \in \mathbb{N}}$ converges to an element of $\mathbb{N}^{\mathbb{N}}$. ⊠

Proposition 2.7. *Suppose that* $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ *is a* $\wedge$*-embedding. Then* $\overline{\pi} \upharpoonright \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$ *is closed.*

Proof. It is sufficient to show that every sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $(\pi(s_n))_{n \in \mathbb{N}}$ converges to $t \frown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$ has a subsequence converging to an element of $\mathbb{N}_*^{\mathbb{N}} \setminus \mathbb{N}^{\mathbb{N}}$. By passing to a subsequence, we can assume that $\pi(s_m) \wedge \pi(s_n) = t$ for all distinct $m, n \in \mathbb{N}$. Let s be the $\sqsubseteq$-minimal element of $\mathbb{N}^{<\mathbb{N}}$ for which $t \sqsubseteq \pi(s)$. Then $s_m \wedge s_n = s$ for all distinct $m, n \in \mathbb{N}$, thus $s_n \to s \frown (\infty)$. ⊠

A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is $\sqsubseteq$-*dense* if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T\ s \sqsubseteq t$. More generally, a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is $\sqsubseteq$-*dense* below $r \in \mathbb{N}^{<\mathbb{N}}$ if $\forall s \in \mathbb{N}^{<\mathbb{N}} \exists t \in T\ r \frown s \sqsubseteq t$.

Proposition 2.8. *Suppose that $T \subseteq \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq T$ or $\pi(\mathbb{N}^{<\mathbb{N}}) \subseteq {\sim}T$.*

Proof. Fix $S \in \{T, {\sim}T\}$ which is $\sqsubseteq$-dense below some $s \in \mathbb{N}^{<\mathbb{N}}$, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_s \cap S$ with the property that $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 2.9. *Suppose that $C \subseteq \mathbb{N}^{\mathbb{N}}$ is a non-meager set with the Baire property. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$.*

Proof. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ for which C is comeager in $\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$, as well as dense open sets $U_n \subseteq \mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}$ with the property that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$. Set $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid \mathcal{N}_t \cap \mathbb{N}^{\mathbb{N}} \subseteq U_n\}$ for all $n \in \mathbb{N}$, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_s \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(\mathbb{N}^n) \subseteq T_n$ for all $n \in \mathbb{N}$ and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

3. Baire measurable functions on $\mathbb{N}^{\mathbb{N}}$

Here we provide a basis for the class of Baire measurable functions from $\mathbb{N}^{\mathbb{N}}$ to separable metric spaces.

Proposition 3.1. *Suppose that X is a second countable topological space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is Baire measurable. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $\phi \circ \overline{\pi}$ is continuous.*

Proof. Fix a comeager set $C \subseteq \mathbb{N}^{\mathbb{N}}$ on which ϕ is continuous, and appeal to Proposition 2.9 to obtain a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{\pi}(\mathbb{N}^{\mathbb{N}}) \subseteq C$. ⊠

Proposition 3.2. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is continuous. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that* $\operatorname{diam} \phi(\mathcal{N}_{\pi(t)}) \to 0$.

Proof. Fix a sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, note that the continuity of ϕ ensures that for all $t \in \mathbb{N}^{<\mathbb{N}}$ the set $T_t = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \operatorname{diam} \phi(\mathcal{N}_s) < \epsilon_t\}$ is $\sqsubseteq$-dense, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\pi(t) \in T_t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Given a countable set I and a topological space X, we say that a sequence $(X_i)_{i \in I}$ of subsets of X *converges* to a point $x \in X$, or $X_i \to x$, if for every open neighborhood U of x, all but finitely many $i \in I$ have the property that $X_i \subseteq U$. We say that $(X_i)_{i \in I}$ is *discrete* if for all $x \in X$ there is an open neighborhood U of x such that all but finitely many $i \in I$ have the property that $U \cap X_i = \emptyset$.

Proposition 3.3. *Suppose that X is a metric space and $\phi\colon \mathbb{N}^\mathbb{N} \to X$ has the property that* $\operatorname{diam} \phi(\mathcal{N}_{t \frown (i)}) \to 0$ *for all $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $(\phi(\mathcal{N}_{\pi(t \frown (i))}))_{i \in \mathbb{N}}$ is convergent or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. For each $t \in \mathbb{N}^{<\mathbb{N}}$, the fact that $\operatorname{diam} \phi(\mathcal{N}_{t \frown (i)}) \to 0$ ensures that there is an injection $\iota_t\colon \mathbb{N} \to \mathbb{N}$ for which $(\phi(\mathcal{N}_{t \frown (\iota_t(i))}))_{i \in \mathbb{N}}$ is convergent or discrete. Define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown (\iota_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

We say that a function $\phi\colon X \to Y$ is *nowhere constant* if there is no non-empty open set $U \subseteq X$ on which ϕ is constant.

Proposition 3.4. *Suppose that X is a metric space and $\phi\colon \mathbb{N}^\mathbb{N} \to X$ is continuous and nowhere constant. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that*

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}} \ \overline{\phi(\mathcal{N}_{\pi(t \frown (i))})} \cap \overline{\textstyle\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \frown (j))})} = \emptyset.$$

Proof. Clearly each $\phi(\mathcal{N}_t)$ is infinite.

Lemma 3.5. *For all $t \in \mathbb{N}^{<\mathbb{N}}$, there is a function $\iota_t\colon \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ such that $(\iota_t(i)(0))_{i \in \mathbb{N}}$ is injective and the closures of $\phi(\mathcal{N}_{t \frown \iota_t(i)})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown \iota_t(j)})$ are disjoint for all $i \in \mathbb{N}$.*

Proof. As each $\phi(\mathcal{N}_{t \frown (i)})$ is infinite, there are extensions $b_i \in \mathbb{N}^\mathbb{N}$ of $t \frown (i)$ such that $\phi(b_i) \notin \{\phi(b_j) \mid j < i\}$ for all $i \in \mathbb{N}$. Fix a subsequence $(a_i)_{i \in \mathbb{N}}$ of $(b_i)_{i \in \mathbb{N}}$ for which $\{\phi(a_i) \mid i \in \mathbb{N}\}$ is discrete. For each $i \in \mathbb{N}$, fix $\epsilon_i > 0$ such that $\phi(a_j) \notin \mathcal{B}(\phi(a_i), \epsilon_i)$ for all $j \in \mathbb{N} \setminus \{i\}$, as well as $\iota_t(i) \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ with $t \frown \iota_t(i) \sqsubseteq a_i$ and $\phi(\mathcal{N}_{t \frown \iota_t(i)}) \subseteq \mathcal{B}(\phi(a_i), \epsilon_i/3)$.

Suppose, towards a contradiction, that there exists $i \in \mathbb{N}$ for which some $x \in X$ is in the closures of $\phi(\mathcal{N}_{t \frown \iota_t(i)})$ and $\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown \iota_t(j)})$. Then there exist $j \in \mathbb{N} \setminus \{i\}$ and $y \in \phi(\mathcal{N}_{t \frown \iota_t(j)})$ with the property that $d(x, y) \leq \epsilon_i/3$, in which case

$$\begin{aligned} d(\phi(a_i), \phi(a_j)) &\leq d(\phi(a_i), x) + d(x, y) + d(y, \phi(a_j)) \\ &< \epsilon_i/3 + \epsilon_i/3 + \epsilon_j/3 \\ &\leq \max\{\epsilon_i, \epsilon_j\}, \end{aligned}$$

so $\phi(a_i) \in \mathcal{B}(\phi(a_j), \epsilon_j)$ or $\phi(a_j) \in \mathcal{B}(\phi(a_i), \epsilon_i)$, a contradiction. ⊠

Define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown \iota_{\pi(t)}(i)$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

We now obtain our main result stabilizing the topological behavior of Baire measurable functions from $\mathbb{N}^\mathbb{N}$ to separable metric spaces.

Theorem 3.6. *Suppose that X is a separable metric space and $\phi\colon \mathbb{N}^{\mathbb{N}} \to X$ is Baire measurable. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$.*

Proof. By Remark 2.5, we are free to replace ϕ by its composition with the extension of any $\wedge$-embedding. For example, by Proposition 3.1, we can assume that ϕ is continuous.

If there exists $s \in \mathbb{N}^{<\mathbb{N}}$ for which $\phi \upharpoonright \mathcal{N}_s$ is constant, then define $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, so $\phi \circ \overline{\pi}$ is constant. Otherwise, Propositions 2.8, 3.2, 3.3, and 3.4 yield a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\operatorname{diam} \phi(\mathcal{N}_{\pi(t)}) \to 0$, $(\phi(\mathcal{N}_{\pi(t \frown (i))}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\phi(\mathcal{N}_{\pi(t \frown (i))})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{\pi(t \frown (j))})} = \emptyset.$$

As $\overline{\pi}(\mathcal{N}_t) \subseteq \mathcal{N}_{\pi(t)}$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that

$$\forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{(\phi \circ \overline{\pi})(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} (\phi \circ \overline{\pi})(\mathcal{N}_{t \frown (j)})} = \emptyset.$$

So by replacing ϕ with $\phi \circ \overline{\pi}$, we can assume that $\operatorname{diam} \phi(\mathcal{N}_t) \to 0$, $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$ or discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, and

$$(\dagger) \qquad \forall i \in \mathbb{N} \forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\phi(\mathcal{N}_{t \frown (i)})} \cap \overline{\bigcup_{j \in \mathbb{N} \setminus \{i\}} \phi(\mathcal{N}_{t \frown (j)})} = \emptyset.$$

To see that ϕ is injective, note that if $a, b \in \mathbb{N}^{\mathbb{N}}$ are distinct, then there is a least $i \in \mathbb{N}$ for which $a(i) \neq b(i)$. Setting $t = a \upharpoonright i = b \upharpoonright i$, it follows from $(\dagger)$ that $\phi(\mathcal{N}_{t \frown (a(i))})$ and $\phi(\mathcal{N}_{t \frown (b(i))})$ are disjoint, thus $\phi(a)$ and $\phi(b)$ are distinct.

We next check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$, then ϕ is a closed continuous embedding. It is sufficient to show that every sequence $(b_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{\mathbb{N}}$ for which $(\phi(b_n))_{n \in \mathbb{N}}$ converges to some $x \in X$ is itself convergent. But a straightforward recursive argument yields $b \in \mathbb{N}^{\mathbb{N}}$ such that x is in the closure of $\phi(\mathcal{N}_{b \upharpoonright i})$ for all $i \in \mathbb{N}$, so $(\dagger)$ ensures that x is not in the closure of $\bigcup_{j \in \mathbb{N} \setminus \{b(i)\}} \phi(\mathcal{N}_{b \upharpoonright i \frown (j)})$ for all $i \in \mathbb{N}$, thus $(b_n \upharpoonright i)_{n \in \mathbb{N}}$ is eventually constant with value $b \upharpoonright i$ for all $i \in \mathbb{N}$, hence $b_n \to b$.

It remains to check that if $(\phi(\mathcal{N}_{t \frown (i)}))_{i \in \mathbb{N}}$ is convergent for all $t \in \mathbb{N}^{<\mathbb{N}}$, then the extension of ϕ to $\mathbb{N}^{\mathbb{N}}_*$ given by $\overline{\phi}(t \frown (\infty)) = \lim_{i \to \infty} \phi(\mathcal{N}_{t \frown (i)})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is a closed continuous embedding. To see that $\overline{\phi}$ is injective, note that if $c, d \in \mathbb{N}^{\mathbb{N}}_*$ are distinct, then there is a least $i \in \mathbb{N}$ with $c(i) \neq d(i)$. By reversing the roles of c and d if necessary, we can assume that $c(i) \neq \infty$. Set $t = c \upharpoonright i = d \upharpoonright i$, and appeal to $(\dagger)$ to see that $\overline{\phi}(c)$ is in the closure of $\phi(\mathcal{N}_{t \frown (c(i))})$ but $\overline{\phi}(d)$ is not, so

$\overline{\phi}(c) \neq \overline{\phi}(d)$. To see that $\overline{\phi}$ is continuous, suppose that $c \in \mathbb{N}^{\mathbb{N}}_*$ and U is an open neighborhood of $\overline{\phi}(c)$, and fix an open neighborhood V of $\overline{\phi}(c)$ whose closure is contained in U. If $c \in \mathbb{N}^{\mathbb{N}}$, then there exists $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_{c \upharpoonright i}) \subseteq V$, thus $\mathcal{N}_{c \upharpoonright i}$ is an open neighborhood of c whose image under $\overline{\phi}$ is contained in U. Otherwise, there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $c = t \frown (\infty)$, as well as $i \in \mathbb{N}$ for which $\phi(\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t \frown (j)}) \subseteq V$. Then $\mathcal{N}_t \setminus \bigcup_{j<i} \mathcal{N}_{t \frown (j)}$ is an open neighborhood of c whose image under $\overline{\phi}$ is contained in U. ⊠

For each topological space X, let c_X denote the unique function from X to the trivial topological space $\{\infty\}$. Given topological spaces $X \subseteq Y$, define $\iota_{X,Y} \colon X \to Y$ by $\iota_{X,Y}(x) = x$ for all $x \in X$.

Proposition 3.7. *Suppose that X is a separable metric space, $\phi \colon \mathbb{N}^{\mathbb{N}} \to X$ is Baire measurable, $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding, and $\phi \circ \overline{\pi}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$. Then there exist $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}},Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\}$ and $\psi \colon \overline{\phi_0(\mathbb{N}^{\mathbb{N}})} \to \overline{\phi(\mathbb{N}^{\mathbb{N}})}$ with the property that $(\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, \psi)$ is a closed continuous embedding of ϕ_0 into ϕ.*

Proof. If $\phi \circ \overline{\pi}$ is constant, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})$. If $\phi \circ \overline{\pi}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{\mathbb{N}},Z}$. ⊠

4. Baire-class-one functions that are not σ-continuous with closed witnesses

Here we strengthen [Sol98, Theorem 3.1] by providing a basis for the class of non-σ-continuous-with-closed-witnesses Baire-class-one functions from analytic metric spaces to separable metric spaces.

Proposition 4.1. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \to X$ has the property that $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that either $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$ or $\phi \circ \overline{\pi}$ is continuous at every point of $\mathbb{N}^{\mathbb{N}}$.*

Proof. We can assume that there is no $s \in \mathbb{N}^{<\mathbb{N}}$ with the property that $\inf\{d(\phi(s \frown b), \phi(s \frown t \frown (\infty))) \mid b \in \mathbb{N}^{\mathbb{N}} \text{ and } t \in \mathbb{N}^{<\mathbb{N}}\} > 0$, since otherwise the $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ has the property that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.

Lemma 4.2. *Suppose that $\epsilon > 0$ and $s \in \mathbb{N}^{<\mathbb{N}}$. Then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ with $d(\phi(s \frown t \frown b), \phi(s \frown t \frown (\infty))) < \epsilon$ for all $b \in \mathbb{N}^{\mathbb{N}}$.*

Proof. Fix $\delta < \epsilon$ and $u \in \mathbb{N}^{<\mathbb{N}}$ with $\operatorname{diam} \phi(\mathcal{N}_{s \frown u} \cap \mathbb{N}^{\mathbb{N}}) < \delta$, and $b \in \mathbb{N}^{\mathbb{N}}$ and $v \in \mathbb{N}^{<\mathbb{N}}$ with $d(\phi(s \frown u \frown b), \phi(s \frown u \frown v \frown (\infty))) < \epsilon - \delta$, and set $t = u \frown v$. ⊠

Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $d(\phi(\pi(t) \frown b), \phi(\pi(t) \frown (\infty))) < \epsilon_{|t|}$ for all $b \in \mathbb{N}^{\mathbb{N}}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, and $\pi(t) \frown (i) \sqsubseteq \pi(t \frown (i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

We say that a metric space is *ϵ-discrete* if all distinct points have distance at least ϵ from one another.

Proposition 4.3. *Suppose that X is a metric space, $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$, $\epsilon > 0$, and $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ with the property that $\phi \circ \overline{\pi}$ is an injection into an ϵ-discrete set or $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is contained in the ϵ-ball around a point of $\phi(\mathcal{N}_t)$.*

Proof. If for no finite set $F \subseteq \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ and extension u of t is it the case that $\phi(\mathcal{N}_u) \subseteq \mathcal{B}(F, \epsilon)$, then fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ with the property that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$, and recursively construct $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $\phi(\pi(t_n) \frown (\infty)) \notin \mathcal{B}(\{\phi(\pi(t_m) \frown (\infty)) \mid m < n\}, \epsilon)$ and $\pi(t'_n) \frown (n) \sqsubseteq \pi(t_n)$ for all $n > 0$, where t'_n is the maximal proper initial segment of t_n.

Otherwise, there exists $x \in \phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ with the property that the set $S = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(x, \epsilon)\}$ is $\sqsubseteq$-dense below some extension u of t, in which case we can recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap S$ with the property that $\pi(v) \frown (i) \sqsubseteq \pi(v \frown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 4.4. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is an injection into an ϵ-discrete set for some $\epsilon > 0$ or* $\operatorname{diam} (\phi \circ \overline{\pi})(\mathcal{N}_t) \to 0$.

Proof. Suppose that for no $\epsilon > 0$ is there a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is an injection into an ϵ-discrete set, fix a sequence $(\epsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively apply Proposition 4.3 to the functions $\phi_t = \phi \circ (\circ_{n < |t|} \overline{\pi_{t \upharpoonright n}})$ to obtain $\wedge$-embeddings $\pi_t \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that $(\phi \circ (\circ_{n \leq |t|} \overline{\pi_{t \upharpoonright n}}))(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is contained in an ϵ_t-ball for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let π be the $\wedge$-embedding obtained from applying Proposition 2.3 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that $\operatorname{diam} (\phi \circ \overline{\pi})(\mathcal{N}_t) \to 0$. ⊠

Define $p \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by setting $p(t \frown (\infty)) = t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. Let $\mathbb{N}^{<\mathbb{N}}_* = \mathbb{N}^{<\mathbb{N}} \cup \{\infty\}$ denote the *one-point compactification* of $\mathbb{N}^{<\mathbb{N}}$.

Theorem 4.5. *Suppose that X is an analytic metric space, Y is a separable metric space, and $\phi\colon X \to Y$ is a Baire-class-one function that is not σ-continuous with closed witnesses. Then there exists $\phi_0 \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ for which there is a closed continuous embedding of $\phi_0 \cup p$ into ϕ.*

Proof. By the Jayne-Rogers theorem (see, for example, [JR82, Theorem 1]), we can assume that ϕ is not G_δ-measurable. Hurewicz's dichotomy theorem for F_σ sets then yields a closed continuous embedding $\psi\colon \mathbb{N}^\mathbb{N}_* \to X$ with $\overline{(\phi \circ \psi)(\mathbb{N}^\mathbb{N})} \cap (\phi \circ \psi)(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$ (see, for example, [CMS, Theorem 4.2]). As $(\psi, \mathrm{id}_{\overline{(\phi\circ\psi)(\mathbb{N}^\mathbb{N}_*)}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ, by replacing the latter with the former, we can assume that $X = \mathbb{N}^\mathbb{N}_*$ and $\overline{\phi(\mathbb{N}^\mathbb{N})} \cap \phi(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$.

By Proposition 3.1, there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is continuous. By composing π with the $\wedge$-embedding given by Proposition 4.1, we can assume that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$ or $\phi \circ \overline{\pi}$ is continuous at every point of $\mathbb{N}^\mathbb{N}$. As ϕ is Baire class one, the former possibility would imply that the pre-images of $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})}$ and $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})}$ under $\phi \circ \overline{\pi}$ are disjoint dense G_δ subsets of $\mathbb{N}^\mathbb{N}_*$, so the latter holds. By Proposition 4.4, we can assume that either there exists $\epsilon > 0$ for which $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is an injection into an ϵ-discrete set, or $\operatorname{diam}(\phi \circ \overline{\pi})(\mathcal{N}_t \cap (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})) \to 0$. As the former possibility contradicts the facts that $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}) \cap (\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$ and $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}) \subseteq \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})}$, it follows that the latter holds. By applying Proposition 4.3 with any $\epsilon > 0$ and $t \in \mathbb{N}^{<\mathbb{N}}$, but replacing the given metric on X by one with respect to which all pairs of distinct points have distance at least ϵ from one another, we can assume that $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is either constant or injective.

Lemma 4.6. *Suppose that $(s_n)_{n\in\mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathbb{N}^\mathbb{N}$ such that $s_n \sqsubseteq b_n$ for all $n \in \mathbb{N}$. Then $d_X((\phi \circ \overline{\pi})(b_n), (\phi \circ \overline{\pi})(s_n \frown (\infty))) \to 0$.*

Proof. Simply note that $(\phi \circ \overline{\pi})(b_n) \in \overline{(\phi \circ \overline{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}))}$ for all $n \in \mathbb{N}$ and $\operatorname{diam}(\phi \circ \overline{\pi})(\mathcal{N}_{s_n} \cap (\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})) \to 0$. ⊠

Along with the facts that $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}) \cap (\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$ and $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}) \subseteq \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})}$, Lemma 4.6 ensures that $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is not constant, and is therefore injective. Along with the fact that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})} \cap (\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}) = \emptyset$, Lemma 4.6 ensures that $(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$ is discrete.

By Theorem 3.6, we can assume that $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$.

We will now complete the proof by showing that there exist $\phi_0 \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\psi \colon \overline{\phi_0(\mathbb{N}^\mathbb{N}_*)} \cup \mathbb{N}^{<\mathbb{N}} \to \overline{\phi(X)}$ for which $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N}_* \to \mathbb{N}^\mathbb{N}_*, \psi)$ is a closed continuous embedding of $\phi_0 \cup p$ into ϕ.

If $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is constant with value $y \in Y$, then set $\phi_0 = c_{\mathbb{N}^\mathbb{N}}$, and note that the extension ψ of $\phi \circ \overline{\pi} \circ p^{-1}$ to $\mathbb{N}^{<\mathbb{N}}_*$ given by $\psi(\infty) = y$ is injective. As Lemma 4.6 ensures that $(\phi \circ \overline{\pi})(s_n \frown (\infty)) \to y$ for every injective sequence $(s_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$, it follows that ψ is continuous, so the compactness of $\mathbb{N}^{<\mathbb{N}}_*$ ensures that ψ is a closed continuous embedding.

If $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is a closed continuous embedding, then set $\phi_0 = \iota_{\mathbb{N}^\mathbb{N},\mathbb{N}^\mathbb{N}}$, and note that the extension ψ of $\phi \circ \overline{\pi} \circ p^{-1}$ to $\mathbb{N}^{\leq\mathbb{N}}$ given by $\psi \restriction \mathbb{N}^\mathbb{N} = (\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is a continuous injection. To see that it is closed, it is enough to show that every injective sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{N}^{\leq\mathbb{N}}$ for which $(\psi(a_n))_{n\in\mathbb{N}}$ converges to some point $y \in Y$ has a subsequence converging to a point of $\mathbb{N}^\mathbb{N}$. As $\mathbb{N}^{\leq\mathbb{N}}_*$ is compact, by passing to a subsequence, we can assume that $(a_n)_{n\in\mathbb{N}}$ converges to a point of $\mathbb{N}^{\leq\mathbb{N}}_*$. As every point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, it therefore converges to a point of $\mathbb{N}^\mathbb{N}_*$. And if there exists $t \in \mathbb{N}^{<\mathbb{N}}$ for which $a_n \to t \frown (\infty)$, then there are extensions $b_n \in \mathbb{N}^\mathbb{N}$ of a_n for all $n \in \mathbb{N}$, in which case $b_n \to t \frown (\infty)$ and $\psi(b_n) \to y$ by Lemma 4.6. Fix $n \in \mathbb{N}$ sufficiently large that $(\phi \circ \overline{\pi})(b_m) \neq y$ for all $m \geq n$, and observe that $\{b_m \mid m \geq n\}$ is a closed subset of $\mathbb{N}^\mathbb{N}$ whose image under $\phi \circ \overline{\pi}$ is not closed, contradicting the fact that $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is closed.

If $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ extends to a closed continuous embedding ψ' on $\mathbb{N}^\mathbb{N}_*$, then set $\phi_0 = \iota_{\mathbb{N}^\mathbb{N},\mathbb{N}^\mathbb{N}_*}$, and note that the extension ψ of $\phi \circ \overline{\pi} \circ p^{-1}$ to $\mathbb{N}^{\leq\mathbb{N}}_*$ given by $\psi \restriction \mathbb{N}^\mathbb{N}_* = \psi' \restriction \mathbb{N}^\mathbb{N}_*$ is injective. To see that it is continuous, suppose that $(t_n)_{n\in\mathbb{N}}$ is an injective sequence of elements of $\mathbb{N}^{<\mathbb{N}}$ converging to $t \frown (\infty)$ for some $t \in \mathbb{N}^{<\mathbb{N}}$, fix $b_n \in \mathcal{N}_{t_n} \cap \mathbb{N}^\mathbb{N}$ for all $n \in \mathbb{N}$, and observe that the continuity of ψ' ensures that $\psi(b_n) \to \psi(t \frown (\infty))$, thus Lemma 4.6 implies that $\psi(t_n) \to \psi(t \frown (\infty))$. As $\mathbb{N}^{\leq\mathbb{N}}_*$ is compact, it follows that ψ is a closed continuous embedding. ⊠

5. Functions on $\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$

Here we provide a basis for the class of all functions from $\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ to analytic metric spaces.

Proposition 5.1. *Suppose that X is a topological space, $\phi \colon \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N} \to X$ is injective, and $x \in X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $x \notin (\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$.*

Proof. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ such that $x \notin \phi(\mathcal{N}_s)$, and define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$. ⊠

Proposition 5.2. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \overline{\pi})(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent or $\{(\phi \circ \overline{\pi})(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$.*

Proof. For each $t \in \mathbb{N}^{<\mathbb{N}}$, there is an injection $\iota_t \colon \mathbb{N} \to \mathbb{N}$ for which $(\phi(t \frown (\iota_t(i), \infty)))_{i \in \mathbb{N}}$ is convergent or $\{\phi(t \frown (\iota_t(i), \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete. Define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ by choosing $\pi(\emptyset) \in \mathbb{N}^{<\mathbb{N}}$ arbitrarily and setting $\pi(t \frown (i)) = \pi(t) \frown (\iota_{\pi(t)}(i))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$, and note that $(\phi \circ \overline{\pi})(t \frown (i, \infty)) = \phi(\pi(t \frown (i)) \frown (\infty)) = \phi(\pi(t) \frown (\iota_{\pi(t)}(i), \infty))$ for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{\mathbb{N}}$. ⊠

Proposition 5.3. *Suppose that X is a metric space, $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$, $F \subseteq X$ is finite, and $t \in \mathbb{N}^{<\mathbb{N}}$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that either $((\phi \circ \overline{\pi})(u \frown (\infty)))_{u \in \mathbb{N}^{<\mathbb{N}}}$ converges to an element of F or the closure of $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F.*

Proof. If the set $S_\epsilon = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(F, \epsilon)\}$ is $\sqsubseteq$-dense below t for all $\epsilon > 0$, then there exist an extension u of t and $x \in F$ such that the set $S_{\epsilon,x} = \{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s \frown (\infty)) \in \mathcal{B}(x, \epsilon)\}$ is $\sqsubseteq$-dense below u for all $\epsilon > 0$. Fix a sequence $(\epsilon_v)_{v \in \mathbb{N}^{<\mathbb{N}}}$ of positive real numbers converging to zero, and recursively construct a function $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ such that $\pi(v) \in S_{\epsilon_v, x}$ for all $v \in \mathbb{N}^{<\mathbb{N}}$ and $\pi(v) \frown (i) \sqsubseteq \pi(v \frown (i))$ for all $i \in \mathbb{N}$ and $v \in \mathbb{N}^{<\mathbb{N}}$, and observe that $(\phi \circ \overline{\pi})(v \frown (\infty)) \to x$.

Otherwise, fix $\epsilon > 0$ and an extension u of t with the property that $\mathcal{N}_u \cap S_\epsilon = \emptyset$, define $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_u \cap \mathbb{N}^{<\mathbb{N}}$ by $\pi(v) = u \frown v$, and note that the closure of $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is disjoint from F. ⊠

For the rest of this section, it will be convenient to fix an enumeration $(t_n)_{n \in \mathbb{N}}$ of $\mathbb{N}^{<\mathbb{N}}$ such that $t_m \sqsubseteq t_n \implies m \leq n$ for all $m, n \in \mathbb{N}$.

Proposition 5.4. *Suppose that X is a metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $((\phi \circ \overline{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges or for no natural numbers $m < n$ is $(\phi \circ \overline{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \overline{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \overline{\pi})(\mathcal{N}_{t_n})$.*

Proof. Suppose that for no $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is the sequence $((\phi \circ \overline{\pi})(t \frown (\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ convergent. By Proposition 5.2, we can assume that $(\phi(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent or $\{\phi(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete for all $t \in \mathbb{N}^{<\mathbb{N}}$. By recursively applying Lemma 5.3 to the functions $\phi_t = \phi \circ (\circ_{k < |t|} \overline{\pi_{t \restriction k}})$, we obtain

$\wedge$-embeddings $\pi_t \colon \mathbb{N}^{<\mathbb{N}} \to \mathcal{N}_t \cap \mathbb{N}^{<\mathbb{N}}$ such that there do not exist natural numbers $m < n$ for which $(\phi \circ (\circ_{k \leq |t_m|} \overline{\pi_{t_m \upharpoonright k}}))(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ (\circ_{k \leq |t_m|} \overline{\pi_{t_m \upharpoonright k}}))(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ (\circ_{k \leq |t_n|} \overline{\pi_{t_n \upharpoonright k}}))(\mathcal{N}_{t_n})$. Let π be the $\wedge$-embedding obtained from applying Proposition 2.3 to $(\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$, and observe that for no natural numbers $m < n$ is it the case that $(\phi \circ \overline{\pi})(t_m \frown (\infty))$ or a limit point of $\{(\phi \circ \overline{\pi})(t_m \frown (i, \infty)) \mid i \in \mathbb{N}\}$ in the closure of $(\phi \circ \overline{\pi})(\mathcal{N}_{t_n})$. ⊠

Theorem 5.5. *Suppose that X is an analytic metric space and $\phi \colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$. Then there is a $\wedge$-embedding $\pi \colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ such that $\phi \circ \overline{\pi}$ is constant, $\phi \circ \overline{\pi}$ extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{\leq \mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\leq \mathbb{N}}$, or $\mathbb{N}^{\leq \mathbb{N}}_*$.*

Proof. As before, we will repeatedly precompose ϕ with appropriate $\wedge$-embeddings, albeit this time so as to stabilize the behavior of the function $\psi = \phi \circ p^{-1}$, as opposed to that of the function ϕ itself. By applying Proposition 4.3 with any $\epsilon > 0$ and $t \in \mathbb{N}^{<\mathbb{N}}$, but replacing the given metric on X by one with respect to which all pairs of distinct points have distance at least ϵ from one another, we can assume that ψ is either constant or injective. As ϕ is constant in the former case, we can assume that we are in the latter.

By Proposition 4.4, we can ensure that $\psi(\mathbb{N}^{<\mathbb{N}})$ is closed and discrete or $\operatorname{diam} \psi(\mathcal{N}_t) \to 0$. As ψ is a closed continuous embedding in the former case, we can assume that we are in the latter.

Let $\overline{\psi}$ be the extension of ψ to the set $D = \mathbb{N}^{<\mathbb{N}} \cup \{b \in \mathbb{N}^{\mathbb{N}} \mid \lim_{i \to \infty} \psi(b \upharpoonright i) \text{ exists}\} \cup \{t \frown (\infty) \in \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \mid \lim_{i \to \infty} \psi(t \frown (i)) \text{ exists}\}$ given by $\overline{\psi}(b) = \lim_{i \to \infty} \psi(b \upharpoonright i)$ and $\overline{\psi}(t \frown (\infty)) = \lim_{i \to \infty} \psi(t \frown (i))$ for all $b \in D \cap \mathbb{N}^{\mathbb{N}}$ and $t \frown (\infty) \in D \cap (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. Note that D could potentially be any set of the form $\mathbb{N}^{<\mathbb{N}} \cup A$, where $A \subseteq \mathbb{N}^{\mathbb{N}}_*$ is analytic (simply consider the inclusion map from $\mathbb{N}^{<\mathbb{N}}$ to $\mathbb{N}^{<\mathbb{N}} \cup A$).

By Proposition 5.2, we can assume that $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \operatorname{dom}(\overline{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$.

As each point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, $\operatorname{diam} \psi(\mathcal{N}_{b \upharpoonright i}) \to 0$ for all $b \in \mathbb{N}^{\mathbb{N}}$, and $\operatorname{diam} \psi(\mathcal{N}_{t \frown (i)}) \to 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, it follows that $\overline{\psi}$ is continuous. To see that $\overline{\psi}$ is closed, it is sufficient show that every injective sequence $(c_n)_{n \in \mathbb{N}}$ of points in the domain of $\overline{\psi}$ for which $(\overline{\psi}(c_n))_{n \in \mathbb{N}}$ is convergent has a subsequence converging to a point in the domain of $\overline{\psi}$. By passing to a subsequence, we can assume that the sequence converges to a point of $\mathbb{N}^{\leq \mathbb{N}}_*$. As each point of $\mathbb{N}^{<\mathbb{N}}$ is isolated, the sequence converges to a point of $\mathbb{N}^{\mathbb{N}}_*$, so the facts that $\operatorname{diam} \psi(\mathcal{N}_{b \upharpoonright i}) \to 0$ for all $b \in \mathbb{N}^{\mathbb{N}}$, $\operatorname{diam} \psi(\mathcal{N}_{t \frown (i)}) \to 0$ for all $t \in \mathbb{N}^{<\mathbb{N}}$, and $\{\psi(t \frown (i)) \mid i \in \mathbb{N}\}$ has a limit point $\implies t \frown (\infty) \in \mathrm{dom}(\overline{\psi})$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ ensure that it converges to a point of the domain of $\overline{\psi}$.

By Proposition 2.8, we can assume that one of the following holds:

(1) $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \subseteq \mathrm{dom}(\overline{\psi})$ and $\forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\psi}(t) = \overline{\psi}(t \frown (\infty))$.
(2) $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \subseteq \mathrm{dom}(\overline{\psi})$ and $\forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\psi}(t) \neq \overline{\psi}(t \frown (\infty))$.
(3) $(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) \cap \mathrm{dom}(\overline{\psi}) = \emptyset$.

As the domain of $\overline{\psi}$ is analytic, so too is its intersection with $\mathbb{N}^{\mathbb{N}}$. It follows that the latter intersection has the Baire property, so Proposition 2.9 allows us to assume that one of the following holds:

(a) The domain of $\overline{\psi}$ is disjoint from $\mathbb{N}^{\mathbb{N}}$.
(b) The domain of $\overline{\psi}$ contains $\mathbb{N}^{\mathbb{N}}$.

In the special case that condition (b) holds, Theorem 3.6 allows us to assume that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is either constant or injective.

Proposition 5.4 allows us to assume that $(\psi(t))_{t \in \mathbb{N}^{<\mathbb{N}}}$ converges to some $x \in X$ or for no natural numbers $m < n$ is $\psi(t_m)$ or $\overline{\psi}(t_m \frown (\infty))$ in the closure of $\psi(\mathcal{N}_{t_n})$. In the former case, Proposition 5.1 allows us to assume that $\psi(\mathbb{N}^{<\mathbb{N}})$ is discrete, so the extension of ψ to $\mathbb{N}^{<\mathbb{N}}_*$ sending ∞ to x is a closed continuous embedding, thus we can assume that we are in the latter.

Lemma 5.6. *Suppose that $c, d \in \mathrm{dom}(\overline{\psi})$ are distinct but $\overline{\psi}(c) = \overline{\psi}(d)$. Then there exists $t \in \mathbb{N}^{<\mathbb{N}}$ such that $\{c, d\} = \{t, t \frown (\infty)\}$.*

Proof. To see that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ is injective, observe that if $m < n$, both $t_m \frown (\infty)$ and $t_n \frown (\infty)$ are in the domain of $\overline{\psi}$, and moreover $\overline{\psi}(t_m \frown (\infty)) = \overline{\psi}(t_n \frown (\infty))$, then $\overline{\psi}(t_m \frown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

To see that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}$ is injective when $\mathbb{N}^{\mathbb{N}}$ is contained in the domain of $\overline{\psi}$, note that otherwise it is constant, and let x be this constant value. Then for each $t \in \mathbb{N}^{<\mathbb{N}}$, there is a sequence $(u_i)_{i \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$ such that $\psi(t \frown (i) \frown (u_i)) \to x$, so the fact that $\mathrm{diam}\, \psi(\mathcal{N}_{t \frown (i)}) \to 0$ ensures that $\overline{\psi}(t \frown (\infty)) = x$, contradicting the fact that $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ is injective.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \psi(\mathbb{N}^{<\mathbb{N}}) = \emptyset$, note that if $b \in \mathrm{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, and $\overline{\psi}(b) = \psi(t)$, then there exist $m < n$ with $t_m = t$ and $t_n \sqsubset b$, so $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

To see that $\overline{\psi}(\mathbb{N}^{\mathbb{N}}) \cap \overline{\psi}(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) = \emptyset$, note that if $b \in \mathrm{dom}(\overline{\psi}) \cap \mathbb{N}^{\mathbb{N}}$, $t \in \mathbb{N}^{<\mathbb{N}}$, $t \frown (\infty) \in \mathrm{dom}(\overline{\psi})$, and $\overline{\psi}(b) = \overline{\psi}(t \frown (\infty))$, then there exist $m < n$ with $t_m = t$ and $t_n \sqsubset b$, in which case $\overline{\psi}(t_m \frown (\infty))$ is in the closure of $\psi(\mathcal{N}_{t_n})$, a contradiction.

Observe finally that if $s, t \in \mathbb{N}^{<\mathbb{N}}$ are distinct, $t \frown (\infty) \in \operatorname{dom}(\overline{\psi})$, and $\psi(s) = \overline{\psi}(t \frown (\infty))$, then there exist $m \neq n$ such that $t_m = s$ and $t_n = t$. Then $\psi(t_m)$ is in the closure of $\psi(\mathcal{N}_{t_n})$ and $\overline{\psi}(t_n \frown (\infty))$ is in $\psi(\mathcal{N}_{t_m})$, a contradiction. ⊠

If (1a) or (1b) holds, then $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\overline{\psi} \upharpoonright \mathbb{N}^{\mathbb{N}}_*$ is an extension of ϕ to a closed continuous embedding. If (2a), (2b), (3a), or (3b) holds, then $\overline{\psi}$ is an extension of ψ to a closed continuous embedding on $\mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\leq\mathbb{N}}_*$, $\mathbb{N}^{<\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}$. ⊠

Proposition 5.7. *Suppose that X is an analytic metric space, $\phi\colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to X$, $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-embedding, and $\phi \circ \overline{\pi}$ is constant, $\phi \circ \overline{\pi}$ extends to a closed continuous embedding on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$, $\mathbb{N}^{\leq\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}_*$. Then there exist $\phi_0 \in \{c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}\} \cup \{\iota_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, Z} \mid Z \in \{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}\}$ and $\psi\colon \overline{\phi_0(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} \to \overline{\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})}$ with the property that $(\overline{\pi} \upharpoonright \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \psi)$ is a closed continuous embedding of ϕ_0 into ϕ.*

Proof. If $\phi \circ \overline{\pi}$ is constant, then set $\phi_0 = c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}$ and let ψ be the unique function from $c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ to $(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. If $\phi \circ \overline{\pi}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, Z}$. And if $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding ψ on $Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}$, then set $\phi_0 = \iota_{\mathbb{N}^{<\mathbb{N}}, Z} \circ p$. ⊠

6. Borel functions that are not Baire class one

Here we provide bases for the classes of non-Baire-class-one Borel functions and non-σ-continuous-with-closed-witnesses Borel functions between analytic metric spaces.

Proposition 6.1. *Suppose that X is a metric space and $\phi\colon \mathbb{N}^{\mathbb{N}}_* \to X$ has the property that $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ is continuous and $\phi(\mathbb{N}^{\mathbb{N}}) \not\subseteq \overline{\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})}$. Then there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ with the property that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})} = \emptyset$.*

Proof. Fix $b \in \mathbb{N}^{\mathbb{N}}$ for which $\phi(b)$ is not in the closure of $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. Then there is an open neighborhood U of $\phi(b)$ disjoint from $\phi(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$, as well as an open neighborhood V of $\phi(b)$ whose closure is contained in U, in which case the continuity of $\phi \upharpoonright \mathbb{N}^{\mathbb{N}}$ yields a proper initial segment s of b for which $\phi(\mathcal{N}_s \cap \mathbb{N}^{\mathbb{N}}) \subseteq V$. Then the $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ given by $\pi(t) = s \frown t$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ is as desired. ⊠

Given $\phi_{\mathbb{N}^{\mathbb{N}}}\colon \mathbb{N}^{\mathbb{N}} \to X$ and $\phi_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}\colon \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \to Y$, let $\phi_{\mathbb{N}^{\mathbb{N}}} \sqcup \phi_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}$ denote the corresponding function from $\mathbb{N}^{\mathbb{N}}_*$ to the disjoint union $X \sqcup Y$.

Theorem 6.2. *Suppose that X and Y are analytic metric spaces and $\phi\colon X \to Y$ is a Borel function that is not Baire class one. Then there exist $\phi_{\mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}},Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}\}$ for which there is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}} \sqcup \phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into ϕ.*

Proof. Hurewicz's dichotomy theorem for F_σ sets yields a closed continuous embedding $\psi\colon \mathbb{N}^\mathbb{N}_* \to X$ with $(\phi \circ \psi)(\mathbb{N}^\mathbb{N}) \cap \overline{(\phi \circ \psi)(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$. As $(\psi, \mathrm{id}_{\overline{(\phi\circ\psi)(\mathbb{N}^\mathbb{N}_*)}})$ is a closed continuous embedding of $\phi \circ \psi$ into ϕ, by replacing the latter with the former, we can assume that $X = \mathbb{N}^\mathbb{N}_*$ and $\phi(\mathbb{N}^\mathbb{N}) \cap \overline{\phi(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$.

By Proposition 3.1, there is a $\wedge$-embedding $\pi\colon \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ for which $(\phi\circ\overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is continuous. By composing π with the $\wedge$-embedding given by Proposition 6.1, we can assume that $\overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N})} \cap \overline{(\phi \circ \overline{\pi})(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} = \emptyset$. By composing π with the $\wedge$-embedding given by Theorem 3.6, we can assume that $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}$ is constant or extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$. And by composing π with the $\wedge$-embedding given by Theorem 5.5, we can assume that $(\phi\circ\overline{\pi}) \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ is constant, $(\phi \circ \overline{\pi}) \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ extends to a closed continuous embedding on $\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$ or $\mathbb{N}^\mathbb{N}_*$, or $\phi \circ \overline{\pi} \circ p^{-1}$ extends to a closed continuous embedding on $\mathbb{N}^{<\mathbb{N}}$, $\mathbb{N}^{<\mathbb{N}}_*$, $\mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^\mathbb{N}$, $\mathbb{N}^{\leq\mathbb{N}}$, or $\mathbb{N}^{\leq\mathbb{N}}_*$.

By Proposition 3.7, there exist $\phi_{\mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\}$ and $\psi_{\mathbb{N}^\mathbb{N}}\colon \overline{\phi_{\mathbb{N}^\mathbb{N}}(\mathbb{N}^\mathbb{N})} \to \overline{\phi(\mathbb{N}^\mathbb{N})}$ for which $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}, \psi_{\mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}}$ into $\phi \restriction \mathbb{N}^\mathbb{N}$. By Proposition 5.7, there exist $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}} \in \{c_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}\} \cup \{\iota_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N},Z} \mid Z \in \{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_*\}\} \cup \{\iota_{\mathbb{N}^{<\mathbb{N}},Z} \circ p \mid Z \in \{\mathbb{N}^{<\mathbb{N}}, \mathbb{N}^{<\mathbb{N}}_*, \mathbb{N}^{\leq\mathbb{N}}_* \setminus \mathbb{N}^\mathbb{N}, \mathbb{N}^{\leq\mathbb{N}}, \mathbb{N}^{\leq\mathbb{N}}_*\}\}$ and $\psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}\colon \overline{\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})} \to \overline{\phi(\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})}$ for which $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}, \psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into $\phi \restriction \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}$. Then $(\overline{\pi} \restriction \mathbb{N}^\mathbb{N}_* \to \mathbb{N}^\mathbb{N}_*, \psi_{\mathbb{N}^\mathbb{N}} \sqcup \psi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}})$ is a closed continuous embedding of $\phi_{\mathbb{N}^\mathbb{N}} \sqcup \phi_{\mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N}}$ into ϕ. ⊠

Theorems 4.5 and 6.2 together provide the promised twenty-seven element basis under closed continuous embeddability for the class of non-σ-continuous-with-closed-witnesses Borel functions between analytic metric spaces.

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