ENDS OF GRAPHED EQUIVALENCE RELATIONS, I

BY

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ABSTRACT

Given a countable Borel equivalence relation E on a Polish space, we show: (1) E admits an endless graphing if and only if E is smooth, (2) E admits a locally finite single-ended graphing if and only if E is aperiodic, (3) E admits a graphing for which there is a Borel way of selecting two ends from each component if and only if E is hyperfinite, and (4) E admits a graphing for which there is a Borel way of selecting a finite set of at least three ends from each component if and only if E is smooth.

1. Introduction

A topological space X is **Polish** if it is separable and completely metrizable. A Borel equivalence relation E on X is **countable** if all of its classes are countable. The descriptive set-theoretic study of such equivalence relations has blossomed over the last several years (see, for example, Jackson-Kechris-Louveau [6]). A Borel graph $\mathscr{G}$ on X is a **graphing** of E if its connected components coincide with the equivalence classes of E. Here we study certain properties of graphings that yield information about their underlying equivalence relations.

A **ray** through $\mathscr{G}$ is an injective sequence $\alpha \in X^{\mathbb{N}}$ such that

$$\forall n \in \mathbb{N}\ ((\alpha(n), \alpha(n+1)) \in \mathscr{G}).$$

We use $[\mathscr{G}]^\infty$ to denote the standard Borel space of all such rays. A graph $\mathscr{T}$ is a **forest** (or **acyclic**) if its connected components are trees. Although these trees are unrooted, we can nevertheless recover their branches as equivalence classes of the associated **tail equivalence relation** $\mathscr{E}_\mathscr{T}$ on $[\mathscr{G}]^\infty$, given by

$$\alpha \mathscr{E}_\mathscr{T} \beta \Leftrightarrow \exists i, j \in \mathbb{N}\, \forall k \in \mathbb{N}\ (\alpha(i+k) = \beta(j+k)).$$

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Generalizing this to graphs, we obtain the relation $\mathscr{E}_{\mathscr{G}}$ of end equivalence. Two rays α, β through $\mathscr{G}$ are **end equivalent** if for every finite set $S \subseteq X$, there is a path from α to β through the graph $\mathscr{G}|(X \setminus S)$. Equivalently, the rays α, β are end equivalent if there is an infinite family $\{\gamma_n\}_{n \in \mathbb{N}}$ of pairwise vertex disjoint paths from α to β. An **end** of $\mathscr{G}$ is an equivalence class of $\mathscr{E}_{\mathscr{G}}$.

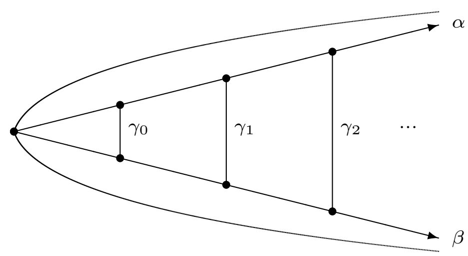

Figure 1: End-equivalent rays and the "infinite ladder" of paths between them.

Our goal here is to explore the connection between countable Borel equivalence relations and the ends of their graphings. In particular, we wish to understand the relationship between the structure of a countable Borel equivalence relation and the ability to select, in a Borel fashion, a given number of ends from each component of one of its graphings.

It should be noted that questions similar to those we study here have been studied in the measure-theoretic setting, e.g., in Adams [1], Blanc [2], Jackson-Kechris-Louveau [6], and Paulin [13]. Although our primary motivation is descriptive, all of our results imply their measure-theoretic counterparts.

In §2, we consider **endless** graphings, i.e., those which have no rays. Recall that E is **smooth** if there are countably many E-invariant Borel sets $B_n \subseteq X$ such that

$$\forall x, y \in X \ (xEy \Leftrightarrow \forall n \in \mathbb{N} \ (x \in B_n \Leftrightarrow y \in B_n)).$$

THEOREM A: *Every Borel equivalence relation on a Polish space that admits an endless graphing is smooth.*

This can be viewed as a generalization of Theorem 4.20 of Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [10], and can be used to affirmatively answer their Question 4.21 (although this question deals only with the case that E is countable).

In §3, we make a brief detour to discuss the problem of finding spanning subtreeings of graphings, which comes up in connection with the arguments of §2. Although we certainly do not offer a general solution to this problem, we do show:

THEOREM B (KECHRIS-MILLER): *For $n \in \mathbb{N}$, every locally countable Borel graph on a Polish space admits a spanning Borel subgraph with no cycles of length $\leq n$.*

In §4, we turn our attention to graphings whose connected components each have but one end. Such graphings are termed **single ended**. Recall that an equivalence relation E is **aperiodic** if each of its equivalence classes is infinite.

THEOREM C: *Every aperiodic countable Borel equivalence relation on a Polish space admits a locally finite single-ended graphing.*

In §5, we consider graphings with the property that there is a Borel way of selecting exactly two ends from each connected component. Recall that an equivalence relation E is **hyperfinite** if it is the union of finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots \subseteq E$. The following fact generalizes a theorem of Paulin [13], which itself generalizes a theorem of Adams [1]:

THEOREM D: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and there is a Borel way of selecting two ends from each $\mathscr{G}$-component. Then E is hyperfinite.*

In §6, we turn our attention to the selection of a finite number of ends, generalizing a result of Blanc [2] which strengthens a result of Paulin [13]:

THEOREM E: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and there is a Borel way of selecting a finite set of at least three ends from each $\mathscr{G}$-component. Then E is smooth.*

2. Endless graphings

As connected components of endless locally finite graphs are finite, only finite Borel equivalence relations admit endless locally finite graphings. The following theorem implies that only smooth Borel equivalence relations admit endless graphings:

THEOREM 2.1: *Suppose that X is a Polish space, E is a Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and $B \subseteq X$ is a Borel subset of X which intersects every ray through $\mathscr{G}$ in only finitely many points. Then $E|B$ is smooth.*

Proof: Suppose, towards a contradiction, that $E|B$ is non-smooth.

LEMMA 2.2: *There is a Borel set $Y \subseteq X$ such that $E|Y$ is a non-smooth hyperfinite equivalence relation, $\mathscr{G}|Y$ is a graphing of $E|Y$, and $B \cap Y$ intersects every equivalence class of $E|Y$.*

Proof: As usual, we use E_0 to denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$xE_0y \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

A **reduction** of E_0 into E is an injection $\pi : 2^{\mathbb{N}} \to X$ such that

$$\forall x, y \in 2^{\mathbb{N}}\ (xE_0y \Leftrightarrow \pi(x)E\pi(y)).$$

As E is non-smooth, Theorem 1.1 of Harrington-Kechris-Louveau [4] ensures that there is a continuous reduction π of E_0 into $E|B$.

We use $[\mathscr{G}]^{<\infty}$ to denote the standard Borel space of $\mathscr{G}$-connected finite subsets of X. Let R denote the set of pairs $(x, \langle S_k \rangle_{k\in\mathbb{N}}) \in 2^{\mathbb{N}} \times ([\mathscr{G}]^{<\infty})^{\mathbb{N}}$ such that:

1. $\forall k \in \mathbb{N}\ (\pi(x) \in S_k)$.
2. $\forall y \in [x]_{E_0}\, \exists k \in \mathbb{N}\ (\pi(y) \in S_k)$.

It is clear that R is Borel, so the Jankov-von Neumann uniformization theorem (see, for example, §18 of Kechris [7]) ensures the existence of a $\sigma(\boldsymbol{\Sigma}^1_1)$-measurable function $S : 2^{\mathbb{N}} \to ([\mathscr{G}]^{<\infty})^{\mathbb{N}}$ whose graph is contained in R. Let μ denote the usual product measure on $2^{\mathbb{N}}$.

Sublemma 2.3: *There is a μ-conull Borel set $A \subseteq 2^{\mathbb{N}}$ such that $S|A$ is Borel.*

Proof: Fix sets $\mathscr{S}_0, \mathscr{S}_1, \ldots$ generating the algebra of Borel subsets of $([\mathscr{G}]^{<\infty})^{\mathbb{N}}$, and for each $n \in \mathbb{N}$, fix Borel sets $A_n \subseteq S^{-1}(\mathscr{S}_n) \subseteq B_n$ such that $\mu(B_n \setminus A_n) = 0$. Now define $A \subseteq 2^{\mathbb{N}}$ by

$$A = 2^{\mathbb{N}} \setminus \left(\bigcup_{n\in\mathbb{N}} B_n \setminus A_n \right),$$

noting that A is μ-conull. As $(S|A)^{-1}(\mathscr{S}_n) = S^{-1}(\mathscr{S}_n) \cap A = A_n \cap A$, it follows that $S|A$ is Borel. ∎

As images of Borel sets under countable-to-one Borel functions are themselves Borel (see, for example, §18 of Kechris [7]), it follows that the set

$$Y = \{x \in X : \exists y \in A\, \exists k \in \mathbb{N}\ (x \in S_k(y))\}$$

is Borel, and it is clear that $\mathscr{G}|Y$ is a graphing of $E|Y$. As the restriction of E_0 to any μ-conull Borel set is non-smooth, it follows that $E|(B \cap Y)$ is non-smooth and hyperfinite, thus so too is $E|Y$. ∎

It is therefore enough to draw out a contradiction under the assumption that E is hyperfinite. A **treeing** of E is a graphing of E which is a forest.

LEMMA 2.4: *Suppose that X is a Polish space, E is a hyperfinite equivalence relation on X, and $\mathscr{G}$ is a graphing of E. Then $\mathscr{G}$ admits a spanning subtreeing.*

Proof: Fix an increasing sequence $F_0 \subseteq F_1 \subseteq \cdots \subseteq E$ of finite Borel equivalence relations such that $F_0 = \Delta(X)$ and $E = \bigcup_{n \in \mathbb{N}} F_n$. Recursively find a decreasing sequence $\mathscr{G} = \mathscr{G}_0 \supseteq \mathscr{G}_1 \supseteq \cdots$ of Borel graphs such that:

1. $\forall n \in \mathbb{N} \, \forall x \in X \; (\mathscr{G}_n | [x]_{F_n}$ is a tree).
2. $\forall n \in \mathbb{N} \; (F_n \cap (\mathscr{G}_n \setminus \mathscr{G}_{n+1}) = \emptyset)$.

Condition (1) ensures that the graph $\mathscr{T} = \bigcap_{n \in \mathbb{N}} \mathscr{G}_n$ is a forest, and condition (2) ensures that the injective $\mathscr{T}$-path between any two E-equivalent points x, y stabilizes at the first stage n for which $x F_n y$, thus $\mathscr{T}$ is a treeing of E. ∎

It is therefore enough to draw out a contradiction under the additional assumption that $\mathscr{T} = \mathscr{G}$ is a treeing of E. We say that $(x_0, x_1) \in \mathscr{T}$ **points towards** $A \subseteq X$ if there is an injective $\mathscr{T}$-path $x_0, x_1, \ldots, x_n$ such that $x_n \in A$. Put $B_0 = B$, set

$$B_{\alpha+1} = \{x \in B_\alpha : \exists y, z \in \mathscr{T}_x \; (y \neq z \text{ and } (x,y),(x,z) \text{ both point towards } B_\alpha)\},$$

where $\mathscr{T}_x = \{y \in X : (x,y) \in \mathscr{T}\}$, and define $B_\lambda = \bigcap_{\alpha < \lambda} B_\alpha$ at limit ordinals.

Let TREE denote the set of trees on $\mathbb{N}$, and let WF $\subseteq$ TREE denote the set of endless trees on $\mathbb{N}$. A betweenness-preserving embedding of $T \in$ TREE into $\mathscr{T}$ is a map $\pi : \mathbb{N} \to X$ such that

$$\forall \ell, m, n \in \mathbb{N} \; \big(\ell \text{ is } T\text{-between } m, n \Leftrightarrow \pi(\ell) \text{ is } \mathscr{T}\text{-between } \pi(m), \pi(n)\big).$$

We write $T \preceq \mathscr{T}$ to denote the existence of a betweenness-preserving embedding of T into $\mathscr{T}$. As $\mathscr{T}$ is Borel, the set

$$A = \{T \in \text{TREE} : T \preceq \mathscr{T}\}$$

is $\mathbf{\Sigma^1_1}$. By the boundedness theorem for WF (see, for example, Theorem 31.2 of Kechris [7], but note that our notation is somewhat different), there exists $\alpha < \omega_1$ such that every $T \in A$ is of rank less than α. Now, a simple induction shows that if $B_\beta \neq \emptyset$, then A contains a tree of rank β, and it follows that $B_\alpha = \emptyset$.

For each equivalence class $[x]_E$ of E, let $\alpha_{[x]_E} \leq \alpha$ be the least ordinal for which $B_{\alpha_{[x]_E}} \cap [x]_E = \emptyset$. As no ray through $\mathscr{T}$ intersects infinitely many points of B, it follows that $\alpha_{[x]_E}$ is not a limit ordinal. Let $\beta_{[x]_E}$ be the predecessor of $\alpha_{[x]_E}$, and observe that the definition of B_α ensures that

$$\forall x \in X \; (1 \leq |B_{\beta_{[x]_E}} \cap [x]_E| \leq 2),$$

from which it easily follows that $E|B$ is smooth, the desired contradiction. ∎

Remark 2.5: A **transversal** is a set $B \subseteq X$ which intersects every equivalence class of E in a single point. Every Borel equivalence relation which admits a Borel transversal is smooth. Although the converse is false, it does hold for countable Borel equivalence relations. It follows that the countable Borel equivalence relations which admit endless graphings are exactly those which have Borel transversals.

Remark 2.6: An equivalence relation E is **treeable** if it admits a treeing. By a result of Hjorth [5] (see also Miller [11]), a treeable equivalence relation is smooth exactly when it admits a Borel transversal. Thus, the treeable Borel equivalence relations which admit endless graphings are exactly those which are smooth.

Remark 2.7: Theorem 2.1 can be viewed as a generalization of Theorem 4.20 of Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [10], and yields an affirmative answer to their Question 4.21, which essentially asks if the equivalence relation induced by an endless locally countable Borel bipartite graph is necessarily smooth.

3. Spanning subtreeings

The following question arises from Lemma 2.4:

Question 3.1: Under what circumstances does a graphing of a countable Borel equivalence relation admit a spanning subtreeing?

It is easily seen that there are graphings which do not admit spanning subtreeings, as there are non-treeable countable Borel equivalence relations, yet every such equivalence relation admits a graphing. Although Question 3.1 remains open, we do know that if the proper cycles of $\mathscr{G}$ are of bounded length, then $\mathscr{G}$ has a spanning subtreeing. More generally, we have:

Theorem 3.2 (Kechris-Miller): *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $n \in \mathbb{N}$. Then every graphing of E admits a spanning subgraphing which has no cycles of length $\leq n$.*

Proof: Given a Borel graph $\mathscr{G}$ on X, set

$$[\mathscr{G}]^{\leq n} = \{S \in [\mathscr{G}]^{<\infty} : |S| \leq n\}.$$

The **intersection graph** induced by $\mathscr{G}$ is the graph $\mathfrak{G}$ on $[\mathscr{G}]^{<\infty}$ given by

$$(S,T) \in \mathfrak{G} \Leftrightarrow (S \neq T \text{ and } S \cap T \neq \emptyset).$$

A **κ-coloring** of a graph G on V is a function $c : V \to C$ such that

$$\forall v, w \in V \ ((v,w) \in G \Rightarrow c(v) \neq c(w)),$$

where C is of cardinality κ. When V and C are Polish and $c : V \to C$ is Borel, we call such a map a **Borel κ-coloring**. The **Borel chromatic number** of G is the least κ for which G admits a Borel κ-coloring. The **degree** of $v \in V$ is $\deg_G(v) = |G_v|$, and G is said to be **bounded** if $\sup_{v\in V} \deg_G(v)$ is finite.

LEMMA 3.3: *Suppose that X is a Polish space, $\mathscr{G}$ is a bounded Borel graph on X, $\mathfrak{G}$ is the corresponding intersection graph, and $n \in \mathbb{N}$. Then $\mathfrak{G}|[\mathscr{G}]^{\leq n}$ has finite Borel chromatic number.*

Proof: As $\mathscr{G}$ is bounded, so too is $\mathfrak{G}|[\mathscr{G}]^{\leq n}$, thus Proposition 4.6 of Kechris-Solecki-Todorcevic [9] ensures that $\mathfrak{G}|[\mathscr{G}]^{\leq n}$ has finite Borel chromatic number.∎

We can now verify a strengthening of the bounded case of the proposition:

LEMMA 3.4: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a bounded graphing of E, $n \in \mathbb{N}$, and $\mathscr{H} \subseteq \mathscr{G}$ is a Borel graph with no cycles of length $\leq n$. Then there is a spanning Borel graph $\mathscr{H} \subseteq \mathscr{H}' \subseteq \mathscr{G}$ which has no cycles of length $\leq n$.*

Proof: Fix a Borel coloring $c : [\mathscr{G}]^{\leq n} \to \{0, 1, \dots, k-1\}$ of $\mathfrak{G}|[\mathscr{G}]^{\leq n}$. We will recursively define a decreasing sequence of spanning subgraphs $\mathscr{G}_i \subseteq \mathscr{G}$, for $i \leq k$, beginning with $\mathscr{G}_0 = \mathscr{G}$. Given $\mathscr{G}_i$, define $\mathscr{C}_i \subseteq [\mathscr{G}_i]^{\leq n}$ by

$$\mathscr{C}_i = \{S \in [\mathscr{G}_i]^{\leq n} : c(S) = i\},$$

and fix a Borel map $S \mapsto G_S$ which assigns to each $S \in \mathscr{C}_i$ a finite graph $G_S \subseteq (\mathscr{G}_i \setminus \mathscr{H})|S$ such that $(\mathscr{G}_i \setminus G_S)|S$ is connected and has no cycles of length $\leq n$. Define

$$\mathscr{G}_{i+1} = \mathscr{G}_i \setminus \bigcup_{S \in \mathscr{C}_i} G_S,$$

noting that $\mathscr{H} \subseteq \mathscr{G}_{i+1} \subseteq \mathscr{G}$ and $\mathscr{G}_{i+1}$ is a spanning subgraph of $\mathscr{G}_i$.

It follows that $\mathscr{H}' = \mathscr{G}_k$ is a spanning subgraph of $\mathscr{G}$ which contains $\mathscr{H}$. Now suppose, towards a contradiction, that there is a cycle of length $\leq n$ in $\mathscr{H}'$. Then there exists a finite set $S \in [\mathscr{H}']^{\leq n}$ which contains such a cycle. Put $i = c(S)$, and note that $S \in [\mathscr{G}_i]^{\leq n}$, thus G_S was defined at stage i. It follows that

$$\mathscr{H}'|S = (\mathscr{H}' \cap G_S)|S \subseteq (\mathscr{G}_i \setminus G_S)|S,$$

which contradicts the fact that the latter set has no cycles of length $\leq n$. ∎

Now suppose that $\mathscr{G}$ is a graphing of E. By Feldman-Moore [3], $\mathscr{G}$ is the union of the graphs of countably many Borel involutions. As a consequence, there is an increasing sequence of bounded Borel graphs $\mathscr{G}_0 \subseteq \mathscr{G}_1 \subseteq \cdots$ whose union is

$\mathscr{G}$. Put $\mathscr{H}_0 = \emptyset$, and given a graph $\mathscr{H}_k \subseteq \mathscr{G}_k$ with no cycles of length $\leq n$, apply Lemma 3.4 to find a spanning Borel subgraph $\mathscr{H}_{k+1} \subseteq \mathscr{G}_k$, with no cycles of length $\leq n$, which contains $\mathscr{H}_k$.

It now follows that $\mathscr{H} = \bigcup_{k \in \mathbb{N}} \mathscr{H}_k$ is a spanning subgraph of $\mathscr{G}$ that contains no cycles of length $\leq n$. ∎

4. Single-ended graphings

Here we verify that the existence of a locally finite single-ended graphing says little about the equivalence relation in question:

THEOREM 4.1: *Suppose that X is a Polish space and E is an aperiodic countable Borel equivalence relation on X. Then E admits a locally finite single-ended graphing.*

Proof: We will obtain the theorem as a corollary of the following fact:

LEMMA 4.2: *There is a partition of X into Borel sets $B_0, B_1, \ldots$ and 2-to-1 Borel surjections $f_n : B_n \to B_{n+1}$ whose graphs are contained in E.*

Proof: As the lemma is a triviality when E is smooth, we can remove E-invariant Borel sets on which E is smooth at countably many stages of the construction.

We begin by associating with each Borel set $B \subseteq X$ an involution $i \in [E]$ whose support is B, off of an E-invariant Borel set on which E is smooth. By Theorem 1 of Feldman-Moore [3], there are Borel involutions $i_k : X \to X$ such that

$$E = \bigcup_{k \in \mathbb{N}} \operatorname{graph}(i_k).$$

Put $i_B^{(0)} = \emptyset$, recursively define partial functions $i_B^{(k+1)}$ by

$$i_B^{(k+1)}(x) = \begin{cases} i_B^{(k)}(x) & \text{if } x \in \operatorname{dom}(i_B^{(k)}), \\ i_k(x) & \text{if } x \neq i_k(x) \text{ and } x, i_k(x) \in B \setminus \operatorname{dom}(i_B^{(k)}), \text{ and} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

and define $i_B : X \to X$ by

$$i_B(x) = \begin{cases} i_B^{(k)}(x) & \text{if } x \in \operatorname{dom}(i_B^{(k)}), \\ x & \text{if } x \notin \bigcup_{k \in \mathbb{N}} \operatorname{dom}(i_B^{(k)}). \end{cases}$$

Noting that $B \setminus \bigcup_{k \in \mathbb{N}} \operatorname{dom}(i_B^{(k)})$ is a partial transversal of E, it follows that i_B is an involution with support B, off of an E-invariant Borel set on which E is smooth.

Now we proceed to the main construction. Fix a Borel linear ordering $\leq$ of X. We will recursively construct Borel sets $A_{n+1}, B_n \subseteq X$ such that

$$\forall n \in \mathbb{N}\ (A_{n+1} \text{ is the disjoint union of the sets } B_{n+1} \text{ and } A_{n+2} = i_{A_{n+1}}(B_{n+1})).$$

After throwing out an E-invariant Borel set on which E is smooth, we can assume that $X = \mathrm{supp}(i_X)$. We begin by setting

$$B_0 = \{x \in X : x < i_X(x)\} \text{ and } A_1 = \{x \in X : x > i_X(x)\}.$$

Now suppose we have found $A_{m+1}, B_m \subseteq X$, for $m \leq n$, as well as $f_m : B_m \to B_{m+1}$, for $m < n$. By throwing out an E-invariant Borel set on which E is smooth, we can assume that $A_{n+1} = \mathrm{supp}(i_{A_{n+1}})$. Now set

$$B_{n+1} = \{x \in X : x < i_{A_{n+1}}(x)\} \text{ and } A_{n+2} = \{x \in X : x > i_{A_{n+1}}(x)\},$$

and define $f_n : B_n \to B_{n+1}$ by

$$f_n(x) = \begin{cases} i_{A_n}(x) & \text{if } i_{A_n}(x) \in B_{n+1}, \\ i_{A_{n+1}} \circ i_{A_n}(x) & \text{otherwise.} \end{cases}$$

Clearly f_n is 2-to-1.

Sublemma 4.3: *The set $A = X \setminus \bigcup_{n\in\mathbb{N}} B_n$ is a partial transversal of E.*

Proof: Suppose, towards a contradiction, that A is not a partial transversal, and let k be the least natural number for which there are distinct $y, z \in A$ such that $i_k(y) = z$. Then there exists $n \in \mathbb{N}$ such that all of the points of the form $i_j(y), i_j(z)$ distinct from y, z, for $j < k$, lie in $B_0 \cup B_1 \cup \cdots \cup B_n$. Then $i_{A_{n+1}}(y) = z$, thus one of y, z lies in B_{n+1}, the desired contradiction. ∎

It follows that, after throwing out one more E-invariant Borel set on which E is smooth, the sets $B_0, B_1, \ldots$ and functions $f_0, f_1, \ldots$ are as desired. ∎

We now proceed to the proof of the theorem. Fix Borel sets $B_0, B_1, \ldots$ and functions $f_n : B_n \to B_{n+1}$ as in Lemma 4.2. By Theorem 3.12 of Jackson-Kechris-Louveau [6] (which is due also to Gaboriau), there are locally finite graphings $\mathscr{G}_n$ of $E|B_n$. Set $f = \bigcup_{n\in\mathbb{N}} f_n$, and define $\mathscr{G}$ on X by

$$\mathscr{G} = \mathrm{graph}(f^{\pm 1}) \cup \bigcup_{n\in\mathbb{N}} \mathscr{G}_n.$$

It is clear that $\mathscr{G}$ is locally finite, so it only remains to verify that for any equivalence class C of E, any two rays α, β through $\mathscr{G}|C$ are end equivalent. Associated with each $S \in [\mathscr{G}|C]^{<\infty}$ is the graph $\mathscr{G}_{\hat{S}} = \mathscr{G}|(C \setminus S)$. We must show that α, β

are $\mathscr{G}_{\hat{S}}$-connected. Note that the set $S' = \bigcup_{n\in\mathbb{N}} f^{-n}(S)$ is finite. It follows that there exists $n \in \mathbb{N}$ such that $\alpha(n), \beta(n) \notin S'$, and there exists $m \in \mathbb{N}$ such that

$$\forall \ell \geq m \ (S' \cap B_\ell = \emptyset).$$

Fix $\ell \geq m$ sufficiently large that $\alpha(n), \beta(n) \in \bigcup_{k<\ell} B_k$, and find iterates $x, y \in B_\ell$ of $\alpha(n), \beta(n)$ under f. Then α, x, y, β are $\mathscr{G}_{\hat{S}}$-connected. ∎

Remark 4.4: Although not every countable Borel equivalence relation admits a bounded graphing, the above argument can be used to show that every aperiodic countable Borel equivalence relation which admits a bounded graphing also admits a bounded single-ended graphing.

As noted by Adams [1], the situation is much different for single-ended treeings. We say that a treeing $\mathscr{T}$ of E is **directable** if there is a Borel $f : X \to X$ such that $\mathscr{T} = \text{graph}(f^{\pm 1})$. That is, E agrees with the **tail equivalence relation** associated with f, given by

$$x E_t(f) y \Leftrightarrow \exists m, n \in \mathbb{N} \ (f^m(x) = f^n(y)).$$

It follows from §1 of Jackson-Kechris-Louveau [6] that $E_t(f)$ is hyperfinite.

Proposition 4.5: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{T}$ is a treeing of E, and there is an $\mathscr{E}_{\mathscr{T}}$-invariant Borel set $\mathscr{B} \subseteq [\mathscr{T}]^\infty$ which selects a single end out of every connected component of $\mathscr{T}$. Then $\mathscr{T}$ is directable, thus E is hyperfinite.*

Proof: Let $f(x)$ be the unique $\mathscr{T}$-neighbor of x which is the initial point of a ray in $\mathscr{B}$ that avoids x. Then graph(f) is $\mathbf{\Sigma^1_1}$, thus f is Borel. As $\mathscr{T} = \text{graph}(f^{\pm 1})$, the proposition follows. ∎

Remark 4.6: There are treeings of hyperfinite equivalence relations for which there is no Borel way of choosing one end. For instance, Adams (unpublished) and others have pointed out that there are undirectable Borel forests of lines. However, there is essentially only one undirectable Borel forest of lines, in the sense that any two such forests are equivalent up to a natural analog of Nadkarni's [12] descriptive notion of Kakutani equivalence.

5. Selecting two ends

In this section, we describe those equivalence relations that admit graphings for which there is a Borel way of selecting exactly two ends from each component:

Theorem 5.1: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and there is a Borel way of selecting two ends from each $\mathscr{G}$-component. Then there is a Borel E-complete section $B \subseteq X$ and a graphing $\mathscr{L} \subseteq \mathscr{G}|B$ of $E|B$ whose components are lines, and E is hyperfinite.*

Proof: Fix a Borel set $\mathscr{B} \subseteq [\mathscr{G}]^\infty$ which selects two ends from each $\mathscr{G}$-component. For each $C \in X/E$ and $S \in [\mathscr{G}|C]^{<\infty}$, set $\mathscr{B}_{\hat{S}} = \mathscr{B} \cap [\mathscr{G}_{\hat{S}}]^\infty$ and define

$$\Phi = \{S \in [\mathscr{G}]^{<\infty} : \exists \alpha, \beta \in \mathscr{B}_{\hat{S}} \ (\alpha, \beta \text{ are not } \mathscr{G}_{\hat{S}}\text{-connected})\}.$$

It is clear that Φ is $\mathbf{\Sigma^1_1}$.

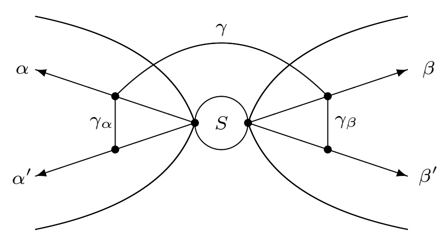

Figure 2: If α, β are $\mathscr{G}_{\hat{S}}$-connected, then so too are α', β'.

Lemma 5.2: *Suppose that $S \in \Phi$ and $\alpha, \beta \in \mathscr{B}_{\hat{S}}$ are end inequivalent. Then there is no path from α to β through $\mathscr{G}_{\hat{S}}$.*

Proof: Suppose, towards a contradiction, that there is a $\mathscr{G}_{\hat{S}}$-path γ from α to β. Fix rays $\alpha' \in [\alpha]_{\mathscr{E}_\mathscr{G}}$ and $\beta' \in [\beta]_{\mathscr{E}_\mathscr{G}}$ through $\mathscr{G}_{\hat{S}}$ which are not $\mathscr{G}_{\hat{S}}$-connected, let γ_α be a $\mathscr{G}_{\hat{S}}$-path from α' to α whose terminal point is the initial point of γ, and let γ_β be a $\mathscr{G}_{\hat{S}}$-path from β to β' whose initial point is the terminal point of γ (see Figure 2). Then $\gamma_\alpha \gamma \gamma_\beta$ is a $\mathscr{G}_{\hat{S}}$-path from α' to β', the desired contradiction. ∎

It follows that $S \in \Phi \Leftrightarrow \forall \alpha, \beta \in \mathscr{B}_{\hat{S}}$ ($\alpha \mathscr{E}_\mathscr{G} \beta$ or α, β are not $\mathscr{G}_{\hat{S}}$-connected), thus Φ is $\mathbf{\Pi^1_1}$, and therefore Borel.

Lemma 5.3: *There is a maximal pairwise disjoint Borel set $\Psi \subseteq \Phi$.*

Proof: As in §3, we define $\mathfrak{G}$ on $[\mathscr{G}]^{<\infty}$ by

$$(S, T) \in \mathfrak{G} \Leftrightarrow S \neq T \text{ and } S \cap T \neq \emptyset.$$

Sublemma 5.4: *There is a Borel coloring $c : [\mathscr{G}]^{<\infty} \to \mathbb{N}$ of $\mathfrak{G}$.*

Proof: Fix an increasing sequence of bounded Borel graphs $\mathscr{G}_n \subseteq \mathscr{G}$ such that $\mathscr{G} = \bigcup_{n \in \mathbb{N}} \mathscr{G}_n$. By Lemma 3.3, there are Borel colorings $c_n : [\mathscr{G}_n]^{\leq n} \to \mathbb{N}$ of $\mathfrak{G}|[\mathscr{G}_n]^{\leq n}$. Fix a bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ and define $c : [\mathscr{G}]^{<\infty} \to \mathbb{N}$ by

$$c(S) = \langle n(S), c_{n(S)}(S) \rangle,$$

where $n(S) \in \mathbb{N}$ is least such that $S \in [\mathscr{G}_n]^{\leq n}$. Clearly c is a coloring of $\mathfrak{G}$. ∎

Now fix a Borel coloring $c : [\mathscr{G}]^{<\infty} \to \mathbb{N}$ of $\mathfrak{G}$. Put $\Psi_0 = \emptyset$, and define

$$\Psi_{n+1} = \Psi_n \cup \{S \in \Phi : c(S) = n \text{ and } \forall T \in \Psi_n \ (S \cap T = \emptyset)\}.$$

It is straightforward to check that the set $\Psi = \bigcup_{n \in \mathbb{N}} \Psi_n$ is as desired. ∎

Now fix a maximal pairwise disjoint Borel set $\Psi \subseteq \Phi$, define $\mathscr{E}$ on Ψ by

$$S \mathscr{E} T \Leftrightarrow \exists x \in X \ (S \cup T \subseteq [x]_E),$$

and let $\mathfrak{T}$ be the set of pairs $(S, T) \in \mathscr{E}$, with $S \neq T$, such that there is $\mathscr{G}$-path from a point of S to a point of T which avoids all other points of $\bigcup \Psi$.

Lemma 5.5: *$\mathfrak{T}$ is a treeing of $\mathscr{E}$ whose vertices each have at most two neighbors.*

Proof: For each $C \in X/E$, set $\mathfrak{T}_C = \mathfrak{T}|C$ and $\Psi_C = \{S \in \Psi : S \subseteq C\}$. We must show that each $\mathfrak{T}_C$ is a tree whose vertices have at most two neighbors.

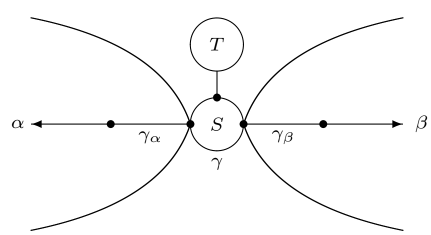

Figure 3: If T is not $\mathscr{G}_{\hat{S}}$-connected to α or β, it cannot disconnect them.

The following sublemma implies that for each $S \in \Psi_C$, every $\mathfrak{T}$-neighbor of S lies in one of the two connected components of $\mathscr{G}_{\hat{S}}$ which contains a ray in $\mathscr{B}_{\hat{S}}$:

Sublemma 5.6: *Suppose that $S, T \in \Psi_C$ and $S \neq T$. Then T is $\mathscr{G}_{\hat{S}}$-connected to a ray in $\mathscr{B}_{\hat{S}}$.*

Proof: Suppose, towards a contradiction, that T is not $\mathscr{G}_{\hat{S}}$-connected to a ray in $\mathscr{B}_{\hat{S}}$. Fix end-inequivalent rays $\alpha, \beta \in \mathscr{B}_{\hat{S}}$, let γ_α be a $\mathscr{G}$-path of minimal length from α to S, let γ_β be a $\mathscr{G}$-path of minimal length from S to β, and let γ be a $\mathscr{G}$-path from the terminal point of γ_α to the initial point of γ_β (see Figure 3). Then $\gamma_\alpha \gamma \gamma_\beta$ is a $\mathscr{G}_{\hat{T}}$-path from α to β, contradicting Lemma 5.2. ∎

The following sublemma implies that each element of Ψ_C has at most two $\mathfrak{T}_C$-neighbors:

Sublemma 5.7: *Suppose that $S, T, U \in \Psi_C$ are distinct and T is $\mathscr{G}_{\hat{S}}$-connected to U. Then exactly one of the following holds:*

1. *Every path from S to T goes through U.*
2. *Every path from S to U goes through T.*

Proof: Fix end-inequivalent rays $\alpha, \beta \in \mathscr{B}_{\hat{S}}$ which avoid $T \cup U$. Sublemma 5.6 ensures that, after reversing the roles of α, β if necessary, we can assume that T and U are both $\mathscr{G}_{\hat{S}}$-connected to β.

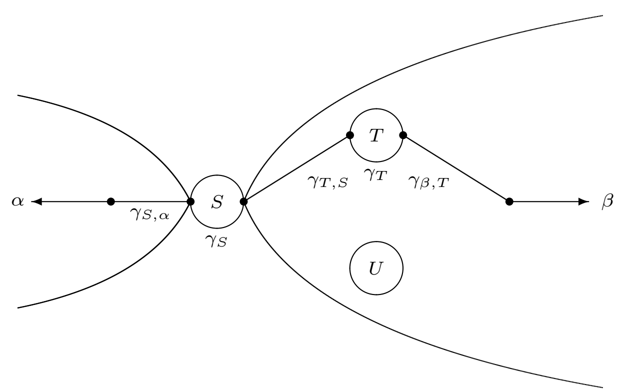

Figure 4: If $\gamma_{T,S}$ avoids U, then α, β are $\mathscr{G}_{\hat{U}}$-connected.

Fix a path $\gamma_{\beta,T}$ of minimal length from β to $T \cup U$. By reversing the roles of T and U if necessary, we can assume that $\gamma_{\beta,T}$ avoids U. Now suppose, towards a contradiction, that there is a $\mathscr{G}$-path from T to S which avoids U. Fix such a path $\gamma_{T,S}$ of minimal length, let γ_T be a $\mathscr{G}$-path through T from the terminal point of $\gamma_{\beta,T}$ to the initial point of $\gamma_{T,S}$, and let $\gamma_{S,\alpha}$ be a $\mathscr{G}$-path of minimal length from the terminal point of $\gamma_{T,S}$ to the initial point of α (see Figure 4). Then $\gamma_{\beta,T} \gamma_T \gamma_{T,S} \gamma_{S,\alpha}$ is a $\mathscr{G}_{\hat{U}}$-path from β to α, which contradicts Lemma 5.2. ∎

It easily follows that $\mathfrak{T}_C$ is connected and acyclic, and the lemma follows. ∎

Now define $A = \{x \in X : \forall S \in \Psi_{[x]_E}\ (S \text{ has two } \mathscr{T}\text{-neighbors})\}$. It follows from the Lusin-Novikov uniformization theorem that A is Borel.

LEMMA 5.8: *There is a Borel complete section $B \subseteq X \setminus A$ for $E|(X \setminus A)$ and a Borel graphing $\mathscr{L} \subseteq \mathscr{G}|(X \setminus A)$ of $E|B$ whose connected components are lines.*

Proof: For each E-class $C \subseteq X \setminus A$, the set of elements of Ψ_C which have exactly one $\mathfrak{T}$-neighbor is either of cardinality 1 or 2. It follows that $E|(X \setminus A)$ is smooth, from which the lemma easily follows. ∎

LEMMA 5.9: *There is a Borel complete section $B \subseteq A$ for $E|B$ and a Borel graphing $\mathscr{L} \subseteq \mathscr{G}|A$ of $E|B$ whose connected components are lines.*

Proof: Fix a Borel function which associates with each pair $(S, T) \in \mathfrak{T}|A$ a $\mathscr{G}$-path $\gamma_{S,T}$ of minimal length connecting S, T, such that $\gamma_{S,T} = \gamma_{T,S}$. Fix also a Borel function which associates with each $S \in \Psi_C$ a $\mathscr{G}$-path γ_S through S whose terminal points agree with those of $\gamma_{S,T}, \gamma_{S,U}$, where T, U are the $\mathfrak{T}$-neighbors of G. It is clear that the forest $\mathscr{L}$ which consists of all edges of paths of the form $\gamma_S, \gamma_{S,T}$, for $(S, T) \in \mathfrak{T}|A$, is as desired. ∎

It follows that there is a Borel E-complete section $B \subseteq X$ and a graphing $\mathscr{L} \subseteq \mathscr{G}|B$ of $E|B$ whose connected components are lines. This then implies that $E|B$ is hyperfinite (see, for example, Remark 6.8 of Kechris-Miller [8]). ∎

REMARK 5.10: Lemma 3.19 of Jackson-Kechris-Louveau [6], which itself builds on an argument of Adams [1], provides a measure-theoretic partial converse for Theorem 5.1. It implies that if μ is a probability measure on X, E is hyperfinite, and $\mathscr{T}$ is a treeing of E, then there is a μ-measurable way of selecting one or two ends from each connected component of $\mathscr{T}$. In fact, this is true for graphings as well, as can be easily seen via Lemma 2.4.

REMARK 5.11: There are treeings $\mathscr{T}$ of hyperfinite equivalence relations for which it is impossible to select one or two ends from each connected component of $\mathscr{T}$. However, there is essentially only one example, in the sense that any two such forests are equivalent up to a natural analog of Nadkarni's [12] descriptive notion of Kakutani equivalence.

6. Selecting finitely many ends

Here we consider Borel graphs for which there is a Borel way of selecting a finite set of at least three ends from each component. We note first the following general fact, which is of interest in its own right:

Proposition 6.1: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\mathscr{G}$ is a graphing of E. Then there is a Borel $\mathscr{E}_{\mathscr{G}}$-complete section $\mathscr{A} \subseteq [\mathscr{G}]^{\infty}$ such that $\mathscr{E}_{\mathscr{G}}|\mathscr{A}$ is countable.*

Proof: Given $S \in [\mathscr{G}]^{<\infty}$ and $\alpha \in [\mathscr{G}]^{\infty}$ which lie in the same E-class, we say that α **eventually settles** into a connected component C of $\mathscr{G}_{\hat{S}}$ if

$$\exists n \in \mathbb{N}\, \forall m \geq n\ (\alpha(m) \in C).$$

We use $C(\alpha, S)$ to denote this connected component.

By repeated use of the Lusin-Novikov uniformization theorem, we can find Borel functions $S_n : X \to [\mathscr{G}]^{<\infty}$ such that

$$\emptyset = S_0(x) \subseteq S_1(x) \subseteq \cdots \text{ and } [x]_E = \bigcup_{n\in\mathbb{N}} S_n(x),$$

for all $x \in X$. Given $x \in X$ and $\alpha \in [\mathscr{G}|[x]_E]^{\infty}$, let $\gamma_0^{(x,\alpha)}$ be the one-point path at x, and given $\gamma_n^{(x,\alpha)}$, let $\gamma_{n+1}^{(x,\alpha)}$ be a $\mathscr{G}$-path of minimal length which begins at the terminal point of the previous path, ends in $C(\alpha, S_{n+1}(x))$, and avoids $S_n(x)$. By again making repeated use of the Lusin-Novikov uniformization theorem, we can ensure that the maps $(x, \alpha) \mapsto \gamma_n^{(x,\alpha)}$ are Borel, thus so too is the map

$$(x, \alpha) \mapsto \beta_{x,\alpha} = \gamma_0^{(x,\alpha)} \gamma_1^{(x,\alpha)} \ldots .$$

As $\beta_{x,\alpha} \in [\alpha]_{\mathscr{E}_{\mathscr{G}}}$ and $\alpha \mathscr{E}_{\mathscr{G}} \alpha' \Rightarrow \beta_{x,\alpha} = \beta_{x,\alpha'}$, it follows that the set

$$\mathscr{A} = \{\alpha \in [\mathscr{G}]^{\infty} : \alpha = \beta_{\alpha(0),\alpha}\}$$

is as desired. ∎

We are now ready for the main result of this section:

Theorem 6.2: *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a graphing of E, and $\mathscr{B} \subseteq [\mathscr{G}]^{\infty}$ is an $\mathscr{E}_{\mathscr{G}}$-invariant Borel set which selects a finite set of at least three ends from each $\mathscr{G}$-component. Then E is smooth.*

Proof: Fix a Borel $\mathscr{A} \subseteq [\mathscr{G}]^{\infty}$ as in Proposition 6.1, and define $\Phi \subseteq [\mathscr{G}]^{<\infty}$ by

$$S \in \Phi \Leftrightarrow \forall \alpha, \beta \in \mathscr{A} \cap \mathscr{B} \cap [\mathscr{G}|[S]_E]^{\infty}\ (\alpha \mathscr{E}_{\mathscr{G}} \beta \text{ or } C(\alpha, S) \neq C(\beta, S)).$$

As each set of the form $\mathscr{A} \cap [\mathscr{G}|[S]_E]^{\infty}$ is countable, it follows that Φ is Borel. By Lemma 5.3, there is a maximal pairwise disjoint Borel set $\Psi \subseteq \Phi$. Note that the maximality of Ψ ensures that for each $x \in X$, there exists $S \in [\mathscr{G}|[x]_E]^{<\infty} \cap \Psi$. Moreover, since Ψ is pairwise disjoint and $\mathscr{B}$ contains at least three ends from the $\mathscr{G}$-component of x, it follows that there is exactly one such S, so E is smooth. ∎

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