Full groups, classification, and equivalence relations

by

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Abstract

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In Chapter I, we study algebraic properties of full groups of automorphisms of σ-complete Boolean algebras. We consider problems of writing automorphisms as compositions of periodic automorphisms and commutators (generalizing work of Fathi [34] and Ryzhikov [69]), as well as problems concerning the connection between normal subgroups of a full group and ideals on the underlying algebra, in the process giving a new proof (joint with David Fremlin) of Shortt's [73] characterization of the normal subgroups of the group of Borel automorphisms of an uncountable Polish space, as well as a characterization of the normal subgroups of full groups of countable Borel equivalence relations which are closed in the uniform topology of Bezuglyi-Dooley-Kwiatkowski [9]. We also characterize the existence of an E-invariant Borel probability measure in terms of a purely algebraic property of $[E]$.

The results of Chapter II include classifications of Borel automorphisms and Borel forests of lines up to the descriptive analog of Kakutani equivalence, along with applications to the study of Borel marriage problems, generalizing and strengthening results of Shelah-Weiss [72], Dougherty-Jackson-Kechris [24], and Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [58]. We also study the sorts of full groups on quotients of the form X/E for which the results of Chapter I do not apply. Actions of such groups satisfy a measureless ergodicity property which we exploit to obtain various classification and rigidity results. In particular, we obtain descriptive analogs of

some results of Connes-Krieger [19] and Feldman-Sutherland-Zimmer [37], answering a question of Bezuglyi.

In Chapter III, we study some descriptive properties of quasi-invariant measures. We prove a general selection theorem, and use this to show a descriptive set-theoretic strengthening of an analog of the Hurewicz ergodic theorem which holds for all countable Borel equivalence relations. This then leads to new proofs of Ditzen's quasi-invariant ergodic decomposition theorem [23] and Nadkarni's [62] characterization of the existence of an E-invariant probability measure, and also gives rise to a quasi-invariant version of Nadkarni's theorem, as well as a version for countable-to-one Borel functions. We close chapter III with results on graphings of countable Borel equivalence relations, strengthening theorems of Adams [1] and Paulin [65].

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Contents

Preface

The work presented here is a part of the descriptive set-theoretic study of countable Borel equivalence relations. At the heart of our explorations is a desire to better understand the descriptive core of various notions which originate in ergodic theory. We have grouped our results into three separate chapters, loosely based on the flavor of the arguments involved. Although each chapter is preceded by an in-depth introduction, we will now provide a brief overview of all three.

Suppose that E is a countable Borel equivalence relation on a Polish space X. The *full group* of E is the group $[E]$ of all Borel automorphisms $f : X \to X$ such that $xEf(x)$, for all $x \in X$. Early work of Dye [27] in the measure-theoretic context has made clear a very strong connection between equivalence relations and the algebraic structure of their full groups, and since then these groups have been the subject of much study within ergodic theory (see Connes-Krieger [19], Feldman-Sutherland-Zimmer [37], and Bezuglyi-Golodets [11]) and topological dynamics (see Giordano-Putnam-Skau [42] and Bezuglyi-Golodets [14]).

In Chapter I, we study algebraic properties of full groups of automorphisms. The results here are quite general, and we spend the vast majority of the chapter working with full groups of automorphisms of σ-complete Boolean algebras, rather than restricting ourselves to Polish spaces. The main observation which yields this generality is that there is a simple property, held by many automorphisms of such algebras, that is sufficient to push through various ideas from the study of orbit equivalence to this general setting. We consider problems of writing automorphisms as compositions of periodic automorphisms and commutators (generalizing work of Fathi [34] and Ryzhikov [69]), as well as problems concerning the connection between normal subgroups of a full group and ideals on the underlying algebra, in the process giving a new proof of Shortt's [73] characterization of the normal subgroups of the group of Borel automorphisms of an uncountable Polish space, as well as a version of Bezuglyi-Golodets's characterization of the uniformly closed normal subgroups of full groups of automorphisms of the Lebesgue measure algebra which holds in the uniform topology of Bezuglyi-Dooley-Kwiatkowski [9] on the group of Borel automorphisms of a Polish space. We also provide a characterization of the existence of an E-invariant Borel probability measure in terms of a purely algebraic property of $[E]$.

The common thread underlying the results of Chapter II is the observation that certain arguments of Dougherty-Jackson-Kechris [24], Harrington-Kechris-Louveau [44], and Shelah-Weiss [72] are sufficiently general so as to allow characterizations of various descriptive set-theoretic objects. Our results here include classifications of Borel automorphisms and Borel forests of lines up to the descriptive analog of Kakutani equivalence, along with applications to the study of Borel marriage problems, generalizing and strengthening results of Shelah-Weiss [72], Dougherty-Jackson-Kechris [24], and Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [58]. We also spend some time studying the sorts of full groups on quotients of the form X/E for which the results of Chapter I do not apply. The automorphisms of such groups satisfy a measureless ergodicity property which we exploit to obtain various classification and rigidity results. In particular, we obtain descriptive analogs of some results of Connes-Krieger [19] and Feldman-Sutherland-Zimmer [37], answering a question of Bezuglyi.

Suppose that μ is a probability measure on the Borel subsets of X. Then μ is *E-invariant* if every element of $[E]$ is measure-preserving, and μ is *E-quasi-invariant* if every element of $[E]$ is non-singular. The study of measured equivalence relations has played an important role in ergodic theory for a long time. Measures also turn out to be very important tools from the descriptive set-theoretic viewpoint. Notable here are papers of Feldman-Moore [36] and Weiss [79] which, in addition to being of great interest in their own right, have helped to bring closer the measure-theoretic and set-theoretic points of view.

In Chapter III, we study the descriptive properties of quasi-invariant measures. The main tool underlying our work here is a general selection theorem which allows us to build finite Borel subequivalence relations which are maximal with respect to whatever local property we desire. Using nothing more than this and our bare hands, we prove a descriptive set-theoretic strengthening of an analog of the Hurewicz ergodic theorem which holds for all countable Borel equivalence relations. This then leads to new proofs of Ditzen's quasi-invariant ergodic decomposition theorem [23] and Nadkarni's [62] characterization of the existence of an E-invariant probability measure, and also gives rise to a quasi-invariant version of Nadkarni's theorem, as well as a version for countable-to-one Borel functions.

Associated with a Borel graph $\mathscr{G}$ on X is an equivalence relation $E_{\mathscr{G}}$ on X,

$$xE_{\mathscr{G}}y \Leftrightarrow \exists x = x_0, x_1, \ldots, x_n = y \, \forall i < n \; ((x_i, x_{i+1}) \in \mathscr{G}).$$

We say that $\mathscr{G}$ *is a graphing of* E if $E = E_{\mathscr{G}}$. A great deal of work has been done to understand the connection between graph-theoretic properties of $\mathscr{G}$ and the structure of the induced equivalence relation $E_{\mathscr{G}}$, both in the contexts of ergodic theory and descriptive set theory (see Connes-Feldman-Weiss [18], Adams [1], Paulin [65], and Jackson-Kechris-Louveau [48]). We close chapter III by using our selection theorem to provide a variety of results on graphings of countable Borel equivalence relations, strengthening theorems of Adams [1] and Paulin [65].

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I am also thankful to many other mathematicians, whose ideas influenced my thesis. Much of Chapter I grew out of conversations with David Fremlin at the Fields Institute during the fall of 2002. Results from Chapters I and II were additionally stimulated by conversations with Sergey Bezuglyi at Caltech during the spring of 2003, and results from Chapter III by the work of Mahendra Nadkarni, with whom I spoke while at Fields. Much of the work in Chapter II is joint with Christian Rosendal.

More generally, I have benefitted greatly from conversations with Jack Feldman, Leo Harrington, Su Gao, Greg Hjorth, Steve Jackson, Alain Louveau, Slawek Solecki, and Vladimir Kanovei. They have always been eager to discuss mathematics, and happy to share their ideas.

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Chapter 1

Full Groups

1 Introduction

Suppose X is a Polish space and E is a countable Borel equivalence relation on X. The *full group of* E is the group $[E]$ of Borel automorphisms $f : X \to X$ such that

$$\forall x \in X \ (xEf(x)).$$

In this chapter, we study various algebraic properties of full groups, as well as their connection to properties of the underlying equivalence relation E.

One natural way that countable Borel equivalence relations arise is via group actions. Given a countable group Γ which acts on X by Borel automorphisms, the *orbit equivalence relation of* Γ is given by

$$xE_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y).$$

The *full group of* Γ, denoted by $[\Gamma]$, is simply the full group of E_Γ^X. Note that $[\Gamma]$ can be expressed entirely in terms of the action of Γ, as the group of Borel automorphisms $f : X \to X$ for which there is a partition of X into Borel sets B_γ such that

$$\forall \gamma \in \Gamma \ (f|B_\gamma = \gamma|B_\gamma).$$

Note that this definition generalizes to group actions on arbitrary Boolean algebras.

One advantage of working within full groups is that the resulting facts are often immediately applicable to other settings. For instance, if Γ acts by measure-preserving, non-singular, or category-preserving transformations, then so too does every element of $[\Gamma]$, as X can be decomposed into countably many pieces where its action agrees with the action of elements of Γ. So if we were to prove, for example, that every Borel automorphism is the composition of three Borel involutions from its full group, then we would automatically obtain the analogous theorems for measure-preserving, non-singular, and category-preserving transformations.

The techniques we use are applicable far beyond these settings, however. Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and π is an automorphism of $\mathfrak{A}$. Although our results do not go through for all such automorphisms, it should be noted that we essentially require only one additional property. We term an element $a \in \mathfrak{A}$ a *π-discrete section* if $a \cdot \pi(a) = \mathbb{O}$. The basic underlying assumption that is essential to our techniques is that all of the automorphisms with which we deal admit maximal discrete sections. The main observation underlying our results is that with maximal discrete sections at our disposal, we can push through many ideas from the study of orbit equivalence to this general setting.

Examples of automorphisms which admit maximal discrete sections include all automorphisms of the Borel subsets of a Polish space X (which can be naturally identified with the Borel automorphisms of X), as well as all automorphisms of any complete Boolean algebra. As a consequence, all of our results apply in these two contexts. Moreover, the inexistence of maximal discrete sections is often sufficient to prove strong negations of theorems which hold true for automorphisms which admit maximal discrete sections.

Although our work throughout this chapter is done in the context of σ-complete Boolean algebras, there is another point of view (recently pointed out to me by John Steel) which more accurately reflects what we do. Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and π is an automorphism of $\mathfrak{A}$ whose powers admits maximal discrete sections, and we wish to prove some algebraic fact about the full group $[\pi]$. For concreteness, let us again suppose that we wish to show that π is the composition of three involutions in $[\pi]$. As it turns out, this can be accomplished by proving that every transitive permutation σ of a countable set X is the composition of three involutions of X *which are definable in a certain restricted language.* The signature for this language consists of a function symbol, as well as a unary relation symbol for each $n \in \mathbb{N}$, the former to be thought of as representing the action of π and the latter representing maximal π^n-discrete sections. In order to obtain the desired result about π we must, in an $L_{\omega_1\omega}$-definable manner, build three involutions of X whose composition is σ. That is, we must be able to sculpt definitions of the involutions we desire out of nothing more than our function and relation symbols, existential and universal quantification, and conjunctions and disjunctions of countable length. We can then interpret the resulting three definitions in the L-structure determined by π and maximal discrete sections for each of its powers, and we will obtain the involutions in $[\pi]$ which we require.

Most of the results of this chapter can be viewed in this way. That is, we are given the problem of checking some algebraic fact about a full group, so we devise a language which is simultaneously sophisticated enough to check the property while weak enough to be interpreted via objects which we can build from the full group.

One more comment about the general structure of this chapter is in order. When deciding upon the language to use in presenting our results, we were faced with a real quandary. To what extent should we substitute readability for generality? In this particular case, the question became one of trading our intuition about purely atomic Boolean algebras for the ability to prove things about all complete Boolean algebras and beyond. Well, this isn't entirely true, in that we could have used Stone spaces to at least recover some of our intuition about the purely atomic case. This too seems problematic, however, as our primary interest is in the algebra of Borel subsets of a Polish space, and it seems rather ridiculous to prove any of the facts in which we are interested for this algebra via its Stone space.

In the end, we have decided to simply prove things in their utmost generality on σ-complete Boolean algebras, for the most part without making any references to Stone spaces. The end result is, unfortunately, that the proofs have come out looking a bit more difficult than they really are, particularly outside of the purely atomic case. For this reason, the reader is strongly encouraged to rely heavily upon his intuition for purely atomic algebras while reading our results. In retrospect, it would have been better to take up the viewpoint suggested by Steel, proving facts about permutation groups in certain restricted languages, and then separately establishing the transfer theorems necessary to push the results to the sorts of algebras we desire. Alas, time is short, and such refinements will have to wait for another day.

In §2, we take up a somewhat detailed exploration of maximal discrete sections. We provide several criteria for the existence of maximal discrete sections, and show that automorphisms of complete Boolean algebras and standard Borel spaces always admit such sections. We also show that the existence of a maximal π-discrete section is a natural generalization of the existence of a support for π, in that it is equivalent to the corresponding autohomeomorphism $\widehat{\pi}$ of the Stone space having clopen support. We close §2 by using maximal discrete section to give a simple proof of a general form of Rokhlin's Lemma which holds for arbitrary finitely additive probability measures on arbitrary Boolean algebras.

In §3, we introduce the full group $[\pi]$ of an automorphism π and describe various notions which originate in the study of orbit equivalence, such as recurrence, smoothness, and complete sections, with an emphasis on their connection to maximal discrete sections. We close with a general characterization of smoothness in Baire spaces.

We spend most of §4-§7 studying the circumstances under which an automorphism of a σ-complete Boolean algebra can be written as the composition of automorphisms of prescribed periods from its full group. Our explorations into such questions originate in a series of discussions with David Fremlin in the fall of 2003, while we were both visiting the Fields Institute. David had shown that every automorphism of a complete Boolean algebra is the composition of eight involutions, and asked if this number could be brought down to three. In §5, we answer this question positively by showing the following:

Theorem. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, π is an automorphism of $\mathfrak{A}$, and every power of π admits a maximal discrete section. Then π is the composition of three involutions from its full group.*

It should be noted that it was David who first pointed out the fact that our arguments, originally intended only for Borel automorphisms of Polish spaces, go through for arbitrary complete Boolean algebras. It should also be noted that, shortly after our success with this problem, David received an email from Peter Biryukov, who pointed out that Ryzhikov [70] had already shown that every automorphism of a complete Boolean algebra is the composition of three involutions. Although his argument is quite different than the one we shall provide, it is worth noting that he too made essential use of maximal discrete sections.

In §4, we characterize the circumstances under which an automorphism is the composition of two involutions from its full group, and we show that the existence of maximal discrete sections is necessary to write an aperiodic automorphism as a composition of periodic automorphisms of its full group. We also briefly discuss the problem of finding Borel automorphisms which are not the composition of two involutions.

In §6, we extend the notion of full group from a single automorphism to a group of automorphisms. Many of the results of §3 go through in this more general setting in a straightforward manner. New here is a notion of *aperiodicity* which makes sense outside of the purely atomic setting. This notion is quite nice in that it is a natural way of generalizing aperiodicity which seems to capture all of our intuition from the purely atomic case. In particular, we show that if we restrict our attention to actions by automorphisms that admit maximal discrete sections, then the aperiodic actions of Γ are exactly those with the property that for all $n \geq 1$, there is a partition of unity into n pieces and an automorphism in $[\Gamma]$ of exact period n which induces a permutation of these pieces.

In §7, we combine ideas of Ryzhikov [69] with our methods to show the following:

Theorem. *Suppose $\mathfrak{A}$ is a σ-complete Boolean algebra, π is an aperiodic automorphism of $\mathfrak{A}$, $n_0 \geq 3$, and $n_1 \geq 2$. Then the following are equivalent:*

1. *There exist $\pi_0, \pi_1 \in [\pi]$ of strict period n_0, n_1 such that $\pi = \pi_0 \circ \pi_1$.*

2. *The powers of π admits maximal discrete sections.*

As part of our proof, we give a general version of Alpern's [3] multiple Rokhlin tower theorem. Using similar methods, we also show that every aperiodic automorphism is a commutator within its full group. Using this, we show the following:

Theorem. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ which admit maximal discrete sections, and G has a subgroup of cardinality strictly less than κ which acts aperiodically. Then every element of G is a commutator.*

These facts generalize and strengthen results of Fathi [33] and Ryzhikov [69]. We close §7 by showing that the existence of a subgroup of cardinality less than κ which acts aperiodically is equivalent to the presence of an aperiodic automorphism in G.

In §8, we study an interesting property which has recently been investigated in the context of infinite permutation groups by Bergman [7], and in the context of more general automorphism groups by Droste-Göbel [25] and Droste-Holland [26]. We actually study two variants of Bergman's original notion:

1. G is *weakly Bergman* if for every increasing, exhaustive sequence $\Gamma_n \nearrow G$ of subsets of G, there exists $k, n \in \mathbb{N}$ such that $\Gamma_n^k = G$.

2. G is *strongly k-Bergman* if for every increasing, exhaustive sequence $\Gamma_n \nearrow G$ of subsets of G, there exists $n \in \mathbb{N}$ such that $\Gamma_n^k = G$. G is *strongly Bergman* if it strongly k-Bergman, for some $k \in \mathbb{N}$.

Both of these properties are quite strong. For instance, even weakly Bergman groups have the property that all of their left-invariant metrics are bounded. We show that the weak Bergman property is shared by every κ-full group of automorphisms of a κ-complete Boolean algebra which admit a maximal discrete section that contains an aperiodic subgroup of cardinality less than κ. We also show that strong Bergmanocity is often ruled out by the existence of invariant probability measures, and ensured by the presence of paradoxical decompositions. This leads to a characterization of the existence of an invariant probability measures for countable Borel equivalence relations in terms of a purely algebraic property of their full group:

Theorem. *Suppose E is an aperiodic countable Borel equivalence relation on a Polish space. Then $[E]$ has the weak Bergman property and the following are equivalent:*

1. *E does not admit an invariant Borel probability measure.*

2. *$[E]$ is strongly Bergman.*

3. *$[E]$ is strongly 16-Bergman.*

In §9, we give a new proof, joint with David Fremlin, of Shortt's characterization of the normal subgroups of the group of Borel automorphisms of a Polish space X. We actually give a technical strengthening of this result, by showing that if $f : X \to X$ is a Borel automorphism with uncountable support, then every Borel automorphism of X is a composition of four conjugates of $f^{\pm 1}$. Getting back to our general setting, we then use ideas of Fremlin [39] to draw a connection between the normal subgroups of an aperiodic full group and ideals on the underlying σ-algebra.

In §10, we make use of this connection. We begin by defining a weak notion of closure for sequences of automorphisms, and characterize the normal subgroups of full groups which satisfy this property. Using this, we obtain a new proof of the characterization of closed normal subgroups of full groups due to Bezuglyi-Golodets [11]. We close the section with an analog of their result which holds for the group of Borel automorphisms of a Polish space X, when equipped with the uniform topology of Bezuglyi-Dooley-Kwiatkowski [9], which is generated by the sets of the form

$$\mathscr{U}(\varphi, \mu_0, \ldots, \mu_n, \epsilon) = \{\psi : \forall i \leq n \ (\mu_i(\mathrm{supp}(\varphi \circ \psi^{-1})) < \epsilon)\},$$

where φ, ψ are Borel automorphisms of X, $\mu_0, \ldots, \mu_n$ are probability measures on X, and $\epsilon > 0$. We show the following:

Theorem. *Suppose that G is a σ-full group of Borel automorphisms of a Polish space, and G contains a countable aperiodic subgroup. Then the uniformly closed normal subgroups of G are exactly those of the form*

$$N = \{\pi \in G : \mathrm{supp}(\pi) \in \mathrm{NULL}_M\},$$

where M is a G-invariant set of probability measures on X.

2 Maximal discrete sections

In this section, we will introduce the many of the concepts which will be important throughout Chapter I. Central here is the notion of a *maximal discrete section* for an automorphism. We show that such sections can always be found in many of the σ-complete Boolean algebras which appear in descriptive set theory and measure theory. We also give an alternate characterization of their existence in terms of the Stone space of the algebra in question. As an application, we show an analog of Rokhlin's Lemma which holds for an arbitrary probability measure on an arbitrary Boolean algebra.

Although we work with automorphisms of general σ-complete Boolean algebras, it will be useful to keep several important examples in mind. The complete Boolean algebras in which we are particularly interested include the algebra of Lebesgue measurable subsets of the reals modulo null sets, as well as the algebra of Baire measurable subsets of a perfect Polish space modulo meager sets. The main incomplete Boolean algebra in which we are interested is the algebra of Borel subsets of an uncountable Polish space X. We are also interested in the various subalgebras obtained by fixing a countable Borel equivalence relation E on X, and restricting our attention to those Borel subsets of X which are E-invariant.

Suppose that $\mathfrak{A}$ is a Boolean algebra. We will use $\mathrm{Aut}(\mathfrak{A})$ to denote the *automorphism group of* $\mathfrak{A}$, and

$$\mathfrak{A}_a = \{b \in \mathfrak{A} : b \leq a\}$$

to denote the *principal ideal induced by* $a \in \mathfrak{A}$. An automorphism π *has a support* if

$$\sum \left\{a \in \mathfrak{A} : \pi | \mathfrak{A}_a = \mathrm{id}\right\}$$

exists, in which case the *support of* π is the complement of this sum.

An element $a \in \mathfrak{A}$ is a π-*discrete section*, or simply π-*discrete*, if $a \cdot \pi(a) = \mathbb{O}$. Note that when $\mathfrak{A}$ is purely atomic, every automorphism of $\mathfrak{A}$ determines a graph $\mathscr{G}$ on the atoms of $\mathfrak{A}$, in which two atoms are neighbors exactly when π carries one to the other. Note that the connected components of this graph are simply the orbits of the atoms of $\mathfrak{A}$ under π, and that $a \in \mathfrak{A}$ is π-discrete exactly when no two $\mathscr{G}$-neighbors are below a.

Proposition 2.1. *Suppose $\mathfrak{A}$ is a Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$, and $a \in \mathfrak{A}$. Then*

$$a \textit{ is } \pi\textit{-discrete} \Leftrightarrow a \textit{ is } \pi^{-1}\textit{-discrete} \Leftrightarrow \forall n \in \mathbb{Z}\ (\pi^n(a) \textit{ is } \pi\textit{-discrete}).$$

Proof. As $a \cdot \pi^{-1}(a) = \pi^{-1}(a \cdot \pi(a))$, it follows that

$$a \text{ is } \pi\text{-discrete} \Leftrightarrow a \text{ is } \pi^{-1}\text{-discrete}.$$

As $\pi^n(a) \cdot \pi^{n+1}(a) = \pi^n(a \cdot \pi(a))$, it follows that if a is π-discrete, then

$$\forall n \in \mathbb{Z}\ (\pi^n(a) \text{ is } \pi\text{-discrete}),$$

which completes the proof of the proposition. ⊣

Discrete sections provide a convenient alternative description of supports:

Proposition 2.2. *Suppose $\mathfrak{A}$ is a Boolean algebra, $a \in \mathfrak{A}$, and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then*

$$\pi|\mathfrak{A}_a \neq \mathrm{id} \Leftrightarrow \exists \mathbb{O} < b \leq a\ (b \textit{ is } \pi\textit{-discrete}).$$

Proof. To see ($\Leftarrow$), simply note that if $b > \mathbb{O}$ is π-discrete, then $\pi(b) \neq b$. To see ($\Rightarrow$), fix $\mathbb{O} < b \leq a$ such that $\pi(b) \neq b$, and note that at least one of $\pi(b) - b$ and $b - \pi(b)$ is non-zero, and both are clearly π-discrete. As

$$b - \pi^{-1}(b) = \pi^{-1}(\pi(b) - b),$$

it follows that $b - \pi^{-1}(b)$ or $b - \pi(b)$ is a non-zero, π-discrete element of $\mathfrak{A}_a$. ⊣

Corollary 2.3. *Suppose $\mathfrak{A}$ is a Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then*

$$\pi \textit{ has a support} \Leftrightarrow \sum \left\{a \in \mathfrak{A} : a \textit{ is } \pi\textit{-discrete}\right\} \textit{ exists},$$

in which case the support of π is this sum.

Proof. Simply observe that by Proposition 2.2, if one of

$$\sum\{a \in \mathfrak{A} : \pi|\mathfrak{A}_a = \mathrm{id}\}, \sum\{a \in \mathfrak{A} : a \text{ is } \pi\text{-discrete}\}$$

exists, then the other is dense in its complement. ⊣

Remark 2.4. An element $a \in \mathfrak{A}$ is *π-invariant* if $\pi(a) = a$. Note that if π has a support, then $\mathrm{supp}(\pi)$ is π-invariant. Also note that if π has a support, then so too does π^{-1} and $\mathrm{supp}(\pi) = \mathrm{supp}(\pi^{-1})$.

Remark 2.5. A *maximal π-discrete section* is a π-discrete section which is not strictly below any other π-discrete section. Note that for all $a \in \mathfrak{A}$,

$$a \text{ is maximal } \pi\text{-discrete} \Leftrightarrow \mathrm{supp}(\pi) - a = \pi^{-1}(a) + \pi(a).$$

Maximal discrete sections will prove to be an important tool throughout this chapter, playing a role similar to that of Rokhlin's Lemma in ergodic theory. Recall that a Boolean algebra is *complete* if every subset of $\mathfrak{A}$ has a least upper bound.

Proposition 2.6. *Every automorphism of a complete Boolean algebra admits a maximal discrete section.*

Proof. Suppose $\mathfrak{A}$ is a complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Let $\langle a_\xi \rangle$ be an increasing sequence of π-discrete sections which is of maximal length, and observe that $a = \sum_\xi a_\xi$ is a maximal π-discrete section. ⊣

Suppose that $\kappa \geq \omega_1$. The Boolean algebra $\mathfrak{A}$ is *κ-complete* if every subset of $\mathfrak{A}$ of cardinality strictly less than κ has a least upper bound. Also, $\mathfrak{A}$ is *σ-complete* if it is ω_1-complete. Suppose $\mathscr{A} \subseteq \mathfrak{A}$. We will use $\mathscr{A}^+$ to denote the non-zero elements of $\mathscr{A}$. A set $\mathscr{B} \subseteq \mathscr{A}$ is *dense in $\mathscr{A}$* if

$$\forall a \in \mathscr{A}^+ \, \exists b \in \mathscr{B}^+ \; (b \leq a).$$

Elements $a, b \in \mathfrak{A}$ are *disjoint* if $a \cdot b = \mathbb{O}$, and *compatible* if $a \cdot b > \mathbb{O}$. A set $\mathscr{B} \subseteq \mathscr{A}$ is *predense* if every $a \in \mathscr{A}$ is *compatible* with some $b \in \mathscr{B}$.

Proposition 2.7. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then the following are equivalent:*

1. *There is a maximal π-discrete section.*

2. *π has a support and it is the sum of three π-discrete sections.*

3. *π has a support and it is the sum of fewer than κ sections which are π-discrete.*

4. *The set of π-discrete sections has a predense subset of cardinality $< \kappa$.*

Proof. To see (1) $\Rightarrow$ (2), note that if a is a maximal π-discrete section, then

$$\mathrm{supp}(\pi) = \pi^{-1}(a) + a + \pi(a),$$

by Remark 2.5. Of course (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is trivial. To see (4) $\Rightarrow$ (1), fix an enumeration $\langle a_\xi \rangle_{\xi<\lambda}$ of a predense subset of the set of π-discrete sections, where $\lambda < \kappa$. We will inductively paste together pieces of the a_ξ's, at stage ξ adding the largest piece of a_ξ that is possible, without destroying the discreteness of the element of $\mathfrak{A}$ that we have built thus far. Put $b_0 = \mathbb{O}$, and recursively define

$$b_{\xi+1} = b_\xi + (a_\xi - (\pi^{-1}(b_\xi) + \pi(b_\xi))),$$

setting $b_\xi = \sum_{\zeta<\xi} b_\zeta$ at limit ordinals. Noting that

$$\begin{aligned} b_{\xi+1} \cdot \pi(b_{\xi+1}) &\leq (b_\xi + (a_\xi - \pi(b_\xi))) \cdot (\pi(b_\xi) + (\pi(a_\xi) - b_\xi)) \\ &\leq b_\xi \cdot \pi(b_\xi) + a_\xi \cdot \pi(a_\xi), \end{aligned}$$

it follows from the obvious induction that $b = b_\lambda$ is π-discrete. Suppose, towards a contradiction, that there is a π-discrete section $c > b$. Then there exists $\xi < \lambda$ with $a_\xi \cdot (c - b) \neq \mathbb{O}$, and it follows from the definition of $b_{\xi+1}$ that

$$\mathbb{O} < a_\xi \cdot (c - b) \leq \pi^{-1}(b_\xi) + \pi(b_\xi),$$

thus $c \cdot (\pi^{-1}(c) + \pi(c)) \neq \mathbb{O}$, contradicting the assumption that c is π-discrete. $\dashv$

Remark 2.8. It is worth noting that our proof of (4) $\Rightarrow$ (1) above shows the stronger fact that every π-discrete section sits below a maximal π-discrete section.

An *antichain* is a pairwise disjoint subset of $\mathfrak{A}$. A Boolean algebra $\mathfrak{A}$ satisfies the *κ-chain condition* if every antichain of $\mathfrak{A}$ is of cardinality strictly less than κ. Also, $\mathfrak{A}$ satisfies the *countable chain condition* if it satisfies the ω_1-chain condition. It is easy to see that if $\mathfrak{A}$ is κ-complete and satisfies the κ-chain condition, then every automorphism of $\mathfrak{A}$ admits a maximal discrete section, for these two assumptions ensure that $\mathfrak{A}$ is complete.

Example 2.9. Suppose that μ is a σ-finite measure on a set X. Then the corresponding measure algebra $\mathfrak{A}_\mu$ satisfies the countable chain condition and is therefore complete. On the other hand, the algebra of Borel subsets of a Polish space is a σ-complete Boolean algebra which neither satisfies the countable chain condition nor is ω_2-complete.

Fortunately, the κ-chain condition has a natural weakening which does not imply completeness, but still ensures the existence of maximal discrete sections. A set $\mathscr{A} \subseteq \mathfrak{A}$ is a *separating family for* $\mathfrak{A}$ if there is a dense set $\mathscr{B} \subseteq \mathfrak{A}$ such that $\mathscr{A}$ separates all disjoint pairs of elements of $\mathscr{B}$, i.e.,

$$\forall b, b' \in \mathscr{B} \ \big(b \cdot b' = \mathbb{0} \Rightarrow \exists a \in \mathscr{A} \ (b \leq a \text{ and } b' \leq \mathbb{1} - a)\big).$$

A Boolean algebra $\mathfrak{A}$ is *purely atomic* if its atoms are dense. In this case,

$$\mathscr{A} \text{ is a separating family for } \mathfrak{A} \Leftrightarrow \mathscr{A} \text{ is a separating family for the atoms of } \mathfrak{A}.$$

Example 2.10. Suppose that $\mathfrak{A} \subseteq \mathscr{P}(2^\kappa)$ is any Boolean algebra which contains the sets of the form

$$X_\alpha = \{x \in 2^\kappa : x_\alpha = 0\},$$

for $\alpha < \kappa$. Then $\mathfrak{A}$ admits a separating family of cardinality κ.

Although unnecessary for our purposes, it is worth noting that any algebra which admits a separating family of cardinality strictly less than κ necessarily satisfies the $2^{<\kappa}$-chain condition.

Proposition 2.11. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra that admits a separating family of cardinality strictly less than κ. Then every automorphism of $\mathfrak{A}$ admits a maximal discrete section.*

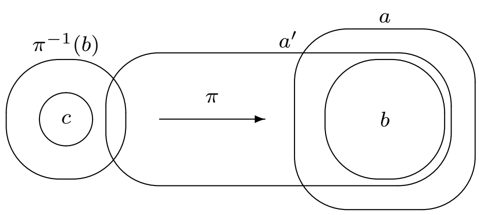

Figure 1.1: An element $a' \in \mathscr{A}$ such that $\mathbb{O} < \pi(c) \leq a \cdot (a' - \pi(a'))$.

Proof. Fix a separating family $\mathscr{A} \subseteq \mathfrak{A}$ of cardinality strictly less than κ, along with the corresponding dense set $\mathscr{B} \subseteq \mathfrak{A}$, and suppose $\pi \in \mathrm{Aut}(\mathfrak{A})$. Of course we may expand $\mathscr{A}$ so as to ensure that it is an algebra which is closed under $\pi^{\pm 1}$. To see that π admits a maximal discrete section, it follows from Proposition 2.7 that it is enough to check $\{a - \pi(a) : a \in \mathscr{A}\}$ is weakly dense in the set of π-discrete sections, or equivalently, that

$$\forall a \in \mathfrak{A}^+ \ \big(a \text{ is } \pi\text{-discrete} \Rightarrow \exists a' \in \mathscr{A} \ (a \cdot (a' - \pi(a')) \neq \mathbb{O})\big). \qquad (\dagger)$$

Suppose that $a \in \mathfrak{A}^+$ is π-discrete, fix non-zero elements $b, c \in \mathscr{B}$ such that $b \leq a$ and $c \leq \pi^{-1}(b)$, and find $a' \in \mathscr{A}$ which separates b from c. Then $\pi(c) \leq a' - \pi(a')$, and $(\dagger)$ follows. $\dashv$

The following fact is a special case of Lemma 4.1 of Kechris-Solecki-Todorcevic [56], whose proof is the source of the arguments provided thus far:

Proposition 2.12 (Kechris-Solecki-Todorcevic). *Every automorphism of the algebra of Borel subsets of a Polish space admits a maximal discrete section.*

Proof. This follows from the fact that Polish topologies are separable and Hausdorff, the observation that any basis for a Hausdorff topology separates points, and Proposition 2.11. $\dashv$

Although Proposition 2.11 establishes the existence of maximal discrete sections for all automorphisms of most of the algebras in which we are interested, we will continue to work in a general setting in which their existence is not guaranteed. One reason for this is our desire to establish the abstract fact that the existence of these sections alone is sufficient to derive many algebraic properties. Moreover, we obtain dichotomy results because of the strength of the inexistence of such sections. A more concrete reason is that one of the sorts of algebras in which we are interested has automorphisms which do not admit maximal discrete sections:

Example 2.13. Suppose $\mathscr{C} = 2^{\mathbb{N}}$ is *Cantor space*, and define E_0 on $\mathscr{C}$ by

$$xE_0y \Leftrightarrow \forall^\infty n \in \mathbb{N}\ (x_n = y_n),$$

where "$\forall^\infty$" is shorthand for "for all but finitely many." Letting $s : \mathscr{C} \to \mathscr{C}$ be the *unilateral shift*,

$$s(x_0x_1\ldots) = x_1x_2\ldots,$$

it is easily verified that s induces an aperiodic automorphism of the algebra $\mathscr{B}_{E_0}$ of E_0-invariant Borel subsets of $\mathscr{C}$. It is also easily verified that any discrete section for s is meager, and therefore cannot be maximal! It is worth noting that there are few such automorphisms. Modulo the open question of whether all $\mathbb{Z} * \mathbb{Z}$-orderable equivalence relations are hyperfinite, the methods of §6-§8 of Chapter II suffice to establish that the least $n > 0$ for which there is no maximal $\pi^{<n}$-discrete section is a complete invariant for conjugacy of aperiodic automorphisms of $\mathscr{B}_{E_0}$ with Borel graphs, when such an n exists.

It will frequently be useful to have an analog of maximal discrete sections for finite collections of automorphisms. For $\Delta \subseteq \mathrm{Aut}(\mathfrak{A})$, an element $a \in \mathfrak{A}$ is Δ-*discrete* if

$$\forall \gamma, \delta \in \Delta\ \big(\gamma \neq \delta \Rightarrow (\delta \cdot a) \cdot (\gamma \cdot a) = \mathbb{O}\big),$$

or equivalently, if

$$\forall \gamma, \delta \in \Delta\ \big(\gamma \neq \delta \Rightarrow \delta \cdot a \text{ is } \gamma\delta^{-1}\text{-discrete}\big).$$

Note that π-discreteness and $\{1, \pi\}$-discreteness are equivalent. More generally,

$$a \text{ is } \pi^{\leq n}\text{-discrete} \Leftrightarrow \forall 1 \leq i \leq n\ \big(a \cdot \pi^i(a) = \mathbb{O}\big),$$

where $\pi^{\leq n}$ is shorthand for the set of automorphisms of the form π^i, with $0 \leq i \leq n$. The natural generalization of Remark 2.5 goes through here:

Proposition 2.14. *Suppose* $\mathfrak{A}$ *is a Boolean algebra,* $\pi \in \mathrm{Aut}(\mathfrak{A})$*,* $a \in \mathfrak{A}$*, and* $n > 0$*. Then*

$$a \textit{ is maximal } \pi^{<n}\textit{-discrete} \Leftrightarrow \prod_{0<i<n} \mathrm{supp}(\pi^i) - a = \sum_{0<|i|<n} \pi^i(a).$$

Proof. It is straightforward to check that

$$a \text{ is } \pi^{<n}\text{-discrete} \Leftrightarrow \sum_{0<|i|<n} \pi^i(a) \leq \prod_{0<i<n} \mathrm{supp}(\pi^i) - a.$$

It is also clear that if equality holds on the right-hand side, then a is maximal $\pi^{<n}$-discrete. So it only remains to check that if the inequality is strict, then a is not maximal $\pi^{<n}$-discrete.

To see this, find a $\pi^{<n}$-discrete section b which is disjoint from

$$\sum_{|i|<n} \pi^i(a).$$

It follows that for all $0 < i < n$,

$$\begin{aligned}
(a+b) \cdot \pi^i(a+b) &= a \cdot \pi^i(a) + a \cdot \pi^i(b) + b \cdot \pi^i(a) + b \cdot \pi^i(b) \\
&= \mathbb{O} + \mathbb{O} + \pi^i(\pi^{-i}(a) \cdot b) + \mathbb{O} \\
&= \mathbb{O},
\end{aligned}$$

thus $a + b$ is a $\pi^{<n}$-discrete section which is properly above a. ⊣

Next, we establish the existence of maximal Δ-discrete sections:

Proposition 2.15. *Suppose* $\mathfrak{A}$ *is a Boolean algebra,* $a \in \mathfrak{A}$*,* $\Delta \subseteq \mathrm{Aut}(\mathfrak{A})$ *is finite, and every element of* $\Delta\Delta^{-1}$ *admits a maximal discrete section. Then there is a maximal* Δ*-discrete element of* $\mathfrak{A}_a$*.*

Proof. We will find a finite family of Δ-discrete elements of $\mathfrak{A}_a$ whose sum is above every Δ-discrete element of $\mathfrak{A}_a$. We will then gradually paste these pieces together, at a given stage adding as much as possible without destroying Δ-discreteness. We will then show that the resulting element of $\mathfrak{A}$ is maximal Δ-discrete.

Fix an enumeration $\langle(\gamma_i, \delta_i)\rangle_{i<n}$ of the pairs of distinct elements of Δ. By Proposition 2.7, there are $\gamma_i\delta_i^{-1}$-discrete sections $a_0^i, a_1^i, a_2^i \in \mathfrak{A}$ such that

$$\forall i < n \left(a_0^i + a_1^i + a_2^i = \mathrm{supp}(\gamma_i\delta_i^{-1})\right).$$

Let s_j be an enumeration of $\{0,1,2\}^n$, and for each $j < 3^n$, set

$$a_j = a \cdot \prod_{i<n} \delta_i^{-1} \cdot a_{s_j(i)}^i.$$

As $\delta \cdot a_j$ is $\gamma\delta^{-1}$-discrete whenever $\gamma \neq \delta$, it follows that a_j is Δ-discrete. Moreover,

$$\begin{aligned} b \text{ is } \Delta\text{-discrete} \quad &\Rightarrow \quad \forall i < n \left(\delta_i \cdot b \text{ is } \gamma_i\delta_i^{-1}\text{-discrete}\right) \\ &\Rightarrow \quad \forall i < n \left(\delta_i \cdot b \leq a_0^i + a_1^i + a_2^i\right) \\ &\Rightarrow \quad \forall i < n \left(b \leq \delta_i^{-1} \cdot (a_0^i + a_1^i + a_2^i)\right) \\ &\Rightarrow \quad b \leq \prod_{i<n} \delta_i^{-1} \cdot (a_0^i + a_1^i + a_2^i), \end{aligned}$$

thus every Δ-discrete section in $\mathfrak{A}_a$ is below $\sum_{j<3^n} a_j$.

Set $b_0 = \mathbb{O}$ and recursively define

$$b_{j+1} = b_j + \left(a_j - \sum_{\gamma\neq\delta} \gamma^{-1}\delta \cdot b_j\right),$$

for $j \leq 3^n$. Noting that for $\gamma \neq \delta$,

$$\begin{aligned} (\gamma \cdot b_{j+1}) \cdot (\delta \cdot b_{j+1}) \quad &= \quad \left(\gamma \cdot b_j + \gamma \cdot \left(a_j - \sum_{\gamma\neq\delta} \gamma^{-1}\delta \cdot b_j\right)\right) \cdot \\ &\qquad \left(\delta \cdot b_j + \delta \cdot \left(a_j - \sum_{\delta\neq\gamma} \delta^{-1}\gamma \cdot b_j\right)\right) \\ &\leq \quad (\gamma \cdot b_j + (\gamma \cdot a_j - \delta \cdot b_j)) \cdot (\delta \cdot b_j + (\delta \cdot a_j - \gamma \cdot b_j)) \\ &\leq \quad (\gamma \cdot b_j) \cdot (\delta \cdot b_j) + (\gamma \cdot b_j) \cdot (\delta \cdot a_j - \gamma \cdot b_j) + \\ &\qquad (\gamma \cdot a_j - \delta \cdot b_j) \cdot (\delta \cdot b_j) + (\gamma \cdot a_j - \delta \cdot b_j) \cdot (\delta \cdot a_j - \gamma \cdot b_j) \\ &\leq \quad (\gamma \cdot b_j) \cdot (\delta \cdot b_j) + (\gamma \cdot a_j) \cdot (\delta \cdot a_j), \end{aligned}$$

it follows from the obvious induction that $b = b_{3^n}$ is Δ-discrete. Suppose, towards a contradiction, that there is a Δ-discrete section $c > b$ in $\mathfrak{A}_a$. Then $c - b$ is non-zero and Δ-discrete, so there exists $j < 3^n$ such that $a_j \cdot (c - b) \neq \mathbb{O}$. It follows from the definition of b_{j+1} that

$$\mathbb{O} < a_j \cdot (c - b) \leq \sum_{\gamma \neq \delta} \gamma^{-1}\delta \cdot b_j,$$

thus there exists distinct $\gamma, \delta \in \Delta$ such that

$$(\gamma \cdot (c - b)) \cdot (\delta \cdot b) \neq \mathbb{O},$$

so $(\gamma \cdot c) \cdot (\delta \cdot c) \neq \mathbb{O}$, which contradicts the assumption that c is Δ-discrete. $\dashv$

Remark 2.16. A $(\lambda^\kappa)^+$-complete Boolean algebra $\mathfrak{A}$ is (κ, λ)-*distributive* if for every sequence $\langle a_{\xi,\eta} \rangle_{\xi < \kappa, \eta < \lambda}$ of elements of $\mathfrak{A}$,

$$\prod_{\xi < \kappa} \sum_{\eta < \lambda} a_{\xi\eta} = \sum_{f \in \lambda^\kappa} \prod_{\xi < \lambda} a_{\xi f(\xi)}.$$

It is straightforward to check that when $\mathfrak{A}$ is $(\kappa, 2)$-distributive, the assumption that Δ is finite in Proposition 2.15 can be weakened to $|\Delta| \leq \kappa$.

It should be noted, however, that such a generalization fails for the algebra of Borel subsets of a Polish space, even when Δ is countably infinite. In the language of §3, this is a simple consequence of the fact that non-smooth automorphisms of this algebra do not admit maximal partial transversals.

Next, we will describe a sense in which the existence of a maximal discrete section for an automorphism can be viewed as a natural strengthening of the existence of a support. We will use $\widehat{\mathfrak{A}}$ to denote the *Stone space* of all ultrafilters on $\mathfrak{A}$, endowed with the topology generated by the basic clopen sets of the form

$$\widehat{a} = \{U \in \widehat{\mathfrak{A}} : a \in U\},$$

where $a \in \mathfrak{A}$. We will denote the *autohomeomorphism of* $\widehat{\mathfrak{A}}$ *corresponding to* π by

$$\widehat{\pi}(U) = \{\pi^{-1}(a) : a \in U\}.$$

The reader is encouraged to look to Fremlin [39] or Bonnet-Monk [16] for background information on Stone spaces. The *support of* $\widehat{\pi}$ is

$$\operatorname{supp}(\widehat{\pi}) = \{U \in \widehat{\mathfrak{A}} : \pi(U) \neq U\},$$

or equivalently, the support of the automorphism of $\mathscr{P}(\widehat{\mathfrak{A}})$ induced by $\widehat{\pi}$. A set $X \subseteq \widehat{\mathfrak{A}}$ is $\widehat{\pi}$-*discrete* if it is discrete with respect to the automorphism of $\mathscr{P}(\widehat{\mathfrak{A}})$ induced by $\widehat{\pi}$. The existence of a maximal π-discrete section has a natural description in terms of the Stone space:

Proposition 2.17. *Suppose that $\mathfrak{A}$ is a Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then the following are equivalent:*

1. *There is a maximal π-discrete element of $\mathfrak{A}$.*
2. *π has a support and $\widehat{\mathrm{supp}(\pi)} = \mathrm{supp}(\widehat{\pi})$.*
3. *$\widehat{\pi}$ has clopen support.*
4. *There is a clopen maximal $\widehat{\pi}$-discrete subset of $\widehat{\mathfrak{A}}$.*

Proof. To see (1) $\Rightarrow$ (2), suppose a is maximal π-discrete and note that $b = \pi^{-1}(a) + a + \pi(a)$ is the support of π, as in the proof of Proposition 2.7. As

$$\widehat{a} \cap \widehat{\pi}^{-1}(\widehat{a}) = \widehat{a \cdot \pi(a)} = \widehat{\mathbb{O}} = \emptyset,$$

it follows that

$$\widehat{b} = \widehat{\pi}^{-1}(\widehat{a}) \cup \widehat{a} \cup \widehat{\pi}(\widehat{a}) \subseteq \mathrm{supp}(\widehat{\pi}).$$

Noting that $\pi|\mathfrak{A}_{\mathbb{1}-b} = \mathrm{id}$ thus $\widehat{\pi}|(\widehat{\mathfrak{A}} \setminus \widehat{b}) = \mathrm{id}$, it follows that $\mathrm{supp}(\widehat{\pi}) = \widehat{b} = \widehat{\mathrm{supp}(\pi)}$.

Of course (2) $\Rightarrow$ (3) is trivial. To see (3) $\Rightarrow$ (4), note that since $\widehat{\mathfrak{A}}$ is zero-dimensional and $\widehat{\pi}$ is continuous, each element of $\mathrm{supp}(\widehat{\pi})$ is contained in a $\widehat{\pi}$-discrete clopen set. As $\mathrm{supp}(\widehat{\pi})$ is compact, it follows that there are $\widehat{\pi}$-discrete clopen sets $\mathscr{U}_0, \ldots, \mathscr{U}_n$ whose union is $\mathrm{supp}(\widehat{\pi})$. The existence of a clopen maximal $\widehat{\pi}$-discrete subset of $\widehat{\mathfrak{A}}$ now follows from applying Proposition 2.7 to the clopen algebra of $\widehat{\mathfrak{A}}$.

To see (4) $\Rightarrow$ (1), let $\widehat{a} \subseteq \widehat{\mathfrak{A}}$ be a clopen maximal $\widehat{\pi}$-discrete set. Then $\widehat{a \cdot \pi(a)} = \widehat{a} \cap \widehat{\pi}^{-1}(\widehat{a}) = \emptyset$, thus $a \cdot \pi(a) = \mathbb{O}$, and it follows that a is π-discrete. Now suppose, towards a contradiction, that there exists $b > a$ with $b \cdot \pi(b) = \mathbb{O}$. Then $\widehat{a} \subsetneq \widehat{b}$ and

$$\widehat{b} \cap \widehat{\pi}^{-1}(\widehat{b}) = \widehat{b \cdot \pi(b)} = \widehat{\mathbb{O}} = \emptyset,$$

contradicting the fact that $\widehat{a}$ is maximal $\widehat{\pi}$-discrete. $\dashv$

We have already mentioned that maximal discrete sections will play a role similar to that of Rokhlin's Lemma within ergodic theory. We will close this section by using maximal discrete sections to give a simple proof of a general version of Rokhlin's Lemma, which neither assumes the invariance of the measure under π nor places restrictions on the Boolean algebra in question. The automorphism π is *fixed-point free* if $\mathrm{supp}(\pi) = \mathbb{1}$, and the automorphism π is *aperiodic* if

$$\forall n \in \mathbb{Z} \left(n \neq 0 \Rightarrow \pi^n \text{ is fixed-point free}\right).$$

The autohomeomorphism $\widehat{\pi}$ is *aperiodic* if the induced automorphism of $\mathscr{P}(\widehat{\mathfrak{A}})$ is aperiodic, or equivalently, if no element of $\widehat{\mathfrak{A}}$ is fixed by a non-zero iterate of π. Note that if every iterate of π admits a maximal discrete section, then Proposition 2.17 ensures that

$$\pi \text{ is aperiodic} \Leftrightarrow \widehat{\pi} \text{ is aperiodic.}$$

Theorem 2.18. *Suppose $\mathfrak{A}$ is a Boolean algebra, μ is a finitely additive probability measure on $\mathfrak{A}$, $\pi \in \mathrm{Aut}(\mathfrak{A})$, and $\widehat{\pi}$ is aperiodic. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$, there is a maximal $\pi^{<n}$-discrete section $a \in \mathfrak{A}$ such that $\mu\left(\sum_{i<n} \pi^i(a)\right) > 1 - \epsilon$.*

Proof. We will find a maximal $\pi^{<k}$-discrete section $d \in \mathfrak{A}$, for some $k \in \mathbb{N}$ sufficiently large, such that $d + \pi^{-1}(d) + \cdots + \pi^{-(n-1)}(d)$ carries very little measure. We will then partition d into finitely many pieces d_{ij} such that

$$\pi^{in+j}(d_{ij}) \leq d \text{ and } \pi(d_{ij}), \ldots, \pi^{in+j-1}(d_{ij}) \text{ are disjoint from } d.$$

The desired section will be the sum of the sections of the form $\pi^{kn}(d_{ij})$, for $k < i$.

The assumption that $\widehat{\pi}$ is aperiodic ensures that each power of $\widehat{\pi}$ has support $\widehat{\mathfrak{A}}$. In particular, it follows that each power of $\widehat{\pi}$ has clopen support, thus each power of π admits a maximal discrete section, by Proposition 2.17. Fix a natural number $m > 1/\epsilon$ and observe that by Proposition 2.15, there is a maximal $\pi^{<mn}$-discrete section $b \in \mathfrak{A}$. As the aperiodicity of $\widehat{\pi}$ implies that each non-zero power of π is fixed-point free, it follows from Proposition 2.14 that

$$\mathbb{1} - b = \prod_{0<i<n} \mathrm{supp}(\pi^i) - b = \sum_{0<|i|<2mn} \pi^i(b),$$

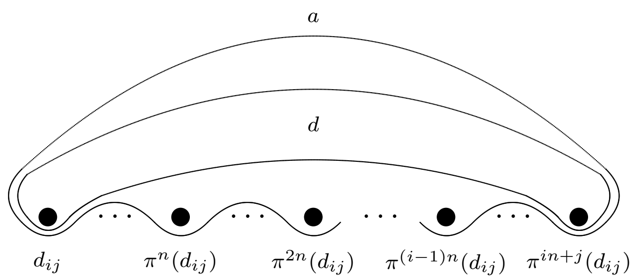

Figure 1.2: The construction of a Rokhlin section from a $\pi^{<mn}$-discrete section.

thus $\sum_{i<2mn-1} \pi^i(b) = \mathbb{1}$. Set $c = \sum_{i<n} \pi^{-i}(b)$ and observe that $\langle \pi^{\ell n}(c) \rangle_{\ell<m}$ is pairwise disjoint, thus $\mu(\pi^{\ell n}(c)) < \epsilon$ for some $\ell < m$. Put $d = \pi^{\ell n}(b)$,

$$d_i = d \cdot \pi^{-i}(d) - \sum_{0<j<i} \pi^{-j}(d),$$

and $d_{ij} = d_{in+j}$, and define

$$a = \sum \left\{ \pi^{kn}(d_{ij}) : k < i \text{ and } j < n \right\}.$$

Note that this is a finite sum, since $d_{ij} = \mathbb{0}$ for all but finitely many values of i, j.

As d is $\pi^{<mn}$-discrete, it easily follows that a is $\pi^{<n}$-discrete. Noting that

$$\sum_{k<2n-1} \pi^k(a) = \mathbb{1},$$

it follows that a is maximal $\pi^{<n}$-discrete. It only remains to note that

$$\mathbb{1} - \pi^{\ell n}(c) \leq \sum_{i<n} \pi^i(a),$$

thus $\mu\left(\sum_{i<n} \pi^i(a)\right) > 1 - \epsilon$. $\dashv$

The use of $\widehat{\pi}$ in the statement of Theorem 2.18 is necessary:

Example 2.19. Suppose that κ is an infinite cardinal and let $\mathfrak{A}_\kappa$ be the κ-complete Boolean algebra which is generated by the singletons contained in κ. Define a map $\mu : \mathfrak{A}_\kappa \to \{0, 1\}$ by

$$\mu(S) = \begin{cases} 0 & \text{if } |S| < \kappa, \\ 1 & \text{otherwise,} \end{cases}$$

noting that μ is a κ-additive probability measure on $\mathfrak{A}_\kappa$. Clearly $\mathfrak{A}_\kappa$ admits aperiodic automorphisms. However, no automorphism of $\mathfrak{A}_\kappa$ admits a discrete section of positive measure, thus the conclusion of Theorem 2.18 must fail. To see that this does not contradict Theorem 2.18, simply note that the autohomeomorphism corresponding to any automorphism of $\mathfrak{A}_\kappa$ is never fixed-point free, and therefore cannot be aperiodic.

3 The full group of an automorphism

In this section, we introduce the full group of an automorphism of a Boolean algebra, and describe several notions with origins in the study of orbit equivalence. In the process, we see several connections between the maximal discrete sections of §2 and these new notions. We close the section with a general characterization of smoothness for the orbit equivalence relations associated with countable groups which act on a *Baire space* by homeomorphisms.

Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra. Given pairwise disjoint sequences $\langle a_i \rangle_{i \in I}$ and $\langle b_i \rangle_{i \in I}$ of elements of $\mathfrak{A}$ and isomorphisms $\pi_i : \mathfrak{A}_{a_i} \to \mathfrak{A}_{b_i}$, such that I is countable and

$$\sum_{i \in I} a_i = \sum_{i \in I} b_i,$$

we will use

$$\pi = \prod_{i \in I} a_i \xrightarrow{\pi_i} b_i$$

to denote the automorphism whose support is contained in $\sum_{i \in I} a_i$ and which satisfies

$$\forall i \in I \ (\pi | \mathfrak{A}_{a_i} = \pi_i).$$

Similarly, given natural numbers j_i, a pairwise disjoint sequence $\langle a_{ij} \rangle_{i \in I, j \leq j_i}$ of elements of $\mathfrak{A}$, and isomorphisms $\pi_{ij} : \mathfrak{A}_{a_{ij}} \to \mathfrak{A}_{a_{ij+1}}$, we will use the *cycle notation*

$$\pi = \prod_{i \in I} (a_{i0} \xrightarrow{\pi_{i0}} a_{i1} \xrightarrow{\pi_{i1}} \cdots \xrightarrow{\pi_{ij_i-1}} a_{ij_i})$$

to denote the automorphism with support below $\sum_{i \in I, j \leq j_i} a_{ij}$ and which satisfies

$$\forall i \in I \forall j < j_i \ (\pi | \mathfrak{A}_{a_{ij}} = \pi_{ij}) \text{ and } \forall i \in I \ (\pi | \mathfrak{A}_{a_{ij_i}} = \pi_{i0}^{-1} \circ \pi_{i1}^{-1} \circ \cdots \circ \pi_{ij_i-1}^{-1}).$$

Note that this notation makes sense even when the a_{ij}'s are not disjoint, as long as the corresponding isomorphisms agree on their intersections.

Now suppose that $\mathfrak{A}$ is a Boolean algebra. Associated with each $\pi \in \mathrm{Aut}(\mathfrak{A})$ is the *full group* $[\pi]$ of automorphisms of $\mathfrak{A}$ of the form

$$\varphi = \prod_{n\in\mathbb{Z}} a_n \xrightarrow{\pi^n} \pi^n(a_n),$$

where $\langle a_n \rangle_{n\in\mathbb{Z}}$ and $\langle \pi^n(a_n) \rangle_{n\in\mathbb{Z}}$ are both partitions of unity. Note that this makes sense even when $\mathfrak{A}$ is not σ-complete!

Such groups often arise in practice when $\mathfrak{A}$ is an algebra of subsets of some set X. In this case, $[\pi]$ is simply the group of all automorphisms $\varphi \in \mathrm{Aut}(\mathfrak{A})$ such that

$$\forall x \in X\, \exists n \in \mathbb{Z}\ (\varphi(x) = \pi^n(x)),$$

where φ, π have been identified with the corresponding permutations of X.

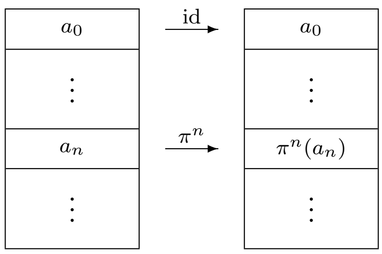

Figure 1.3: An element of the full group of π.

Proposition 3.1. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and every power of π admits a maximal discrete section. Then every element of $[\pi]$ admits a maximal discrete section.*

Proof. Suppose that $\varphi \in [\pi]$. By Proposition 2.7, it is enough to find countably many φ-discrete sections whose sum is $\mathrm{supp}(\varphi)$. Fix a partition of unity $\langle a_n \rangle_{n\in\mathbb{Z}}$ such that

$$\varphi = \prod_{n\in\mathbb{Z}} a_n \xrightarrow{\pi^n} \pi^n(a_n),$$

fix maximal π^n-discrete sections $b_n \in \mathfrak{A}$, and observe that

$$\operatorname{supp}(\varphi) = \sum_{n \in \mathbb{Z}} a_n \cdot \operatorname{supp}(\pi^n) \leq \sum_{m,n \in \mathbb{Z}} a_n \cdot \pi^m(b_n).$$

Now for each $m, n \in \mathbb{Z}$,

$$\begin{aligned}(a_n \cdot \pi^m(b_n)) \cdot \varphi(a_n \cdot \pi^m(b_n)) &= a_n \cdot \pi^m(b_n) \cdot \pi^n(a_n) \cdot \pi^{m+n}(b_n) \\ &\leq \pi^m(b_n \cdot \pi^n(b_n)) \\ &= \mathbb{0},\end{aligned}$$

thus $a_n \cdot \pi^m(b_n)$ is φ-discrete. ⊣

The *π-saturation* of $a \in \mathfrak{A}$ is $[a]_\pi = \sum_{n \in \mathbb{Z}} \pi^n(a)$, and a is a *π-complete section* if $[a]_\pi = \mathbb{1}$. When $\mathfrak{A}$ is purely atomic, a complete section is a section which contains at least one point of the orbit of each atom. Many of the arguments to come can be modified so as to use π-discrete complete sections instead of maximal π-discrete sections. The following fact shows that yields no greater generality:

Proposition 3.2. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \operatorname{Aut}(\mathfrak{A})$, $n \in \mathbb{Z}$, and there is a π^n-discrete, π-complete section. Then π^n admits a maximal discrete section.*

Proof. Suppose $a \in \mathfrak{A}$ is a π^n-discrete, π-complete section, and note that each iterate $\pi^k(a)$ of a is π^n-discrete and $\sum_{k \in \mathbb{N}} \pi^k(a) = \mathbb{1}$. It now follows from Proposition 2.7 that π^n admits a maximal discrete section. ⊣

Corollary 3.3. *Suppose $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \operatorname{Aut}(\mathfrak{A})$, and for each $n \in \mathbb{N}$ there is a π^n-discrete, π-complete section. Then every element of $[\pi]$ admits a maximal discrete section.*

Proof. This follows directly from Propositions 3.1 and 3.2. ⊣

A *partial π-transversal* is an element $a \in \mathfrak{A}$ such that

$$\forall n \in \mathbb{N} \left(\pi^n | \mathfrak{A}_{a \cdot \pi^n(a)} = \mathrm{id}\right).$$

When $\mathfrak{A}$ is purely atomic, a partial transversal is a section which contains at most one point of the orbit of each atom. A *π-transversal* is a partial π-transversal which is also a π-complete section. An automorphism π is *smooth* if it admits a transversal.

Proposition 3.4. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$ is smooth. Then every element of $[\pi]$ is smooth, and therefore admits a maximal discrete section.*

Proof. To see that every $\varphi \in [\pi]$ is smooth, fix a π-transversal $a_0 \in \mathfrak{A}$, and note that each $a_n = \pi^n(a_0)$ is also a π-transversal. It follows that each is a partial φ-transversal. We will paste them together into a full transversal. Fix an enumeration k_n of $\mathbb{Z}$, set $b_0 = \mathbb{O}$, recursively define

$$b_{n+1} = b_n + \left(a_{k_n} - \sum_{m<n} [b_m]_\varphi \right),$$

and set $b = \sum_{n\in\mathbb{N}} b_n$.

To see that every smooth automorphism φ admits a maximal discrete section, fix a φ-transversal $a \in \mathfrak{A}$. Now note that φ has support $b = [a - \pi(a)]_\varphi$ and a is a φ-discrete complete section for $\varphi|\mathfrak{A}_b$, and apply Proposition 3.2. $\dashv$

The automorphism π is *periodic* if $\forall a > \mathbb{O}\, \exists \mathbb{O} < b \leq a\, \exists n > 0\ (\pi^n|\mathfrak{A}_b = \mathrm{id})$.

Proposition 3.5. *Suppose $\mathfrak{A}$ is a σ-complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$ is periodic. Then π is smooth $\Leftrightarrow \forall \varphi \in [\pi]$ (φ admits a maximal discrete section).*

Proof. Of course, ($\Rightarrow$) follows from Proposition 3.4. To see ($\Leftarrow$), note that each π^n has a support, by Proposition 2.7. An automorphism φ is of *exact period n* if

$$\varphi^n = \mathrm{id} \text{ and } \forall 0 < i < n\ (\pi^i \text{ is fixed-point free}).$$

For each $n > 0$, the *exact period n part of π* is

$$a_n = \prod_{0<i<n} \mathrm{supp}(\pi^i) - \mathrm{supp}(\pi^n).$$

Note that each a_n is π-invariant. Also, observe that the periodicity of π ensures that these sections form a partition of unity. Let b_n be a maximal $\pi^{<n}$-discrete section,

and note that $a_n \cdot b_n$ is a transversal of $\pi|\mathfrak{A}_{a_n}$. It follows that

$$\sum_{n>0} a_n \cdot b_n$$

is a π-transversal. ⊣

An element $a \in \mathfrak{A}$ is *doubly π-recurrent* if $a \leq \sum_{n>0} \pi^n(a)$ and $a \leq \sum_{n>0} \pi^{-n}(a)$.

Proposition 3.6. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $a \in \mathfrak{A}$, and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then there is a π-invariant $b \in \mathfrak{A}$ such that $\pi|\mathfrak{A}_b$ is smooth and $a - b$ is doubly recurrent for $\pi|\mathfrak{A}_{\mathbb{1}-b}$.*

Proof. The element b is the π-saturation of the sum of the *leftmost piece of* a,

$$\ell = a - \sum_{n>0} \pi^n(a),$$

and the *rightmost piece of* a,

$$r = a - \sum_{n>0} \pi^{-n}(a).$$

Clearly, $\ell + (r - [\ell]_\pi)$ is a transversal of $\pi|\mathfrak{A}_b$, and $a - b$ is doubly recurrent. ⊣

A much stronger fact holds in complete Boolean algebras:

Proposition 3.7. *Suppose that $\mathfrak{A}$ is a complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$. Then there is a π-invariant $a \in \mathfrak{A}$ such that $\pi|\mathfrak{A}_a$ is smooth and every element of $\mathfrak{A}_{\mathbb{1}-a}$ is doubly π-recurrent.*

Proof. Fix a maximal collection of partial π-transversals a_ξ with pairwise disjoint saturations, and put $a = \sum_\xi [a_\xi]_\pi$. Clearly $\sum_\xi a_\xi$ is a transversal for $\pi|\mathfrak{A}_a$. It follows from maximality that there is no non-zero π-transversal in $\mathfrak{A}_{\mathbb{1}-a}$, and it follows from this and Proposition 3.6 that every element of $\mathfrak{A}_{\mathbb{1}-a}$ is doubly recurrent. ⊣

Remark 3.8. On the other hand, no non-smooth automorphism of a purely atomic algebra can satisfy Proposition 3.7. This is because any candidate for such a b can be enlarged by adding the saturation of an atom of $\mathfrak{A}$.

Next we will give a characterization of smoothness, in a certain setting, which will hopefully provide some intuition for this notion (and will certainly simplify our work in the examples to come). A *Baire space* is a topological space in which the intersection of countable many dense open sets is dense. Suppose Γ is a group which acts on X by homeomorphisms. Then the *orbit equivalence relation* associated with the action is given by

$$xE_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y).$$

The *orbit* of a point is its E_Γ^X-class. A *complete section for* E_Γ^X for E_Γ^X is a set $B \subseteq X$ which intersects every class of E_Γ^X, a *partial transversal for* E_Γ^X is a set $B \subseteq X$ which intersects every class of E_Γ^X in at most one point, and a *transversal of* E_Γ^X is a complete section which is also a partial transversal. We will use "$\forall^*$" to denote, "for comeagerly many."

Proposition 3.9. *Suppose that X is a Baire space and Γ is a countable group which acts on X by homeomorphisms. The following are equivalent:*

1. *E_Γ^X admits a Baire measurable transversal.*

2. *$\forall^* x \in X$ $\big($$x$ belongs to an open partial transversal of $E_\Gamma^X$$\big)$.*

If X is a complete metric space, then these are equivalent to:

3. *$\forall^* x \in X$ $\big($$x$ is not a limit point of $[x]_\Gamma$$\big)$.*

If Γ acts by isometries, then these are equivalent to:

4. *$\forall^* x \in X$ $\big($$[x]_\Gamma$ is closed$\big)$.*

Proof. To see (1) $\Rightarrow$ (2), suppose B is a Baire measurable transversal of E_Γ^X. To see that comeagerly many points belongs to an open partial transversal of E_Γ^X, it clearly suffices to check that every non-empty open set $\mathscr{U}$ contains a non-empty open partial transversal of E_Γ^X. As $X = \bigcup_{\gamma\in\Gamma} \gamma \cdot B$, it follows that by replacing B with its image under an element of Γ, we may assume that B is non-meager in $\mathscr{U}$, thus comeager in some non-empty open set $\mathscr{V} \subseteq \mathscr{U}$. We claim that $\mathscr{V}$ must be a partial transversal of E_Γ^X. For if not, then there exists $\gamma \in \Gamma$ such that

$$\mathscr{W} = (\mathscr{V} \cap \gamma^{-1} \cdot \mathscr{V}) \cap \operatorname{supp}(\gamma)$$

is non-empty. But then $\gamma \cdot (\mathscr{W} \cap B)$ is a non-meager subset of $\mathscr{V}$ which is disjoint from B, contradicting the fact that B is comeager in $\mathscr{V}$.

To see (2) $\Rightarrow$ (1), suppose that comeagerly many points are contained in an open partial transversal of E_Γ^X, and let $\mathscr{U}$ be a maximal open partial transversal of E_Γ^X. To see that E_Γ^X admits a Baire measurable transversal, it clearly suffices to check that $[\mathscr{U}]_\Gamma$ is dense. If this is not the case, then there is a non-empty open set $\mathscr{V} \subseteq X \setminus [\mathscr{U}]_\Gamma$. Fix $x \in \mathscr{V}$ which is contained in a non-empty open partial transversal $\mathscr{W}$ of E_Γ^X, and observe that $\mathscr{U} \cup (\mathscr{V} \cap \mathscr{W})$ is an open partial transversal of E_Γ^X, contradicting the maximality of $\mathscr{U}$.

Clearly (2) $\Rightarrow$ (3). To see that (3) $\Rightarrow$ (2) when X is a complete metric space, it suffices to check that if there are non-meagerly many points which are not contained in an open transversal of E_Γ^X, then there are non-meagerly many points which are limit points of their orbit. Fix a non-empty open set $\mathscr{U}$ such that

$$\forall^* x \in \mathscr{U} \ (x \text{ is not contained in an open partial transversal of } E_\Gamma^X),$$

and note that $\mathscr{U}$ does not contain an open partial transversal. For each $n > 0$, put

$$\begin{aligned} \mathscr{U}_n &= \bigcup_{\gamma \in \Gamma} \{x \in X : 0 < d(x, \gamma \cdot x) < 1/n\} \\ &= \{x \in X : \exists y \in [x]_\Gamma \ (0 < d(x,y) < 1/n)\}, \end{aligned}$$

noting that each $\mathscr{U}_n$ is open. It suffices to check that each $\mathscr{U}_n$ is dense in $\mathscr{U}$, for then $G = \bigcap_{n>0} \mathscr{U}_n$ is a G_δ set which is dense in $\mathscr{U}$, all of whose elements are limit points of their orbits. If $\mathscr{U}_n$ is not dense, then there is a non-empty open set $\mathscr{V} \subseteq \mathscr{U} \setminus \mathscr{U}_n$, and any open subset of $\mathscr{V}$ of diameter $< 1/n$ is clearly a partial transversal, contradicting the fact that $\mathscr{U}$ does not contain any open partial transversals.

To see (4) $\Rightarrow$ (3), it is enough to observe that if $[x]_\Gamma$ is closed, then x is not a limit point of $[x]_\Gamma$. Suppose, towards a contradiction, that x is a limit point of $[x]_\Gamma$. Then every point of $[x]_\Gamma$ is a limit point of $[x]_\Gamma$, since Γ acts by homeomorphisms. As no perfect set is countable, this contradicts the fact that Γ is countable.

It remains to check that (3) $\Rightarrow$ (4) when X is a complete metric space and the elements of Γ are isometries. It is enough to show that if $[x]_\Gamma$ is not closed, then x is a limit point of $[x]_\Gamma$. Noting that for all $\epsilon > 0$ there exist $\gamma, \delta \in \Gamma$ such that $0 < d(\gamma \cdot x, \delta \cdot x) < \epsilon$, or equivalently, $0 < d(x, \gamma^{-1}\delta \cdot x) < \epsilon$, it follows that x is a limit point of $[x]_\Gamma$. $\dashv$

Corollary 3.10. *Suppose that X is a Baire space with a non-Baire measurable subset and Γ is a countable group of homeomorphisms with a dense orbit. Then E_Γ^X has no Baire measurable transversal.*

Proof. Since there is a dense E_Γ^X-class, the only non-empty open partial transversals are the open singletons. Put $\mathscr{U} = \{x \in X : \{x\} \text{ is open}\}$. Clearly $\mathscr{U}$ is open and Γ-invariant. Suppose, towards a contradiction, that E_Γ^X has a Baire measurable transversal. Then comeagerly many points are contained in open partial transversals of E_Γ^X, thus $\mathscr{U}$ is dense. It follows that every set $Y \subseteq X$ is the union of a meager set $Y \setminus \mathscr{U}$ and an open set $Y \cap \mathscr{U}$ and is therefore Baire measurable, contradicting the fact that X has a subset which is not Baire measurable. $\dashv$

Remark 3.11. Even when X is an uncountable Polish space, the existence of a non-Baire measurable subset of X is not automatic. For example, the set $X = 2^{\leq \mathbb{N}}$ when equipped with the metric

$$d(x, y) = \begin{cases} 1/2^n & \text{if } n \text{ is least such that } x_n \neq y_n, \\ 0 & \text{if } x = y, \end{cases}$$

forms a Polish space whose subsets are all Baire measurable. In general, a Polish space X has a non-Baire measurable subset exactly when it is *generically uncountable*, that is, when no countable subset of X is comeager.

4 Compositions of two involutions

Armed with the results of §2 and §3, we are now ready to begin our study of the circumstances under which an automorphism can be written as a product of periodic automorphisms from its full group. We begin by showing that an automorphism is a composition of two involutions from its full group exactly when it is smooth. We then show that if an aperiodic automorphism can be written as a product of periodic automorphisms from its full group, then every element of its full group admits maximal discrete sections. We close the section with a discussion of the problem of finding automorphisms of σ-complete Boolean algebras which are not the composition of a pair of involutions. We give a general setting in which an automorphism of a rooted set-theoretic tree is such a composition, and we also show that any isometry of a Polish metric space is a composition of two Borel involutions. Finally, we describe how Bratteli-Vershik diagrams naturally lead to the Chacón automorphism, which del Junco [20] has shown is not the composition of two measure-preserving involutions.

Suppose that $\mathfrak{A}$ is a Boolean algebra. An automorphism $\pi \in \mathrm{Aut}(\mathfrak{A})$ is an *involution* if $\pi^2 = \mathrm{id}$, and automorphisms $\varphi, \psi \in \mathrm{Aut}(\mathfrak{A})$ are *conjugate* if there is another automorphism $\pi \in \mathrm{Aut}(\mathfrak{A})$ such that $\pi \circ \varphi = \psi \circ \pi$.

Proposition 4.1. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and π is an automorphism of $\mathfrak{A}$. Then the following are equivalent:*

1. *π is smooth.*

2. *π is the composition of two involutions from its full group.*

3. *π is conjugate to its inverse via an element of its full group.*

Proof. To see (1) $\Rightarrow$ (2), suppose $a_0 \in \mathfrak{A}$ is a π-transversal and put $a_n = \pi^n(a_0)$. The fact that a_0 is a π-transversal ensures that $\pi^{m-n}|\mathfrak{A}_{a_m \cdot a_n} = \mathrm{id}$. In particular, it follows that $\pi^{-2m}|\mathfrak{A}_{a_m \cdot a_n} = \pi^{-2n}|\mathfrak{A}_{a_m \cdot a_n}$, so that, using our cycle notation,

$$\iota_0 = \prod_{n \geq 0} (a_n \xrightarrow{\pi^{-2n}} a_{-n}) \text{ and } \iota_1 = \prod_{n \geq 0} (a_n \xrightarrow{\pi^{-2n-1}} a_{-n-1})$$

are involutions from $[\pi]$. It only remains to note that

$$\begin{aligned}\iota_0 \circ \iota_1 &= \prod_{n\geq 0}(a_n \xrightarrow{\pi^{-2n}} a_{-n}) \circ \prod_{n\geq 0}(a_n \xrightarrow{\pi^{-2n-1}} a_{-n-1}) \\ &= \prod_{n\geq 0}(a_n \xrightarrow{\pi^{2n+2}\circ\pi^{-2n-1}} a_{n+1}) \\ &= \prod_{n\geq 0}(a_n \xrightarrow{\pi} a_{n+1}) \\ &= \pi.\end{aligned}$$

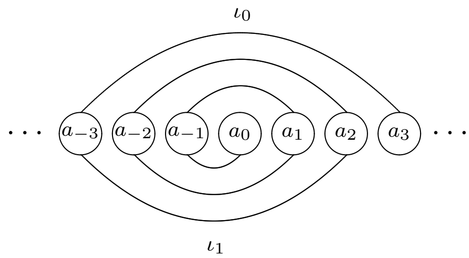

Figure 1.4: The action of ι_0, ι_1 on $\mathfrak{A}$.

To see (2) $\Rightarrow$ (3), suppose that $\pi = \iota_0 \circ \iota_1$ and $\iota_0 \in [\pi]$, and observe that

$$\begin{aligned}\iota_0 \circ \pi &= \iota_0 \circ \iota_0 \circ \iota_1 \\ &= \iota_1 \\ &= \iota_1 \circ \iota_0 \circ \iota_0 \\ &= \pi^{-1} \circ \iota_0.\end{aligned}$$

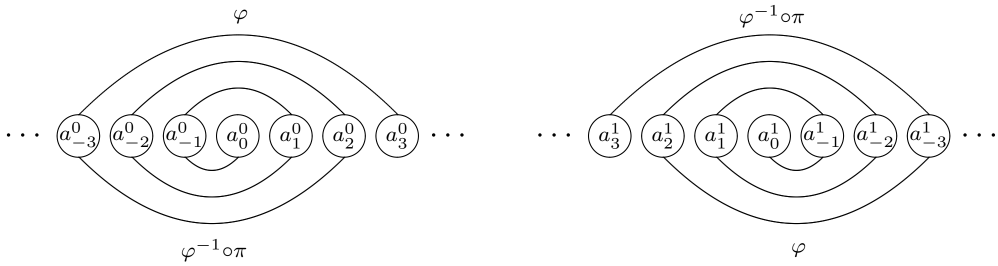

Figure 1.5: The action of φ on $\mathfrak{A}$.

To see (3) $\Rightarrow$ (1), suppose π is conjugate to its inverse via $\varphi \in [\pi]$, fix a_n^k with

$$\varphi = \prod_{n\in\mathbb{Z}, 0\leq k\leq 1} (a_n^k \xrightarrow{\pi^{2n+k}} \pi^{2n+k}(a_n^k)),$$

and note that for all $n \in \mathbb{Z}$, $k \in \{0,1\}$, and $a \leq a_n^k$,

$$\begin{aligned} \varphi \circ \pi(a) &= \pi^{-1} \circ \varphi(a) \\ &= \pi^{-1} \circ \pi^{2n+k}(a) \\ &= \pi^{2(n-1)+k} \circ \pi(a), \end{aligned}$$

thus $\varphi | \mathfrak{A}_{\pi(a_n^k)} = \pi^{2(n-1)+k} | \mathfrak{A}_{\pi(a_n^k)}$. It follows that $\pi(a_n^k) \leq a_{n-1}^k$, and since π is an automorphism, it must be the case that $\pi(a_n^k) = a_{n-1}^k$. It then follows that

$$a = a_0^0 + a_0^1$$

is a partial π-transversal. To see that a is a π-complete section and therefore a π-transversal, it only remains to note that for each $a \in \mathfrak{A}^+$, there exists $k \in \{0,1\}$ and $n \in \mathbb{N}$ with $a \cdot a_n^k \neq \mathbb{O}$, thus $\pi^n(a) \cdot a_0^k = \pi^n(a \cdot a_n^k) \neq \mathbb{O}$. ⊣

Remark 4.2. We have recently realized that this argument is identical to that behind the proof of Lemma 2.5 of Truss [76], although he was working in a somewhat different context.

It follows from Propositions 3.4 and 4.1 that if π is the composition of two involutions from its full group, then every element of $[\pi]$ admits a maximal discrete section. For aperiodic automorphisms, this is a special case of a more general fact:

Proposition 4.3. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and π is an aperiodic automorphism of $\mathfrak{A}$ which is the composition of periodic automorphisms from its full group. Then every element of $[\pi]$ admits a maximal discrete section.*

Proof. A permutation τ of $\mathbb{Z}$ is of *bounded period* if there exists $n > 0$ such that $\pi^n = \mathrm{id}$, a permutation τ of $\mathbb{Z}$ is of *bounded displacement* if

$$\sup_{n \in \mathbb{Z}} |\tau(n) - n| < \infty,$$

and the *average displacement* of a permutation τ of $\mathbb{Z}$ is

$$d(\tau) = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{|k| \leq n} \tau(k) - k.$$

At the heart of our proof is the following simple fact about the successor on $\mathbb{Z}$:

Lemma 4.4. *The successor on $\mathbb{Z}$ is not the composition of permutations of bounded periods and bounded displacements.*

Proof. Clearly the successor has average displacement 1. We will simply show that the average displacement of a composition of permutations of bounded periods and bounded displacements is 0.

Suppose that $\tau_0, \ldots, \tau_m$ is a collection of permutations of bounded period N and bounded displacement d. Note that for each $k, n \in \mathbb{N}$ with $|k| \leq n - dN$, the entire τ_i-orbit of k is contained in $[-n, n]$. As the sum of the displacements over a finite orbit is 0, it follows that

$$\begin{aligned} |d(\tau_i)| &= \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{|k|\leq n} \tau_i(k) - k \right| \\ &= \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{|k|\leq n-dN} \tau_i(k) - k + \sum_{n-dN<|k|\leq n} \tau_i(k) - k \right| \\ &\leq \lim_{n\to\infty} \frac{2d^2N}{2n+1}, \end{aligned}$$

thus $d(\tau_i) = 0$.

Setting $\tau_{m+1} = \mathrm{id}$ and noting that

$$\tau_{i+1} \circ \cdots \circ \tau_{m+1}([-n, n]) \subseteq [-n - md, n + md],$$

it follows that

$$\begin{aligned}
|d(\tau_0 \circ \cdots \circ \tau_m)| &= \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{|k|\le n} \tau_0 \circ \cdots \circ \tau_m(k) - k \right| \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{|k|\le n, i\le m} \tau_i \circ \cdots \circ \tau_m(k) - \tau_{i+1} \circ \cdots \circ \tau_{m+1}(k) \right| \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{k\in\tau_{i+1}\circ\cdots\circ\tau_{m+1}([-n,n]), i\le m} \tau_i(k) - k \right| \\
&\le \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{k\in[-n-md, n+md], i\le m} \tau_i(k) - k \right| \\
&\le \lim_{n\to\infty} \frac{1}{2n+1} \left(2md(m+1)d + \left| \sum_{|k|\le n, i\le m} \tau_i(k) - k \right| \right) \\
&\le \lim_{n\to\infty} \frac{1}{2n+1} \left| \sum_{|k|\le n, i\le m} \tau_i(k) - k \right| \\
&= |d(\tau_0)| + \cdots + |d(\tau_m)| \\
&= 0,
\end{aligned}$$

It follows that the composition of the τ_i's is not the successor. $\dashv$

Now we return to the task of seeing that each element of $[\pi]$ admits a maximal discrete section. By Corollary 3.3, it is enough to show that for each $n > 0$, there is a π^n-discrete, π-complete section. For this, it is clearly enough to show that there is a collection of finitely many π^n-discrete sections whose sum is a π-complete section.

Suppose that $\pi_i \in [\pi]$ are periodic and $\pi = \pi_0 \circ \cdots \pi_m$. Fix $a_\ell^{(i)}, b_\ell^{(i)}$ with

$$\pi_i = \prod_{\ell\in\mathbb{Z}} a_\ell^{(i)} \xrightarrow{\pi^\ell} \pi^\ell(a_\ell^{(i)}) \text{ and } \pi_i^{-1} = \prod_{\ell\in\mathbb{Z}} b_\ell^{(i)} \xrightarrow{\pi^\ell} \pi^\ell(b_\ell^{(i)}),$$

and for $s, t : \{0, \dots m\} \times \{0, \dots, n-1\} \to \mathbb{Z}$, let $a_{st} \in \mathfrak{A}$ be maximal such that

$$\forall i \le m \, \forall j < n \left(\pi_i | \mathfrak{A}_{\pi^j(a_{st})} = \pi^{s(i,j)} | \mathfrak{A}_{\pi^j(a_{st})} \text{ and } \pi_i^{-1} | \mathfrak{A}_{\pi^j(a_{st})} = \pi^{t(i,j)} | \mathfrak{A}_{\pi^j(a_{st})} \right).$$

That is, set

$$a_{st} = \prod_{i\le m, j<n} \pi^{-j}(a_{s(i,j)}^{(i)}) \cdot \prod_{i\le m, j<n} \pi^{-j}(b_{t(i,j)}^{(i)}).$$

Clearly, the set of such sections is predense, thus $\sum_{s,t} a_{st} = \mathbb{1}$. We claim that

$$\mathscr{A} = \{a_{st} - \pi^n(a_{st})\}_{s,t:\{0,\dots m\}\times\{0,\dots,n-1\}\to\mathbb{Z}}$$

is the desired family of π^n-discrete sections whose sum is a π-complete section.

Suppose, towards a contradiction, that $a = \mathbb{1} - [\sum \mathscr{A}]_\Gamma$ is non-zero. Note that

$$a \cdot a_{st} \leq \pi^n(a_{st}),$$

thus $a \cdot a_{st} = \pi^n(a \cdot a_{st})$. Letting $l\%n$ denote the *remainder when* l *is divided by* n, it follows that for all $i \leq m$, $\ell \in \mathbb{Z}$, and $s, t : \{0, \dots m\} \times \{0, \dots, n-1\} \to \mathbb{Z}$,

$$\pi_i|\mathfrak{A}_{\pi^\ell(a\cdot a_{st})} = \pi^{s(i,\ell\%n)}|\mathfrak{A}_{\pi^\ell(a\cdot a_{st})} \text{ and } \pi_i^{-1}|\mathfrak{A}_{\pi^\ell(a\cdot a_{st})} = \pi^{t(i,\ell\%n)}|\mathfrak{A}_{\pi^\ell(a\cdot a_{st})}.$$

Fix $s, t : \{0, \dots, m\} \times \{0, \dots, n-1\} \to \mathbb{Z}$ such that $b = a \cdot a_{st}$ is non-zero, and for each $i \leq m$, define $\tau_i : \mathbb{Z} \to \mathbb{Z}$ by

$$\tau_i(\ell) = \ell + s(i, \ell\%n).$$

We will show that each τ_i is a permutation of $\mathbb{Z}$ of bounded period and displacement and such that $\tau_0 \circ \dots \circ \tau_m$ is the successor function on $\mathbb{Z}$, which contradicts Lemma 4.4.

Lemma 4.5. *Each* τ_i *is injective.*

Proof. Suppose, towards a contradiction, that there exists $\ell_0, \ell_1 \in \mathbb{Z}$ such that $\tau(\ell_0) = \tau(\ell_1)$, and fix $c \leq b$ such that $\pi^{\ell_0}(c) \cdot \pi^{\ell_1}(c) = \mathbb{0}$. Noting that for all $\ell \in \mathbb{Z}$,

$$\pi_i(\pi^\ell(c)) = \pi^{\tau_i(\ell)-\ell} \circ \pi^\ell(c) = \pi^{\tau_i(\ell)}(c),$$

it follows that $\pi_i(\pi^{\ell_0}(c)) = \pi_i(\pi^{\ell_1}(c))$, the desired contradiction. $\dashv$

Lemma 4.6. *Each* τ_i *is surjective.*

Proof. We must show that each $\ell \in \mathbb{Z}$ is in the range of τ. Noting that

$$\pi^{-1}|\mathfrak{A}_{\pi^\ell(b)} = \pi^{t(i,\ell)}|\mathfrak{A}_{\pi^\ell(b)},$$

it follows that

$$\pi|\mathfrak{A}_{\pi^{t(i,\ell)+\ell}(b)} = \pi^{-t(i,\ell)}|\mathfrak{A}_{\pi^{t(i,\ell)+\ell}(b)}.$$

As $\pi|\mathfrak{A}_{\pi^{t(i,\ell)+\ell}(b)} = \pi^{s(i,(t(i,\ell)+\ell)\%n)}|\mathfrak{A}_{\pi^{t(i,\ell)+\ell}(b)}$, it follows that

$$s(i,(t(i,\ell)+\ell)\%n) = -t(i,\ell),$$

thus

$$\begin{aligned}\tau_i(t(i,\ell)+\ell) &= (t(i,\ell)+\ell) - s(i,(t(i,\ell)+\ell)\%n)\\ &= (t(i,\ell)+\ell) - t(i,\ell)\\ &= \ell.\end{aligned}$$

It follows that τ_i is surjective. ⊣

Lemma 4.7. *Each τ_i is of bounded period and bounded displacement.*

Proof. It is clear that τ_i is of bounded displacement $\max_{j<n}|s(i,j)|$. To see that τ_i is of bounded period, recursively define $d_i : \mathbb{Z}\times\mathbb{N}\to\mathbb{Z}$ by putting $d_i(\ell,0)=0$ and

$$d_i(\ell,k+1) = d_i(\ell,k) + s(i,(\ell+d_i(\ell,k))\%n),$$

noting that

$$\pi_i^k|\mathfrak{A}_{\pi^\ell(b)} = \pi^{d_i(\ell,k)}|\mathfrak{A}_{\pi^\ell(b)}.$$

Also note that by the obvious induction, $d_i(\ell,k)$ only depends on $\ell\%n$ and k. Finally, note that

$$\tau_i^k(\ell) = \ell + d_i(\ell,k),$$

by one more straightforward induction.

As π_i is periodic, for each $j<n$ we can find $k_j>0$ and $\mathbb{O} < b_j \leq \pi^j(b)$ with

$$\pi_i^{k_j}|\mathfrak{A}_{b_j} = \mathrm{id}.$$

It then follows that $\pi^{d_i(j,k_j)}|\mathfrak{A}_{b_j} = \mathrm{id}$, thus $d_i(j,k_j)=0$, so

$$\forall\ell\in\mathbb{Z}\ (\ell\equiv j \pmod n \Rightarrow \tau_i^{k_j}(\ell)=\ell).$$

Setting $k=\prod_{j<n}k_j$, it follows that $\tau_i^k = \mathrm{id}$. ⊣

Lemma 4.8. $\tau = \tau_0 \circ \cdots \circ \tau_m$.

Proof. Recursively define $d : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$ by putting $d(\ell, m) = s(m, \ell \% n)$ and

$$d(\ell, i-1) = d(\ell, i) + s(i-1, (\ell + d(\ell, i))\% n),$$

noting that for all $i \leq m$,

$$\pi_i \circ \cdots \circ \pi_m | \mathfrak{A}_{\pi^\ell(b)} = \pi^{d(\ell,i)} | \mathfrak{A}_{\pi^\ell(b)}.$$

In particular, the assumption that $\pi = \pi_0 \circ \cdots \circ \pi_m$ implies that

$$\forall \ell \in \mathbb{Z}\ (d(\ell, 0) = 1).$$

As $\tau_0 \circ \cdots \circ \tau_m(\ell) = \ell + d(\ell, 0) = \ell + 1$, the lemma follows. $\dashv$

We will close this section with a few words about a related question:

Question 4.9. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra. Which automorphisms of $\mathfrak{A}$ are the composition of two involutions? Which automorphisms of $\mathfrak{A}$ are conjugate to their inverses?*

The answer to these questions are surprisingly evasive! Even for the measure-preserving automorphisms of the complete Boolean algebra $\mathscr{LM}$ of Lebesgue measurable subsets of $[0, 1]$ mod null, simply finding *some* automorphism which is not the composition of two involutions is sufficiently difficult that Halmos-von Neumann [43] once suggested that there might not be any such automorphisms. Nevertheless, such automorphisms were constructed in the early 1950's in Anzai [5], and in the late 1970's del Junco [20] showed that the generic measure-preserving automorphism is not conjugate to its inverse.

Similar problems arise in looking for Borel automorphisms of uncountable Polish spaces which are not conjugate to their inverses. One of the simplest Borel automorphisms is the *odometer* σ on *Cantor space* $\mathscr{C} = 2^{\mathbb{N}}$, which is given by

$$\sigma(x) = \begin{cases} 0^n 1y & \text{if } x = 1^n 0y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

Intuitively, σ is "addition by 10^∞ with right carry." As the odometer is clearly an isometry with dense orbits, it follows from Corollary 3.10 that it is non-smooth, and it follows from the proof of Proposition 4.1 that it is not conjugate to its inverse via an element of its full group. However, the odometer *is* conjugate to its inverse via the map $x \mapsto \overline{x}$ which flips each digit of x. This is really just a symptom of a much more general phenomenon:

Proposition 4.10. *Every isometry of a Polish space is the composition of two Borel involutions.*

Proof. As the proposition is obvious for countable Polish spaces, we may assume that the underlying space is $\mathscr{C}$ and that the Polish metric d on $\mathscr{C}$ is compatible with its usual Borel structure. Define

$$E = \{(x, y) \in \mathscr{C}^2 : x \in \overline{[y]_f}\},$$

and note that for all $(x, y) \in E$,

$$\begin{aligned} z \in \overline{[x]_f} \quad &\Rightarrow \quad \forall \epsilon > 0 \, \exists m, n \in \mathbb{Z} \, \big(d(y, f^m(x)), d(z, f^n(x)) < \epsilon\big) \\ &\Rightarrow \quad \forall \epsilon > 0 \, \exists m, n \in \mathbb{Z} \, \big(d(f^m(y), x), d(f^n(z), x) < \epsilon\big) \\ &\Rightarrow \quad \forall \epsilon > 0 \, \exists n \in \mathbb{Z} \, (d(z, f^n(y)) < \epsilon) \\ &\Rightarrow \quad z \in \overline{[y]_f}, \end{aligned}$$

from which it follows that E is an equivalence relation whose classes are the sets of the form $\overline{[x]_f}$.

Let $\ell(x)$ be the lexicographically minimal element of $\overline{[x]_f}$, and put $Y = \ell(X)$. For each $y \in Y$, define $i_y : [y]_f \to [y]_f$ by $i(f^n(y)) = f^{-n}(y)$. It is easily verified that i_y is an involution isometry of $([y]_f, d|[y]_f)$ which anticommutes with $f|[y]_f$. Moreover, i_y has a unique extension to an involution isometry $i_y : \overline{[y]_f} \to \overline{[y]_f}$ which anticommutes with $f|\overline{[y]_f}$. It follows that the map $x \mapsto i_{\ell(x)}(x)$ is a Borel involution which anticommutes with f. $\dashv$

Remark 4.11. (Clemens) It is worth noting that the above argument shows that every isometry with dense orbits is the composition of two involution isometries. The assumption of dense orbits is necessary, however, as it is easy to build a countable Polish metric space in which the equivalence relation E mentioned above has 2 classes, but whose isometries consist of the powers of a single aperiodic isometry, and thus has no non-trivial involutions.

On the other hand, the assumption of dense orbits can be dropped if we restrict our attention to certain tree metrics. A *tree* is a pair $(T, \leq)$, where T is any set and $\leq$ is a well-founded partial order on T. The *height of* $t \in T$ is the order type of $\leq |\{s \in T : s < t\}$, and the *height of* $(T, \leq)$ is the least ordinal α for which $(T, \leq)$ has no nodes of height α. I will use $\mathrm{Aut}(T, \leq)$ to denote the set of automorphisms of $(T, \leq)$. A *branch point* for $\pi \in \mathrm{Aut}(T, \leq)$ is a node $t \in T$ whose π-orbit is of strictly greater cardinality than that of all of its $\leq$-predecessors, or equivalently, a node $t \in T$ whose π-orbit includes one of his siblings.

Proposition 4.12. *Suppose* $(T, \leq)$ *is a tree,* $\pi \in \mathrm{Aut}(T, \leq)$*, and every element of* T *has only finitely many* $\leq$*-predecessors which are branch points. Then* T *is the composition of two involutions. In particular, every automorphism of a tree of height* $\leq \omega$ *is the composition of two involutions.*

Proof. Let $\alpha(t)$ denote the height of $t \in T$ and set $T_\alpha = \{t \in T : \alpha(t) = \alpha\}$. Let A_0 be a transversal for $\pi|T_0$, and suppose we have constructed a decreasing sequence $\langle A_\beta \rangle_{\beta<\alpha}$ such that

1. $\forall \beta < \alpha \left(A_\beta \text{ is a transversal for } \pi|T_\beta\right)$.
2. $\forall \beta < \alpha \, \forall t \in A_\beta \, \forall s < t \left(s \in A_{\alpha(s)}\right)$.

We claim that $B_\alpha = \{t \in T_\alpha : \forall s < t \ (s \in A_\beta)\}$ is a complete section for $\pi|T_\alpha$. To see this, note that for each $t \in T_\alpha$, it follows from the fact that t has only finitely many π-branching $\leq$-predecessors that there exists $s < t$ such that

$$\forall s < r < t \ (r \text{ is not } \pi\text{-branching}).$$

Fixing $n \in \mathbb{Z}$ such that $\pi^n(s) \in A_{\alpha(s)}$, it follows that $\pi^n(r) \in A_{\alpha(r)}$ for all $r < t$, thus $\pi^n(t) \in B_\alpha$. It follows that there is a transversal $A_\alpha \subseteq B_\alpha$ of $\pi|T_\alpha$.

Associate with each $t \in T$ a natural number $n(t)$ such that $\pi^{n(t)}(t) \in A_{\alpha(t)}$. It is easily verified that the map $t \mapsto \pi^{2n(t)}(t)$ is an involution of $(T, \leq)$ which anticommutes with π. ⊣

Remark 4.13. Note that when T is of height ω, the requirement that each node has only finitely many predecessors which are branch points is superfluous. It follows that every isometry of Cantor space and every isometry of *Baire space* $\mathscr{N} = \mathbb{N}^{\mathbb{N}}$ is the composition of two involution isometries.

Remark 4.14. It should also be noted that there are automorphisms of trees of height $\omega + 1$ which are not the composition of two involutions. To see this, let T be the complete binary tree and set

$$T' = \{(x, \beta) \in \mathscr{C} \times \mathbf{ORD} : \beta < \aleph_{\alpha(x)}\},$$

where $\alpha : \mathscr{C} \to \mathbf{ORD}$ is a mapping from Cantor space into the ordinals such that $x, y \in \mathscr{C}$ lie in the same orbit of σ exactly when $\alpha(x) = \alpha(y)$. We will show that no $\pi \in \mathrm{Aut}(T \cup T', \subseteq)$ whose restriction to T induces the odometer can be conjugate to its inverse. First note that the isometry of $\mathscr{C}$ induced by the restriction of any element of $\mathrm{Aut}(T \cup T', \subseteq)$ to T is in $[\sigma]$. Now suppose, towards a contradiction, that there is an automorphism $\varphi \in \mathrm{Aut}(T \cup T', \subseteq)$ such that $\varphi \circ \pi = \pi^{-1} \circ \varphi$. Letting ψ be the isometry of $\mathscr{C}$ induced by $\varphi | T$, a simple category argument shows that ψ is of the form $x \mapsto x_0 + \overline{x}$, for some fixed $x_0 \in \mathscr{C}$, on a σ-invariant comeager subset of $\mathscr{C}$, where $+$ denotes addition with right carry, contradicting the fact that $\psi \in [\sigma]$.

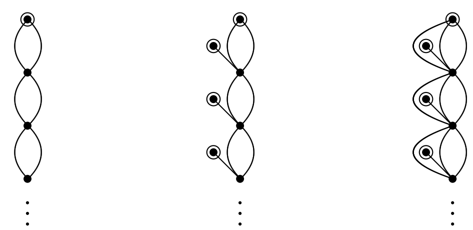

Figure 1.6: Some natural modifications of the odometer.

Despite Propositions 4.10 and 4.12, there is a natural path from the odometer to an automorphism which is not Borel conjugate to its inverse. We will use $\mathscr{D}_0$, $\mathscr{D}_1$, and $\mathscr{D}_2$ to denote the above diagrams. Associated with each diagram $\mathscr{D}_i$ is the space X_i of *downward firing paths* which begin at a circled node in $\mathscr{D}_i$. We will identify X_0, X_1, and X_2 with $\mathscr{C}$, $\mathbb{N} \times \mathscr{C}$, and $\mathbb{N} \times 3^{\mathbb{N}}$.

Except for the fact that each X_i has a leftmost and/or rightmost path, these diagrams are examples of the *Borel-Bratteli diagrams* of Bezuglyi-Dooley-Kwiatkowski [9]. As noted there, one can associate with every such diagram a *Vershik automorphism*, which is obtained by replacing the minimal initial segment of a path which is not a sequence of rightmost edges with the leftmost path whose final edge is to the right of the final edge of the initial segment. As the above diagrams are not quite Borel-Bratteli diagrams, their associated Vershik map is merely a Borel partial function, not an automorphism. However, their restrictions to the (co-countable) set of non-eventually constant sequences are Borel automorphisms.

The Vershik map corresponding to $\mathscr{D}_0$ is the map σ_0 which sends $1^n 0 y$ to $0^n 1 y$ and is undefined at 1^∞. Away from 1^∞, this *is* the odometer. The diagram $\mathscr{D}_0$ alone makes it abundantly clear that the restriction of σ_0 is conjugate to its inverse, for the map which reflects downward firing paths around the vertical axis of symmetry of $\mathscr{D}_0$ clearly anticommutes with the Vershik map.

One natural strategy for building a Borel automorphism which is not conjugate to its inverse is to modify $\mathscr{D}_0$ so as to destroy this symmetry. It is clear that any such modification must involve changing infinitely many levels of $\mathscr{D}_0$, since otherwise it is easy to come up with another diagram with an isomorphic Vershik automorphism, but which possesses the same sort of symmetry as $\mathscr{D}_0$. Thus $\mathscr{D}_1$ is the simplest natural candidate. The Vershik map corresponding to $\mathscr{D}_1$ is

$$\sigma_1(n, x) = \begin{cases} (m, 1y) & \text{if } n = 0 \text{ and } x = 1^m 0 y, \\ (n-1, 0x) & \text{if } n > 0. \end{cases}$$

However, the restriction of σ_1 can also be seen to be conjugate to its inverse, as one can paste together the reflection used to show that the odometer is conjugate to its inverse with another reflection which lives in the new piece of $\mathscr{D}_1$:

$$i(n,x) = \begin{cases} (0,\overline{x}) & \text{if } n = 0, \\ (m+1, 0^{n-1}1\overline{y}) & \text{if } n > 0 \text{ and } x = 0^m 1y. \end{cases}$$

It is easy (albeit somewhat tedious) to verify that i is an involution which anticommutes with σ_1 on the non-eventually constant sequences.

The next simplest candidate which is not already ruled out by the above sorts of remarks is $\mathscr{D}_2$. The Vershik map corresponding to $\mathscr{D}_2$ is

$$\sigma_2(n,x) = \begin{cases} (0, 0^m 1y) & \text{if } n = 0 \text{ and } x = 2^m 0y, \\ (m+1, y) & \text{if } n = 0 \text{ and } x = 2^m 1y, \\ (0, 0^{n-1}2x) & \text{if } n > 0. \end{cases}$$

This is the inverse of the *Chacón automorphism*. In del Junco-Rahe-Swanson [21], it is shown that the Chacón automorphism is not conjugate to its inverse via a measure-preserving transformation. As this automorphism has a unique ergodic measure, it follows that it is not conjugate to its inverse via a Borel automorphism, since any such automorphism would necessarily preserve measure.

5 Compositions of three involutions

In this section, we show that an automorphism of a σ-complete Boolean algebra, whose powers admit maximal discrete sections, can always be written as a product of 3 involutions from its full group. In the aperiodic case, it then follows from Proposition 4.3 that the existence of maximal discrete sections for the powers of π is a necessary and sufficient condition for writing π as the composition of n involutions from its full group, for any $n \geq 3$.

Suppose that $\mathfrak{A}$ is a purely atomic Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$ is aperiodic. One can visualize any $\varphi \in [\pi]$ as a collection of directed arcs which lie above the π-orbits of atoms of $\mathfrak{A}$. An automorphism

$$\varphi = \prod_{n \in \mathbb{Z}} a_n \xrightarrow{\pi^n} \pi^n(a_n)$$

in $[\pi]$ is *non-crossing* if none of these arcs cross, or equivalently, if

$$\forall n \in \mathbb{Z}\, \forall m \in (0, n) \left(\pi^m(a_n) \leq \sum_{\ell \in (0,n)} a_{\ell - m} \right), \tag{†}$$

where $(0, n)$ is shorthand for $\{m \in \mathbb{Z} : 0 < m < n \text{ or } n < m < 0\}$.

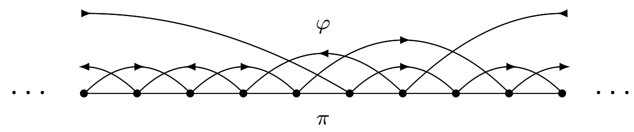

Figure 1.7: The arcs associated with the action of φ on the π-orbit of an atom of $\mathfrak{A}$.

An arc associated with $\varphi \in [\pi]$ is an *outer arc* if the φ-orbit of the points connected by the arc lies below it, and $a \leq a_n$ is the *base of an outer n-arc* if

$$\forall m \in \mathbb{N}\, \exists \ell \in [0, n] \left(\varphi^m | \mathfrak{A}_a = \pi^\ell | \mathfrak{A}_a \right).$$

We say that b *is covered by* φ if every atom of b sits below an outer arc, i.e.,

$$b \leq \sum \{\pi^m(a) : \exists n \in \mathbb{N}\ (m \in (0, n) \text{ and } a \text{ is the base of an outer } n\text{-arc})\}, \tag{‡}$$

and φ is *covering* if it covers $\mathbb{1}$. It should be noted that if b is covered by φ then, in fact, every atom below b is covered by infinitely many outer arcs.

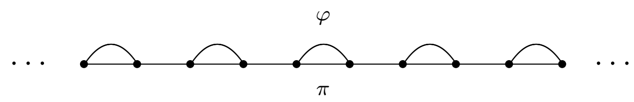

Figure 1.8: The arcs associated with the action of a non-covering involution.

Even when $\mathfrak{A}$ is not purely atomic, we will take (†) and (‡) to be the official definitions of non-crossing and covering. It should be noted that non-crossing covering automorphisms are necessarily periodic, although we will have little need for this.

Proposition 5.1. *Suppose that $\mathfrak{A}$ is a Boolean algebra and $\pi \in \operatorname{Aut}(\mathfrak{A})$ is aperiodic.*

1. *If φ is a non-crossing covering automorphism, then $\varphi \circ \pi$ is periodic.*

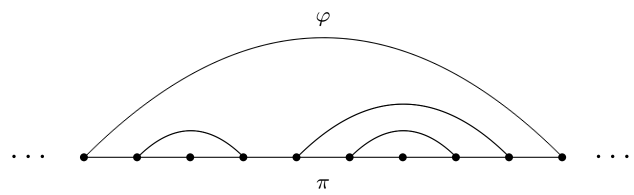

Figure 1.9: The arcs associated with the action of a non-crossing covering involution.

2. *If $\mathfrak{A}$ is σ-complete and π admits a non-crossing covering automorphism, then every element of $[\pi]$ admits a maximal discrete section.*

Proof. To see (1) suppose, towards a contradiction, that there is a non-zero $\varphi \circ \pi$-invariant $a \in \mathfrak{A}$ such that $\varphi \circ \pi | \mathfrak{A}_a$ is aperiodic. As φ is covering, there is a non-zero $b \leq a$ and positive natural numbers $m < n$ such that $\pi^{-m}(b)$ is the base of an outer n-arc. We will simply show that there is a non-zero element of $\mathfrak{A}_b$ whose orbit under $\varphi \circ \pi$ is trapped under this arc. As φ is non-crossing,

$$\forall \mathbb{0} < c \leq b \, \forall \ell \in \mathbb{Z} \, \forall k \notin [-m, n-m] \, \big((\varphi \circ \pi)^{\ell} | \mathfrak{A}_c \neq \pi^k | \mathfrak{A}_c \big).$$

In particular, we can recursively build a decreasing sequence $b_0 < \cdots < b_{n+1}$ of non-zero elements of $\mathfrak{A}_b$ and a sequence of integers $k_0, \ldots, k_{n+1} \in [-m, n-m]$ with

$$\forall 0 \leq \ell \leq n+1 \, \big((\varphi \circ \pi)^{\ell} | \mathfrak{A}_{b_\ell} = \pi^{k_\ell} | \mathfrak{A}_{b_\ell} \big).$$

As two of the k_i's must be equal, it follows that for some $1 \leq \ell \leq n+1$,

$$(\varphi \circ \pi)^{\ell} | \mathfrak{A}_{b_{n-1}} = \mathrm{id},$$

contradicting the the aperiodicity of $\varphi \circ \pi | \mathfrak{A}_a$.

To see (2), suppose φ is a non-crossing covering automorphism, fix a_n such that

$$\varphi(a) = \prod_{n \in \mathbb{Z}} a_n \xrightarrow{\pi^n} \pi^n(a_n),$$

and let

$$b_n = \sum_{|m| > n} a_m \cdot \pi^{-m}(a_0 + a_{-1} + \cdots + a_{-m})$$

be the piece of $\mathfrak{A}$ which is the base of an outer m-arc, where $|m| > n$. Note that each b_n is $\pi^{\leq n}$-discrete because φ is non-crossing. As the base of an outer n-arc can only be covered by an outer m-arc if $|m| > |n|$, the obvious induction shows that each b_n is a complete section. It follows that the sections of the form $\pi^m(b_n)$, with $m \in \mathbb{Z}$, are π^n-discrete sections which sum to $\operatorname{supp}(\pi^n)$. By Proposition 2.7, each π^n admits a maximal discrete section. By Proposition 3.1, every element of $[\pi]$ admits a maximal discrete section. $\dashv$

Corollary 5.2. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and $\pi \in \operatorname{Aut}(\mathfrak{A})$ admits a non-crossing covering involution. Then π is the composition of three involutions from its full group.*

Proof. Suppose $\iota_0 \in [\pi]$ is a non-crossing covering involution. By Proposition 5.1, $\iota_0 \circ \pi$ is periodic and every element of $[\iota_0 \circ \pi] \subseteq [\pi]$ admits a maximal discrete section. By Proposition 3.5, $\iota_0 \circ \pi$ is smooth. By Proposition 4.1, there are involutions $\iota_1, \iota_2 \in [\iota_0 \circ \pi]$ such that $\iota_0 \circ \pi = \iota_1 \circ \iota_2$, or equivalently, $\pi = \iota_0^{-1} \circ \iota_1 \circ \iota_2$. $\dashv$

Now that the problem of writing an automorphism π as a composition of three involutions in $[\pi]$ has been reduced to the problem of finding non-crossing covering involutions, it is time to embark upon the solution to this auxiliary problem.

An element $a \in \mathfrak{A}$ has *large gaps* (with respect to an automorphism π of $\mathfrak{A}$) if

$$\forall n \in \mathbb{N}\, \exists b \leq a \ \big([a]_\pi = [b]_\pi \text{ and } a \cdot (\pi(b) + \pi^2(b) + \cdots + \pi^n(b)) = \mathbb{O}\big).$$

Proposition 5.3. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and $\pi \in \operatorname{Aut}(\mathfrak{A})$ is aperiodic and admits a complete section with large gaps. Then π admits a non-crossing covering involution.*

Proof. Suppose that $\mathfrak{A}$ is purely atomic. Given any complete section $a \in \mathfrak{A}$, one can think of the orbits of atoms of $\mathfrak{A}$ as lying on a two dimensional mountain range. The *height* of an atom $b \in \mathfrak{A}$ is the least $h(b) \in \mathbb{N}$ such that either $\pi^{h(b)}(b) \leq a$ or $\pi^{-h(b)}(b) \leq a$. If $\pi^{h(b)}(b) \leq a$ and $\pi^{-h(b)}(b) \not\leq a$, then b is in the downward sloping piece of the range. If $\pi^{h(b)}(b) \not\leq a$ and $\pi^{-h(b)}(b) \leq a$, then b is in the upward sloping piece of the range. If $\pi^{h(b)}(b) \leq a$ and $\pi^{-h(b)}(b) \leq a$, then b is on the peak of a mountain in the range. For $n > 0$,

$$a_n = \pi^n(a) - \sum_{i \in [0,n)} \pi^i(a) - \sum_{i \in [0,n]} \pi^{-i}(a).$$

is the part of $\mathfrak{A}$ at height n which lies on the upward sloping piece of the range. For $n < 0$, the same formula defines the part of $\mathfrak{A}$ at height $|n|$ which lies on the downward sloping piece.

Now suppose that $a \in \mathfrak{A}$ is a complete section with large gaps. Note that this guarantees that $\sum_{m>n} a_m$ is a complete section, for all $n \in \mathbb{N}$. As the construction of a non-crossing covering involution is straightforward when π is smooth (an "infinite rainbow" centered on a transversal will certainly do the job), we may assume that each $\sum_{m>n} a_m$ is doubly recurrent. That is, any atom of $\mathfrak{A}$ can travel along its orbit to an atom of greater height by moving in either direction.

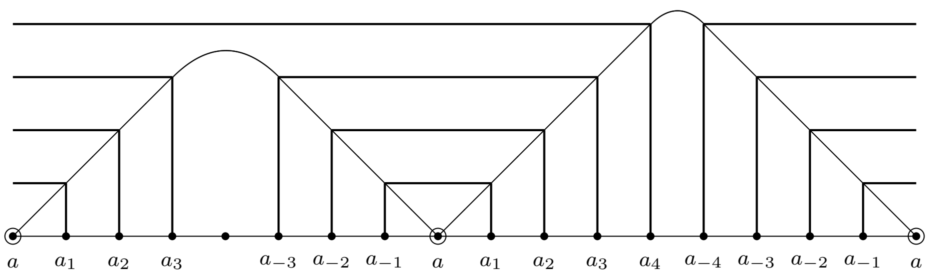

Figure 1.10: The induced mountain atop the π-orbit of an atom.

There is now a non-crossing covering involution right in front of us. Namely, the map which fixes the atoms on the peaks of the mountains and sends any other atom of $\mathfrak{A}$ to the unique atom at the same height which he can see from his perch on the mountain. The fact that there are arbitrarily tall mountains in either direction ensures that such a point exists, and the visibility condition ensures that the resulting involution is non-crossing. Formally, define

$$\iota = \prod_{n>0} (a_n \xrightarrow{\pi^{-2n}} a_{-n}).$$

It is clear that $\iota \in [\pi]$ is an involution. Noting that

$$\forall n \in \mathbb{N} \ (a_{n+1} \leq \pi(a_n) \text{ and } a_{-n-1} \leq \pi^{-1}(a_{-n})),$$

it easily follows that ι is non-crossing, and the fact that each $\sum_{m>n} a_m$ is doubly recurrent ensures that ι is covering. It now only remains to remark that, with the exception of the intuition behind the definition of ι, we have not actually used our assumption that $\mathfrak{A}$ is purely atomic. $\dashv$

Corollary 5.4. *Every aperiodic automorphism of a σ-complete Boolean algebra which admits a complete section with large gaps is the composition of three involutions from its full group.*

There are many Boolean algebras whose automorphisms admit complete sections with large gaps. A *measure algebra* is a pair $(\mathfrak{A}, \mu)$, where $\mathfrak{A}$ is a σ-complete Boolean algebra and μ is a measure on $\mathfrak{A}$ which is strictly positive on $\mathfrak{A}^+$. A measure algebra is *semi-finite* if the elements of finite measure are dense in $\mathfrak{A}$.

Proposition 5.5. *Every aperiodic automorphism of a semi-finite complete measure algebra admits a complete section with large gaps.*

Proof. Let $\mathscr{A} \subseteq \mathfrak{A}$ be a maximal set of elements of $\mathfrak{A}$ with large gaps such that

$$\forall a, b \in \mathscr{A} \ (a \neq b \Rightarrow [a]_\pi \cdot [b]_\pi = \mathbb{O}),$$

and let $a \in \mathfrak{A}$ be the least upper bound of $\mathscr{A}$. Clearly a has large gaps. Suppose, towards a contradiction, that a is not a complete section. Then there is a non-zero $b \leq \mathbb{1} - [a]_\pi$ of finite measure. For each $n \in \mathbb{N}$, let b_n be a maximal $\pi^{<n \cdot 3^n}$-discrete section and fix $i < 3^n$ such that

$$c_n = \sum_{in \leq j < (i+1)n} b \cdot \pi^j(b_n)$$

is of measure at most $\mu(b)/3^n$. Now set $c = \mathbb{1} - \sum_{n \in \mathbb{N}} c_n$, noting that $\mu(c) \geq \mu(b)/2$. It follows that $d = [c]_\pi$ is non-zero and c is a complete section for $\pi | \mathfrak{A}_d$ with large gaps, contradicting the maximality of $\mathscr{A}$. $\dashv$

Our task is not yet complete, however, for there are automorphisms of σ-complete Boolean algebras which do not admit complete sections with large gaps. Let $\mathscr{BP}$ denote the unique atomless, σ-complete Boolean algebra which has a countable dense subalgebra (i.e., the Baire measurable subsets of a perfect Polish space mod meager).

Proposition 5.6. *Suppose that f is an isometry of a complete ultrametric space. The following are equivalent:*

1. *f admits a Baire measurable transversal.*
2. *f admits a Baire measurable complete section with large gaps.*

Proof. The proof of (1) $\Rightarrow$ (2) is straightforward. To see $\neg(1) \Rightarrow \neg(2)$, suppose B is a Baire measurable complete section, and find an open ball $\mathscr{U}$ which does not contain a non-empty open partial transversal. As $X = \bigcup_{n \in \mathbb{Z}} f^n(B)$, it follows that by replacing B with its image under some iterate of f and shrinking $\mathscr{U}$, we may assume that B is comeager in $\mathscr{U}$. Then $f^n(\mathscr{U}) \cap \mathscr{U} \neq \emptyset$ for some $n \neq 0$. As any two balls of equal radius in an ultrametric space are either disjoint or identical, it follows that $f^n(\mathscr{U}) = \mathscr{U}$, thus

$$A = \bigcap_{k \in \mathbb{Z}} f^{kn}(B)$$

is comeager in $\mathscr{U}$. Now note that for all $C \subseteq [A]_\pi$, at least one of $\pi(C), \ldots, \pi^n(C)$ must intersect A, and therefore must intersect B. It follows that B does not have large gaps. ⊣

Corollary 5.7. *There is an automorphism of the homogeneous complete Boolean algebra $\mathscr{BP}$ which does not admit a complete section with large gaps. There is an automorphism of the σ-complete Boolean algebra of Borel subsets of an uncountable Polish space which does not admit a complete section with large gaps.*

Proof. By Corollary 3.10, the odometer does not admit a Baire measurable transversal, and thus cannot admit a Baire measurable complete section with large gaps. It follows that the automorphisms of the above Boolean algebras which are induced by the odometer do not have complete sections with large gaps. ⊣

Finally, it is time to complete the task at hand:

Theorem 5.8. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$, and every element of $[\pi]$ admits a maximal discrete section. Then π is the composition of three involutions from $[\pi]$.*

Proof. First note that by Propositions 3.5 and 4.1, the restriction of π to each of its exact period n parts is the composition of two involutions. Thus we may assume that π is aperiodic. Suppose that $a \in \mathfrak{A}$ is doubly π-recurrent. Note that for each $k > 0$, there is a maximal $a_k^\pi \leq a$ such that $\pi(a_k^\pi), \ldots, \pi^{k-1}(a_k^\pi)$ are disjoint from a and $\pi^k(a_k^\pi) \leq a$. Explicitly, this section is given by

$$a_k^\pi = a \cdot \left(\pi^{-k}(a) - \sum_{1 \leq \ell < k} \pi^{-\ell}(a) \right).$$

The *induced automorphism* of $\mathfrak{A}_a$ is given by

$$\pi_a = \prod_{k \in \mathbb{N}} a_k^\pi \xrightarrow{\pi^k} \pi^k(a_k^\pi).$$

Fix a decreasing sequence of elements $a_n \in \mathfrak{A}$ such that $a_0 = \mathbb{1}$ and a_{n+1} is maximal $\pi_{a_n}^{<3}$-discrete. Set $b_n = \pi_{a_n}(a_{n+1})$, $c_n = \pi_{a_n}^{-1}(a_{n+1})$, $\varphi_n = \pi_{a_n}^{-1} \circ \pi_{a_{n+1}} \circ \pi_{a_n}^{-1}$, and

$$\iota = \prod_{n>0} (b_n \xrightarrow{\varphi_n} c_n).$$

It is easily verified that ι is a non-crossing involution, but it need not be covering. Nevertheless,

$$a = \sum_{n>0} \pi(b_n) + \cdots + \pi^{3^n-1}(b_n)$$

is covered by ι. As $\mathbb{1} - a$ has large gaps and we have already produced a covering non-crossing involution off of $[\mathbb{1} - a]_\pi$, it follows from Proposition 5.3 that π admits a non-crossing covering involution, thus π is the composition of three involutions from its full group, by Corollary 5.2. ⊣

6 The full group of a group of automorphisms

Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms. When no confusion will result, we will occasionally identify an element $\gamma \in \Gamma$ with the automorphism by which it acts. The *full group* of Γ is the group $[\Gamma]$ of automorphisms of $\mathfrak{A}$ of the form

$$\pi = \prod_{\gamma \in \Gamma} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma,$$

where $\langle a_\gamma \rangle_{\gamma \in \Gamma}$ and $\langle \gamma \cdot a_\gamma \rangle_{\gamma \in \Gamma}$ are both partitions of unity.

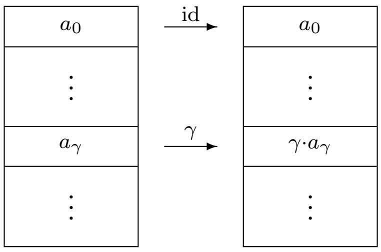

Figure 1.11: An element of the full group of Γ.

Proposition 6.1. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections. Then every element of $[\Gamma]$ admits a maximal discrete section.*

Proof. Suppose $\varphi \in [\Gamma]$, fix a partition of unity $\langle a_\gamma \rangle_{\gamma \in \Gamma}$ such that $\varphi = \prod_{\gamma \in \Gamma} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma$, fix maximal γ-discrete sections $b_\gamma \in \mathfrak{A}$, and observe that

$$\operatorname{supp}(\varphi) = \sum_{\gamma \in \Gamma} a_\gamma \cdot \operatorname{supp}(\gamma) \leq \sum_{\gamma \in \Gamma} a_\gamma \cdot (\gamma^{-1} \cdot b_\gamma + b_\gamma + \gamma \cdot b_\gamma).$$

Now for each $\gamma \in \Gamma$ and $n \in \mathbb{Z}$,

$$\begin{aligned} (a_\gamma \cdot (\gamma^n \cdot b_\gamma)) \cdot \varphi(a_\gamma \cdot (\gamma^n \cdot b_\gamma)) &= a_\gamma \cdot (\gamma^n \cdot b_\gamma) \cdot (\gamma \cdot a_\gamma) \cdot (\gamma^{n+1} \cdot b_\gamma) \\ &\leq \gamma^n \cdot (b_\gamma \cdot (\gamma \cdot b_\gamma)) \\ &= \mathbb{O}, \end{aligned}$$

thus each $a_\gamma \cdot (\gamma \cdot b_\gamma)$ is φ-discrete, so φ admits a maximal discrete section, by Proposition 2.7. $\dashv$

The Γ-*saturation* of $a \in \mathfrak{A}$ is $[a]_\Gamma = \sum_{\gamma \in \Gamma} \gamma \cdot a$ and a is a Γ-*complete section* if $[a]_\Gamma = \mathbb{1}$. A *partial* Γ-*transversal* is an element $a \in \mathfrak{A}$ such that

$$\forall \gamma \in \Gamma \ \big(\gamma | \mathfrak{A}_{a \cdot (\gamma \cdot a)} = \mathrm{id}\big),$$

and a Γ-*transversal* is a partial Γ-transversal which is also a Γ-complete section.

Proposition 6.2. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms. Then the following are equivalent:*

1. Γ *admits a transversal.*

2. The set of partial Γ*-transversals has a predense subset of cardinality* $< \kappa$*.*

Proof. To see (1) $\Rightarrow$ (2), it is enough to check that if $a \in \mathfrak{A}$ is a partial Γ-transversal, then for each $\delta \in \Gamma$, the image $\delta \cdot a$ is also a partial Γ-transversal. To see this, suppose that $\gamma \in \Gamma$, and observe that if $b \leq (\delta \cdot a) \cdot (\gamma \cdot (\delta \cdot a))$, then $\delta^{-1} \cdot b \leq a \cdot (\delta^{-1}\gamma\delta \cdot a)$. As a is a partial Γ-transversal, it follows that

$$\delta^{-1}\gamma\delta \cdot (\delta^{-1} \cdot b) = \delta^{-1} \cdot b,$$

thus

$$\begin{aligned} \gamma \cdot b &= \delta \cdot (\delta^{-1}\gamma\delta \cdot (\delta^{-1} \cdot b)) \\ &= \delta \cdot (\delta^{-1} \cdot b) \\ &= b, \end{aligned}$$

hence $\delta \cdot a$ is a partial Γ-transversal.

To see (2) $\Rightarrow$ (1), suppose that $\langle a_\xi \rangle_{\xi < \lambda}$ is a predense sequence of partial Γ-transversals, with $\lambda < \kappa$. Put $b_0 = \mathbb{O}$ and recursively define

$$b_{\xi+1} = b_\xi + (a_\xi - \Gamma \cdot b_\xi),$$

and $b_\xi = \sum_{\zeta < \xi} b_\zeta$ when ξ is a limit ordinal. Noting that

$$\begin{aligned} b_{\xi+1} \cdot (\gamma \cdot b_{\xi+1}) &= (b_\xi + (a_\xi - [b_\xi]_\Gamma)) \cdot (\gamma \cdot (b_\xi + (a_\xi - [b_\xi]_\Gamma))) \\ &\leq b_\xi \cdot (\gamma \cdot b_\xi) + a_\xi \cdot (\gamma \cdot a_\xi), \end{aligned}$$

it follows from the obvious induction that b_λ is a partial Γ-transversal. To see that b_λ is a Γ-complete section, note that for any non-zero $a \in \mathfrak{A}$ there exists $\xi < \lambda$ such that $a \cdot a_\xi \neq \mathbb{O}$. It then follows from the definition of $b_{\xi+1}$ that $a \cdot (b_{\xi+1} + [b_\xi]_\Gamma) \neq \mathbb{O}$, thus $a \cdot (\Gamma \cdot b_\lambda) \neq \mathbb{O}$. $\dashv$

The action of the group Γ on $\mathfrak{A}$ is *smooth* if it admits a transversal.

Proposition 6.3. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts smoothly on $\mathfrak{A}$. Then every element of $[\Gamma]$ admits a maximal discrete section.*

Proof. By Proposition 6.1, it is enough to show that each $\gamma \in \Gamma$ admits a maximal discrete section. Fix a Γ-transversal $a \in \mathfrak{A}$, and note that $\mathrm{supp}(\gamma)$ is the sum of the γ-discrete sections of the form $(\delta \cdot a) - \gamma \cdot (\delta \cdot a)$, thus γ admits a maximal discrete section by Proposition 2.7. $\dashv$

The Γ-*orbit* of $a \in \mathfrak{A}$ is the set $\{\gamma \cdot a : \gamma \in \Gamma\}$. An element $a \in \mathfrak{A}$ is Γ-*periodic* if for densely many $b \in \mathfrak{A}$ there exists $n \in \mathbb{N}$ such that for all $c \leq b$, every pairwise disjoint set of non-zero elements of $\mathfrak{A}_a$ contained in the Γ-orbit of c is of cardinality at most n. The action of Γ is *periodic* if $\mathbb{1}$ is Γ-periodic. When $\mathfrak{A}$ is purely atomic, this simply says that the Γ-orbit of every atom is finite.

Proposition 6.4. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections. If the action of Γ admits a periodic complete section, then the action of Γ on $\mathfrak{A}$ is smooth.*

Proof. Suppose that $\Delta \subseteq \Gamma$ is finite and contains 1. A Δ-discrete section $b \in \mathfrak{A}$ is a *local Δ-witness to the Γ-periodicity of a* if $[b]_\Delta \leq a$ and for all $c \leq b$, every pairwise disjoint subset of the Γ-orbit of c contained in $\mathfrak{A}_a$ is of cardinality at most $|\Delta|$. It immediately follows that $a \in \mathfrak{A}$ is Γ-periodic exactly when the local witnesses to periodicity are dense below a. The remainder of the proof hinges on the following connection between local witnesses and partial transversals:

Lemma 6.5. *Suppose that $\mathfrak{A}$ is a Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms, and $\Delta \subseteq \Gamma$ is finite and contains 1. Then every local Δ-witness to periodicity is a partial Γ-transversal.*

Proof. Suppose, towards a contradiction, that b is a local Δ-witness to periodicity and there exists $\gamma \in \Gamma$ such that

$$\gamma^{-1} | \mathfrak{A}_{b \cdot (\gamma^{-1} \cdot b)} \neq \mathrm{id}.$$

It then follows from Proposition 2.2 that there is a non-zero γ^{-1}-discrete section $c \leq b \cdot (\gamma^{-1} \cdot b)$. Then $c \cdot (\gamma \cdot c) = \mathbb{O}$ by Proposition 2.1, and since $\gamma \cdot c \leq b$, it follows that $(\gamma \cdot c) \cdot (\delta \cdot c) = \mathbb{O}$ for all $\delta \in \Delta$, and this contradicts the fact that b is a local Δ-witness to Γ-periodicity. $\dashv$

Before completing the proof, it will be useful to make a slight detour and consider the corresponding notion of Γ-aperiodicity. An element $a \in \mathfrak{A}$ is Γ-*aperiodic* if for all $n \in \mathbb{N}$, there are densely many non-zero $b \in \mathfrak{A}_a$ whose Γ-orbits contain a pairwise disjoint subset of $\mathfrak{A}_a$ of cardinality n. Clearly, a is Γ-aperiodic exactly when there are no local witnesses to the periodicity of a. Also note that when $\mathfrak{A}$ is purely atomic, aperiodicity means that $\mathfrak{A}_a$ contains infinitely many elements of the Γ-orbit of every atom in $\mathfrak{A}_a$. Equivalently, a is Γ-aperiodic if for every atom $b \leq a$ there is an infinite set $\Delta \subseteq \Gamma$ such that $\{\delta \cdot b\}_{\delta \in \Delta} \subseteq \mathfrak{A}_a$ is pairwise disjoint.

Lemma 6.6. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms which have supports, and $a \in \mathfrak{A}$. Then there is a Γ-aperiodic $a_\infty^\Gamma \leq a$ such that $a - a_\infty^\Gamma$ is Γ-periodic.*

Proof. For each finite set $\Delta \subseteq \Gamma$ containing 1 and $\gamma \in \Gamma$, define

$$a_\Delta^\gamma = \prod_{\delta \in \Delta} a \cdot (\gamma^{-1} \cdot a) \cdot \operatorname{supp}(\gamma^{-1}\delta).$$

We claim that there are no non-zero local Δ-witnesses $b \leq a_\Delta^\gamma$ to the Γ-periodicity of a. For if b were such a witness, then we could find $c \leq b$ such that $\gamma \cdot c$ is disjoint from $[c]_\Delta$, contradicting the definition of a local Δ-witness to Γ-periodicity.

It follows that there are no such witnesses below

$$a_\Delta^\Gamma = \sum_{\gamma \in \Gamma} a_\Delta^\gamma,$$

thus there can be no non-zero local witnesses to the Γ-periodicity of a below

$$a_\infty^\Gamma = \prod_{\Delta \subseteq \Gamma \text{ finite}} a_\Delta^\Gamma.$$

On the other hand, it is clear that for any non-zero $b \leq a - a_\Delta^\gamma$, either $a \cdot (\gamma \cdot b) = \mathbb{O}$ or there exists $\delta \in \Delta$ such that $(\gamma \cdot b) \cdot (\delta \cdot b) \neq \mathbb{O}$. It follows that

$$a - a_\Delta^\Gamma = \prod_{\gamma \in \Gamma} a - a_\Delta^\gamma$$

is a local Δ-witness to the Γ-periodicity of a, and therefore of $a - a^\Gamma_\infty$. As

$$\begin{aligned} a - a^\Gamma_\infty &= a - \prod_{\Delta \subseteq \Gamma \text{ finite}} a^\Gamma_\Delta \\ &= \sum_{\Delta \subseteq \Gamma \text{ finite}} a - a^\Gamma_\Delta, \end{aligned}$$

it follows that $a - a^\Gamma_\infty$ is Γ-periodic. ⊣

Now suppose that $a \in \mathfrak{A}$ is a Γ-periodic complete section. To see that the action of Γ is smooth, it is enough to find a collection of fewer than κ partial Γ-transversals which are predense below a, by Proposition 6.2 and the fact that a is a Γ-complete section. By Lemma 6.5, it is therefore enough to find a collection of fewer than κ local witnesses to periodicity which are predense below a.

By the proof of Proposition 2.15, for each finite $\Delta \subseteq \Gamma$ containing 1 we can find finitely many Δ-discrete sections $a^\Delta_0, \ldots, a^\Delta_n \in \mathfrak{A}$ whose sum is $\mathbb{1}$. We claim that the set of sections of the form

$$b^\Delta_i = a^\Delta_i \cdot (a - a^\Gamma_\Delta)$$

is as desired. To see this, suppose that $b \leq a$ is non-zero, and find a finite set $\Delta \subseteq \Gamma$ containing 1 and a non-zero local Δ-witness $c \leq b$ to the Γ-periodicity of a. As c is Δ-discrete and $c \leq a - a^\Gamma_\Delta$, it follows that $b^\Delta_i \cdot c \neq \mathbb{0}$ for some i, and this completes the proof. ⊣

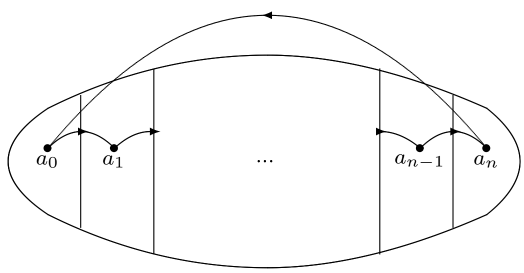

Figure 1.12: The action of a smooth automorphism of strict period $n + 1$ on its support.

An automorphism π is of *strict period* n if

$$\{a \in \mathfrak{A} : \forall i < n \ (\pi^i(a) \neq a) \text{ and } \pi^n(a) = a\}$$

is dense in $\mathfrak{A}_{\mathrm{supp}(\pi)}$. When $\mathfrak{A}$ is purely atomic, this means that the orbit of every atom is of cardinality 1 or n. It is important here that automorphisms of strict period n can have atoms whose orbits are of cardinality 1! We will say that an automorphism is of *exact period* n if it is fixed-point free and of strict period n.

Proposition 6.7. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, $a \in \mathfrak{A}$ is Γ-aperiodic, and $n \in \mathbb{N}$. Then there is a partition $b_0, \ldots, b_n$ of a and an automorphism $\pi \in [\Gamma]$ of strict period $n+1$ such that*

$$\forall i < n\ (\pi(b_i) = b_{i+1}) \text{ and } \pi(b_n) = b_0.$$

Proof. We will begin by showing that the desired map can be built off of a Γ-invariant section on which the action of Γ is smooth.

Lemma 6.8. *There is a sequence of pairs (a_ξ, Δ_ξ) of length strictly less than κ such that each Δ_ξ contains 1 and is of cardinality $n+1$, and for each $\Delta \subseteq \Gamma$ of cardinality $n+1$, $\{a_\xi : \Delta = \Delta_\xi\}$ is predense in the Δ-discrete elements of $\mathfrak{A}_a$.*

Proof. As Γ has fewer than κ finite subsets, it is enough to check that for each finite set $\Delta \subseteq \Gamma$, there is a finite set of Δ-discrete sections which is predense in the set of all Δ-discrete sections. This is exactly what is shown in the second paragraph of the proof of Proposition 2.15. $\dashv$

Fix such a sequence (a_ξ, Δ_ξ). We will recursively build up $a_0, \ldots, a_n$ and π, beginning with

$$a_0^{(0)} = a_1^{(0)} = \cdots = a_n^{(0)} = \mathbb{O}.$$

At stage ξ, we fix an enumeration $\langle\, \delta_i^{(\xi)} \,\rangle_{i \leq n}$ of Δ_ξ, with $\delta_0^{(\xi)} = 1$. We then consider the maximal element of $\mathfrak{A}_{a_\xi}$ whose images under the elements of Δ_ξ lie below a and are disjoint from all of the sections that were constructed before stage ξ. This element is simply

$$a_0^{(\xi)} = a_\xi \cdot \left(\prod_{\delta \in \Delta_\xi} \delta^{-1} \cdot a \right) - \sum_{\eta < \xi, i \leq n, \delta \in \Delta_\xi} \delta^{-1} \cdot a_i^{(\eta)}.$$

We then set $a_i^{(\xi)} = \delta_i^{(\xi)} \cdot a_0^{(\xi)}$, for each $0 < i \leq n$.

As soon as this process has been completed, we set $a_i = \sum_\xi a_i^{(\xi)}$, $a' = \sum_i a_i$, and

$$\pi = \prod_\xi (a_0^{(\xi)} \xrightarrow{\delta_1^{(\xi)}} a_1^{(\xi)} \xrightarrow{\delta_2^{(\xi)}(\delta_1^{(\xi)})^{-1}} \cdots \xrightarrow{\delta_n^{(\xi)}(\delta_{n-1}^{(\xi)})^{-1}} a_n^{(\xi)}).$$

It is clear that π is an automorphism of strict period $n+1$ that sends a_i to a_{i+1} and has support a', but it could be the case that $a' < a$.

Nevertheless, $a - a'$ is Γ-periodic. To see this suppose, towards a contradiction, that there is a finite set $\Delta \subseteq \Gamma$ of cardinality $n+1$ containing 1 and $b \leq a - a'$ such that $\{\delta \cdot b\}_{\delta \in \Delta}$ is a pairwise disjoint set of sections below $a - a'$. Then there exists ξ such that $\Delta_\xi = \Delta$ and $c = b \cdot a_\xi$ is non-zero, and it follows that

$$a_0^{(\xi)} \cdot c \neq \mathbb{O} \text{ or } \exists \eta < \xi \, \exists i \leq n \, \exists \delta \in \Delta_\xi \; ((\delta^{-1} \cdot a_i^{(\eta)}) \cdot c \neq \mathbb{O}),$$

contradicting the fact that $c \leq b \leq a - a'$ and $\delta \cdot c \leq \delta \cdot b \leq a - a'$.

It follows from Proposition 6.4 that the action of Γ on $\mathfrak{A}_{[a-a']_\Gamma}$ is smooth. So it only remains to handle the case that the action of Γ on $\mathfrak{A}$ is smooth. For this, we will need the following fact:

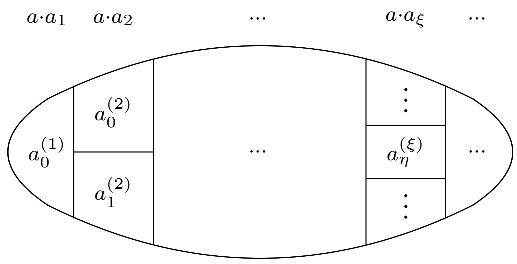

Figure 1.13: When Γ acts smoothly, a can be partitioned into partial Γ-transversals.

Lemma 6.9. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality $\lambda < \kappa$ which acts smoothly on $\mathfrak{A}$ by automorphisms, and $a \in \mathfrak{A}$. Then there is a partition of unity into Γ-invariant sections a_ξ, where ξ ranges over cardinals $\leq \lambda$, and partitions of each $a \cdot a_\xi$ into ξ transversals for the action of Γ on $\mathfrak{A}_{a_\xi}$.*

Proof. We begin with a sublemma:

Sublemma 6.10. *For all $b \leq a$, there is a maximal partial Γ-transversal in $\mathfrak{A}_{a-b}$.*

Proof. Fix an enumeration γ_ξ of Γ. Fix a Γ-transversal $c \in \mathfrak{A}$, and set

$$c_\xi = (\gamma_\xi \cdot c) \cdot (a - b),$$

noting that these sections are partial Γ-transversals which sum to $a-b$. We will define an increasing sequence of sections d_ξ, at stage ξ adding as much of c_ξ as possible while maintaining that d_ξ is a partial transversal. That is, we take $d_0 = \mathbb{O}$,

$$d_{\xi+1} = d_\xi + (c_\xi - [d_\xi]_\Gamma),$$

and $d_\xi = \sum_{\eta<\xi} d_\eta$ at limit ordinals. Noting that

$$\begin{aligned} d_{\xi+1} \cdot (\gamma \cdot d_{\xi+1}) &= (d_\xi + (c_\xi - [d_\xi]_\Gamma)) \cdot (\gamma \cdot d_\xi + \gamma \cdot (c_\xi - [d_\xi]_\Gamma)) \\ &= d_\xi \cdot (\gamma \cdot d_\xi) + d_\xi \cdot (\gamma \cdot (c_\xi - [d_\xi]_\Gamma)) + \\ &\quad\; (c_\xi - [d_\xi]_\Gamma) \cdot (\gamma \cdot d_\xi) + (c_\xi - [d_\xi]_\Gamma) \cdot (\gamma \cdot (c_\xi - [d_\xi]_\Gamma)) \\ &\leq d_\xi \cdot (\gamma \cdot d_\xi) + c_\xi \cdot (\gamma \cdot c_\xi), \end{aligned}$$

it follows from the obvious induction that $d = \sum_\xi d_\xi$ is a partial Γ-transversal. Now suppose, towards a contradiction, that there is a partial Γ-transversal $e \in \mathfrak{A}_{a-b}$ with $d < e$. Then there exists ξ such that

$$c_\xi \cdot (e - d) \neq \mathbb{O},$$

thus $d_{\xi+1} \cdot (e - d) \neq \mathbb{O}$, a contradiction. $\dashv$

Now fix an enumeration γ_ξ of Γ and a Γ-transversal $b \in \mathfrak{A}$. Put $b_0 = a \cdot b$, and repeatedly apply Sublemma 6.10 so as to produce a sequence of sections b_ξ, with b_ξ a maximal partial Γ-transversal in $\mathfrak{A}_a$ which is disjoint from $\sum_{\eta<\xi} b_\eta$.

It follows from the proof of Sublemma 6.10 that the b_ξ's may be chosen so that

$$a = \sum_{\xi<\lambda} b_\xi.$$

This is because we can use the same transversal c in constructing each of these sections, and by doing so, we will guarantee that

$$\gamma_\xi \cdot c \leq \sum_{\eta\leq\xi} b_\eta.$$

In addition to this, we will we need one more fact:

Sublemma 6.11. *For all $\eta < \xi$, $[b_\xi]_\Gamma \leq [b_\eta]_\Gamma$.*

Proof. Setting $d = [b_\xi]_\Gamma - [b_\eta]_\Gamma$, it is clear that $e = b_\eta + b_\xi \cdot d$ is in $\mathfrak{A}_a$ and disjoint from $\sum_{\zeta<\eta} b_\zeta$. Noting that

$$\begin{aligned} e \cdot (\gamma \cdot e) &= (b_\eta + b_\xi \cdot d) \cdot (\gamma \cdot b_\eta + \gamma \cdot (b_\xi \cdot d)) \\ &= b_\eta \cdot (\gamma \cdot b_\eta) + b_\eta \cdot (\gamma \cdot (b_\xi \cdot d)) + (b_\xi \cdot d) \cdot (\gamma \cdot b_\eta) + (b_\xi \cdot d) \cdot (\gamma \cdot (b_\xi \cdot d)) \\ &= b_\eta \cdot (\gamma \cdot b_\eta) + (b_\xi \cdot d) \cdot (\gamma \cdot (b_\xi \cdot d)) \\ &\leq b_\eta \cdot (\gamma \cdot b_\eta) + b_\xi \cdot (\gamma \cdot b_\xi), \end{aligned}$$

it follows that e is a partial Γ-transversal, thus by maximality we have that $b_\eta = e$, hence $d = \mathbb{O}$, and it follows that $[b_\xi]_\Gamma \leq [b_\eta]_\Gamma$. ⊣

As the b_ξ's are built up, they gradually cover more and more of a. Although it is not necessarily the case, it could happen that non-zero Γ-invariant elements of $\mathfrak{A}$ are entirely covered before all of the b_ξ's are constructed. That is, it might be that $[b_\xi]_\Gamma < [b_\eta]_\Gamma$, for some $\eta < \xi$. In the purely atomic case, this corresponds to the case when the intersection of a with the Γ-orbit of some atom is of cardinality less than $|\Gamma|$. For each cardinal $\xi \leq \lambda$, let a_ξ be the maximal element of $\mathfrak{A}$ which is covered for the first time at an ordinal stage of cardinality ξ. That is, set

$$a_\xi = \prod_{\eta<\xi} [b_\eta]_\Gamma - \prod_{\eta<\xi^+} [b_\eta]_\Gamma,$$

noting that even a_λ makes sense, since $b_\eta = \mathbb{O}$ for all $\eta \geq \lambda$. It remains to check that each $a \cdot a_\xi$ can be partitioned into ξ transversals for the action of Γ on a_ξ. As the sections of the form

$$a \cdot ([b_\eta]_\Gamma - [b_{\eta+1}]_\Gamma),$$

with $\xi \leq \eta < \xi^+$, partition $a \cdot a_\xi$, it is enough to show that each of these can be partitioned into ξ transversals for the action of Γ on $[b_\eta]_\Gamma - [b_{\eta+1}]_\Gamma$. As

$$\langle b_\zeta \cdot ([b_\eta]_\Gamma - [b_{\eta+1}]_\Gamma) \rangle_{\zeta\leq\eta}$$

provides such a partition, this completes the proof of the lemma. ⊣

Now let a_ξ be as in Lemma 6.9. We claim that $a_n = \mathbb{O}$, for each $n \in \mathbb{N}$. To see this, simply note that if $a_n \neq \mathbb{O}$, then we can find a non-zero $b \leq a_n$ and a finite set $\Delta \subseteq \Gamma$ containing 1 for which b is a local Δ-witness to periodicity, contradicting the Γ-aperiodicity of a.

For each cardinal ξ, fix transversals $b_\eta^{(\xi)}$, with $\eta < \xi$, for the action of Γ on a_ξ which partition $a \cdot a_\xi$. We claim that there are elements of $[\Gamma]$ such that

$$\pi_\eta^{(\xi)}(b_\eta^{(\xi)}) = b_{\eta+1}^{(\xi)}.$$

Granting this, set

$$S = \{\lambda + i(n+1) : \lambda \text{ is a limit ordinal and } i \in \mathbb{N}\},$$

put $b_0 = \sum_{\eta<\xi,\eta\in S} a_\eta^{(\xi)}$, define

$$\pi = \prod_{\eta<\xi,\eta\in S} (a_\eta^{(\xi)} \xrightarrow{\pi_\eta^{(\xi)}} a_{\eta+1}^{(\xi)} \xrightarrow{\pi_{\eta+1}^{(\xi)}} \cdots \xrightarrow{\pi_{\eta+n-1}^{(\xi)}} a_{\eta+n}^{(\xi)}),$$

and put $b_i = \pi^i(a_0)$, for $i \leq n$. It easily follows that π is an element of Γ of exact period $n+1$ which carries b_i to b_{i+1}, for $i < n$. It only remains to check the following:

Lemma 6.12. *Suppose $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, and a, b are Γ-transversals. Then there exists $\pi \in [\Gamma]$ such that $\pi(a) = b$.*

Proof. As we can always take π to be the identity on $\mathfrak{A}_{a \cdot b}$, we may assume that $a \cdot b = \mathbb{O}$. Fix an enumeration γ_ξ of Γ. We will gradually build up the domain and range of π, at stage ξ including the largest remaining piece of a which γ_ξ maps into the remaining piece of b. That is, we take

$$a_\xi = \left(a - \sum_{\eta<\xi} a_\eta\right) \cdot \left(\gamma_\xi^{-1} \cdot \left(b - \sum_{\eta<\xi} b_\eta\right)\right) \text{ and } b_\xi = \gamma_\xi \cdot a_\xi.$$

Set $a' = \sum_\xi a_\xi$ and $b' = \sum_\xi b_\xi$, noting that these elements lie below a and b. In fact, if $c \leq a$ then there exists ξ with $c \cdot (\gamma_\xi^{-1} \cdot b) \neq \mathbb{O}$, and it follows that

$$c \cdot a_\eta \neq \mathbb{O},$$

for some $\eta \leq \xi$. Thus every $c \leq a$ has a non-zero intersection with a', so $a' = a$. It similarly follows that $b' = b$, thus

$$\pi = \prod_{\xi} (a_\xi \xrightarrow{\gamma_\xi} b_\xi)$$

is the desired element of $[\Gamma]$. ⊣

7 Compositions of periodic automorphisms

In this section, we will give a complete the characterization of the circumstances under which an aperiodic automorphism π of a σ-complete Boolean algebra $\mathfrak{A}$ is the composition of two automorphisms of prescribed strict periods from its full group. We have already seen that the existence of maximal discrete sections for the powers of π is necessary, and here we will show that it is sufficient, as long as we are not trying to write the automorphism in question as the composition of two involutions. We will also use similar ideas to show that if G is a σ-full group of automorphisms which contains at least one aperiodic automorphism, then every element of G is a commutator. We actually show this with the apparently weaker hypothesis that G is κ-complete and contains a subgroup of cardinality less than κ which acts aperiodically, but we then show that this implies that G contains an aperiodic automorphism. This latter result is the natural generalization of the fact that every aperiodic countable Borel equivalence relation contains an aperiodic hyperfinite subequivalence relation.

As was the case with involutions, we first consider the smooth case:

Proposition 7.1. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra and $\pi \in \mathrm{Aut}(\mathfrak{A})$ is aperiodic and smooth. Then for every $n_0, n_1 \geq 2$, there are automorphisms $\pi_i \in [\pi]$ such that each π_i is of strict period n_i and $\pi = \pi_0 \circ \pi_1$.*

Proof. It is enough to show that the successor on $\mathbb{Z}$ is the composition of permutations τ_i which are of strict period n_i, for then we can obtain the desired automorphisms by fixing a π-transversal $a_0 \in \mathfrak{A}$, setting $a_n = \pi^n(a_0)$, and defining

$$\pi_i = \prod_{n \in \mathbb{Z}} a_n \xrightarrow{\pi^{\tau_i(n)-n}} a_{\tau_i(n)}.$$

We will build the τ_i's recursively. At each stage, we will simply compose our approximation to each τ_i with an appropriately chosen cycle. The following lemma describes one way of selecting these additional cycles (see Figure 1.14 for a depiction of their action on $\mathbb{Z}$):

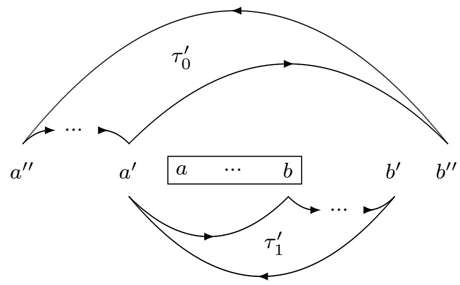

Figure 1.14: Extending τ_i to τ_i'.

Lemma 7.2. *Suppose $n_i \geq 2$, $a \leq b$, τ_i is a permutation of $[a,b]$ of strict period n_i, $\tau_1(b) = b$, and*

$$\tau_0 \circ \tau_1 = (a\ a+1\ \cdots\ b).$$

Set $a' = a - 1$, $b' = b + (n_1 - 2)$, $a'' = a' - (n_0 - 2)$, $b'' = b' + 1$, and define

$$\tau_0' = (a''\ a''+1\ \cdots\ a'\ b'') \circ \tau_0 \text{ and } \tau_1' = (a'\ b\ b+1\ \cdots\ b') \circ \tau_1.$$

Then $\tau_0' \circ \tau_1' = (a''\ a''+1\ \cdots\ b'')$.

Proof. Suppose that $a'' \leq k \leq b''$. We break the task of checking that

$$\tau_0' \circ \tau_1' = (a''\ a''+1\ \cdots\ b'')$$

into several cases:

1. If $a'' \leq k < a'$, then $\tau_1'(k) = k$, thus $\tau_0' \circ \tau_1'(k) = \tau_0'(k) = k + 1$.

2. If $k = a'$, then $\tau_1'(k) = b$, thus $\tau_0' \circ \tau_1'(k) = \tau_0'(b) = \tau_0(b) = a = k + 1$.

3. If $a \leq k < b$, then $\tau_1'(k) = \tau_1(k)$, thus $\tau_0' \circ \tau_1'(k) = \tau_0' \circ \tau_1(k) = \tau_0 \circ \tau_1(k) = k + 1$.

4. If $b \leq k < b'$, then $\tau_1'(k) = k + 1$, thus $\tau_0' \circ \tau_1'(k) = \tau_0'(k + 1) = k + 1$.

5. If $k = b'$, then $\tau_1'(k) = a'$, thus $\tau_0' \circ \tau_1'(k) = \tau_0'(a') = b''$.

6. If $k = b''$, then $\tau_1'(k) = b''$, thus $\tau_0' \circ \tau_1'(k) = \tau_0'(b'') = a''$.

It follows that $\tau_0' \circ \tau_1' = (a''\ a''+1\ \cdots\ b'')$, as desired. $\dashv$

Using the above notation, set $a_0 = b_0 = 0$, $\tau_i^{(0)} = \mathrm{id}$, $a_{k+1} = a_k''$, $b_{k+1} = b_k''$, and

$$\tau_i^{(k+1)} = (\tau_i^{(k)})'.$$

It follows that $\tau_i^{(k)}$ is a permutation of $\{-k(n_0-1), \ldots, k(n_1-1)\}$ such that

$$\tau_0^{(k)} \circ \tau_1^{(k)} = (-k(n_0-1)\ \cdots\ k(n_1-1)).$$

The primary remaining observation is that for all $n \in \mathbb{Z}$ and all $k_0, k_1 > |n|$,

$$(\tau_i^{(k_0)})^{\pm 1}(n) = (\tau_i^{(k_1)})^{\pm 1}(n),$$

by the definition of $\tau_i^{(k)}$. In particular, it makes sense to define $\tau_i : \mathbb{Z} \to \mathbb{Z}$ by

$$\tau_i(n) = \lim_{k \to \infty} \tau_i^{(k)}(n).$$

We claim that τ_0, τ_1 are permutations of $\mathbb{Z}$ whose composition is the successor. To see that τ_i is injective, suppose that $\tau_i(m) = \tau_i(n)$, and fix $k > \max(|m|, |n|)$. Then

$$\tau_i^{(k)}(m) = \tau_i(m) = \tau_i(n) = \tau_i^{(k)}(n),$$

thus $m = n$, since $\tau_i^{(k)}$ is injective. To see that τ_i is surjective, fix $n \in \mathbb{N}$ and set $k = |n| + 2$ and

$$m = (\tau_i^{(k)})^{-1}(n).$$

As $|m| \leq k - 1$ by the definition of $\tau_i^{(k)}$, it follows that

$$\tau_i(m) = \tau_i^{(k)}(m) = n.$$

To see that $\tau_0 \circ \tau_1$ is the successor function, suppose that $n \in \mathbb{N}$, fix

$$k > \max(|n|, |\tau_1^{(|n|+1)}(n)|),$$

and observe that

$$\begin{aligned}\tau_0 \circ \tau_1(n) &= \tau_0 \circ \tau_1^{(k)}(n) \\ &= \tau_0^{(k)} \circ \tau_1^{(k)}(n) \\ &= n+1,\end{aligned}$$

which completes the proof of the proposition. ⊣

Remark 7.3. It will be important later on to note that the cycle $(0\ \ldots\ k(n_0+n_1-2))$ is the composition of permutations τ_i which are of strict period n_i. Of course this follows easily from the above proof, as the permutations obtained at the k^{th} stage of the construction satisfy

$$\tau_0^{(k)} \circ \tau_1^{(k)} = (-k(n_0-1)\ \cdots\ k(n_1-1)),$$

thus we can obtain the desired permutations by conjugating $\tau_i^{(k)}$ through the map $\varphi : \{-k(n_0-1), \ldots, k(n_1-1)\} \to \{0, \ldots, k(n_0+n_1-2)\}$ which is given by

$$\varphi(n) = n + k(n_0-1).$$

Remark 7.4. If $n_i \geq 3$, then $\tau_{1-i}^{(k+1)}$ has at least one more fixed point than $\tau_{1-i}^{(k)}$, thus has at least $k+1$ fixed points. This will be quite important in the arguments to come! Proposition 4.1 tells us that whatever we have in store had better not work in the case that $n_0 = n_1 = 2$. The fact that the number of fixed points does not (indeed, cannot) increase in this case is where our argument breaks down.

Remark 7.5. When $n_0, n_1 \geq 3$, k of the fixed points $\ell_1^{(0)}, \ldots, \ell_k^{(0)}, \ell_1^{(1)}, \ldots, \ell_k^{(1)}$ of the automorphisms

$$\varphi \circ \tau_0^{(k)} \circ \varphi^{-1}, \varphi \circ \tau_1^{(k)} \circ \varphi^{-1}$$

can be chosen so that they are interspersed, in the sense that

$$0 < \ell_1^{(0)} < \ell_1^{(1)} < \cdots < \ell_k^{(0)} < \ell_k^{(1)}.$$

This follows from a simple inductive argument, although is perhaps most easily seen by examining Figure 1.15.

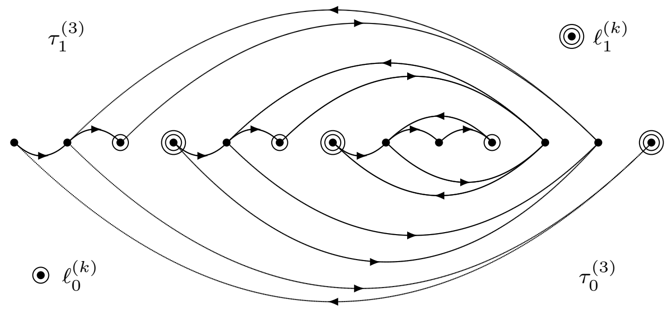

Figure 1.15: The action of $\tau_i^{(3)}$ when $n_0 = n_1 = 3$.

Remark 7.6. It will also be important to note that $\tau_0^{(k)}, \tau_1^{(k)}$ are conjugate. This follows from the fact that each of these permutations has the same cycle type.

We can already obtain an approximation to the sort of result we desire:

Proposition 7.7. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$ is aperiodic, and every element of $[\pi]$ admits a maximal discrete section. Then for all $n_0, n_1, n_2 \geq 2$, there are automorphisms $\pi_i \in [\pi]$ of strict period n_i such that*

$$\pi = \pi_0 \circ \pi_1 \circ \pi_2.$$

Proof. Find a maximal $\pi^{<n_0}$-discrete section $a \in \mathfrak{A}$ and a covering non-crossing involution ι for π_a. Let

$$b = \sum_{n>0} a - \mathrm{supp}(\pi^{-n} \circ \iota) = a \cdot \prod_{n \geq 0} \mathrm{supp}(\pi^n \circ \iota)$$

be the piece of $\mathfrak{A}$ which is moved in the "same direction" by ι and π, and define

$$\pi_0 = (b \xrightarrow{\pi} \pi(b) \xrightarrow{\pi} \cdots \xrightarrow{\pi} \pi^{n_0-2}(b) \xrightarrow{\iota \circ \pi^{-(n_0-2)}} \iota(b)).$$

It is clear that π_0 is a non-crossing covering automorphism of strict period n_0, since it has the same outer arcs as ι. By Lemma 5.1, it follows that $\varphi = \pi_0 \circ \pi$ is periodic. As a is $\pi^{<n_0}$-discrete, it also follows that

$$\forall c \in \mathfrak{A} \ (\pi_0 | \mathfrak{A}_c \neq \pi^{-1} | \mathfrak{A}_c),$$

thus φ is fixed-point free.

By Ryzhikov [69], for each $k \geq 2$ there is a finite group $G_k = \langle g_1^{(k)}, g_2^{(k)} \rangle$, where

$$g_k = g_1^{(k)} g_2^{(k)}$$

is of order k and $g_i^{(k)}$ is of order n_i. We will group the φ-orbits of cardinality k into sets of cardinality $|G_k|/k$, and copy the action of G_k on itself by left multiplication over to these sets in such a way that the action of g_k on G_k goes to the action of φ. The push-forwards of the actions of the generators of G_k will then give rise to the desired automorphisms $\pi_1, \pi_2 \in [\pi]$.

Set $H_k = \langle g_k \rangle$ and fix representatives $h_i^{(k)} \in G_k$, with $i < |G_k|/k$, for the left cosets of H_k within G_k. Let a_k be the period k part of φ. As the special case of Proposition 7.7 in which π is smooth follows from Proposition 7.1, we may assume that each a_k is π-aperiodic. Let b_k be a transversal for the action of φ on $\mathfrak{A}_{a_k}$, noting that b_k is also π-aperiodic. It follows from Proposition 6.7 that there exists $\psi_k \in \mathrm{Aut}(\mathfrak{A}_{b_k})$ of exact period $|G_k|/k$. Let c_k be a ψ_k-transversal. Put

$$X_k = \{(m, n) \in \mathbb{N}^2 : m < k \text{ and } n < |G_k|/k\},$$

noting that the elements of the form $\varphi^m \circ \psi_k^n(c_k)$, with $(m, n) \in X_k$, partition a_k. For $i \in \{1, 2\}$ and $(m, n) \in X_k$, let (m', n') be the unique element of X_k such that

$$g_i^{(k)} g_k^m h_k^{(n)} = g_k^{m'} h_k^{(n')},$$

and define

$$\pi_i^{(k)} = \prod_{(m,n) \in X_k} \varphi^m \circ \psi_k^n(c_k) \xrightarrow{\varphi^{m'} \circ \psi_k^{n'} \circ \psi_k^{-n} \circ \varphi^{-m}} \varphi^{m'} \circ \psi_k^{n'}(c_k).$$

It is clear that $\pi_i^{(k)}$ is of exact period $|g_i^{(k)}| = n_i$, thus so too is

$$\pi_i = \prod_{k \geq 2} a_k \xrightarrow{\pi_i^{(k)}} a_k.$$

As $g_k = g_1^{(k)} \circ g_2^{(k)}$, it follows that $\varphi = \pi_1 \circ \pi_2$, thus $\pi = \pi_0^{-1} \circ \pi_1 \circ \pi_2$. $\dashv$

In order to describe the circumstances under which we can write π as a product of *two* elements of its full group of prescribed periods, we must first make a slight detour, and show a version of Alpern's [3] multiple Rokhlin tower theorem.

Suppose that $\pi \in \mathrm{Aut}(\mathfrak{A})$ and $a \in \mathfrak{A}$ is doubly recurrent. As in the proof of Theorem 5.8, we can partition a into countably many sections a_n^π, where a_n^π is the maximal $b \leq a$ such that $\pi(b), \ldots, \pi^{n-1}(b)$ are disjoint from a and $\pi^n(b) \leq a$. Explicitly, these sections are given by

$$a_n^\pi = a \cdot \pi^{-n}(a) - \sum_{0<i<n} \pi^{-i}(a),$$

and the associated *induced automorphism of* $\mathfrak{A}_a$ is given by

$$\pi_a = \prod_{n>0} a_n^\pi \xrightarrow{\pi^n} \pi^n(a_n^\pi).$$

The section a is *n-spaced* if $a = a_n^\pi$. Such sections are very useful in constructions involving automorphisms. Unfortunately, not every automorphism admits complete sections of this sort!

Example 7.8. The *Bernoulli shift* on $X = 2^{\mathbb{Z}}$ is the bilateral shift,

$$f(\langle x_n \rangle_{n\in\mathbb{Z}}) = \langle x_{n+1} \rangle_{n\in\mathbb{N}}.$$

We claim that for all $n \geq 2$, f does not admit an n-spaced Baire measurable complete section. Suppose, towards a contradiction, that $B \subseteq X$ is such a section. As f sends meager sets to meager sets and

$$X = \bigcup_{n\in\mathbb{Z}} f^n(B),$$

it follows that B is non-meager, thus comeager in some basic clopen neighborhood $\mathscr{N}_s$, with $s : [-k, k] \to \{0, 1\}$.

We will show that B is comeager. Of course, it is enough to check that each $s' : [-k', k] \to \{0, 1\}$ can be extended to an s'' such that B is comeager in $\mathscr{N}_{s''}$. Fix $m \in \mathbb{N}$ sufficiently large that

$$mn - k > k',$$

and define $s'' : [-k', k'] \cup [mn - k, mn + k] \to \{0, 1\}$ by

$$s''(i) = \begin{cases} s'(i) & \text{if } -k' \leq i \leq k, \\ s(i - mn) & \text{otherwise.} \end{cases}$$

As $\mathscr{N}_{s''} \subseteq f^{mn}(\mathscr{N}_s)$ and B is comeager in $\mathscr{N}_s$ and n-spaced, it follows that B is comeager in $\mathscr{N}_{s''}$.

As B is comeager, it follows that B contains comeagerly many full orbits of f. In particular, B cannot be n-spaced, the desired contradiction.

To make up for this deficiency, we will work with a slightly more general sort of section. Suppose that $S \subseteq \mathbb{N}$ is finite. Then $a \in \mathfrak{A}$ is *S-spaced* if

$$a = \sum_{n \in S} a_n^\pi.$$

When $\mathfrak{A}$ is purely atomic, $a \in \mathfrak{A}$ is S-spaced exactly when for each atom $b \leq a$, the least $k > 0$ such that $\pi^k(b) \leq a$ is in S (i.e., the size of the gap in-between successive atoms is always in S).

Now we are ready for the promised version of Alpern's theorem [3]:

Proposition 7.9. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, π is an aperiodic automorphism of $\mathfrak{A}$ whose powers admit maximal discrete sections, and $m, n \in \mathbb{N}$ are relatively prime. Then π admits an $\{m, n\}$-spaced complete section.*

Proof. Fix $x, y \in \mathbb{N}$ such that

$$ny - mx = 1,$$

and find a maximal $\pi^{<(mx)^2}$-discrete section $a \in \mathfrak{A}$, noting that

$$a = \sum_{(mx)^2 \leq k < 2(mx)^2} a_k^\pi.$$

Setting $S' = \{(q, r) \in \mathbb{N}^2 : 0 < r < mx \leq q < 2mx\}$, it follows that the sections

$$a_{qr} = a^\pi_{(mx)q+r} = a^\pi_{(mx)(q-r)+(ny)r},$$

with $(q, r) \in S'$, partition a. Set $b_{qr} = \pi^{mx(q-r)}(a_{qr})$, put

$$a'_{qr} = a_{qr} + \pi^m(a_{qr}) + \pi^{2m}(a_{qr}) + \cdots + \pi^{(x(q-r)-1)m}(a_{qr})$$

and

$$b'_{qr} = b_{qr} + \pi^n(b_{qr}) + \pi^{2n}(b_{qr}) + \cdots + \pi^{(y(q-r)-1)n}(b_{qr}),$$

and define

$$c = \sum_{(q,r)\in S'} a'_{qr} + b'_{qr}.$$

Noting that

$$\forall (q, r) \in S' \ (a'_{qr} \leq c^{\pi}_m \text{ and } b'_{qr} \leq c^{\pi}_n),$$

it follows that c is the desired S-spaced complete section. $\dashv$

Remark 7.10. When $S \subseteq \mathbb{N}$ is finite, the same idea can be used to show that π admits an S-spaced complete section $\Leftrightarrow$ π admits a $\gcd(S)$-spaced complete section.

Now we are ready to turn to the problem of writing an automorphism as a composition of two automorphisms from its full group of prescribed periods. This was originally accomplished in the case of the Lebesgue measure algebra by Ryzhikov [69]. Although the proof we give is a bit different than his, the basic idea of using involution results to reduce the problem to the finite case is taken directly from Ryzhikov [69] and [70].

Theorem 7.11. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$ is aperiodic, and every element of $[\pi]$ admits a maximal discrete section. Then for all $n_0 \geq 2$ and $n_1 \geq 3$, there are automorphisms $\pi_i \in [\pi]$ such that each π_i is of strict period n_i and $\pi = \pi_0 \circ \pi_1$.*

Proof. Let $m_0 = 3(n_0 + n_1 - 2)$ and $m_1 = (m_0 + 1)(n_0 + n_1 - 2)$, noting that $m_0 + 1$ and $m_1 + 1$ are relatively prime. By Remark 7.4, there are permutations

$$\tau^{(j)}_i \in S_{\{0,\ldots,m_j\}},$$

of strict period n_i, such that $\tau_0^{(j)}$ has fixed points $0 < \ell_0 < \ell_1 < \ell_2 \leq m_0$ which are independent of j, and

$$\tau_0^{(j)} \circ \tau_1^{(j)} = (0 \ \cdots \ m_j).$$

It follows from Proposition 7.9 that π admits an $\{m_0+1, m_1+1\}$-spaced complete section $a \in \mathfrak{A}$. By Proposition 7.7, there are automorphisms $\pi'_i \in [\pi_a]$ of strict period n_0 such that

$$\pi_a = \pi'_0 \circ \pi'_1 \circ \pi'_2.$$

Let $\varphi \in [\pi]$ be the automorphism of strict period n given by

$$\varphi = \prod_{0 \leq i \leq 2} \pi^{\ell_i}(a) \xrightarrow{\pi^{\ell_i} \circ (\pi'_i)^{-1} \circ \pi^{-\ell_i}} \pi^{\ell_i}(a),$$

and set $\psi = \varphi \circ \pi$. By the *period n part of φ*, we mean

$$\prod_{0<m<n} \operatorname{supp}(\pi^m) - \operatorname{supp}(\pi^n).$$

Let a_j be the period $m_j + 1$ part of ψ.

Lemma 7.12. $a_0 + a_1 = \mathbb{1}$.

Proof. Set $b_j = (\pi'_0 \circ \pi'_1 \circ \pi'_2)(a^\pi_{m_j+1})$, and note that

$$\forall \ell < \ell_0 \ (\psi^\ell | \mathfrak{A}_{b_j} = \pi^\ell | \mathfrak{A}_{b_j}),$$

thus $\psi^{\ell_0} | \mathfrak{A}_{b_j} = \pi^{\ell_0} \circ (\pi'_0)^{-1} | \mathfrak{A}_{b_j}$. It follows that for all $\ell_0 \leq \ell < \ell_1$,

$$\psi^\ell | \mathfrak{A}_{b_j} = \pi^\ell \circ (\pi'_0)^{-1} | \mathfrak{A}_{b_j},$$

thus $\psi^{\ell_1} | \mathfrak{A}_{b_j} = \pi^{\ell_1} \circ (\pi'_1)^{-1} \circ (\pi'_0)^{-1} | \mathfrak{A}_{b_j})$. It then follows that for all $\ell_1 \leq \ell < \ell_2$,

$$\psi^\ell | \mathfrak{A}_{b_j} = \pi^\ell \circ (\pi'_1)^{-1} \circ (\pi'_0)^{-1} | \mathfrak{A}_{b_j},$$

thus

$$\begin{aligned} \psi^{\ell_2} | \mathfrak{A}_{b_j} &= \pi^{\ell_2} \circ (\pi'_2)^{-1} \circ (\pi'_1)^{-1} \circ (\pi'_0)^{-1} | \mathfrak{A}_{b_j} \\ &= \pi^{\ell_2} \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1} | \mathfrak{A}_{b_j}. \end{aligned}$$

It follows that $\psi^{\ell_2}(b_j) = \pi^{\ell_2}(a^{\pi}_{m_j+1})$, thus for all $\ell_2 \leq \ell \leq m_j + 1$,

$$\psi^{\ell}|\mathfrak{A}_{b_j} = \pi^{\ell} \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_j}.$$

In particular, we have

$$\begin{aligned}
\psi^{m_j+1}|\mathfrak{A}_{b_j} &= \pi^{m_j+1} \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_j} \\
&= \pi_a \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_j} \\
&= (\pi'_0 \circ \pi'_1 \circ \pi'_2) \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_j} \\
&= \mathrm{id}.
\end{aligned}$$

Noting that for all $\ell \leq m_j + 1$,

$$\psi^{\ell}(b_j) = \pi^{\ell}(b_j),$$

it follows that $\psi|\mathfrak{A}_{[b_j]_\varphi}$ is of exact period $m_j + 1$ and $[b_0]_\psi + [b_1]_\psi = \mathbb{1}$. ⊣

Said somewhat differently, this means that $\psi|\mathfrak{A}_{[a_j]_\psi}$ is of exact period $m_j + 1$ and

$$\psi = \prod_{j \in \{0,1\}} (a_j \xrightarrow{\psi} \psi(a_j) \xrightarrow{\psi} \cdots \xrightarrow{\psi} \psi^{m_j}(a_j)).$$

Next, we will take advantage of the fact that we already know how to write cycles of length $m_i + 1$ in the desired form. For each $i \in \{0, 1\}$, set

$$\psi_i = \prod_{0 \leq j \leq 1, 0 \leq k \leq m_j} \psi^k(a \cdot a_j) \xrightarrow{\psi^{\tau_i^{(j)}(k)-k}} \psi^{\tau_i^{(j)}(k)}(a \cdot a_j).$$

As ψ_i is of strict period n_i and

$$\varphi \circ \pi = \psi = \psi_0 \circ \psi_1,$$

it follows that $\pi = (\varphi^{-1} \circ \psi_0) \circ \psi_1$. As the supports of ψ_0 and φ are disjoint, it follows that $\pi_0 = \varphi^{-1} \circ \psi_0$ and $\pi_1 = \psi_1$ are as desired. ⊣

Remark 7.13. Of course, the same proof can be used to show the corresponding theorem when $n_0 \geq 3$ and $n_1 \geq 2$. Alternatively, this version can be obtained as a corollary by applying Theorem 7.11 to find automorphisms $\pi_0^*, \pi_1^* \in [\pi]$ of strict periods n_1, n_0 such that

$$\pi^{-1} = \pi_0^* \circ \pi_1^*,$$

and then taking $\pi_0 = (\pi_1^*)^{-1}$ and $\pi_1 = (\pi_0^*)^{-1}$.

By slightly modifying the above argument, a stronger result can be obtained when $n_0 = n_1$. Recall that $\pi_0, \pi_1 \in [\pi]$ are *conjugate* if there exists $\varphi \in [\pi]$ such that $\varphi \circ \pi_0 \circ \varphi^{-1} = \pi_1$.

Theorem 7.14. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$ is aperiodic, every element of $[\pi]$ admits a maximal discrete section, and $n \geq 3$. Then there are automorphisms $\pi_i \in [\pi]$, of strict period n, such that $\pi = \pi_0 \circ \pi_1$ and π_0, π_1 are conjugate via an element of $[\pi]$.*

Proof. Let $m_0 = 3(n + n - 2) = 6(n - 1)$ and $m_1 = (m_0 + 1)(n + n - 2) = 2(m_0 + 1)(n - 1)$, noting that $m_0 + 1, m_1 + 1$ are relatively prime. By Remark 7.5, there are natural numbers

$$0 < \ell_0^{(1)} < \ell_0^{(0)} < \ell_1^{(1)} < \ell_1^{(0)} < \ell_2^{(1)} < \ell_2^{(0)} \leq m_0,$$

and permutations $\tau_i^{(j)}$ of $\{0, \ldots, m_j\}$ of strict period n, where $\tau_i^{(j)}(\ell_j^{(i)}) = \ell_j^{(i)}$ and

$$\tau_0^{(j)} \circ \tau_1^{(j)} = (0 \ \cdots \ m_j).$$

It follows from Proposition 7.9 that π admits an $\{m_0+1, m_1+1\}$-spaced complete section $a \in \mathfrak{A}$. By Proposition 7.7, there are automorphisms $\pi_i' \in [\pi_a]$ of strict period n such that

$$\pi_a = \pi_0' \circ \pi_1' \circ \pi_2'.$$

Moreover, we may assume that the support of each π_i' is π-aperiodic. This is because Proposition 6.4 ensures that π is smooth on the piece where this fails, and the proof of Proposition 7.1 clearly produces conjugate transformations when $n_0 = n_1$.

Let c_i be a transversal for the action of π_i' on $\mathfrak{A}_{\mathrm{supp}(\pi_i')}$, note that each c_i is π-aperiodic, and apply Proposition 6.7 to find involutions $\iota_i \in [\pi]$ with support c_i. Let d_i be a maximal ι_i-discrete section, and set

$$e_i^{(0)} = [d_i]_{\pi_i'} \text{ and } e_i^{(1)} = [\iota_i(d_i)]_{\pi_i'}.$$

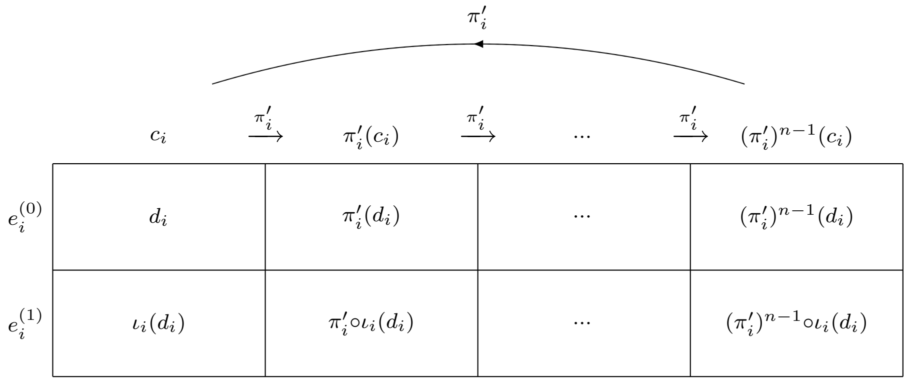

Figure 1.16: The partition of $\mathrm{supp}(\pi_i') \leq a$ into the various images of d_i.

Now define $\varphi_j \in [\pi]$ by

$$\varphi_j = \prod_{0 \leq i \leq 2} \pi^{\ell_i^{(j)}}(a) \xrightarrow{\pi^{\ell_i^{(j)}} \circ (\pi_i' | e_i^{(j)})^{-1} \circ \pi^{-\ell_i^{(j)}}} \pi^{\ell_i^{(j)}}(a),$$

set $\psi = \varphi_1 \circ \varphi_0 \circ \pi$, and let a_j be the period $m_j + 1$ part of ψ.

Lemma 7.15. $a_0 + a_1 = \mathbb{1}$.

Proof. For each $s : \{0, 1, 2, 3\} \to \{0, 1\}$, set

$$b_s = e_0^{(s(0))} \cdot \pi_0'(e_1^{(s(1))}) \cdot \pi_0' \circ \pi_1'(e_2^{(s(2))}) \cdot (\pi_0' \circ \pi_1' \circ \pi_2')(a^{\pi}_{m_{s(3)}+1}),$$

and note that

$$\forall \ell < \ell_0^{(s(0))} \ (\psi^{\ell} | \mathfrak{A}_{b_s} = \pi^{\ell} | \mathfrak{A}_{b_s}),$$

thus $\psi^{\ell_0^{(s(0))}} | \mathfrak{A}_{b_s} = \pi^{\ell_0^{(s(0))}} \circ (\pi_0')^{-1} | \mathfrak{A}_{b_s}$. It follows that for all $\ell_0^{(s(0))} \leq \ell < \ell_1^{(s(1))}$,

$$\psi^{\ell} | \mathfrak{A}_{b_s} = \pi^{\ell} \circ (\pi_0')^{-1} | \mathfrak{A}_{b_s},$$

thus $\psi^{\ell_1^{(s(1))}} | \mathfrak{A}_{b_s} = \pi^{\ell_1^{(s(1))}} \circ (\pi_1')^{-1} \circ (\pi_0')^{-1} | \mathfrak{A}_{b_s})$. It follows that for all $\ell_1^{(s(1))} \leq \ell < \ell_2^{(s(2))}$,

$$\psi^{\ell} | \mathfrak{A}_{b_s} = \pi^{\ell} \circ (\pi_1')^{-1} \circ (\pi_0')^{-1} | \mathfrak{A}_{b_s},$$

thus

$$\begin{aligned} \psi^{\ell_2^{(s(2))}} | \mathfrak{A}_{b_s} &= \pi^{\ell_2^{(s(2))}} \circ (\pi_2')^{-1} \circ (\pi_1')^{-1} \circ (\pi_0')^{-1} | \mathfrak{A}_{b_s} \\ &= \pi^{\ell_2^{(s(2))}} \circ (\pi_0' \circ \pi_1' \circ \pi_2')^{-1} | \mathfrak{A}_{b_s}. \end{aligned}$$

It follows that $\psi^{\ell_2^{(s(2))}}(b_s) = \pi^{\ell_2^{(s(2))}}(a^{\pi}_{m_{s(3)}+1})$, thus for all $\ell_2^{(s)} \leq \ell \leq m_{s(3)} + 1$,

$$\psi^{\ell}|\mathfrak{A}_{b_s} = \pi^{\ell} \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_s}.$$

In particular, we have

$$\begin{aligned}
\psi^{m_{s(3)}+1}|\mathfrak{A}_{b_s} &= \pi^{m_{s(3)}+1} \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_s} \\
&= \pi_a \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_s} \\
&= (\pi'_0 \circ \pi'_1 \circ \pi'_2) \circ (\pi'_0 \circ \pi'_1 \circ \pi'_2)^{-1}|\mathfrak{A}_{b_s} \\
&= \text{id}.
\end{aligned}$$

Noting that for all $\ell \leq m_{s(3)} + 1$,

$$\psi^{\ell}(b_s) = \pi^{\ell}(b_s),$$

it follows that $\psi|\mathfrak{A}_{[b_s]_\psi}$ is of exact period $m_{s(3)} + 1$ and $\sum_{s:\{0,1,2,3\}\to\{0,1\}}[b_s]_\psi = \mathbb{1}$. $\dashv$

It is once more time to take advantage of the fact that we already know how to write cycles of length $m_i + 1$ in the desired form. For each $i \in \{0, 1\}$, set

$$\psi_i = \prod_{0\leq j\leq 1, 0\leq k\leq m_j} \psi^k(a \cdot b_j) \xrightarrow{\psi^{\tau_i^{(j)}(k)-k}} \psi^{\tau_i^{(j)}(k)}(a \cdot b_j),$$

and note that the ψ_i's are conjugate automorphisms of strict period n and

$$\varphi_1 \circ \varphi_0 \circ \pi = \psi = \psi_0 \circ \psi_1.$$

As φ_0, φ_1 have disjoint supports and therefore commute, it follows that

$$\begin{aligned}
\pi &= \varphi_0^{-1} \circ \varphi_1^{-1} \circ \psi_0 \circ \psi_1 \\
&= \varphi_0^{-1} \circ \varphi_1^{-1} \circ \psi_0 \circ \varphi_1 \circ \varphi_1^{-1} \circ \psi_1 \\
&= \varphi_1^{-1} \circ (\varphi_0^{-1} \circ \psi_0) \circ \varphi_1 \circ (\varphi_1^{-1} \circ \psi_1).
\end{aligned}$$

As φ_i, ψ_i have disjoint supports and therefore commute, it follows that

$$\pi_0 = \varphi_1^{-1} \circ (\varphi_0^{-1} \circ \psi_0) \circ \varphi_1$$

and $\pi_1 = \varphi_1^{-1} \circ \psi_1$ are automorphisms of strict period n and $\pi = \pi_0 \circ \pi_1$.

To see that π_0 and π_1 are conjugate, it is enough to check that $\varphi_0^{-1} \circ \psi_0$ and $\varphi_1^{-1} \circ \varphi_1$ are conjugate, and for this, it is enough to show that φ_0 is conjugate to φ_1 and ψ_0 is conjugate to ψ_1. The former part follows from the fact that each ι_i easily extends to a conjugacy of $\pi_i' | e_i^{(0)}$ and $\pi_i' | e_i^{(1)}$ (see Figure 1.16), and the latter fact follows from the fact that the permutations of $\mathbb{Z}$ from which the ψ_i's are built are conjugate, which itself follows from the Remark 7.6. $\dashv$

Recall that $g \in G$ is a *commutator* if it is of the form $g = [g_0, g_1]$, where

$$[g_0, g_1] = g_0 g_1 g_0^{-1} g_1^{-1}.$$

Note that g is a commutator exactly when it is of the form $g_0 g_1$, where g_0 and g_1^{-1} are conjugate. As every smooth automorphism is conjugate to its inverse, it now follows that if π is aperiodic and the elements of $[\pi]$ admit maximal discrete sections, then every element of $[\pi]$ is a commutator. This is a special case of a more general fact:

Theorem 7.16. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts aperiodically on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections. Then every element of $[\Gamma]$ is a commutator.*

Proof. Fix $\pi \in [\Gamma]$, let a_∞ be the aperiodic part of π, let

$$a_{\text{even}} = \sum_{n \in \mathbb{N}} \prod_{0<k<2n} \operatorname{supp}(\pi^k) - \operatorname{supp}(\pi^{2n})$$

be the evenly periodic part of π, and let

$$a_{\text{odd}} = \sum_{n \in \mathbb{N}} \prod_{0<k<2n+1} \operatorname{supp}(\pi^k) - \operatorname{supp}(\pi^{2n+1})$$

be the oddly periodic part of π. Fix a transversal $b_{\text{even}} \in \mathfrak{A}$ for $\pi | \mathfrak{A}_{a_{\text{even}}}$. By the proof of Proposition 6.7, there is an involution $\iota_{\text{even}} \in [\Gamma]$ such that:

1. $\operatorname{supp}(\iota_{\text{even}}) \leq b_{\text{even}}$.

2. $c = b_{\text{even}} - \operatorname{supp}(\iota_{\text{even}})$ is a partial Γ-transversal.

3. $[c]_\Gamma$ is below the Γ-periodic part of a_{even}.

Set $d_\infty = [c]_\Gamma \cdot [a_\infty]_\Gamma$. It easily follows from the fact that c is a partial Γ-transversal that there exists $\varphi \in [\Gamma]$ such that

$$\varphi(c \cdot d_\infty) \leq a_\infty.$$

Set $e_0 = [c \cdot d_\infty]_\pi + [\varphi(c \cdot d_\infty)]_\pi$, $e_1 = [c]_\Gamma - [a_\infty]_\Gamma$, $e_2 = \mathbb{1} - (e_0 + e_1)$. We will now complete the proof via several lemmas.

Lemma 7.17. $\pi|\mathfrak{A}_{e_0}$ *is a commutator in* $[\Gamma|\mathfrak{A}_{e_0}]$.

Proof. Set $e = [c \cdot d_\infty]_\pi$. It follows from Proposition 4.1 that there are involutions $\iota_0, \iota_1 \in [\Gamma|\mathfrak{A}_e]$ whose composition is $\pi|\mathfrak{A}_e$. Now, set $f = [\varphi(c \cdot d_\infty)]_\pi$. It follows from a simple modification of the proof of Proposition 7.1 that there are automorphisms $\varphi_0, \varphi_1 \in [\Gamma|\mathfrak{A}_f]$ such that:

1. $\pi|\mathfrak{A}_f = \varphi_0 \circ \varphi_1$.

2. For each $i \in \{1, 2, 3\}$ and $j \in \{0, 1\}$, the exact period i part of φ_j is an aperiodic complete section for $\pi|\mathfrak{A}_f$.

3. For each $j \in \{0, 1\}$, the parts of φ_j of exact periods 1,2, and 3 partition f.

Setting $\pi_j = \iota_j \circ \varphi_j$, it follows that $\pi|\mathfrak{A}_{e_0} = \pi_0 \circ \pi_1$. Now fix transversals $g_i^{(j)}$ for the action of π_j on its part of exact period i. The smoothness of the action of Γ on $\mathfrak{A}_{e_0}$ coupled with the Γ-aperiodicity of each of these sections guarantees that for each $i \in \{1, 2, 3\}$, there is a map $\varphi_i \in [\Gamma]$ such that

$$\varphi_i(g_i^{(0)}) = g_i^{(1)}.$$

It only remains to note that $\prod_{i \in \{1,2,3\}} g_i^{(0)} \xrightarrow{\varphi_i} g_i^{(1)}$ can easily be extended to a conjugacy of π_0, π_1 within $[\Gamma]$. ⊣

Lemma 7.18. $\pi|\mathfrak{A}_{e_1}$ *is a commutator in* $[\Gamma|\mathfrak{A}_{e_1}]$.

Proof. As $[c]_\Gamma$ lies below the Γ-periodic part of a_{even}, it follows that e_1 lies below the Γ-aperiodic part of a_{odd}. By Proposition 4.1, we can find involutions $\iota_0, \iota_1 \in [\Gamma|\mathfrak{A}_{e_1}]$ such that

$$\pi|\mathfrak{A}_{e_1} = \iota_0 \circ \iota_1.$$

Coupled with the fact that e_1 lies below the Γ-aperiodic part of a_{odd}, the proof of Proposition 4.1 implies that for each $k \in \{0, 1\}$, the exact period 1 and 2 parts of ι_k are aperiodic complete sections for $\pi|\mathfrak{A}_{e_1}$. As $\pi|\mathfrak{A}_{e_1}$ is smooth, a simple argument then shows that ι_0 and ι_1 are conjugate. ⊣

Lemma 7.19. *$\pi|\mathfrak{A}_{e_2}$ is a commutator in $[\Gamma|\mathfrak{A}_{e_2}]$.*

Proof. We still must handle the principal ideal below $\mathbb{1} - [c]_\Gamma$. We will deal with the principal ideals below the evenly periodic, oddly periodic, and aperiodic parts of this section separately. Of course, Theorem 7.14 gives us the latter bit, so we need only handle the evenly and oddly periodic parts. As the involutions produced by applying the proof of Proposition 4.1 to an oddly periodic automorphism φ are conjugate within $[\varphi]$, it follows that we need only concern ourselves with $a_{\mathrm{even}} - (e_0 + e_1)$.

Fix a transversal d_0 for $\iota|\mathfrak{A}_{a_{\mathrm{even}}-(e_0+e_1)}$ and put $d_1 = \iota(d_0)$, noting that d_0, d_1 partition $a_{\mathrm{even}} - (e_0 + e_1)$. By the proof of Proposition 4.1, for each $k \in \{0, 1\}$ there is a pair of involutions

$$\iota_0^{(k)}, \iota_1^{(k)} \in [\pi|\mathfrak{A}_{[d_k]_\pi}]$$

such that

1. $\pi|\mathfrak{A}_{[d_0]_\pi} = \iota_0^{(0)} \circ \iota_1^{(0)}$ and $\pi|\mathfrak{A}_{[d_1]_\pi} = \iota_0^{(1)} \circ \iota_1^{(1)}$.
2. $\iota_0^{(0)}$ and $\iota_1^{(1)}$ are fixed-point free.
3. $\mathrm{supp}(\iota_1^{(0)}) = [d_0]_\pi - d_0'$ and $\mathrm{supp}(\iota_0^{(1)}) = [d_1]_\pi - d_1'$, where
   (a) d_0' is a disjoint sum of two partial transversals of $\pi|[d_0]_\pi$.
   (b) d_1' is a disjoint sum of two partial transversals of $\pi|[d_1]_\pi$.

In particular, (3) implies there is an automorphism in $[\Gamma]$ which carries d_0 to d_1, thus

$$\iota_0 = \prod_{k \in \{0,1\}} [d_k]_\pi \xrightarrow{\iota_0^{(k)}} [d_k]_\pi \text{ and } \iota_1 = \prod_{k \in \{0,1\}} [d_k]_\pi \xrightarrow{\iota_1^{(k)}} [d_k]_\pi$$

are conjugate, and clearly $\pi|\mathfrak{A}_{a_{\mathrm{even}}-[c]_\Gamma} = \iota_0 \circ \iota_1$, so $\pi|\mathfrak{A}_{a_{\mathrm{even}}-[c]_\Gamma}$ is a commutator. ⊣

Remark 7.20. The above proof shows the slightly stronger statement that if the support of π is Γ-aperiodic, then π_0 and π_1 can be chosen so that their supports are contained in the support of π.

This result can be generalized once more. A group $G \leq \mathrm{Aut}(\mathfrak{A})$ is κ*-full* if

$$\forall \Gamma \subseteq G \ (|\Gamma| < \kappa \Rightarrow [\Gamma] \subseteq G).$$

As usual, we will say that G is σ*-full* if it is ω_1-full.

Corollary 7.21. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ that admit maximal discrete sections, and G has a subgroup of cardinality strictly less than κ which acts aperiodically. Then every element of G is a commutator.*

Proof. Fix an automorphism $\pi \in G$, let $\Gamma \leq G$ be a group of cardinality strictly less than κ which acts aperiodically, and note that the group $\Delta = \langle \Gamma, \pi \rangle$ is also of cardinality strictly less than κ and acts aperiodically. It follows that π is a commutator within $[\Delta] \leq G$. ⊣

As every infinite Polish space has an aperiodic automorphism, we have the following:

Theorem 7.22. *Every Borel automorphism of an infinite Polish space X is a commutator within the group of all Borel automorphisms of X.*

Finally, it should be noted that the existence of an aperiodic subgroup of cardinality strictly less than κ is equivalent to the existence of an aperiodic automorphism of G. This is a consequence of the following fact:

Proposition 7.23. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts aperiodically on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections. Then $[\Gamma]$ contains an aperiodic automorphism.*

Proof. The *full semigroup of* Γ is the semigroup $[\![\Gamma]\!]$ of isomorphisms $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$, with $a, b \in \mathfrak{A}$, which are of the form

$$\pi = \prod_{\gamma \in \Gamma} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma,$$

in which $\langle a_\gamma \rangle_{\gamma \in \Gamma}$ is a partition of a and $\langle \gamma \cdot a_\gamma \rangle_{\gamma \in \Gamma}$ is a partition of b.

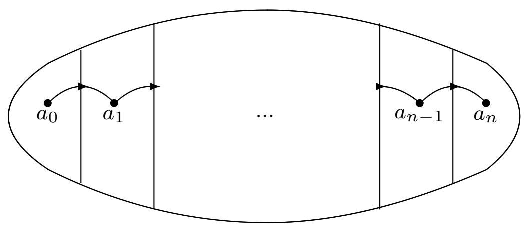

Figure 1.17: The action of an acyclic element of $[\![\Gamma]\!]$ of period $n + 1$.

The map $\varphi \in [\![\Gamma]\!]$ is *acyclic of period* n if there is a partition of unity $a_1, \ldots, a_n$ such that $\mathrm{dom}(\varphi) = \mathbb{1} - a_n$, $\mathrm{rng}(\varphi) = \mathbb{1} - a_1$, and $\forall 1 \leq i < n \ (\varphi(a_i) = a_{i+1})$.

Lemma 7.24. *Suppose that $\varphi : \mathfrak{A}_a \to \mathfrak{A}_b$ is in $[\![\Gamma]\!]$ and acyclic of period n. Then there exists $a' \leq a$, $b' \leq b$, and a map $\psi : \mathfrak{A}_{a'} \to \mathfrak{A}_{b'}$ in $[\![\Gamma]\!]$ which is acyclic of period $2n$ and extends φ.*

Proof. As $\mathbb{1}$ is Γ-aperiodic, so too is a. By Proposition 6.7, there is an involution $\iota \in [\Gamma]$ such that $\mathrm{supp}(\iota) = a$. Let $a' \leq a$ be a maximal ι-discrete section, and observe that the map obtained by composing φ with the partial map

$$\varphi^{n-1}(a') \xrightarrow{\iota \circ \varphi^{-(n-1)}} \iota(a')$$

is the desired extension of φ. $\dashv$

Set $a_0 = \mathbb{1}$ and $\varphi_0 = \emptyset$, and repeatedly apply Lemma 7.24 so as to obtain a sequence of maps $\varphi_n \in [\![\Gamma]\!]$ and sections $a_n = \mathbb{1} - \mathrm{rng}(\varphi_n)$ and $b_n = \mathbb{1} - \mathrm{dom}(\varphi_n)$ such that:

1. $\varphi_{n+1} \in [\![\Gamma]\!]$ is acyclic of period 2^{n+1} and extends φ_n.

2. $a_{n+1} \leq a_n$ and $b_{n+1} \leq b_n$.

Finally, let $\varphi \in [\![\Gamma]\!]$ be the minimal extension of the φ_n's, which is given by

$$\varphi(a) = \sum_{n \in \mathbb{N}} \varphi_n(a \cdot \mathrm{dom}(\varphi_n)).$$

Set $a = \prod_{n \in \mathbb{N}} a_n = \mathbb{1} - \mathrm{rng}(\varphi_n)$, $b = \prod_{n \in \mathbb{N}} b_n = \mathbb{1} - \mathrm{dom}(\varphi_n)$, and

$$c = \mathbb{1} - ([a]_\varphi + [b]_\varphi).$$

It is clear that $\varphi|\mathfrak{A}_c$ is aperiodic, so it only remains to find aperiodic automorphisms $\varphi_a \in [\Gamma|\mathfrak{A}_{[a]_\pi}]$ and $\varphi_b \in [\Gamma|\mathfrak{A}_{[b]_\pi}]$. Fix a function $f : \mathbb{N} \to \mathbb{N}$ which has exactly one orbit. Noting that $\langle \varphi^k(a) \rangle_{k \in \mathbb{N}}$ and $\langle \varphi^{-k}(a) \rangle_{k \in \mathbb{N}}$ are both pairwise disjoint, it follows that the maps

$$\varphi_a = \prod_{k \in \mathbb{N}} \varphi^k(a) \xrightarrow{\varphi^{f(k)-k}} \varphi^{f(k)}(a) \text{ and } \varphi_b = \prod_{k \in \mathbb{N}} \varphi^{-k}(b) \xrightarrow{\varphi^{k-f(k)}} \varphi^{-f(k)}(b)$$

are the desired aperiodic automorphisms. ⊣

8 Bergman's property

In this section, we study a group-theoretic property which was originally discovered to hold for the group of permutations of an infinite set in Bergman [7]. We show that this property holds for a wide variety of full groups. We also describe a strengthening of this property which holds for full groups which admit paradoxical decompositions, but fails for many which admit invariant probability measures. This leads to a characterization of the existence of an invariant probability measure for a countable Borel equivalence relation in terms of a purely algebraic feature of its full group.

A group G has the *weak Bergman property* if for every increasing, exhaustive sequence $\langle \Delta_n \rangle_{n \in \mathbb{N}}$ of subsets of G, there exists $n \in \mathbb{N}$ such that $G = \Delta_n^n$.

Proposition 8.1. *All left-invariant metrics on weakly Bergman groups are bounded.*

Proof. Suppose d is a left-invariant metric on a weakly Bergman group G. Set

$$\Delta_n = \{g \in G : d(1_G, g) \leq n\},$$

and note that since G has the weak Bergman property, there is a natural number $n \in \mathbb{N}$ such that $\Delta_n^n = G$. As d is left-invariant, it follows that

$$\begin{aligned} d(1_G, g_1 \cdots g_n) &\leq d(1_G, g_1) + d(g_1, g_1 g_2) + \cdots + d(g_1 g_2 \cdots g_{n-1}, g_1 g_2 \cdots g_n) \\ &= d(1_G, g_1) + d(1_G, g_2) + \cdots d(1_G, g_n) \\ &\leq n^2, \end{aligned}$$

thus $G = \Delta_{n^2}$. ⊣

Remark 8.2. It follows that if G is weakly Bergman and $X \subseteq G$ is a set of generators for G, then every element of G is a product of boundedly many elements of $X^{\pm 1}$. That is, every Cayley graph of a weakly Bergman group is of bounded diameter.

In Bergman [7], it was shown that the group of all permutations of an infinite set is weakly Bergman. A wide class of automorphism groups share this property:

Proposition 8.3. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ which admit maximal discrete sections, and G has a subgroup of cardinality strictly less than κ which acts aperiodically. Then G has the weak Bergman property.*

Proof. We must show that for every increasing, exhaustive sequence of sets $\Delta_n \subseteq G$, there exists $n \in \mathbb{N}$ such that $G = \Delta_n^n$. Note that by replacing Δ_n with $\Delta_n \cap \Delta_n^{-1}$, we may assume that each Δ_n is symmetric. For $\Delta \subseteq G$ and $a \in \mathfrak{A}$, we will use $\Delta|\mathfrak{A}_a$ to denote $\{\pi|\mathfrak{A}_a : \pi \in \Delta \text{ and } \pi(a) = a\}$.

Lemma 8.4. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, G is a σ-full group of automorphisms of $\mathfrak{A}$, $a \in \mathfrak{A}$, $\langle a_n \rangle_{n \in \mathbb{N}}$ is a pairwise disjoint sequence of elements of $\mathfrak{A}_a$, and $\langle \Lambda_n \rangle_{n \in \mathbb{N}}$ is an increasing, exhaustive sequence of subsets of $G|\mathfrak{A}_a$. Then $\forall^\infty n \in \mathbb{N}\ (G|\mathfrak{A}_{a_n} = \Lambda_n|\mathfrak{A}_{a_n})$.*

Proof. Suppose, towards a contradiction, that there is an increasing sequence of natural numbers k_n and automorphisms $\pi_n \in G|\mathfrak{A}_{a_n}$ with $\pi_{k_n} \notin \Lambda_{k_n}|\mathfrak{A}_{a_{k_n}}$. Then

$$\pi = \prod_{n \in \mathbb{N}} a_n \xrightarrow{\pi_n} a_n$$

is in $(G|\mathfrak{A}_a) \setminus \bigcup_{n \in \mathbb{N}} \Lambda_n$, a contradiction. ⊣

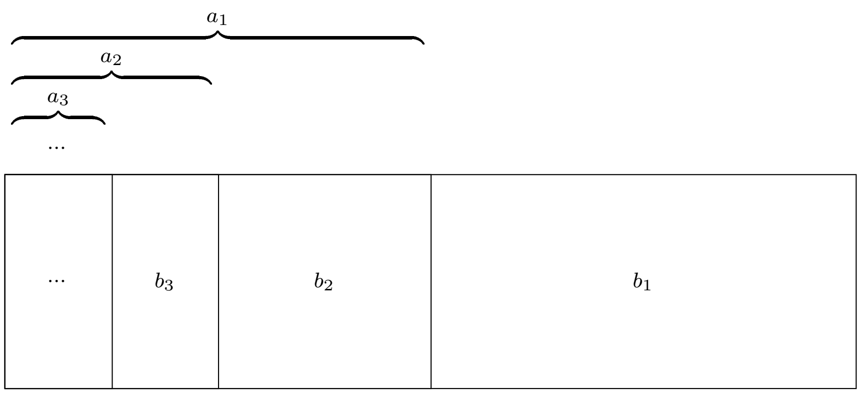

Figure 1.18: The partition of unity generated by the ι_n's.

Now fix a group $\Gamma \leq G$ of cardinality strictly less than κ which acts aperiodically, and set $a_0 = \mathbb{1}$. Given a Γ-aperiodic section $a_n \in \mathfrak{A}$, apply Proposition 6.7 to find an involution $\iota_{n+1} \in G$ with support a_n, and fix a maximal ι_{n+1}-discrete section $a_{n+1} \in \mathfrak{A}$. Set $b_{n+1} = \iota_{n+1}(a_{n+1})$, noting that the b_n's are pairwise disjoint. It follows from Lemma 8.4 that there exists $n_0 > 1$ such that

$$\forall n \geq n_0 \ (G|\mathfrak{A}_{b_n} = \Delta_n|\mathfrak{A}_{b_n}).$$

For each $n > n_0$, set $c_n = \iota_{n_0}(b_n)$, noting that the c_n's are pairwise disjoint and below b_{n_0}. Now observe that the sets of the form

$$\Lambda_n = \{\pi|\mathfrak{A}_{b_{n_0}} : \pi \in \Delta_n \text{ and } \operatorname{supp}(\pi) \leq b_{n_0}\}$$

forms an increasing, exhaustive sequence of subsets of $G|\mathfrak{A}_{b_{n_0}}$. It follows from Lemma 8.4 that there exists $n_1 > n_0$ such that

$$\forall n \geq n_1 \ (G|\mathfrak{A}_{c_n} = \Lambda_n|\mathfrak{A}_{c_n}).$$

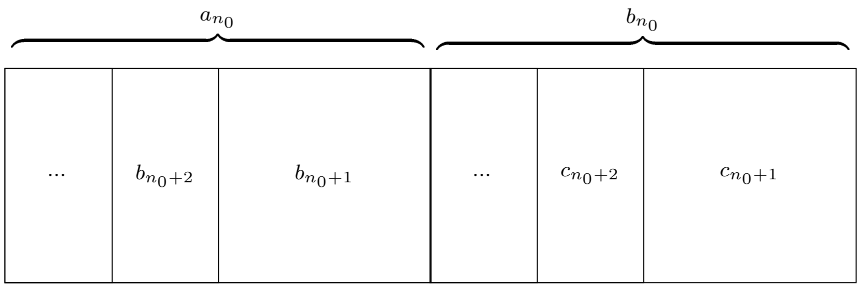

Figure 1.19: The partition of a_{n_0-1} into the b_n's and c_n's.

Lemma 8.5. *Suppose* $\pi \in G$ *and* $\operatorname{supp}(\pi) \leq c_{n_1}$. *Then there exists* $\pi_k \in \Delta_{n_k}$ *with*

$$\pi = [\pi_0, \pi_1].$$

Proof. By Remark 7.20, there are automorphisms $\varphi_0, \varphi_1 \in G$ which are supported by c_{n_1} and satisfy $\pi = [\varphi_0, \varphi_1]$. It follows from the construction of n_0, n_1 that there are automorphisms $\pi_k \in \Delta_{n_k}$ such that:

1. $\pi_0|\mathfrak{A}_{c_{n_0}} = \varphi_0|\mathfrak{A}_{c_{n_0}}$ and $\pi_0|\mathfrak{A}_{b_{n_0}-c_{n_1}} = \mathrm{id}$.

2. $\pi_1|\mathfrak{A}_{c_{n_1}} = \varphi_1|\mathfrak{A}_{c_{n_1}}$ and $\pi_1|\mathfrak{A}_{\mathbb{1}-b_{n_0}} = \mathrm{id}$.

Noting that b_{n_0} and c_{n_1} are π_0-invariant and π_1-invariant, it follows that

(a) $[\pi_0, \pi_1]|\mathfrak{A}_{c_{n_1}} = [\varphi_0, \varphi_1] = \pi|\mathfrak{A}_{c_{n_1}}$,

(b) $[\pi_0, \pi_1]|\mathfrak{A}_{b_{n_0}-c_{n_1}} = \pi_1 \circ \pi_1^{-1}|\mathfrak{A}_{b_{n_0}-c_{n_1}} = \mathrm{id}$, and

(c) $[\pi_0, \pi_1]|\mathfrak{A}_{\mathbb{1}-b_{n_0}} = \pi_0 \circ \pi_0^{-1}|\mathfrak{A}_{\mathbb{1}-b_{n_0}} = \mathrm{id}$,

thus $\pi = [\pi_0, \pi_1]$. $\dashv$

Now find an involution $\iota \in G$ with support c_{n_1} and let c be a maximal ι-discrete section. Let $\pi_1, \ldots, \pi_k$ be an enumeration of the automorphisms of the form

$$\left(c \xrightarrow{\iota_0^{\delta_0}\circ\cdots\circ\iota_{n_1}^{\delta_{n_1}}} \iota_0^{\delta_0} \circ \cdots \circ \iota_{n_1}^{\delta_{n_1}}(c)\right)\left(\iota(c) \xrightarrow{\iota_0^{\epsilon_0}\circ\cdots\circ\iota_{n_1}^{\epsilon_{n_1}}} \iota_0^{\epsilon_0} \circ \cdots \circ \iota_{n_1}^{\epsilon_{n_1}} \circ \iota(c)\right),$$

where $\forall i \leq n_1$ $(\delta_i, \epsilon_i \in \{0, 1\})$, and fix $n \geq n_1$ sufficiently large that each π_i is in Δ_n.

Note that any involution which is supported by a section of the form $\pi_i(c)+\pi_j(c)$ is a conjugate of an involution supported by c_{n_1} via one of the π_k's. It follows that every such involution is in Δ_n^3.

Finally, observe that any involution $\iota \in G$ is a product of the k^2 involutions

$$\iota_{ij} = \big(\pi_i(c) \cdot \iota^{-1}(\pi_j(c)) \xrightarrow{\iota} \pi_j(c) \cdot \iota^{-1}(\pi_i(c))\big),$$

thus $\Delta_n^{3k^2}$ contains every involution in G, so $G = \Delta_n^{9k^2}$, by Theorem 5.8. $\dashv$

A group G is *strongly k-Bergman* if for every increasing, exhaustive sequence $\langle \Delta_n \rangle_{n\in\mathbb{N}}$ of subsets of G, there exists $n \in \mathbb{N}$ such that $\Delta_n^k = G$. A group G is *strongly Bergman* if it is strongly k-Bergman, for some $k \in \mathbb{N}$. In Bergman [7], it is shown that infinite permutation groups are strongly 17-Bergman. For the sort of groups in which we are interested, however, there is an impediment to this stronger property:

Proposition 8.6. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, Γ is a countable group that acts aperiodically on $\mathfrak{A}$ by automorphisms which admit maximal discrete sections, and μ is a Γ-invariant probability measure on $\mathfrak{A}$. Then $[\Gamma]$ does not have the strong Bergman property.*

Proof. We must show that for each $k > 0$, $[\Gamma]$ is not strongly k-Bergman. That is, for each $k > 0$, we must find an increasing, exhaustive sequence $\langle \Delta_n \rangle_{n\in\mathbb{N}}$ of subsets of $[\Gamma]$ such that

$$\forall n \in \mathbb{N}\ (\Delta_n^{k-1} \neq [\Gamma]).$$

Recall that the *full semigroup of* Γ is the set $[\![\Gamma]\!]$ of isomorphisms $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$, with $a, b \in \mathfrak{A}$, which are of the form

$$\pi = \prod_{\gamma\in\Gamma} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma,$$

in which $\langle a_\gamma \rangle_{\gamma\in\Gamma}$ is a partition of a and $\langle \gamma \cdot a_\gamma \rangle_{\gamma\in\Gamma}$ is a partition of b.

Suppose that $\pi, \varphi \in [\![\Gamma]\!]$, and note that there is a maximal $a \le \mathrm{dom}(\pi) \cdot \mathrm{dom}(\varphi)$ on which $\pi|\mathfrak{A}_a = \varphi|\mathfrak{A}_a$. This section is given by

$$e(\pi, \varphi) = \mathrm{dom}(\pi) \cdot \mathrm{dom}(\varphi) - \mathrm{supp}(\pi \circ \varphi^{-1}|\mathfrak{A}_{\pi(\mathrm{dom}(\varphi))\cdot \mathrm{rng}(\varphi)}).$$

With this in mind, we may think of

$$d(\pi, \varphi) = \mu(\mathbb{1} - e(\pi, \varphi))$$

as the *distance between* π, φ. More generally, when $\pi \in [\Gamma]$ and $\Delta \subseteq [\![\Gamma]\!]$, we will use

$$d(\pi, \Delta) = \inf_{\delta \in \Delta} d(\pi, \Delta)$$

to denote the *distance from* π *to* Δ.

Fix an increasing, exhaustive sequence of finite sets $\Gamma_n \subseteq \Gamma$, and put

$$\Delta_n = \{\varphi \in [\Gamma] : d(\gamma, [\![\Gamma_n]\!]) < 1/k\}.$$

Lemma 8.7. $\langle \Delta_n \rangle_{n \in \mathbb{N}}$ *is an increasing, exhaustive sequence of subsets of* $[\Gamma]$.

Proof. It is clear that $\Delta_n \subseteq \Delta_{n+1}$. To see that $[\Gamma] = \bigcup_{n \in \mathbb{N}} \Delta_n$, fix

$$\pi = \prod_{\gamma \in \Gamma} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma$$

in $[\Gamma]$, and choose $n \in \mathbb{N}$ sufficiently large that

$$a = \sum_{\gamma \in \Gamma_n} a_\gamma$$

is of measure at least $1 - 1/k$. Let $\varphi \in [\![\Gamma_n]\!]$ be the partial map

$$\varphi = \prod_{\gamma \in \Gamma_n} a_\gamma \xrightarrow{\gamma} \gamma \cdot a_\gamma,$$

and observe that $d(\pi, \varphi) < 1/k$, thus $\pi \in \Delta_n$. ⊣

It remains to check that each Δ_n^{k-1} is strictly contained in $[\Gamma]$. We begin by noting that the definition of Δ_n places a serious limitation on elements of $[\![\Gamma_n^k]\!]$:

Lemma 8.8. *Suppose that* $\pi_1, \ldots, \pi_{k-1} \in \Delta_n$. *Then* $d(\pi_1 \circ \cdots \circ \pi_{k-1}, [\![\Gamma_n^k]\!]) < 1 - 1/k$.

Proof. For each $1 \leq i \leq k-1$, fix $\varphi_i \in [\![\Gamma_n]\!]$ such that

$$d(\pi_i, \varphi_i) < 1/k,$$

and observe that

$$\begin{aligned} e(\pi_1 \circ \cdots \circ \pi_{k-1}, \varphi_1 \circ \cdots \circ \varphi_{k-1}) &\geq e(\pi_{k-1}, \varphi_{k-1}) \cdot \pi_{k-1}^{-1}(e(\pi_{k-2}, \varphi_{k-2})) \cdot \\ &\quad \cdots \cdot (\pi_2 \circ \cdots \circ \pi_{k-1})^{-1}(e(\pi_1, \varphi_1)), \end{aligned}$$

thus

$$\begin{aligned} \mathbb{1} - e(\pi_1 \circ \cdots \circ \pi_{k-1}, \varphi_1 \circ \cdots \circ \varphi_{k-1}) &\leq (\mathbb{1} - e(\pi_{k-1}, \varphi_{k-1})) + \\ &\quad \cdots + (\mathbb{1} - (\pi_2 \circ \cdots \circ \pi_{k-1})^{-1}(e(\pi_1, \varphi_1))). \end{aligned}$$

As μ is Γ-invariant, it follows that

$$d(\pi_1 \circ \cdots \circ \pi_{k-1}, \varphi_1 \circ \cdots \circ \varphi_{k-1}) \leq d(\pi_{k-1}, \varphi_{k-1}) + \cdots + d(\pi_1, \varphi_1),$$

and this latter quantity is strictly less than $1 - 1/k$. ⊣

In particular, our task will be complete if we can find $\pi \in [\Gamma]$ such that

$$d(\pi, [\![\Gamma_n^{k-1}]\!]) \geq 1 - 1/k.$$

Thus, the following lemma completes the proof:

Lemma 8.9. *Suppose that $\Delta \subseteq [\Gamma]$ is finite, $n \in \mathbb{N}$, and $\epsilon > 0$. Then there is an automorphism $\pi \in [\Gamma]$ such that $d(\pi, [\![\Delta]\!]) \geq 1 - \epsilon$.*

Proof. By Proposition 7.23, there is an aperiodic automorphism $\pi \in [\Gamma]$. Suppose, towards a contradiction, that for each $n \in \mathbb{N}$, there exists $\varphi_n \in [\![\Delta]\!]$ with

$$d(\pi^n, \varphi_n) < 1 - \epsilon.$$

Then for each $n \in \mathbb{N}$, we can find pairwise disjoint sections $a_\delta^{(n)}$, whose sum is of measure at least ϵ, such that for all $\delta \in \Delta$,

$$\varphi_n | \mathfrak{A}_{a_\delta^{(n)}} = \delta | \mathfrak{A}_{a_\delta^{(n)}}.$$

It follows that we can find natural numbers $m < n$ and $\delta \in \Delta$ such that

$$a = a_\delta^{(m)} \cdot a_\delta^{(n)}$$

is of positive measure. In particular, a is non-zero and

$$\pi^m|\mathfrak{A}_a = \pi^n|\mathfrak{A}_a,$$

thus $\pi^{n-m}|\mathfrak{A}_{\pi^m(a)} = \mathrm{id}$, contradicting the aperiodicity of π. ⊣

Remark 8.10. The automorphism produced by the proof of Lemma 8.9 is aperiodic. In fact, we could have built an aperiodic automorphism $\pi \in [\Gamma]$ such that

$$d(\pi, [\![\Delta]\!]) = 1.$$

This follows from a straightforward modification of the proof of Proposition 7.23.

There is also a natural condition which ensures strong Bergmanocity, and even allows us to substantially weaken the assumption of the existence of maximal discrete sections. Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra and Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms. We will write $a \approx b$ to indicate the existence of an isomorphism $\pi : \mathfrak{A}_a \to \mathfrak{A}_b$ in $[\![\Gamma]\!]$. The action of Γ is *paradoxical* if there is a partition of unity into two elements $a \approx b \approx \mathbb{1}$.

Proposition 8.11. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ whose elements are all products of k commutators, and G has a subgroup of cardinality strictly less than κ that acts paradoxically. Then G is strongly $(12k+4)$-Bergman.*

Proof. We must show that for every increasing, exhaustive sequence of sets $\Delta_n \subseteq G$, there exists $n \in \mathbb{N}$ such that $G = \Delta_n^{12k+4}$. Note that by replacing Δ_n with $\Delta_n \cap \Delta_n^{-1}$, we may assume that each Δ_n is symmetric.

Fix a group $\Gamma \leq G$ of cardinality strictly less than κ which acts paradoxically, fix an increasing, exhaustive sequence of sets of $\Delta_n \subseteq G$, and set $a_0 = \mathbb{1}$. Given a Γ-paradoxical section $a_n \in \mathfrak{A}$, an appeal to paradoxicality yields an involution $\iota_{n+1} \in G$ with support a_n and Γ-paradoxical maximal discrete section $a_{n+1} \in \mathfrak{A}$. Set $b_{n+1} = \iota_{n+1}(a_{n+1})$, noting that the b_n's are pairwise disjoint. It follows from Lemma 8.4 that there exists $n_0 > 1$ such that

$$\forall n \geq n_0 \ (G|\mathfrak{A}_{b_n} = \Delta_n|\mathfrak{A}_{b_n}).$$

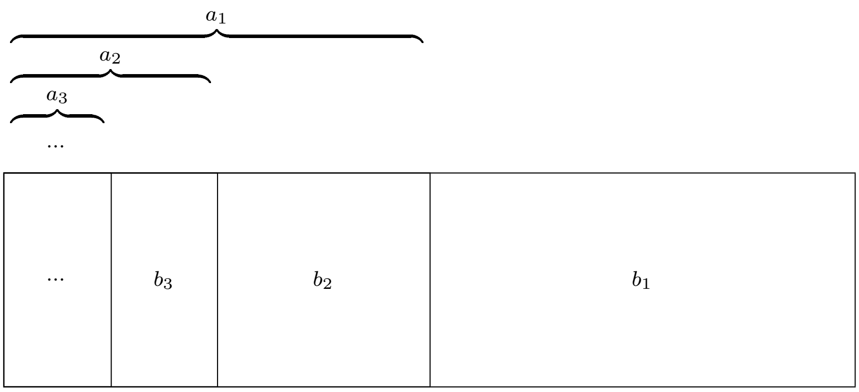

Figure 1.20: The partition of unity generated by the ι_n's.

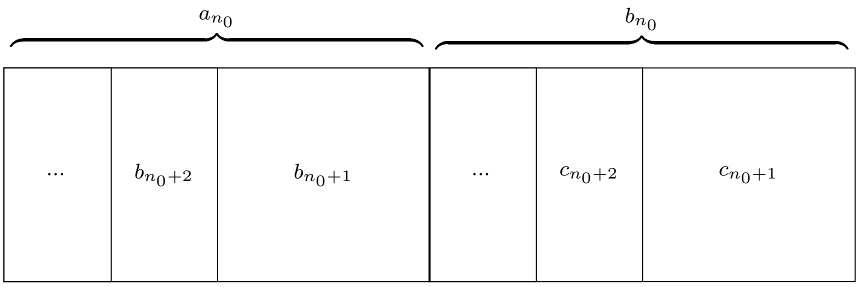

Figure 1.21: The partition of a_{n_0-1} into the b_n's and c_n's.

For each $n > n_0$, set $c_n = \iota_{n_0}(b_n)$, noting that the c_n's are pairwise disjoint and below b_{n_0}. Now observe that the sets of the form

$$\Lambda_n = \{\pi|\mathfrak{A}_{b_{n_0}} : \pi \in \Delta_n \text{ and } \operatorname{supp}(\pi) \leq b_{n_0}\}$$

form an increasing, exhaustive sequence of subsets of $G|\mathfrak{A}_{b_{n_0}}$. It follows from Lemma 8.4 that there exists $n_1 > n_0$ such that

$$\forall n \geq n_1 \ (G|\mathfrak{A}_{c_n} = \Lambda_n|\mathfrak{A}_{c_n}).$$

Lemma 8.12. *Suppose* $\pi \in G$ *and* $\operatorname{supp}(\pi) \leq b$. *Then there exists* $\pi_i^{(\ell)} \in \Delta_{n_i}$ *with*

$$\pi = \prod_{\ell < k} [\pi_0^{(\ell)}, \pi_1^{(\ell)}].$$

Proof. Fix $\pi \in G$ which is supported by b and note that since $b_{n_1} \approx \mathbb{1}$,

$$\pi = \prod_{\ell<k} [\varphi_0^{(\ell)}, \varphi_1^{(\ell)}],$$

where $\pi_i^{(\ell)} \in \mathrm{Aut}(\mathfrak{A}_b)$. Now find automorphisms $\pi_i^{(\ell)} \in \Delta_{n_i}$ such that

1. $\pi_i^{(\ell)}$ agrees with $\varphi_i^{(\ell)}$ on $\mathfrak{A}_b$.
2. $\pi_0^{(\ell)} | \mathfrak{A}_{b_{n_0}-b} = \mathrm{id}$.
3. $\pi_1^{(\ell)} | \mathfrak{A}_{\mathbb{1}-b_{n_0}} = \mathrm{id}$.

As these conditions easily imply that $[\pi_0^{(\ell)}, \pi_1^{(\ell)}]$ is supported by b and

$$[\pi_0^{(\ell)}, \pi_1^{(\ell)}] | \mathfrak{A}_b = [\varphi_0^{(\ell)}, \varphi_1^{(\ell)}],$$

it follows that $\pi = \prod_{\ell<k} [\pi_0^{(\ell)}, \pi_1^{(\ell)}]$. $\dashv$

It follows that every automorphism which is supported by b is the product of $4k$ elements of Δ_{n_1}. Now put $c_0 = \mathbb{1} - b$ and partition b into $c_1 \approx c_2 \approx b$. As $b \approx \mathbb{1} \approx \mathbb{1} - b$, it follows that $c_0 \approx c_1 \approx c_2$. By fixing $n_2 \geq n_1$ sufficiently large, we can ensure that Δ_{n_2} includes an involution which swaps any pair of elements of this partition and is the identity on the remaining element. In particular, it follows that every automorphism which is the identity on c_i, for $i \in \{1, 2\}$, is the product of $4k+2$ elements of Δ_{n_2}. We now need one more lemma:

Lemma 8.13. *Suppose that $\mathfrak{A}$ is a σ-complete Boolean algebra, $\pi \in \mathrm{Aut}(\mathfrak{A})$,*

$$a_0, a_1, a_2 \in \mathfrak{A}$$

forms a partition of unity, and $a_0 + a_1$ is a doubly recurrent π-complete section. Then there exists $\pi_i \in [\pi]$ with $\pi_i | \mathfrak{A}_{a_i} = \mathrm{id}$ and $\pi = \pi_0 \circ \pi_1 \circ \pi_2$.

Proof. Let π_2 be the automorphism of $\mathfrak{A}$ which is given by

$$\pi_2 = (a_0 + a_1) \xrightarrow{\pi_{a_0+a_1}} (a_0 + a_1),$$

and for each $i \in \{0,1\}$, let $a_i^{(n)}$ be the maximal section $a \leq a_0 + a_1$ such that

$$\forall m < n \ (\pi^m(a) \leq a_2) \text{ and } \pi^n(a) \leq a_{1-i}.$$

Explicitly, $a_i^{(n)}$ is given by

$$a_i^n = \pi(a_0 + a_1) \cdot \pi^{-n}(a_{1-i}) \cdot \prod_{m<n} \pi^{-m}(a_2).$$

Note that $\{\pi^m(a_i^n) : 0 \leq i \leq 1 \text{ and } m < n\}$ partitions a_2 and $\{\pi^n(a_i^n) : n \in \mathbb{N}\}$ partitions $a_{1-i} \cdot \pi(a_2)$. For each $i \in \{0,1\}$, put

$$\pi_i = \prod_{n>0} (a_i^n \xrightarrow{\pi} \pi(a_i^n) \xrightarrow{\pi} \cdots \xrightarrow{\pi} \pi^n(a_i^n)),$$

noting that π_0 and π_1 have disjoint supports. As $\pi_i|\mathfrak{A}_{a_i} = \mathrm{id}$ is clear, it only remains to check the following:

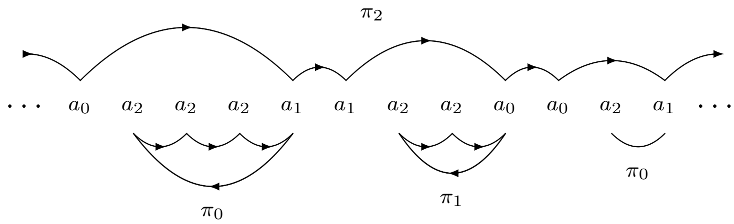

Figure 1.22: The action of the π_i's on the π-orbit of an atom of $\mathfrak{A}$.

Sublemma 8.14. $\pi = \pi_0 \circ \pi_1 \circ \pi_2$.

Proof. The proof breaks into three cases:

1. To see that $\pi|\mathfrak{A}_{a_2} = \pi_0 \circ \pi_1 \circ \pi_2|\mathfrak{A}_{a_2}$, fix natural numbers $m < n$, suppose that $a \leq \pi^m(a_i^n)$, and note that $\pi_2(a) = a$, thus

$$\begin{aligned} \pi_0 \circ \pi_1 \circ \pi_2(a) &= \pi_0 \circ \pi_1(a) \\ &= \pi_0 \circ \pi^{m+1}(a) \\ &= \pi^{m+1}(a), \end{aligned}$$

where the final equality follows from the fact that π_0, π_1 have disjoint supports.

2. To see that $\pi|\mathfrak{A}_{(a_0+a_1)\cdot\pi^{-1}(a_0+a_1)} = \pi_0 \circ \pi_1 \circ \pi_2|\mathfrak{A}_{(a_0+a_1)\cdot\pi^{-1}(a_0+a_1)}$, suppose that

$$a \leq (a_0 + a_1) \cdot \pi^{-1}(a_0 + a_1),$$

and note that $\pi_2(a) = \pi(a)$ is disjoint from $\pi(a_2)$, thus fixed by π_0 and π_1.

3. To see that $\pi|\mathfrak{A}_{(a_0+a_1)\cdot\pi^{-1}(a_2)} = \pi_0 \circ \pi_1 \circ \pi_2|\mathfrak{A}_{(a_0+a_1)\cdot\pi^{-1}(a_2)}$, suppose that

$$a \leq (a_0 + a_1) \cdot \pi^{-1}(a_i^n),$$

and observe that

$$\begin{aligned} \pi_0 \circ \pi_1 \circ \pi_2(a) &= \pi_0 \circ \pi_1 \circ \pi^{n+1}(a) \\ &= \pi^{-n} \circ \pi^{n+1}(a) \\ &= \pi(a). \end{aligned}$$

Noting that

$$a_2 + (a_0 + a_1) \cdot \pi^{-1}(a_0 + a_1) + (a_0 + a_1) \cdot \pi^{-1}(a_2) = \mathbb{1},$$

it follows that $\pi = \pi_0 \circ \pi_1 \circ \pi_2$. $\dashv$

Now suppose that $\pi \in G$. Clearly, there is a partition of unity into three π-invariant pieces on which c_0+c_1, c_0+c_2, and c_1+c_2 are doubly recurrent, respectively, and the proposition follows. $\dashv$

Remark 8.15. The hypotheses of Proposition 8.11 fall well short of those of Theorem 5.8. However, Bergmanocity nevertheless ensures that if every element of G is the composition of finitely many involutions, then there is a bound on the number of involutions necessary.

In particular, Proposition 8.11 implies that a wide variety of Boolean algebras have strongly Bergman automorphism groups:

Corollary 8.16. *The following groups are strongly 16-Bergman:*

1. *The group of permutations of an infinite set.*

2. The group of Borel automorphisms of an infinite Polish space.

3. The group of automorphisms of a Boolean algebra of the form

$$\mathfrak{A} = \prod_{\alpha} \mathfrak{A}_\alpha,$$

where each $\mathfrak{A}_\alpha$ is an infinite weakly homogeneous complete Boolean algebra.

Proof. It is straightforward to see that any aperiodic smooth automorphism acts paradoxically, so we just need to check that each of these algebras admits such an automorphism. This is straightforward for (1) and (2), so only (3) remains.

Clearly, it is enough to show that each infinite weakly homogeneous complete Boolean algebra $\mathfrak{A}$ admits an aperiodic smooth automorphism. If $\mathfrak{A}$ has an atom, then weak homogeneity ensures that $\mathfrak{A}$ is purely atomic, and completeness ensures that $\mathfrak{A}$ is isomorphic to the power set algebra. As $\mathfrak{A}$ is infinite, it must be the power set algebra of an infinite set, and it follows that any aperiodic permutation of the atoms of $\mathfrak{A}$ induces the desired smooth aperiodic automorphism.

We are left with the case that $\mathfrak{A}$ is atomless. By a theorem of Koppelberg and Solovay (see Theorem 4.1 of Rubin-Štěpánek [66]), $\mathfrak{A}$ is of the form $\mathfrak{B}^\kappa$, where $\mathfrak{B}$ is a homogeneous complete Boolean algebra. So it is enough to show that $\mathfrak{B}$ admits a smooth aperiodic automorphism. As $\mathfrak{A}$ is atomless, so too is $\mathfrak{B}$. It then that follows that there is a partition of unity $\langle b_n \rangle_{n \in \mathbb{Z}}$ and isomorphisms $\pi_n : \mathfrak{B}_{b_n} \rightarrow \mathfrak{B}_{b_{n+1}}$, thus

$$\pi = \prod_{n \in \mathbb{Z}} b_n \xrightarrow{\pi_n} b_{n+1}$$

is the desired smooth aperiodic automorphism of $\mathfrak{B}$. ⊣

Remark 8.17. It follows from Corollary 8.16 and Maharam's Theorem that the automorphism group of every atomless semi-finite complete measure algebra is strongly 16-Bergman.

There are circumstances under which the observations we have made thus far yield a simple algebraic characterization of the existence of an invariant probability measure. Suppose E is a countable Borel equivalence relation on a Polish space X.

By Feldman-Moore [36], there is a countable group Γ of Borel automorphisms of X whose associated *orbit equivalence relation*,

$$xE_\Gamma^X y \Leftrightarrow \exists\gamma \in \Gamma\ (\gamma \cdot x = y),$$

is E. The *full group of* E is the group $[E]$ of Borel automorphisms of X whose graphs are contained in E. This group can be identified with the full group of Γ, when Γ is viewed as acting on the algebra of Borel subsets of X.

Theorem 8.18. *Suppose that E is an aperiodic countable Borel equivalence relation on a Polish space. Then $[E]$ has the weak Bergman property, and exactly one of the following holds:*

1. *E admits an invariant Borel probability measure.*

2. *$[E]$ is strongly Bergman.*

Moreover, if (2) holds then $[E]$ is strongly 16-Bergman.

Proof. Let Γ be a countable group of Borel automorphisms such that $E = E_\Gamma^X$. By a result of Becker-Kechris [6] which itself hinges on a theorem of Nadkarni [61], a countable Borel equivalence relation E admits an invariant probability measure exactly when the action of Γ is not paradoxical. The theorem now follows from Proposition 8.6, Theorem 7.16, and Proposition 8.11. $\dashv$

9 Normal subgroups

In this section, we study the normal subgroup structure of full groups. We begin with a new proof of Shortt's [73] theorem characterizing the normal subgroups of the group of Borel automorphisms of an uncountable Polish space. We then move on to show a version of Theorem 381S of Fremlin [39], which describes the normal subgroups of a full group in terms of invariant ideals on the underlying algebra. Fremlin's result only goes through for actions with no partial transversals, which automatically rules out many of the full groups in which we are interested, such as the full group of a countable Borel equivalence relation. Our version of the result goes through for an arbitrary aperiodic action, although its conclusion is (necessarily) a bit weaker than that of Fremlin's theorem.

We begin with the new proof of Shortt's theorem:

Theorem 9.1 (Shortt). *The group of Borel automorphisms of an uncountable Polish space has exactly three proper normal subgroups: the automorphisms of finite support and even cycle type, the automorphisms of finite support, and the automorphisms of countable support.*

Proof (Fremlin-Miller). Suppose that N is a proper normal subgroup of the group of all Borel automorphisms. We will begin with the case that every element of N has countable support. Fix a countably infinite set $S = \{x_n\}_{n \in \mathbb{N}}$ of elements of X, and note that each Borel automorphism $f \in N$ for which $f(S) = S$ induces a permutation of the x_n's, and therefore a permutation τ_f of the naturals which index them. As there is a Borel isomorphism between any two countably infinite Borel subsets of X, it follows that

$$N' = \{\tau_f : f \in N \text{ and } f(S) = S\}$$

does not depend on the choice of S. As N is normal in the group of all Borel automorphisms, it follows that N' is normal in S_∞. As the only proper normal subgroups of S_∞ are the group of permutations of finite support and even cycle type and the group of permutations of finite support, it follows that N must be one of the groups mentioned in the statement of the theorem.

Now, fix a Borel automorphism $f : X \to X$ with uncountable support. It only remains to check that the normal closure of f is necessarily the group of all Borel automorphisms of X. For this, we will need the following lemma:

Lemma 9.2. *There is a Borel automorphism $g : X \to X$ such that the commutator $[f, g]$ is an involution with uncountable, co-uncountable support.*

Proof. For each $n \in \mathbb{N} \cup \{\infty\}$, define $X_n \subseteq X$ by

$$X_n = \{x \in X : |[x]_f| = n\},$$

and set $X_{\geq 4} = X \setminus (X_2 \cup X_3)$. The proof now breaks into three cases:

1. X_2 is uncountable: We will build a copy of Figure 1.23. Find a Borel transversal $B \subseteq X_2$ of $E^{X_2}_{\langle f \rangle}$, find a Borel set $B' \subseteq B$ such that both B' and $B \setminus B'$ are uncountable, find a partition of B' into uncountable Borel sets $B_0, B_2 \subseteq B'$, put $B_{2i+1} = f(B_{2i})$ for $i \in \{0, 1\}$, find a Borel automorphism $h : B_0 \to B_2$, and put
$$g(x) = \begin{cases} h(x) & \text{if } x \in B_0, \\ h^{-1}(x) & \text{if } x \in B_2, \\ x & \text{otherwise.} \end{cases}$$
Setting $X'_2 = [B']_f$, it follows that the $E^{X'_2}_{\langle f,g \rangle}$-class of every $x \in X'_2$ consists of

$$\begin{tikzcd} B_2 \arrow[r, leftrightarrow, "f"] \arrow[d, leftrightarrow, "g"'] & B_3 \\ B_0 \arrow[r, leftrightarrow, "f"'] & B_1 \end{tikzcd}$$

Figure 1.23: The action of $f, g : X \to X$ on X'_2.

exactly one point from each B_i. Letting x_i denote this element, it is clear that
$$f|[x]_{\langle f,g \rangle} = (x_0\ x_1)(x_2\ x_3) \text{ and } g|[x]_{\langle f,g \rangle} = (x_0\ x_2),$$
thus
$$\begin{aligned} [f,g]|[x]_{\langle f,g \rangle} &= [(x_0\ x_1)(x_2\ x_3), (x_0\ x_2)] \\ &= (x_0\ x_1)(x_2\ x_3)(x_0\ x_2) \circ (x_0\ x_1)(x_2\ x_3)(x_0\ x_2) \\ &= (x_3\ x_2\ x_1\ x_0)(x_3\ x_2\ x_1\ x_0) \\ &= (x_0\ x_2)(x_1\ x_3), \end{aligned}$$
hence $[f, g]$ is an involution with uncountable, co-uncountable support.

2. X_3 is uncountable: We will build a copy of Figure 1.24. Find a Borel transversal $B \subseteq X_3$ of $E^{X_3}_{\langle f \rangle}$, find a partition of B into uncountable Borel sets B_0, B_3, put
$$B_{3i+j} = f^j(B_{3i}),$$

for $i \in \{0,1\}$ and $j \in \{1,2\}$, find a Borel isomorphism $h : B_0 \to B_3$, and put

$$g(x) = \begin{cases} h(x) & \text{if } x \in B_0, \\ h^{-1}(x) & \text{if } x \in B_3, \\ x & \text{otherwise.} \end{cases}$$

Setting $X_3' = [B]_f$, it follows that the $E^{X_3}_{\langle f,g\rangle}$-class of every $x \in X_3'$ consists of

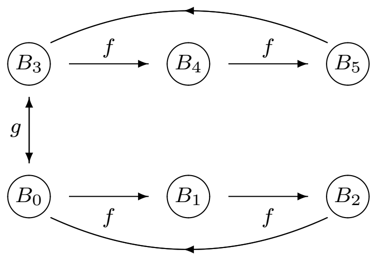

Figure 1.24: The action of $f, g : X \to X$ on X_3'.

exactly one point from each B_i. Letting x_i denote this element, it is clear that

$$f|[x]_{\langle f,g\rangle} = (x_0\ x_1\ x_2)(x_3\ x_4\ x_5) \text{ and } g|[x]_{\langle f,g\rangle} = (x_0\ x_3),$$

thus

$$\begin{aligned} [f,g]|[x]_{\langle f,g\rangle} &= [(x_0\ x_1\ x_2)(x_3\ x_4\ x_5), (x_0\ x_3)] \\ &= (x_0\ x_1\ x_2)(x_3\ x_4\ x_5)(x_0\ x_3) \circ (x_2\ x_1\ x_0)(x_5\ x_4\ x_3)(x_0\ x_3) \\ &= (x_0\ x_4\ x_5\ x_3\ x_1\ x_2)(x_5\ x_4\ x_3\ x_2\ x_1\ x_0) \\ &= (x_0\ x_3)(x_1\ x_4), \end{aligned}$$

hence $[f,g]$ is an involution with uncountable, co-uncountable support.

3. $X_{\geq 4}$ is uncountable: We will build a copy of Figure 1.25. Find a Borel maximal $f^{\leq 3}$-discrete section $B \subseteq X_{\geq 4}$, find an uncountable, co-uncountable Borel set $B_0 \subseteq B$, put $B_i = f^i(B)$ for $i \in \{1,2,3\}$, and define

$$g(x) = \begin{cases} f^2(x) & \text{if } x \in B_0, \\ f^{-2}(x) & \text{if } x \in B_2, \\ x & \text{otherwise.} \end{cases}$$

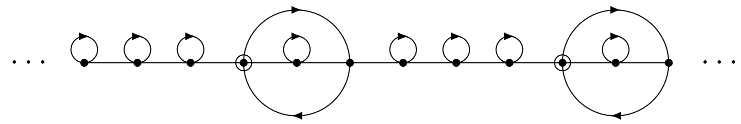

Figure 1.25: The action of g on an orbit of f.

Noting that

$$
\begin{aligned}
f \circ g \circ f^{-1}(x) &= f\left(\begin{cases} f^2 \circ f^{-1}(x) & \text{if } f^{-1}(x) \in B_0, \\ f^{-2} \circ f^{-1}(x) & \text{if } f^{-1}(x) \in B_2, \\ f^{-1}(x) & \text{otherwise} \end{cases}\right) \\
&= \begin{cases} f^2(x) & \text{if } x \in B_1, \\ f^{-2}(x) & \text{if } x \in B_3, \\ x & \text{otherwise,} \end{cases}
\end{aligned}
$$

it follows that

$$
[f,g](x) = (f \circ g \circ f^{-1}) \circ g^{-1}(x) = \begin{cases} f^2(x) & \text{if } x \in B_0 \cup B_1, \\ f^{-2}(x) & \text{if } x \in B_2 \cup B_3, \\ x & \text{otherwise,} \end{cases}
$$

hence $[f,g]$ is an involution with uncountable, co-uncountable support.

As one of X_2, X_3, and $X_{\geq 4}$ is uncountable, this completes the proof of the lemma. ⊣

Next, note that any two Borel involutions with uncountable, co-uncountable support are conjugate, since any Borel isomorphism between transversals can be easily extended to the desired conjugacy. In particular, it follows that if g is as above and $i : X \to X$ is any Borel involution with uncountable, co-uncountable support, then there is a Borel automorphism $h : X \to X$ such that

$$
i = h \circ [f,g] \circ h^{-1} = (h \circ f \circ h^{-1}) \circ ((h \circ g) \circ f^{-1} \circ (h \circ g)^{-1}),
$$

thus i is the composition of a conjugate of f and a conjugate of f^{-1}.

As it is easy to see that every Borel involution is a composition of two Borel involutions which have uncountable, co-uncountable support, it follows from Theorem 5.8 that every Borel automorphism is a composition of twelve conjugates of $f^{\pm 1}$, thus the normal closure of f contains every Borel automorphism. ⊣

Remark 9.3. It is not hard to modify the proof of Theorem 9.1 so as to see that if $f : X \to X$ has uncountable support, then every Borel automorphism of X is a composition of six conjugates of $f^{\pm 1}$. To see this, it is enough to check that every Borel automorphism is a composition of three Borel involutions with uncountable, co-uncountable support. This follows from a fairly straightforward modification of the proof of Theorem 5.8. (The main observation here is that if $i : X \to X$ a non-crossing covering Borel involution then, by removing all of the arcs associated with i which have no arcs below them, we obtain a non-crossing covering involution with uncountably many fixed-points.)

Remark 9.4. With a little more work, we can show that if $f : X \to X$ has uncountable support, then every Borel automorphism of X is a composition of four conjugates of $f^{\pm 1}$. To see this, partition X into two uncountable f-invariant Borel sets, one on which f is smooth and one on which f is aperiodic. On the smooth part, we can repeat the argument we have provided thus far, noting that by Proposition 4.1, the restriction of f to this piece is the composition of two involutions. To handle the aperiodic part, note that the proof of Lemma 9.2 can be modified so as to show that there is a Borel automorphism $g : X \to X$ such that the commutator $[f, g]$ is of strict period 3 and has uncountable, co-uncountable support. Once this has been accomplished, it only remains to observe that by the proof of Theorem 7.14, every aperiodic Borel automorphism is the composition of two automorphisms of strict period 3 which have uncountable, co-uncountable support.

Remark 9.5. As noted in Moran [60], four is best possible. For if x_0, x_1, x_2 are distinct elements of an infinite set X, then $(x_0\ x_1\ x_2)$ is not the composition of three fixed-point free involutions of X.

Now it is time to move on to the main result of this section. Suppose that $\mathfrak{A}$ is a

κ-complete Boolean algebra and G is a κ-full group of automorphisms of $\mathfrak{A}$. Ideally, we would like to show that every normal subgroup of G is of the form

$$N = \{\pi \in G : \operatorname{supp}(\pi) \in \mathscr{I}\},$$

where $\mathscr{I}$ is a G-invariant ideal on $\mathfrak{A}$. In the special case that $\mathfrak{A}$ is a complete Boolean algebra and the action of G admits no non-trivial partial transversals, this is the exact content of Theorem 381S of Fremlin [39]. Unfortunately, the requirement that the action of G admits no non-trivial partial transversals automatically excludes purely atomic Boolean algebras. In particular, this theorem tells us nothing about subgroups of the group of Borel automorphisms of a Polish space.

Worse still, in all but the most trivial of cases, the existence of a partial G-transversal gives rise to normal subgroups of G which are not of the desired form. In order to get around this problem, we will seek only to characterize the elements of $N \trianglelefteq G$ whose supports are in some sense large. Define

$$G_\infty = \{\pi \in G : \exists \Gamma \leq G \ (|\Gamma| < \kappa \text{ and } \operatorname{supp}(\pi) \text{ is } \Gamma\text{-aperiodic})\},$$

and for each $N \trianglelefteq G$, let $\mathscr{I}_N$ be the G-invariant ideal which is generated by the supports of elements of $G_\infty \cap N$.

Theorem 9.6. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ that admit maximal discrete sections, and $N \trianglelefteq G$. Then*

$$\{\pi \in G : \operatorname{supp}(\pi) \in \mathscr{I}_N\} \leq N,$$

thus $G_\infty \cap N = \{\pi \in G_\infty : \operatorname{supp}(\pi) \in \mathscr{I}_N\}$.

Proof. We will begin with a series of observations that, in the spirit of the proof of Theorem 9.1, will reduce the theorem to a question about involutions.

Lemma 9.7. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, and $\varphi \in [\Gamma]$ has Γ-aperiodic support. Then there are involutions $\iota_0, \iota_1 \in [\Gamma]$ such that*

$$\operatorname{supp}(\varphi) \leq \operatorname{supp}([\varphi, \iota_0]) + \operatorname{supp}(\iota_1 \circ [\varphi, \iota_0] \circ \iota_1^{-1}),$$

and $[\varphi, \iota_0]$ is an involution with Γ-aperiodic support.

Proof. For each $n \geq 1$, we will use

$$a_n^\varphi = \sum_{1 \leq i < n} \operatorname{supp}(\varphi^i) - \operatorname{supp}(\varphi^n)$$

to denote the *period n part of φ*, and we will use

$$a_\infty^\varphi = \prod_{n \geq 1} \operatorname{supp}(\varphi^n)$$

to denote the *aperiodic part of φ*.

We will begin by defining involutions on $\mathfrak{A}_2 = \mathfrak{A}_{a_2^\varphi}$, $\mathfrak{A}_3 = \mathfrak{A}_{a_3^\varphi}$, and $\mathfrak{A}_4 = \mathfrak{A}_{a_{\geq 4}^\varphi}$:

1. On $\mathfrak{A}_2$: We will build a copy of Figure 1.26. Fix a φ-transversal $c_2 \in \mathfrak{A}_2$, and note that by the first half of the proof of Proposition 6.7, there is an involution $\iota_2 \in [\Gamma]$ such that $\operatorname{supp}(\iota_2) \leq c_2$ and

$$c_2 - \operatorname{supp}(\iota_2)$$

is a partial Γ-transversal. Fix a maximal ι_2-discrete section $c_{00}^{(2)}$, set

$$c_{jk}^{(2)} = \varphi^j \circ \iota_2^k(c_{00}^{(2)}),$$

for $0 \leq j, k \leq 1$, and note that

$$\iota_2 = (c_{00}^{(2)} \xrightarrow{\iota_2} c_{01}^{(2)}) \text{ and } \varphi = (c_{00}^{(2)} \xrightarrow{\varphi} c_{10}^{(2)})(c_{01}^{(2)} \xrightarrow{\varphi} c_{11}^{(2)}),$$

thus

$$\begin{aligned}
[\varphi, \iota_2] &= (\varphi \circ \iota_2)^2 \\
&= ((c_{00}^{(2)} \xrightarrow{\varphi} c_{10}^{(2)})(c_{01}^{(2)} \xrightarrow{\varphi} c_{11}^{(2)})(c_{00}^{(2)} \xrightarrow{\iota_2} c_{01}^{(2)}))^2 \\
&= (c_{00}^{(2)} \xrightarrow{\varphi \circ \iota_2} c_{11}^{(2)} \xrightarrow{\varphi} c_{01}^{(2)} \xrightarrow{\varphi \circ \iota_2} c_{10}^{(2)})^2 \\
&= (c_{00}^{(2)} \xrightarrow{\iota_2} c_{01}^{(2)})(c_{10}^{(2)} \xrightarrow{\varphi \circ \iota_2 \circ \varphi} c_{11}^{(2)}).
\end{aligned}$$

It follows that $[\varphi, \iota_2]$ is an involution and

$$\operatorname{supp}([\varphi, \iota_2]) = c_{00}^{(2)} + c_{01}^{(2)} + c_{10}^{(2)} + c_{11}^{(2)}.$$

$$\begin{tikzcd}
c^{(2)}_{01} \arrow[r, leftrightarrow, "\varphi"] \arrow[d, leftrightarrow, "\iota_2"'] & c^{(2)}_{11} \\
c^{(2)}_{00} \arrow[r, leftrightarrow, "\varphi"'] & c^{(2)}_{10}
\end{tikzcd}$$

Figure 1.26: The action of φ, ι_2 on the $c^{(2)}_{jk}$'s.

Also, it will be important later to note that

$$a_2^{\varphi} - \mathrm{supp}(\iota_2)$$

is the sum of the two partial Γ-transversals $c_2 - \mathrm{supp}(\iota_2)$ and $\varphi(c_2 - \mathrm{supp}(\iota_2))$.

2. On $\mathfrak{A}_3$: We will build a copy of Figure 1.27. Let c_3 be a transversal of the action of φ on $\mathfrak{A}_3$, and note that by the first half of the proof of Proposition 6.7, there is an involution $\iota_3 \in [\Gamma]$ such that $\mathrm{supp}(\iota_3) \leq c_3$ and

$$c_3 - \mathrm{supp}(\iota_3)$$

is a partial Γ-transversal. Fix a maximal ι_3-discrete section $c^{(3)}_{00}$, set

$$c^{(3)}_{jk} = \varphi^j \circ \iota_3^k(c^{(3)}_{00}),$$

for $0 \leq j \leq 2, 0 \leq k \leq 1$, and note that

$$\iota_3 = (c^{(3)}_{00} \xrightarrow{\iota_3} c^{(3)}_{01}) \text{ and } \varphi = (c^{(3)}_{00} \xrightarrow{\varphi} c^{(3)}_{10} \xrightarrow{\varphi} c^{(3)}_{20})(c^{(3)}_{01} \xrightarrow{\varphi} c^{(3)}_{11} \xrightarrow{\varphi} c^{(3)}_{21}),$$

thus

$$\begin{aligned}
[\varphi, \iota_3] &= (c^{(3)}_{00} \xrightarrow{\varphi} c^{(3)}_{10} \xrightarrow{\varphi} c^{(3)}_{20})(c^{(3)}_{01} \xrightarrow{\varphi} c^{(3)}_{11} \xrightarrow{\varphi} c^{(3)}_{21})(c^{(3)}_{00} \xrightarrow{\iota_3} c^{(3)}_{01}) \\
&\quad (c^{(3)}_{20} \xrightarrow{\varphi^{-1}} c^{(3)}_{10} \xrightarrow{\varphi^{-1}} c^{(3)}_{00})(c^{(3)}_{21} \xrightarrow{\varphi^{-1}} c^{(3)}_{11} \xrightarrow{\varphi^{-1}} c^{(3)}_{01})(c^{(3)}_{00} \xrightarrow{\iota_3} c^{(3)}_{01}) \\
&= (c^{(3)}_{00} \xrightarrow{\varphi\circ\iota_3} c^{(3)}_{11} \xrightarrow{\varphi} c^{(3)}_{21} \xrightarrow{\varphi} c^{(3)}_{01} \xrightarrow{\varphi\circ\iota_3} c^{(3)}_{10} \xrightarrow{\varphi} c^{(3)}_{20}) \\
&\quad (c^{(3)}_{00} \xrightarrow{\varphi^{-1}\circ\iota_3} c^{(3)}_{21} \xrightarrow{\varphi^{-1}} c^{(3)}_{11} \xrightarrow{\varphi^{-1}} c^{(3)}_{01} \xrightarrow{\varphi^{-1}\circ\iota_3} c^{(3)}_{20} \xrightarrow{\varphi^{-1}} c^{(3)}_{10}) \\
&= (c^{(3)}_{00} \xrightarrow{\iota_3} c^{(3)}_{01})(c^{(3)}_{10} \xrightarrow{\varphi\circ\iota_3\circ\varphi^{-1}} c^{(3)}_{11}).
\end{aligned}$$

It follows that $[\varphi, \iota_3]$ is an involution and

$$\operatorname{supp}([\varphi, \iota_3]) = c_{00}^{(3)} + c_{01}^{(3)} + c_{10}^{(3)} + c_{11}^{(3)}.$$

It will later be important to note that $a_3^\varphi - \operatorname{supp}(\iota_3)$ is the sum of $c_{20}^{(3)} + c_{21}^{(3)}$ and the three partial Γ-transversals of the form

$$\varphi^i(c_3 - \operatorname{supp}(\iota_3)),$$

with $i \in \{0, 1, 2\}$.

$$\begin{tikzcd}
c_{01}^{(3)} \arrow[r, "\varphi"] \arrow[d, leftrightarrow, "\iota_3"'] & c_{11}^{(3)} \arrow[r, "\varphi"] & c_{21}^{(3)} \arrow[ll, bend right] \\
c_{00}^{(3)} \arrow[r, "\varphi"'] & c_{10}^{(3)} \arrow[r, "\varphi"'] & c_{20}^{(3)} \arrow[ll, bend left]
\end{tikzcd}$$

Figure 1.27: The action of φ, ι_3 on the $c_{jk}^{(3)}$'s.

3. On $\mathfrak{A}_4$: We will build a copy of Figure 1.28. Fix a maximal $\varphi^{\leq 3}$-discrete section $c_0^{(4)} \in \mathfrak{A}_4$, set $c_i^{(4)} = \varphi^i(c_4)$ for $i \in \{1, 2, 3\}$, and define

$$\iota_4 = (c_0^{(4)} \xrightarrow{\varphi^2} c_2^{(4)}).$$

Noting that $\varphi \circ \iota_4 \circ \varphi^{-1} = (\varphi(c_0^{(4)}) \xrightarrow{\varphi^2} \varphi(c_2^{(4)})) = (c_1^{(4)} \xrightarrow{\varphi^2} c_3^{(4)})$, it follows that

$$[\varphi, \iota_4] = (c_0^{(4)} \xrightarrow{\varphi^2} c_2^{(4)})(c_1^{(4)} \xrightarrow{\varphi^2} c_3^{(4)}),$$

thus $[\varphi, \iota_4]$ is an involution and

$$\operatorname{supp}([\varphi, \iota_4]) = c_0^{(4)} + c_1^{(4)} + c_2^{(4)} + c_3^{(4)}.$$

Let $\iota_0 \in [\Gamma]$ be the involution which agrees with ι_k on $\mathfrak{A}_k$, for $k \in \{2, 3, 4\}$. It only remains to find an involution $\iota_1 \in [\Gamma]$ which sends

$$a = \big(a_2^\varphi - \operatorname{supp}([\varphi, \iota_2])\big) + \big(a_3^\varphi - \operatorname{supp}([\varphi, \iota_3])\big) + \big(a_{\geq 4}^\varphi - \operatorname{supp}([\varphi, \iota_4])\big)$$

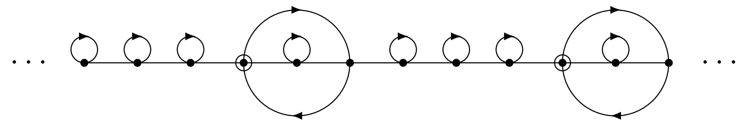

Figure 1.28: The action of ι_4 on a φ-orbit of $\mathfrak{A}_{\geq 4}$ in the purely atomic case.

into

$$b = \operatorname{supp}([\varphi, \iota_2]) + \operatorname{supp}([\varphi, \iota_3]) + \operatorname{supp}([\varphi, \iota_{\geq 4}]).$$

As $\operatorname{supp}(\varphi)$ is Γ-aperiodic, so too is

$$c = c_{00}^{(2)} + c_{00}^{(3)} + c_0^{(4)}.$$

By the proof of Proposition 6.12, there is an involution $\iota' \in [\Gamma]$ which sends

$$d = \sum_{0 \leq i \leq 1} \varphi^i(c_2 - \operatorname{supp}(\iota_2)) + \sum_{0 \leq i \leq 2} \varphi^i(c_3 - \operatorname{supp}(\iota_3))$$

into c. It follows that

$$\iota_1 = (c \xrightarrow{\iota'} d) \prod_{1 \leq i \leq 2} (c_{2i}^{(3)} \xrightarrow{\varphi^{-1}} c_{1i}^{(3)}) \prod_{1 \leq i \leq 3} (\varphi^3(c_i^{(4)}) \xrightarrow{\varphi^{-3}} c_i^{(4)})$$

is as desired. $\dashv$

It now follows that if $\operatorname{supp}(\pi) \in \mathscr{I}_N$, then π is contained in a group $\Gamma \leq G$ of cardinality strictly less than κ which also contains finitely many involutions in N with Γ-aperiodic supports which cover the support of π. By Theorem 5.8, π is itself the composition of three involutions whose supports are below the support of π. So it only remains to prove the following:

Lemma 9.8. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, $N \trianglelefteq [\Gamma]$, $\iota \in [\Gamma]$ is an involution, and there are finitely many involutions $\iota_k \in N$ with Γ-aperiodic support such that*

$$\operatorname{supp}(\iota) \leq \sum_k \operatorname{supp}(\iota_k).$$

Then $\iota \in N$.

Proof. For each involution $\iota' \in [\Gamma]$, put

$$e(\iota, \iota') = \sum\{a \in \mathfrak{A} : \iota|\mathfrak{A}_a = \iota'|\mathfrak{A}_a\} = \mathbb{1} - \operatorname{supp}(\iota' \circ \iota^{-1}).$$

We claim that, by expanding the list of ι_k's, we can ensure that

$$\operatorname{supp}(\iota) \leq \sum_k e(\iota, \iota_k).$$

To see this, it is enough to observe the following:

Sublemma 9.9. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, and $\iota, \iota' \in [\Gamma]$ are involutions. Then there are conjugates $\iota'_0, \iota'_1, \iota'_2, \iota'_3 \in [\Gamma]$ of ι' such that*

$$\operatorname{supp}(\iota) \cdot \operatorname{supp}(\iota') \leq \sum_k e(\iota, \iota'_k).$$

Proof. Let $a_0, a_1 \in \mathfrak{A}$ be a partition of $\operatorname{supp}(\iota)$ into maximal ι-discrete sections, and let $a'_0, a'_1 \in \mathfrak{A}$ be a partition of $\operatorname{supp}(\iota')$ into maximal ι'-discrete sections. Set $a_{jk} = a_j \cdot a'_k$, noting that

$$\operatorname{supp}(\iota) \cdot \operatorname{supp}(\iota') \leq \sum_{0 \leq j,k \leq 1} a_{jk}.$$

Setting

$$\iota_{jk} = (\iota(a_{jk}) \xrightarrow{\iota' \circ \iota} \iota'(a_{jk})),$$

it follows that for all $a \leq a_{jk}$,

$$\begin{aligned} \iota_{jk} \circ \iota' \circ \iota_{jk}(a) &= \iota_{jk} \circ \iota'(a) \\ &= \iota \circ \iota' \circ \iota'(a) \\ &= \iota(a), \end{aligned}$$

thus

$$a_{jk} \leq e(\iota, \iota_{jk} \circ \iota' \circ \iota_{jk}),$$

hence the involutions of the form $\iota_{jk} \circ \iota' \circ \iota_{jk}$ are as desired. $\dashv$

When $a \in \mathfrak{A}$ is π-invariant, we will use

$$\pi|a = a \xrightarrow{\pi} a$$

to denote the automorphism of $\mathfrak{A}$ which is supported by a and agrees with π on $\mathfrak{A}_a$.

Sublemma 9.10. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, Γ is a group of cardinality strictly less than κ which acts on $\mathfrak{A}$ by automorphisms that admit maximal discrete sections, $N \trianglelefteq [\Gamma]$, $\iota \in N$ is an involution with Γ-aperiodic support, and $a \in \mathfrak{A}$ is ι-invariant. Then $\iota|a \in N$.*

Proof. Of course, we may assume $a \leq \mathrm{supp}(\pi)$. There are essentially two cases:

1. a is Γ-periodic: Let b be a maximal ι-discrete section, note that $b - a$ is Γ-aperiodic, and apply Proposition 6.7 to find an involution $\varphi \in [\Gamma]$ such that $\mathrm{supp}(\varphi) = b - a$. Let b_{00} be a maximal φ-discrete section, put

$$b_{jk} = \iota^j \circ \varphi^k(b_{00}),$$

for $0 \leq j, k \leq 1$, and note that

$$\iota|(\mathbb{1} - a) = (b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11}) \text{ and } \varphi = (b_{00} \xrightarrow{\varphi} b_{01}).$$

$$\begin{tikzcd}
b_{01} \arrow[r, leftrightarrow, "\iota"] \arrow[d, leftrightarrow, "\varphi"'] & b_{11} \arrow[dl, leftrightarrow, "\psi"'] \\
b_{00} \arrow[r, leftrightarrow, "\iota"'] & b_{10}
\end{tikzcd}$$

Figure 1.29: The action of ι, φ, ψ on $\mathfrak{A}_{\mathbb{1}-a}$.

Set $\psi = (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})$, and observe that

$$\begin{aligned}
\varphi \circ \iota \circ \varphi^{-1}|(\mathbb{1} - a) &= (b_{00} \xrightarrow{\varphi} b_{01})(b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11})(b_{00} \xrightarrow{\varphi} b_{01}) \\
&= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{01} \xrightarrow{\iota\circ\varphi} b_{10}),
\end{aligned}$$

and

$$\begin{aligned}
\psi \circ \iota \circ \psi^{-1}|(\mathbb{1} - a) &= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11})(b_{00} \xrightarrow{\iota\circ\varphi} b_{11}) \\
&= (b_{00} \xrightarrow{\varphi} b_{01})(b_{10} \xrightarrow{\iota\circ\varphi\circ\iota} b_{11}),
\end{aligned}$$

thus

$$\begin{aligned}
(\varphi \circ \iota \circ \varphi^{-1}) \circ (\psi \circ \iota \circ \psi^{-1})|(\mathbb{1} - a) &= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{01} \xrightarrow{\iota\circ\varphi} b_{10}) \\
&\quad\ (b_{00} \xrightarrow{\varphi} b_{01})(b_{10} \xrightarrow{\iota\circ\varphi\circ\iota} b_{11}) \\
&= (b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11}) \\
&= \iota|(\mathbb{1} - a),
\end{aligned}$$

and it follows that $\iota \circ (\varphi \circ \iota \circ \varphi^{-1}) \circ (\psi \circ \iota \circ \psi^{-1}) = \iota|a$.

2. a is Γ-aperiodic: Let b be a maximal ι-discrete section, note that $a \cdot b$ is Γ-aperiodic, and apply Proposition 6.7 to find an involution $\varphi \in [\Gamma]$ such that $\mathrm{supp}(\varphi) = a \cdot b$. Let b_{00} be a maximal φ-discrete section, put

$$b_{jk} = \iota^j \circ \varphi^k(b_{00}),$$

for $0 \leq j, k \leq 1$, and note that

$$\iota|a = (b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11}) \text{ and } \varphi = (b_{00} \xrightarrow{\varphi} b_{01}).$$

$$\begin{tikzcd}
b_{01} \arrow[r, leftrightarrow, "\iota"] \arrow[d, leftrightarrow, "\varphi"'] & b_{11} \arrow[dl, leftrightarrow, "\psi"'] \\
b_{00} \arrow[r, leftrightarrow, "\iota"'] & b_{10}
\end{tikzcd}$$

Figure 1.30: The action of ι, φ, ψ on $\mathfrak{A}_a$.

Set $\psi = (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})$, and observe that

$$\begin{aligned}
\varphi \circ \iota \circ \varphi^{-1}|a &= (b_{00} \xrightarrow{\varphi} b_{01})(b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11})(b_{00} \xrightarrow{\varphi} b_{01}) \\
&= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{01} \xrightarrow{\iota\circ\varphi} b_{10}),
\end{aligned}$$

and

$$\begin{aligned}\psi \circ \iota \circ \psi^{-1}|a &= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11})(b_{00} \xrightarrow{\iota\circ\varphi} b_{11}) \\ &= (b_{00} \xrightarrow{\varphi} b_{01})(b_{10} \xrightarrow{\iota\circ\varphi\circ\iota} b_{11}),\end{aligned}$$

thus

$$\begin{aligned}(\varphi \circ \iota \circ \varphi^{-1}) \circ (\psi \circ \iota \circ \psi^{-1})|a &= (b_{00} \xrightarrow{\iota\circ\varphi} b_{11})(b_{01} \xrightarrow{\iota\circ\varphi} b_{10}) \\ &\quad\ (b_{00} \xrightarrow{\varphi} b_{01})(b_{10} \xrightarrow{\iota\circ\varphi\circ\iota} b_{11}) \\ &= (b_{00} \xrightarrow{\iota} b_{10})(b_{01} \xrightarrow{\iota} b_{11}) \\ &= \iota|a,\end{aligned}$$

and it follows that $(\varphi \circ \iota \circ \varphi^{-1}) \circ (\psi \circ \iota \circ \psi^{-1}) = \iota|a$.

For the general case, simply note that $\iota|a$ is the product of $\iota|a^{\Gamma}_{\infty}$ and $\iota|(\mathbb{1} - a^{\Gamma}_{\infty})$, and that these automorphisms are in N by the above arguments. $\dashv$

To complete the proof of the lemma, and thus the theorem, put

$$a_k = e(\iota, \iota_k) - \sum_{\ell<k} e(\iota, \iota_\ell),$$

and observe that ι is the product of the automorphisms of the form $\iota_k|a_k$. $\dashv$

10 Closed normal subgroups

In this section, we use Theorem 9.6 to completely characterize the normal subgroups of a full group which satisfy a certain closure condition. We then use this to give a new proof of Bezuglyi-Golodets's [11] characterization of closed normal subgroups of full groups of probability algebras. We also give a version of this theorem for the group of Borel automorphisms of an uncountable Polish space, when equipped with the uniform topology of Bezuglyi-Dooley-Kwiatkowski [9].

Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra. A sequence $\langle \pi_n \rangle_{n\in\mathbb{N}}$ of automorphisms of $\mathfrak{A}$ *discretely converges to* π if for densely many $a \in \mathfrak{A}$,

$$\forall^{\infty} n \in \mathbb{N}\ (\pi_n(a) = \pi(a)).$$

When $\mathfrak{A}$ is purely atomic, this simply means that for each atom $a \in \mathfrak{A}$, the value of $\pi_n(a)$ eventually settles down to $\pi(a)$. A set $\mathscr{A} \subseteq \mathrm{Aut}(\mathfrak{A})$ is *discretely σ-closed* if the limit of every discretely convergent sequence in $\mathscr{A}^{\mathbb{N}}$ is also in $\mathscr{A}$.

Theorem 10.1. *Suppose that $\mathfrak{A}$ is a κ-complete Boolean algebra, G is a κ-full group of automorphisms of $\mathfrak{A}$ that admit maximal discrete sections, and G has a subgroup of cardinality strictly less than κ that acts aperiodically. Then the discretely σ-closed normal subgroups of G are exactly the groups of the form*

$$N = \{\pi \in G : \mathrm{supp}(\pi) \in \mathscr{I}\},$$

where $\mathscr{I}$ is a G-invariant σ-ideal on $\mathfrak{A}$.

Proof. First note that if $\mathscr{I}$ is a G-invariant σ-ideal on $\mathfrak{A}$, then

$$N = \{\pi \in G : \mathrm{supp}(\pi) \in \mathscr{I}\}$$

is clearly a normal subgroup of G. Moreover, if $\langle \pi_n \rangle_{n \in \mathbb{N}}$ is a sequence of elements of N which discretely converges to π, then

$$\mathrm{supp}(\pi) \leq \sum_{n \in \mathbb{N}} \mathrm{supp}(\pi_n),$$

thus $\pi \in N$, and it follows that N is discretely σ-closed.

It remains to show that if $N \trianglelefteq G$ and $\mathscr{I}$ is the G-invariant σ-ideal generated by the supports of elements of N, then

$$N = \{\pi \in G : \mathrm{supp}(\pi) \in \mathscr{I}\}.$$

For this, we will need the following lemma:

Lemma 10.2. *Suppose that $\varphi \in N$. Then there exists $\pi \in N$ with $\mathrm{supp}(\varphi) \leq \mathrm{supp}(\pi)$ and an aperiodic group $\Gamma \leq G$ of cardinality strictly less than κ which contains φ and π and for which π has Γ-aperiodic support.*

Proof. Let Γ be a subgroup of G of cardinality strictly less than κ that acts aperiodically and contains φ. Let a be the Γ-saturation of the Γ-periodic part of

$\operatorname{supp}(\varphi)$. As the action of Γ on $\mathfrak{A}$ is aperiodic, it follows that there is an involution $\iota \in [\Gamma]$ such that $\operatorname{supp}(\iota) \leq a$ and

$$\iota(a \cdot \operatorname{supp}(\varphi)) \leq a - \operatorname{supp}(\varphi).$$

Setting $\pi_0 = [\varphi, \iota]$, it follows that

$$a \cdot \operatorname{supp}(\varphi) \leq \operatorname{supp}(\pi_0) \leq a,$$

and $\operatorname{supp}(\pi_0)$ is Γ-periodic.

Let $\langle \iota_n \rangle_{n \in \mathbb{N}}$ be a sequence of involutions with supports in $\mathfrak{A}_a$ such that the sections of the form $\iota_n(\operatorname{supp}(\pi_0))$ are pairwise disjoint, and define

$$\pi_n = \varphi \circ (\iota_1 \circ \pi_0 \circ \iota_1^{-1}) \circ \cdots \circ (\iota_n \circ \pi_0 \circ \iota_n^{-1}).$$

It is clear that $\langle \pi_n \rangle_{n>0}$ is a sequence of elements of N which is discretely convergent to an automorphism $\pi \in [\Gamma]$ with Γ-aperiodic support, thus $\pi \in N$. As

$$\operatorname{supp}(\varphi) \leq \operatorname{supp}(\pi),$$

the lemma follows. $\dashv$

It follows from Theorem 9.6 that

$$N = \{\pi \in G : \operatorname{supp}(\pi) \in \mathscr{I}_N\},$$

so it only remains to check that $\mathscr{I}_N$ is a σ-ideal. That is, we must check that if $\pi_0, \pi_1, \ldots \in N$ and

$$\operatorname{supp}(\pi) \leq \sum_{n \in \mathbb{N}} \operatorname{supp}(\pi_n),$$

then $\operatorname{supp}(\pi) \in N$.

By Lemma 10.2, we may assume that each $\operatorname{supp}(\pi_n)$ is Γ-aperiodic. Let ι_0 be an involution which has the same support as π_0, and recursively find ι_{n+1} such that

$$\operatorname{supp}(\iota_{n+1}) \leq \operatorname{supp}(\pi_{n+1}) - \sum_{m \leq n} \operatorname{supp}(\iota_m),$$

and

$$\left(\operatorname{supp}(\pi_{n+1}) - \sum_{m\leq n} \operatorname{supp}(\iota_m)\right) - \operatorname{supp}(\iota_{n+1})$$

is a partial Γ-transversal. Setting

$$\varphi_n = \iota_0 \circ \cdots \circ \iota_n,$$

it is clear that the φ_n's are discretely convergent to an involution $\iota \in N$, and that

$$\sum_{n\in\mathbb{N}} \operatorname{supp}(\pi_n) - \operatorname{supp}(\iota)$$

is a partial Γ-transversal. It follows that there is an involution $\iota' \in [\Gamma]$ such that

$$\sum_{n\in\mathbb{Z}} \operatorname{supp}(\pi_n) \leq \operatorname{supp}(\iota) + \operatorname{supp}(\iota' \circ \iota \circ \iota'),$$

thus $\operatorname{supp}(\pi) \in \mathscr{I}_N$. ⊣

Corollary 10.3. *Suppose that $\mathfrak{A}$ is a complete Boolean algebra which satisfies the countable chain condition and G is a full subgroup of* $\operatorname{Aut}(\mathfrak{A})$ *which acts aperiodically. Then the discretely σ-closed normal subgroups of G are exactly the groups of the form*

$$N = \{\pi \in G : \operatorname{supp}(\pi) \leq a\},$$

where a is a G-invariant element of $\mathfrak{A}$.

Proof. Simply note that every σ-ideal on $\mathfrak{A}$ is of the form $\mathscr{I} = \mathfrak{A}_a$, for some $a \in \mathfrak{A}$, and apply Theorem 10.1. ⊣

We will say that a measure algebra $(\mathfrak{A}, \mu)$ is a *probability algebra* if μ is a probability measure. Note that every probability algebra necessarily satisfies the countable chain condition, and is therefore complete. Associated with $(\mathfrak{A}, \mu)$ is the *uniform topology on* $\operatorname{Aut}(\mathfrak{A})$, which is generated by the metric

$$d(\varphi, \psi) = \mu(\operatorname{supp}(\varphi \circ \psi^{-1})).$$

We now are ready for our new proof of a theorem a theorem of Bezuglyi-Golodets [11], which itself generalizes the special case when Γ acts by measure-preserving automorphisms, due to Dye [27].

Theorem 10.4 (Bezuglyi-Golodets). *Suppose that* $(\mathfrak{A}, \mu)$ *is a probability algebra and* G *is a full group which acts aperiodically on* $\mathfrak{A}$ *by automorphisms. Then the normal subgroups of* G *which are closed in the uniform topology are exactly the groups of the form*

$$N = \{\pi \in G : \text{supp}(\pi) \leq a\},$$

where $a \in \mathfrak{A}$ *is* Γ*-invariant.*

Proof (Miller). Noting that any discretely convergent sequence is necessarily convergent in the uniform topology, it follows from Corollary 10.3 that every normal subgroup is of the desired form. It only remains to check that each group of the form

$$N = \{\pi \in G : \text{supp}(\pi) \leq a\}$$

is closed in the uniform topology, or equivalently, that its complement is open. To see this, note that if $\pi \notin N$, then $\epsilon = \mu(\text{supp}(\pi) - a)$ is positive, and the ϵ-ball centered at π avoids N. ⊣

Let $\mathscr{B}$ be the σ-complete Boolean algebra of Borel subsets of an uncountable Polish space X, and let $P(X)$ denote the standard Borel space of probability measures on X. In Bezuglyi-Dooley-Kwiatkowski [9], the *uniform topology on* $\text{Aut}(\mathscr{B})$ is defined as the topology which is generated by the basic open sets of the form

$$\mathscr{U}(\varphi, \mu_0, \ldots, \mu_n, \epsilon) = \{\psi \in \text{Aut}(\mathscr{B}) : \forall i \leq n \ (\mu_i(\text{supp}(\varphi \circ \psi^{-1})) < \epsilon)\},$$

where $\varphi \in \text{Aut}(\mathscr{B})$, $\mu_0, \ldots, \mu_n \in P(X)$, and $\epsilon > 0$. We will close this section by proving an analog of Theorem 10.4 for the group of Borel automorphisms of X, equipped with the uniform topology. Given a group G of Borel automorphisms and a set $M \subseteq P(X)$, we will say that M is *G-invariant* if

$$\forall \mu \in M \ (g_* \mu \in M).$$

Let NULL_μ denote the ideal of null Borel subsets of X, and for $M \subseteq P(X)$, put

$$\text{NULL}_M = \bigcap_{\mu \in M} \text{NULL}_\mu.$$

Theorem 10.5. *Suppose that G is a σ-full group of automorphisms of $\mathscr{B}$ which contains a countable aperiodic subgroup. Then the uniformly closed normal subgroups of G are exactly those of the form*

$$N = \{\pi \in G : \operatorname{supp}(\pi) \in \mathrm{NULL}_M\},$$

where $M \subseteq P(X)$ is G-invariant.

Proof. First, we must check that all subsets of G of the desired form are uniformly closed normal subgroups. Suppose that $M \subseteq P(X)$, and set

$$N = \{\pi \in G : \operatorname{supp}(\pi) \in \mathrm{NULL}_M\}.$$

Clearly $\mathrm{id} \in N$ and $\pi \in N \Rightarrow \pi^{-1} \in N$. As

$$\operatorname{supp}(\varphi \circ \psi) \leq \operatorname{supp}(\varphi) + \operatorname{supp}(\psi),$$

it follows that N is a subgroup of G. As

$$\operatorname{supp}(\psi \circ \varphi \circ \psi^{-1}) = \psi(\operatorname{supp}(\varphi)),$$

it follows that if M is G-invariant, then N is a normal subgroup of G. To see that N is uniformly closed, note that if $\pi \notin N$ then there exists a probability measure $\mu \in M$ such that

$$\mu(\operatorname{supp}(\pi)) \geq \epsilon,$$

for some $\epsilon > 0$. Noting that if $\varphi \in N$ then

$$\mu(\operatorname{supp}(\pi) - \operatorname{supp}(\varphi)) \geq \epsilon,$$

it follows that $\mu(\operatorname{supp}(\pi \circ \varphi^{-1})) \geq \epsilon$, thus $\mathscr{U}(\pi, \mu, \epsilon)$ is an open neighborhood of π which avoids N.

It remains to check that every uniformly closed $N \trianglelefteq G$ is of the desired form. As every discretely convergent sequence is uniformly convergent, it follows from Theorem 10.1 that there is a G-invariant σ-ideal $\mathscr{I}$ on $\mathscr{B}$ such that

$$N = \{\pi \in G : \operatorname{supp}(\pi) \in \mathscr{I}\}.$$

Let $M \subseteq P(X)$ be the set of all probability measures μ on X for which $\mathscr{I} \subseteq \mathrm{NULL}_\mu$. It is clear that M is G-invariant and $\mathscr{I} \subseteq \mathrm{NULL}_M$.

Now suppose, towards a contradiction, that there exists $\pi \in G \setminus N$ such that

$$\mathrm{supp}(\pi) \in \mathrm{NULL}_M.$$

As N is uniformly closed, there exist $\mu_1, \ldots, \mu_n \in P(X)$ and $\epsilon > 0$ such that

$$\mathscr{U}(\pi, \mu_1, \ldots, \mu_n, \epsilon) \cap N = \emptyset.$$

Put $b_0 = \mathbb{0}$, and given $i < n$ and $b_i \in \mathscr{I}$, find $b_{i+1} \in \mathscr{B}$ which is of maximal μ_{i+1}-measure with the property that $b_i \leq b_{i+1}$ and $b_{i+1} \in \mathscr{I}$. Set $b = b_n$, and for each $1 \leq i \leq n$ with $\mu_i(b) < 1$, let ν_i be the relative probability measure

$$\nu_i(a) = \mu_i(a - b)/\mu_i(\mathbb{1} - b).$$

Immediately we obtain the following:

Lemma 10.6. *For all $1 \leq i \leq n$, either $\mu_i(b) = 1$ or $\nu_i \in M$.*

As $b \in \mathscr{I}$ and $\mathscr{I}$ is G-invariant, it follows that $[b]_\pi \in \mathscr{I}$, so the automorphism

$$\varphi = \pi | [b]_\pi$$

is in N. Noting that for all $1 \leq i \leq n$,

$$\begin{aligned}
\mu_i(\mathrm{supp}(\pi \circ \varphi^{-1})) &= \mu_i(\mathrm{supp}(\pi \circ \varphi^{-1}) - b) \\
&\leq \mu_i((\mathrm{supp}(\pi) + \mathrm{supp}(\varphi)) - b) \\
&\leq \mu_i(\mathrm{supp}(\pi) - b) + \mu_i(\mathrm{supp}(\varphi) - b) \\
&= 0,
\end{aligned}$$

thus $\varphi \in \mathscr{U}(\pi, \mu_1, \ldots, \mu_n, \epsilon)$, we now have the desired contradiction. $\dashv$

Chapter 2

Some classification problems

1 Introduction

In this chapter, we modify techniques of Shelah-Weiss [72], Harrington-Kechris-Louveau [44], and Dougherty-Jackson-Kechris [24] so as to produce a variety of new descriptive set-theoretic classification results. We focus on three basic topics: order-preserving embeddings of Borel functions, betweenness-preserving embeddability of Borel forests of lines, and G-action embeddings of group actions on quotient spaces of the form X/E, where X is a Polish space and E is a countable Borel equivalence relation on X.

Associated with each Borel function $f : X \to X$ is a Borel equivalence relation $E_t(f)$ and a Borel assignment of partial ordering $\leq_f$ to each $E_t(f)$-class, given by

$$xE_t(f)y \Leftrightarrow \exists m, n \in \mathbb{N}\ (f^m(x) = f^n(y)) \text{ and } x \leq_f y \Leftrightarrow \exists n \geq 0\ (f^n(x) = y).$$

Given Borel functions $f : X \to X$ and $g : Y \to Y$, an *order-preserving embedding* of f into g is a Borel injection $\pi : X \to Y$ such that for all $x, x' \in X$,

$$xE_t(f)x' \Leftrightarrow \pi(x)E_t(g)\pi(x') \text{ and } x \leq_f x' \Leftrightarrow \pi(x) \leq_g \pi(x').$$

The odometer is the isometry of $2^{\mathbb{N}}$ which adds 1 to the 0^{th} digit of x, and then carries right. More precisely, it is given by

$$\sigma(x) = \begin{cases} 0^n 1 y & \text{if } x = 1^n 0 y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

In §2, we show the following:

Theorem. *Suppose that X is a Polish space and $f : X \to X$ is a Borel function. Then exactly one of the following holds:*

1. *X can be partitioned into countably many Borel $\leq_f$-antichains.*

2. *There is an order-preserving Borel embedding of the odometer into f.*

This generalizes the result of Shelah-Weiss [72] and strengthens special cases of Harrington-Kechris-Louveau [44] and Kanovei [50].

Suppose that $f : X \to X$ is a Borel automorphism and $A \subseteq X$ is Borel. We say that A is a *complete section for f* if it intersects every orbit of f, and that A is *doubly recurrent for f* if for all $x \in A$, there exists $m < 0 < n$ such that $f^m(x), f^n(x) \in A$. Associated with any such set $A \subseteq X$ is the *induced automorphism of A*,

$$f_A(x) = f^{n(x)}(x),$$

where $n(x)$ is the least natural number n for which $f^n(x) \in A$. Two Borel automorphisms $f : X \to X$ and $g : Y \to Y$ are *descriptive Kakutani equivalent* if there are doubly recurrent Borel complete sections $A \subseteq X$ and $B \subseteq Y$ and a Borel isomorphism of f_A and g_B. This notion was introduced by Nadkarni [61], who asked if any two Borel automorphisms of finite rank are descriptive Kakutani equivalent. This is the analog of the measure-theoretic notion of Kakutani [49], who originally conjectured that in the measure-theoretic context, all automorphisms should be Kakutani equivalent (this later turned out to be false).

The results of §3 are joint with Christian Rosendal. We strengthen the argument of Dougherty-Jackson-Kechris [24] so as to show that every Borel automorphism order-preservingly embeds into the odometer. This leads to the following:

Theorem (Miller-Rosendal). *Suppose that X is a Polish space and $f, g : X \to X$ are non-smooth aperiodic Borel automorphisms of X. Then f, g are descriptive Kakutani equivalent.*

In particular, this answers Nadkarni's [61] question and provides a positive answer to the descriptive version of Kakutani's original conjecture. Actually, our proof gives a somewhat stronger result, and we show that further strengthenings of it are impeded by the complexities which arise in the measure-theoretic context. Christian Rosendal has noted that our result also gives the corresponding result for Borel $\mathbb{R}$-flows. We close the section with an answer to a related question of Louveau. We use the argument of Eigen-Hajian-Weiss [31] to show that any collection of non-smooth Borel automorphisms which are minimal for the relation of Borel conjugacy must be of size continuum.

A *Borel forest of lines* is a Borel forest $\mathscr{L} \subseteq X^2$ whose connected components are trees of vertex degree 2. Clearly, if $f : X \to X$ is an aperiodic Borel automorphism, then $\text{graph}(f) \cup \text{graph}(f^{-1})$ is Borel forest of lines. However, Scot Adams has pointed out there are Borel forests of lines which are not of this form. We term such forests *undirectable*.

Although there is no notion of order-preserving embedding for Borel forests of lines, there is something very close — a betweenness-preserving embedding. In §4, we produce a combinatorially simple undirectable line $\mathscr{L}_0$, by weaving together pairs of orbits of the odometer. We then alter the combinatorics of the Shelah-Weiss [72] argument so as to show that $\mathscr{L}_0$ is the minimal undirectable Borel forest of lines. We then apply this result to the study of 2-regular Borel marriage problems. We obtain a new proof of Laczkovich's [59] result that there is a 2-regular Borel marriage problem with a solution but no universally measurable solution. A corollary of our result for $\mathscr{L}_0$ is that every 2-regular Borel marriage problem with a universally measurable solution has a Borel solution. We also use this result to obtain a positive answer to a question of Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [58], which deals with the connection between definable solutions to Borel marriage problems and the existence of certain sorts of ergodic probability measures.

In §5, we define the corresponding notion of Kakutani equivalence for Borel forests of lines, and modify the Dougherty-Jackson-Kechris [24] argument once more so as to show that any two undirectable Borel forests of lines are Kakutani equivalent (again, this is joint with Christian Rosendal). In particular, this leads to the following:

Theorem (Miller-Rosendal). *Up to Kakutani equivalence, there are exactly three Borel forests of lines on Polish spaces. In order of betweenness-preserving Borel embeddability, these are:*

1. *Forests which are induced by smooth Borel automorphisms.*

2. *Forests which are induced by non-smooth Borel automorphisms.*

3. *Forests which are not induced by Borel automorphisms at all.*

Much as one can associate with any non-smooth Borel equivalence relation E the σ-ideal of Borel sets on which E is smooth, one can associate with any undirectable Borel forest of lines $\mathscr{L}$ the σ-ideal of Borel sets on which $\mathscr{L}$ is directable. We close §5 by showing that these directability ideals are genuinely new, in the sense that no such ideal is also the smoothness ideal of any Borel equivalence relation. This generalizes and strengthens a result of Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [58].

In §6, we turn our attention to group actions on quotient spaces of the form X/E, where X is Polish and E is a countable Borel equivalence relation. We introduce a descriptive notion of ergodicity for such actions which has strong strong ties to the measure-theoretic study of normalizers of full groups (see Connes-Krieger [19], Bezuglyi-Golodets [12] and [13], Bezuglyi [8], and Feldman-Sutherland-Zimmer [37]). This study of this notion is partially motivated by the desire to understand the sorts of actions for which the results of Chapter I do not apply. When the group in question is cyclic, say $\Gamma = \langle \pi \rangle$, then the ergodicity of the action of Γ is equivalent to the inexistence of a maximal π-discrete section.

In §7, we embark upon the task of proving the descriptive analogs of various results from ergodic theory. We associate with each countable group G an equivalence relation $E_0(G)$ on $X_0(G) = G^{\mathbb{N}}$ and a smooth action of G on $X_0(G)$ which factors over the quotient to an ergodic action of G. We then show the following:

Theorem. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and G is a countable group which acts freely and in a Borel fashion on X/E. Then exactly one of the following holds:*

1. *The action of G on X/E is not ergodic.*

2. *There is a continuous embedding of $X_0(G)$ into X which induces a G-action embedding of $X_0(G)/E_0(G)$ into X/E.*

As a corollary, we see that the descriptive notion of ergodicity for an action of G on X/E is equivalent to the existence of an E-ergodic probability measure for which the action of G has a non-singular lifting. From this, we obtain the following:

Theorem. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, G is a countable group which acts ergodically and freely on X/E, L is a countable signature, and $\mathscr{K}$ is a class of countable L-structures which is closed under isomorphism. Then the following are equivalent:*

1. *There is a definable assignment of $\mathscr{K}$-structures to the orbits of G.*

2. *There is a right-invariant $\mathscr{K}$-structure on G.*

As a special case, we obtain the following rigidity theorem:

Theorem. *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, G and H are countable groups which act freely and ergodically on X/E and Y/F, and $E_G \cong E_H$. Then $G \cong H$.*

In §8, we describe a special case in which the conclusion of this theorem can be substantially strengthened:

Theorem. *Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, G and H are countable groups which act freely and ergodically on X/E and Y/F, $E_G \cong E_H$, and these equivalence relations have hyperfinite liftings. Then $G \cong H$ and the actions are Borel isomorphic.*

For actions of finite groups, this has the following consequence:

Theorem. *Suppose that X and Y are Polish spaces, E and F are hyperfinite Borel equivalence relations on X and Y, G and H are finite groups which act freely and ergodically on X/E and Y/F, and $E_G \cong E_H$. Then $G \cong H$ and the actions are Borel isomorphic.*

This answers a question of Bezuglyi.

2 Order-preserving embeddability of σ

Suppose that X and Y are Polish spaces and $f : X \to X$ and $g : Y \to Y$ are Borel functions. The *quasi-ordering induced by f* is given by

$$x \leq_f y \Leftrightarrow \exists n \in \mathbb{N}\ (f^n(x) = y),$$

and the *tail equivalence relation induced by f* is given by

$$x E_t(f) y \Leftrightarrow \exists m, n \in \mathbb{N}\ (f^m(x) = f^n(y)).$$

Note that $E_t(f)$ is the transitive closure of the comparability relation of $\leq_f$.

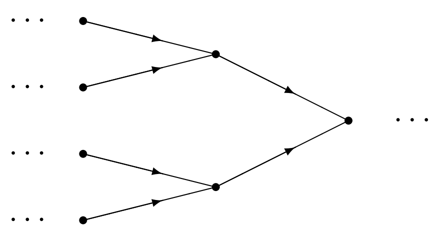

Figure 2.1: A piece of an $E_t(f)$-class whose points are $\leq_f$-ascending from left to right.

A *reduction* of $R \subseteq X^2$ into $S \subseteq Y^2$ is a map $\pi : X \to Y$ such that

$$\forall x, x' \in X\ ((x, x') \in R \Leftrightarrow (\pi(x), \pi(x')) \in S).$$

An *order-preserving embedding of f into g* is an injection $\pi : X \to Y$ which is simultaneously a reduction of $E_t(f)$ into $E_t(g)$ and a reduction of $\leq_f$ into $\leq_g$. We will use $f \leq_{\mathscr{O}} g$ to denote the existence of an order-preserving Borel embedding of f into g. A function f is *aperiodic* if

$$\forall x \, \forall n > 0 \; (f^n(x) \neq x).$$

The notion of order-preserving embeddability is a bit simpler for such functions:

Proposition 2.1. *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel function. Then the following are equivalent:*

1. *π is an order-preserving embedding of f into g.*

2. *π is a reduction of $\leq_f$ into $\leq_g$.*

Moreover, if f is injective and g is aperiodic, then these are equivalent to:

3. *$\forall x, x' \in X \; (x <_f x' \Rightarrow \pi(x) <_g \pi(x')$ and $\pi(x) E_t(g) \pi(x') \Rightarrow x E_t(f) x')$.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3), so it is enough to show (2) $\Rightarrow$ (1) and if f is injective and g is aperiodic, then (3) $\Rightarrow$ (2).

We will begin with (2) $\Rightarrow$ (1). Suppose that f is aperiodic and π is a reduction of $\leq_f$ into $\leq_g$. To see that π is injective, simply note that if $\pi(x) = \pi(x')$ then

$$\pi(x) \leq_g \pi(x') \leq_g \pi(x),$$

thus $x \leq_f x' \leq_f x$. As f is aperiodic, this implies that $x = x'$.

It remains to check that π is a reduction of $E_t(f)$ into $E_t(g)$, i.e.,

$$\forall x, x' \in X \; (x E_t(f) x' \Leftrightarrow \pi(x) E_t(g) \pi(x')).$$

To see ($\Rightarrow$), suppose that $x E_t(f) x'$ and find $x'' \in X$ such that $x, x' \leq_f x''$. Then

$$\pi(x), \pi(x') \leq_g \pi(x''),$$

thus $\pi(x) E_t(g) \pi(x')$. To see ($\Leftarrow$), suppose $\pi(x) E_t(g) \pi(x')$ and find $n \in \mathbb{N}$ such that

$$\pi(x), \pi(x') \leq_g g^n \circ \pi(x).$$

Note that the injectivity of π and the aperiodicity of f ensure that

$$\pi(x) <_g \pi \circ f(x) <_g \cdots <_g \pi \circ f^n(x),$$

thus $g^n \circ \pi(x) \leq_g \pi \circ f^n(x)$. It follows that

$$\pi(x') \leq_g \pi \circ f^n(x),$$

thus $x' \leq_f f^n(x)$, hence $xE_t(f)x'$.

It only remains to check (3) $\Rightarrow$ (2). Suppose that f is injective, g is aperiodic, and $\pi(x) \leq_g \pi(x')$. Then $\pi(x)E_t(g)\pi(x')$, so $xE_t(f)x'$. Now the only way that (2) can fail is if $x' <_f x$, but this implies that $\pi(x') <_g \pi(x)$, a contradiction. $\dashv$

A countable Borel equivalence relation E is *smooth* if there is a Borel set B which intersects every E-class in exactly one point. Such a set is called a *transversal* of E. The canonical example of a non-smooth equivalence relation is the equivalence relation on *Cantor space* $\mathscr{C} = 2^{\mathbb{N}}$ given by

$$xE_0y \Leftrightarrow \forall^\infty n \in \mathbb{N}\ (x_n = y_n),$$

where "$\forall^\infty$" means "for all but finitely many."

Proposition 2.2. *E_0 is not smooth.*

Proof. Suppose, towards a contradiction, that B is a Borel transversal of E_0. For each $n \in \mathbb{N}$, let i_n be the involution of $\mathscr{C}$ which flips the n^{th} digit, i.e.,

$$[i_n(x)]_k = \begin{cases} 1 - x_k & \text{if } k = n, \\ x_k & \text{otherwise.} \end{cases}$$

As each of these functions sends meager sets to meager sets and the sets of the form $i_{n_0} \circ \cdots \circ i_{n_\ell}(B)$ cover $\mathscr{C}$, B is non-meager. Fix $s \in 2^{<\mathbb{N}}$ such that B is comeager in $\mathscr{N}_s$, set $n = |s|$, and observe that B and $i_n(B)$ are disjoint sets which are both comeager in $\mathscr{N}_s$, the desired contradiction. $\dashv$

As in Kanovei [50], we will use $\leq_0$ to denote the following partial ordering of $\mathscr{C}$:

$$x \leq_0 y \Leftrightarrow (x = y \text{ or } \exists n \in \mathbb{N}\ (x_n < y_n \text{ and } \forall m > n\ (x_m = y_m))).$$

We will also use $\leq_0$ to denote the similarly defined linear ordering of 2^k given by

$$x \leq_0 y \Leftrightarrow (x = y \text{ or } \exists n < k\ (x_n < y_n \text{ and } \forall n < m < k\ (x_m = y_m))).$$

Note that if $x, y \in \mathscr{C}$ agree from their k^{th} digit on, then

$$x \leq_0 y \Leftrightarrow (x|k) \leq_0 (y|k).$$

The *odometer* is the isometry of $\mathscr{C}$ given by

$$\sigma(x) = \begin{cases} 0^n 1 y & \text{if } x = 1^n 0 y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

Thus the odometer is the map which adds 1 to the 0^{th} digit of x, and then carries right. Let $\mathscr{C}_0$ denote the set of non-eventually constant elements of $\mathscr{C}$.

Proposition 2.3. *$E_t(\sigma) = (E_0|\mathscr{C}_0) \cup (\mathscr{C} \setminus \mathscr{C}_0)^2$ and $\leq_\sigma = \leq_0 \cup ([1^\infty]_{E_0} \times [0^\infty]_{E_0})$.*

Proof. We will show that $E_t(\sigma)|\mathscr{C}_0 = E_0|\mathscr{C}_0$ and $\leq_\sigma |\mathscr{C}_0 = \leq_0 |\mathscr{C}_0$, and the rest follows easily from the observation that $\sigma(1^\infty) = 0^\infty$. I claim that for $n > 0$,

$$\forall x \in \mathscr{C}_0\ \left(\{\sigma^i(0^n x)\}_{i<2^n} = \{sx\}_{s \in 2^n} \text{ and } \sigma^{2^n - 1}(0^n x) = 1^n x\right).$$

The proof is by induction on n. The case $n = 1$ is trivial since $\sigma(0x) = \sigma(1x)$ is clear from the definition of σ. Now suppose the claim has been proven up to n. Then

$$\begin{aligned} \sigma^{2^n}(0^{n+1}x) &= \sigma \circ \sigma^{2^n - 1}(0^n 0 x) \\ &= \sigma(1^n 0 x) \\ &= 0^n 1 x. \end{aligned}$$

It follows that $\sigma^{2^{n+1}-1}(0^{n+1}x) = \sigma^{2^n - 1}(0^n 1 x) = 1^{n+1} x$ and

$$\begin{aligned} \{\sigma^i(0^{n+1}x)\}_{i<2^{n+1}} &= \{\sigma^i(0^n 0 x)\}_{i<2^n} \cup \{\sigma^i(0^n 1 x)\}_{i<2^n} \\ &= \{s0x\}_{s\in 2^n} \cup \{s1x\}_{s \in 2^n} \\ &= \{sx\}_{s \in 2^{n+1}}, \end{aligned}$$

which completes the proof of the claim.

It immediately follows that $E_0|\mathscr{C}_0 \subseteq E_t(\sigma)|\mathscr{C}_0$. To see the reverse inclusion, note that it is enough to show $xE_0\sigma^{\pm 1}(x)$ for all $x \in \mathscr{C}_0$, and this follows trivially from the definition of σ.

Now fix $x, y \in \mathscr{C}_0$. It is clear that $x <_0 \sigma(x)$, thus $x <_\sigma y \Rightarrow x <_0 y$. It remains to check that if $x <_0 y$, then $x <_\sigma y$. It already follows from the last paragraph that $xE_t(\sigma)y$, so the only way this can fail is if $y \leq_\sigma x$. But it follows from our previous observation that this implies $y \leq_0 x$, a contradiction. $\dashv$

An *antichain* for a partial order $\leq$ is a set of pairwise $\leq$-incomparable elements. The goal of this section is to prove the following:

Theorem 2.4. *Suppose that X is a Polish space and $f : X \to X$ is a Borel function. Then exactly one of the following holds:*

1. *X can be partitioned into countably many Borel $\leq_f$-antichains.*

2. *There is an order-preserving Borel embedding of the odometer into f.*

We will actually show the version of Theorem 2.4 in which (2) is replaced with, "$\leq_0$ continuously embeds into $\leq_f$," but this is clearly sufficient by the above remarks.

Before getting to the proof, we will briefly discuss how Theorem 2.4 sits in relation to similar known results. The main ingredient of Shelah-Weiss [72] is the special case of Theorem 2.4 in which f is a Borel automorphism. Actually, in their proof (1) is replaced with the statement, "$E_t(f)$ is smooth," but these are equivalent:

Proposition 2.5. *Suppose that X is a Polish space and $f : X \to X$ is a Borel automorphism. Then X can be partitioned into countably many Borel $\leq_f$-antichains $\Leftrightarrow E_t(f)$ is smooth.*

Proof. To see ($\Rightarrow$), suppose A_n are Borel antichains which partition X, let

$$[Y]_f = \{x \in X : \exists y \in Y \ (xE_t(f)y)\}$$

denote the $E_t(f)$-*saturation* of $Y \subseteq X$, set

$$B_n = A_n \setminus \bigcup_{m<n} [A_m]_f,$$

and note that $B = \bigcup_n B_n$ is a Borel transversal of $E_t(f)$.

To see ($\Leftarrow$), suppose that B is a Borel transversal of $E_t(f)$, let $\langle k_n \rangle_{n \in \mathbb{N}}$ be an enumeration of $\mathbb{Z}$, and observe that the sets of the form

$$A_n = f^n(B),$$

for $n \in \mathbb{Z}$, give the desired partition X into Borel $\leq_f$-antichains. $\dashv$

If f is many-to-one, however, then these conditions are *not* equivalent:

Example 2.6. Consider the *shift* on

$$[\mathbb{Z}]^{\mathbb{N}} = \{x \in \mathbb{Z}^{\mathbb{N}} : \forall n \in \mathbb{N}\ (x_n < x_{n+1})\},$$

given by $s(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_{n+1} \rangle_{n \in \mathbb{N}}$. It is clear that

$$X_k = \{x \in [\mathbb{Z}]^{\mathbb{N}} : x_0 = k\}$$

partitions $[\mathbb{Z}]^{\mathbb{N}}$ into countably many Borel $\leq_s$-antichains. On the other hand, $E_t(s)$ is not smooth. To see this, note that E_0 reduces into $E_t(s)$ via the injection $\pi(\langle x_n \rangle_{n \in \mathbb{N}}) = \langle x_n + 2n \rangle_{n \in \mathbb{N}}$. Now suppose, towards a contradiction, that B is a Borel transversal of $E_t(s)$. It immediately follows that $\pi^{-1}(B)$ is a Borel transversal of E_0, which contradicts Proposition 2.2.

The main result of Kanovei [50] (and its subsequent strengthenings due to Louveau) lives in a much broader setting than Theorem 2.4, in that it describes when $\leq_0$ embeds into an arbitrary Borel partial order. However, when restricted to partial orders of the form $\leq_f$, where f is a Borel function, it is the version of Theorem 2.4 in which (1) is replaced with the statement, "$\leq_f$ is contained in a Borel linear ordering of X." This can be easily recovered from Theorem 2.4:

Proposition 2.7. *Suppose that X is a Polish space, $\leq$ is a Borel partial ordering of X, and there is a partition of X into countably many Borel $\leq$-antichains. Then $\leq$ can be extended to a Borel linear ordering of X.*

Proof. Given $x \in X$ and an antichain $A \subseteq$, we will write $x < A$ whenever x lies below some element of A. Suppose that A_n are Borel antichains which partition X, and for each $s \in \{0,2\}^{<\mathbb{N}}$, define

$$B_s = \left\{x \in A_{|s|} : \forall n < |s| \ (x < A_n \Leftrightarrow s_n = 0)\right\}.$$

Let $s(x) = s1$, where s is the unique element of $\{0,2\}^{<\mathbb{N}}$ with $x \in B_s$, and note that

$$\forall x, y \in X \ (x < y \Rightarrow s(x) <_{\text{lex}} s(y)).$$

For each $s \in \{0,2\}^{<\mathbb{N}}$, fix a Borel linear ordering $<_s$ of B_s. It follows that

$$x \prec y \Leftrightarrow (s(x) <_{\text{lex}} s(y) \text{ or } \exists s \in \{0,2\}^{<\mathbb{N}} \ (x <_s y))$$

defines the desired extension of $<$. ⊣

The structure of the proof of Theorem 2.4 is quite similar to that of the proof given in Shelah-Weiss [72], in that there are three basic tools that need to be identified: a σ-ideal, a game, and a coloring. Unsurprisingly, the σ-ideal here is simply the family of sets which can be covered with a countable family of disjoint Borel antichains. Unlike the situation in Shelah-Weiss [72], however, some amount of descriptive set theory appears necessary to show that this σ-ideal possesses various closure properties that are essential to the proof. The game, which is necessarily different than the one used in Shelah-Weiss [72], is a Choquet-like game in which player II makes fewer commitments than usual at each stage. The coloring is similar to that used in Shelah-Weiss [72], although it is interesting to note that it is not always possible to build countable Borel colorings of the graphs associated with Borel functions in the same way that one can for Borel automorphisms (see Kechris-Solecki-Todorcevic [56]), and it is only the fact that colorings of somewhat smaller graphs are all that is necessary that saves the proof.

I should add that the change of topology results which Weiss [79] later used to simplify the proof of Shelah-Weiss [72] can also be used to simplify the proof of Theorem 2.4 that I will give, but only in the special case that f sends Borel sets to Borel sets. This is because Borel functions do *not* necessarily send Borel sets to

Borel sets, and it is impossible to find a Polish topology on a Polish space X whose associated Borel sets properly extend the usual Borel subsets of X (see, for example, Exercise 25.19 of [51]). Nevertheless, the extra bit of descriptive set theory necessary to prove the full version of Theorem 2.4 is quite minimal, and certainly pales in comparison to that which is used in Harrington-Kechris-Louveau [44].

Suppose X is a Polish space and $f : X \to X$ is a Borel function. The *recurrent part* of a set $Y \subseteq X$ is given by

$$\mathrm{rec}(Y) = \{y \in Y : \exists^\infty n \in \mathbb{N}\ (f^n(y) \in Y)\} = \bigcap_{n \in \mathbb{N}} f^{\leq -n}(Y),$$

where "$\exists^\infty$" is shorthand for "there exists infinitely many," and "$f^{\leq -n}(Y)$" is shorthand for "$\bigcup_{m \geq n} f^{-m}(Y)$." The set $Y \subseteq X$ is *nowhere recurrent* if $\mathrm{rec}(Y) = \emptyset$, and the set $Y \subseteq X$ is *strongly nowhere recurrent* if

$$\exists n \in \mathbb{N}\ (Y \cap f^{\leq -n}(Y) = \emptyset).$$

Proposition 2.8. *Suppose that X is a Polish space, $f : X \to X$ is an aperiodic Borel function, and $Y \subseteq X$. Then the following are equivalent:*

1. *Y can be covered with countably many nowhere recurrent analytic sets.*
2. *Y can be covered with countably many strongly nowhere recurrent analytic sets.*
3. *Y can be covered with countably many nowhere recurrent Borel sets.*
4. *Y can be covered with countably many pairwise disjoint Borel antichains.*

Proof. To see (1) $\Rightarrow$ (2), it suffices to show that every nowhere recurrent analytic set A can be covered with countably many strongly nowhere recurrent analytic sets. As $\mathbf{\Sigma^1_1}$ obeys the generalized separation property (see 35.1 of [K]), there is a sequence of Borel sets $B_n \supseteq f^{\leq -n}(A)$ whose intersection is empty. It follows that $A_n = A \setminus B_n$ is a countable collection of strongly nowhere recurrent analytic sets which cover A.

To see (2) $\Rightarrow$ (3), it suffices to show that every strongly nowhere recurrent analytic set A can be covered with a nowhere recurrent Borel set. Fix a natural number n such that $A \cap f^{\leq -n}(A) = \emptyset$, noting that $A \cap f^{\geq n}(A) = \emptyset$ as well. As $\mathbf{\Sigma^1_1}$ obeys the separation property (see 14.7 of [K]), we can find a Borel set $B \supseteq A$ which is disjoint from $f^{\geq n}(A)$, and then $B \setminus \mathrm{rec}(B)$ is a nowhere recurrent Borel set containing A.

To see (3) $\Rightarrow$ (4), it suffices to show that every nowhere recurrent Borel set B can be partitioned into countably many Borel antichains, and it is clear that

$$B_n = \{x \in B : f^n(x) \in B \text{ and } \forall m > n\ (f^m(x) \notin B)\}$$

defines such a partition. As (4) $\Rightarrow$ (1) is trivial, this completes the proof. $\dashv$

Remark 2.9. Christian Rosendal has pointed out that by using the first reflection principle (see 35.10 of Kechris [51]), one can easily see that every nowhere recurrent analytic set is contained in a nowhere recurrent Borel set. We will not need this strengthening of (1) $\Rightarrow$ (3), however.

We will use $\mathscr{I}$ to denote the σ-ideal generated by Borel antichains.

Proposition 2.10. *Suppose that X is a Polish space, $f : X \to X$ is Borel, and $A \subseteq X$ is analytic. Then $A \in \mathscr{I} \Leftrightarrow f^{-1}(A) \in \mathscr{I} \Leftrightarrow \mathrm{rec}(A) \in \mathscr{I}$.*

Proof. To see $A \in \mathscr{I} \Rightarrow f^{-1}(A) \in \mathscr{I}$, simply note that pre-images of nowhere recurrent sets are nowhere recurrent. To see $f^{-1}(A) \in \mathscr{I} \Rightarrow \mathrm{rec}(A) \in \mathscr{I}$, note that

$$\mathrm{rec}(A) \subseteq \bigcup_{n>0} f^{-n}(A).$$

It follows from our previous observation that each $f^{-n}(A) \in \mathscr{I}$, thus $\mathrm{rec}(A) \in \mathscr{I}$. It remains to check that $\mathrm{rec}(A) \in \mathscr{I} \Rightarrow A \in \mathscr{I}$. Suppose $\mathrm{rec}(A) \in \mathscr{I}$, find nowhere recurrent Borel sets B_n which cover $\mathrm{rec}(A)$, and note that $A \setminus \bigcup_n B_n$ is a nowhere recurrent analytic set, thus $A \in \mathscr{I}$. $\dashv$

Corollary 2.11. *Suppose that X is a Polish space, $f : X \to X$ is Borel, and $A \subseteq X$ is analytic. Then $A \notin \mathscr{I} \Leftrightarrow \exists^\infty k > 0\ (A \cap f^{-k}(A) \notin \mathscr{I})$.*

Proof. It is enough to show ($\Rightarrow$). If $A \notin \mathscr{I}$, then $\mathrm{rec}(A) \notin \mathscr{I}$ by Proposition 2.10. Fix $n \in \mathbb{N}$. As $\mathrm{rec}(A) \subseteq \bigcup_{k>n} A \cap f^{-k}(A)$, it follows that $A \cap f^{-k}(A) \notin \mathscr{I}$ for some $k > n$. $\dashv$

It should be noted that one *can* have $A \in \mathscr{I}$ and $f(A) \notin \mathscr{I}$:

Example 2.12. Set $X = \mathscr{C} \times \mathbb{N}$, put $X_0 = \{(x, n) \in X : n = 0\}$, and define $f : X \to X$ by

$$f(x,n) = \begin{cases} (\sigma(x), 0) & \text{if } n = 0, \\ (x, n-1) & \text{otherwise.} \end{cases}$$

Then $A = X \setminus X_0$ is nowhere recurrent, thus $A \in \mathscr{I}$, but $f(A) = X \notin \mathscr{I}$ by Propositions 2.2 and 2.5.

It is because of the existence of such sets that the game which appears in Shelah-Weiss [72] cannot be used here. Instead, we will use the Choquet-like game $\mathfrak{G}$ which is given by

$$\begin{array}{llllll} \text{I} & \emptyset \neq A_0 \in \mathbf{\Sigma}^1_1 & & \emptyset \neq A_1 \in \mathbf{\Sigma}^1_1 & & \cdots \\ \text{II} & & \mathscr{A}_0 \subseteq \mathbf{\Sigma}^1_1, |\mathscr{A}_0| \leq \aleph_0 & & \mathscr{A}_1 \subseteq \mathbf{\Sigma}^1_1, |\mathscr{A}_1| \leq \aleph_0 & \end{array},$$

subject to the requirements that $A_n = \bigcup \mathscr{A}_n$, each element of each $\mathscr{A}_n$ is of diameter $\leq 1/(n+1)$, and each A_{n+1} is contained in some element of $\mathscr{A}_n$, and in which player II wins if $\bigcap_n A_n \neq \emptyset$.

Proposition 2.13. *Player II has a winning strategy in* $\mathfrak{G}$.

Proof. Given $Y \subseteq X \times \mathscr{N}$, we will use $p[Y]$ to denote the *projection* of Y onto X. We will assume that $\mathscr{N}$ is endowed with its usual metric, X is endowed with a Polish metric which is compatible with its underlying Polish topology, and $X \times \mathscr{N}$ is endowed with the product metric. This will be important because any sequence of sets $Y_n \subseteq X \times \mathscr{N}$ whose diameter is vanishing with respect to the product metric gives rise to another decreasing sequence of sets $p[Y_n] \subseteq X$ whose diameter is vanishing with respect to the metric on X.

Now we will describe the winning strategy for player II. After player I plays A_0, player II fixes a closed set $C_0 \subseteq X \times \mathscr{N}$ such that $A_0 = p[C_0]$, finds a countable set $\mathscr{C}_0$ of closed sets of diameter ≤ 1 whose union is C_0, and then plays $\mathscr{A}_0 = \{p[C_0'] : C_0' \in \mathscr{C}_0\}$.

After player I plays A_1, player II fixes closed sets $C_{11} \subseteq X \times \mathscr{N}$ and $C_{10} \in \mathscr{C}_0$ such that

$$A_1 = p[C_{11}] \subseteq p[C_{10}],$$

finds countable sets $\mathscr{C}_{1i}$, for $i \leq 1$, of closed sets of diameter $\leq 1/2$ whose union is C_{1i}, and plays

$$\mathscr{A}_1 = \{p[C'_0] \cap p[C'_1] : \forall i \leq 1 \ (C'_i \in \mathscr{C}_{1i})\}.$$

Player II simply continues in this fashion. After player I plays A_{n+1}, player II fixes closed sets $C_{n+1n+1} \subseteq X \times \mathscr{N}$ and $C_{n+1i} \in \mathscr{C}_{ni}$, for $i \leq n$, such that

$$A_{n+1} = p[C_{n+1n+1}] \subseteq \bigcap_{i \leq n} p[C_{n+1i}],$$

finds countable sets $\mathscr{C}_{n+1i}$, for $i \leq n+1$, of closed sets of diameter $\leq 1/(n+1)$ whose union is C_{n+1i}, and plays

$$\mathscr{A}_{n+1} = \left\{ \bigcap_{i \leq n+1} p[C'_i] : \forall i \leq n+1 \ (C'_i \in \mathscr{C}_{n+1i}) \right\}.$$

It is clear that as long as player I does his part, this strategy leads to a valid run of $\mathfrak{G}$. Fix $i \in \mathbb{N}$, and note that $\langle C_{ni} \rangle_{i \leq n}$ is a decreasing sequence of closed sets with vanishing diameter, so there is a single point (x_i, y_i) in their intersection. As $\langle p[C_{ni}] \rangle_{i \leq n}$ also has vanishing diameter, it follows that x_i is the unique element of $\bigcap_{i \leq n} p[C_{ni}]$, and moreover, that the value of x_i does not depend on i. It easily follows that x_i is the unique element of $\bigcap_n A_n$, thus the strategy we have described is a winning strategy for player II. $\dashv$

In addition to this winning strategy for $\mathfrak{G}$, we will also need a way of coloring certain sorts of graphs. The *graph associated with* $f : X \to X$ is given by

$$(x, y) \in \mathscr{G}_f \Leftrightarrow (x \neq y \text{ and } (x = f(y) \text{ or } y = f(x))).$$

Given any graph $\mathscr{G}$ on X, we will use $\mathscr{G}^{<n}$ to denote the thickened graph in which two distinct points are neighbors if they are of distance less than n from one another with respect to the graph metric on $\mathscr{G}$. A set $B \subseteq X$ is *$\mathscr{G}$-discrete* if no point of B has a $\mathscr{G}$-neighbor which also lies in B. A *coloring* of $\mathscr{G}$ is a function $c : X \to I$ such that $c(x) \neq c(y)$ whenever $(x, y) \in \mathscr{G}$, or equivalently, a function $c : X \to I$ such that the pre-image of any singleton is $\mathscr{G}$-discrete. A *κ-coloring* is a coloring $c : X \to I$ with $\kappa = |I|$.

Ideally, we would like to show that each $\mathscr{G}_f^{<n}$ admits a Borel $\aleph_0$-coloring. While this is true when f is finite-to-one (see Lemma 1.17 of [48]), it is false in general (see Proposition 6.2 of [56]). So instead, we will work with a somewhat smaller graph. Let $E_0(f)$ denote the subequivalence relation of $E_t(f)$ which is given by

$$xE_0(f)y \Leftrightarrow \exists n \in \mathbb{N}\ (f^n(x) = f^n(y)),$$

and set $\mathscr{G}_n(f) = \mathscr{G}_f^{<n} \setminus E_0(f)$.

Proposition 2.14. *Suppose that X is a Polish space, $f : X \to X$ is a Borel function, and $n \in \mathbb{N}$. Then $\mathscr{G}_n(f)$ admits a Borel $\aleph_0$-coloring.*

Proof. Assume without loss of generality that $X = \mathscr{C}$. Let $i(x) \in \mathbb{N}$ be least such that

$$\forall \ell, m < n\ (f^\ell(x) \neq f^m(x) \Rightarrow f^\ell(x)|i(x) \neq f^m(x)|i(x)),$$

and put $\pi(x) = \langle f^m(x)|i(x)\rangle_{m<n}$. Now suppose $\pi(x) = \pi(y)$ and $(x, y) \in \mathscr{G}_f^{<n}$, and find $\ell, m < n$ with $f^\ell(x) = f^m(y)$. As $i(x) = i(y)$, it follows that

$$f^\ell(x)|i(x) = f^m(y)|i(y) = f^m(x)|i(x),$$

thus $\ell = m$, so $xE_0(f)y$. Hence, π is a Borel $\aleph_0$-coloring of $\mathscr{G}_n(f)$. $\dashv$

A function $f : X \to X$ is *eventually periodic* if

$$\forall x \in X\, \forall^\infty n \in \mathbb{N}\, \exists m > n\ (f^m(x) = f^n(x)).$$

It is easy to see that one can always find an $E_t(f)$-invariant Borel set $B \subseteq X$ such that $f|B$ is aperiodic and $f|(X \setminus B)$ is eventually periodic. Thus, the following fact will allow us to concentrate on aperiodic functions:

Proposition 2.15. *Suppose that X is a Polish space and $f : X \to X$ is an eventually periodic Borel function. Then X is in the associated ideal $\mathscr{I}$.*

Proof. Note that $B = \{x \in X : \exists n > 0\ (x = f^n(x))\}$ is a Borel $E_t(f)$-complete section. Fix any Borel linear ordering $<$ of X, and define

$$B_n = \{x \in B : x \text{ is the } n^{\text{th}} \text{ element of } [x]_E \cap B \text{ with respect to } <\}.$$

Now put $B' = f^{-1}(B) \setminus B$, and observe that $\langle B_n \rangle_{n \in \mathbb{N}} \cup \langle f^{-n}(B') \rangle_{n \in \mathbb{N}}$ partitions X into Borel $\leq_f$-antichains, thus $X \in \mathscr{I}$. $\dashv$

The following fact completes the proof of Theorem 2.4:

Proposition 2.16. *Suppose that X is a Polish space and $f : X \to X$ is a Borel function whose associated ideal $\mathscr{I}$ does not contain X. Then $\leq_0$ continuously reduces into $\leq_f$.*

Proof. By Lemma 2.15, we may assume that f is aperiodic. Let φ be a winning strategy for player II in $\mathfrak{G}$. We will recursively define a decreasing sequence of $\mathscr{I}$-positive clopen sets A_n and an increasing sequence of natural numbers k_n, beginning with $A_0 = X$ and $k_0 = 1$. At stage n, we will have defined $\mathscr{I}$-positive analytic sets $A_0 \supseteq B_1 \supseteq A_1 \supseteq \cdots \supseteq B_n \supseteq A_n$, which should be viewed as increasingly accurate approximations to the left branch of the desired embedding. We will also have found a sequence of natural numbers $k_0 < k_1 < \cdots < k_n$.

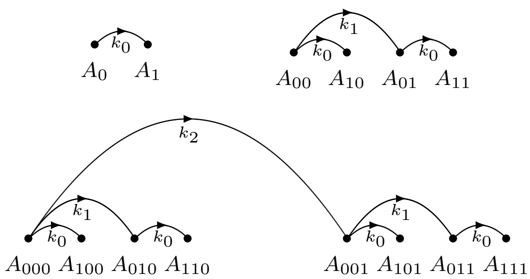

Figure 2.2: The first three stages of the construction of $\pi : \mathscr{C} \hookrightarrow X$.

We proceed to the next step via three separate stages:

1. **Thinning out A_n:** Fix a Borel coloring $c : \mathscr{G}_{3k_n}(f) \to \mathbb{N}$, find $i \in \mathbb{N}$ with
$$B = \{x \in A_n : c(x) = i\}$$
$\mathscr{I}$-positive, and set $B_{n+1} = B$.

2. **Playing the game:** Given a function $f : 2^{n+1} \to \mathscr{P}(X)$ such that
$$\forall s \in 2^{n+1} \ \left(f(s) \in \varphi(B_0, f^{s_0 k_0}(B_1), \ldots, f^{\sum_{i \leq n} s_i k_i}(B_{n+1})) \right),$$

define $B_f \subseteq B_{n+1}$ by
$$B_f = \bigcap_{s \in 2^{n+1}} f^{-\sum_{i \leq n} s_i k_i}(B_s).$$
As there are only countably many such functions and $B = \bigcup_f B_f$, it follows that we can find such an f for which B_f is $\mathscr{I}$-positive.

3. **Finding the next link:** For each $k > 3k_n$, set
$$C_k = \{x \in B_f : f^k(x) \in A_n\}.$$
We claim that there exists $k > 3k_n$ such that $C_k \notin \mathscr{I}$. To see this, it is enough to note that
$$C = B_f \setminus \bigcup_{k > 3k_n} C_k$$
is strongly nowhere recurrent, and therefore $\mathscr{I}$-null. So fix such a $k \in \mathbb{N}$, define $k_{n+1} = k$, and set $A_{n+1} = C_k$.

Once the recursion is complete, we obtain a decreasing sequence of $\mathscr{I}$-positive analytic sets $A_n \subseteq X$ and natural numbers $k_n \in \mathbb{N}$ such that:

(a) For each $n \in \mathbb{N}$, the set A_{n+1} is $\mathscr{G}_{3k_n}(f)$-discrete.

(b) For each $x \in \mathscr{C}$, the sequence
$$\begin{array}{llllll} \mathrm{I} & B_0 & & \ldots & f^{\sum_{i \leq n} k_i x_i}(B_{n+1}) & & \ldots \\ \mathrm{II} & & \varphi(B_0) & & & \varphi(B_0, \ldots, f^{\sum_{i \leq n} k_i x_i}(B_{n+1})) & \end{array}$$
constitutes a valid run of $\mathfrak{G}$. In particular, $f^{\sum_{i \leq n} k_i x_i}(A_{n+1})$ is of diameter $\leq 1/(n+1)$ and
$$\bigcap_{n \in \mathbb{N}} f^{\sum_{i \leq n} k_i x_i}(A_{n+1})$$
is a singleton. Let $\pi(x)$ denote the unique element of this intersection.

(c) For each $n \in \mathbb{N}$, $f^{k_{n+1}}(A_{n+1}) \subseteq A_n$.

It follows from (b) that $\pi : \mathscr{C} \to X$ is a continuous injection, so it only remains to check that π is a reduction of $\leq_0$ into $\leq_f$, i.e., that for all $x, y \in \mathscr{C}$,
$$x \leq_0 y \Leftrightarrow \pi(x) \leq_f \pi(y).$$

To see ($\Rightarrow$), it is enough to check that $x <_0 y \Rightarrow \pi(x) <_f \pi(y)$. So suppose that $x <_0 y$, and find $n \in \mathbb{N}$, $s, t \in 2^{n+1}$ with $s_n < t_n$, and $w \in \mathscr{C}$ such that $x = sw$ and $y = tw$. Now put $z = 0^{n+1}w$, and note that for all $m \geq n$,

$$f^{\sum_{i \leq n} s_i k_i} \circ f^{\sum_{i \leq m} z_i k_i}(B_{m+1}) = f^{\sum_{i \leq m} x_i k_i}(B_{m+1}).$$

In particular, as $f^{\sum_{i \leq n} s_i k_i} \circ \pi(z)$ lies in the set on the left-hand side of this equality, it must also lie in the set on the right-hand side. It then follows from the definition of π that

$$f^{\sum_{i \leq n} s_i k_i} \circ \pi(z) = \pi(x).$$

Of course, a similar argument shows that

$$f^{\sum_{i \leq n} t_i k_i} \circ \pi(z) = \pi(y).$$

Now, the fact that $s_n < t_n$ ensures that

$$\ell = \sum_{i \leq n} t_i k_i - \sum_{i \leq n} s_i k_i$$

is strictly greater than 0, and it follows that

$$\begin{aligned} f^{\ell} \circ \pi(x) &= f^{\ell} \circ f^{\sum_{i \leq n} s_i k_i} \circ \pi(z) \\ &= f^{\sum_{i \leq n} t_i k_i} \circ \pi(z) \\ &= \pi(y), \end{aligned}$$

thus $\pi(x) <_0 \pi(y)$.

It only remains to show ($\Leftarrow$). Note that it is enough to show that

$$\pi(x) \leq_f \pi(y) \Rightarrow x E_0 y,$$

since then the only way ($\Leftarrow$) could fail is if $\pi(x) \leq_f \pi(y)$ and $y <_0 x$, and our proof of ($\Rightarrow$) implies that $y <_0 x \Rightarrow \pi(y) <_f \pi(y)$, a contradiction. We will actually show the contrapositive. More precisely, we will show that if $x_n \neq y_n$, then

$$\forall i_x, i_y < k_n/2 \; (f^{i_x} \circ \pi(x) \neq f^{i_y} \circ \pi(y)).$$

Suppose, towards a contradiction, that $x_n < y_n$ but we can find $i_x, i_y < k_n/2$ with

$$f^{i_x} \circ \pi(x) = f^{i_y} \circ \pi(y).$$

By the definition of π, we can find $u, v \in B_{n+1}$ such that

$$f^{\sum_{i \leq n} x_i k_i}(u) = \pi(x) \text{ and } f^{\sum_{i \leq n} y_i k_i}(v) = \pi(y).$$

Setting $i_u = i_x + \sum_{i \leq n} x_i k_i$ and $i_v = i_y + \sum_{i \leq n} y_i k_i$, we have that

$$f^{i_u}(u) = f^{i_v}(v).$$

Noting that $i_u, i_v < 3k_n$ and B_{n+1} is $\mathscr{G}_{3k_n}(f)$-discrete, it follows that $uE_0(f)v$. As f is aperiodic and $f^{i_u}(u) = f^{i_v}(v)$, this implies that $i_u = i_v$. That is,

$$i_x + \sum_{i \leq n} x_i k_i = i_y + \sum_{i \leq n} y_i k_i.$$

As $x_n = 0$ and $y_n = 1$, this means that

$$k_n = (i_x - i_y) + \sum_{i < n} (x_i - y_i) k_i,$$

contradicting the fact that the sum on the right is bounded above by $k_n/2 + k_n/3$. $\dashv$

3 Kakutani equivalence

The following fact comes from a result of Jackson-Kechris-Louveau [48]:

Proposition 3.1. *Suppose $\mathscr{G}$ is a locally countable Borel graph which admits a Borel $\aleph_0$-coloring. Then there is a Borel maximal $\mathscr{G}$-discrete set.*

Proof. For $Y \subseteq X$, I will use

$$\mathscr{G}(Y) = \{x \in X : \exists y \in Y \ ((x, y) \in \mathscr{G})\}$$

to denote the set of $\mathscr{G}$-neighbors of points of Y. Note that if Y is Borel, then the local countability of $\mathscr{G}$ ensures that $\mathscr{G}(Y)$ is also Borel. Fix a Borel $\aleph_0$-coloring $c : X \to \mathbb{N}$ of $\mathscr{G}$, put $B_0 = c^{-1}(\{0\})$, and recursively define B_n by

$$B_{n+1} = B_n \cup (c^{-1}(\{n+1\}) \setminus \mathscr{G}(B_n)).$$

It is clear that each B_n is $\mathscr{G}$-discrete, thus so too is $B = \bigcup_n B_n$. Now note that if $x \notin B$, then $x \notin B_{c(x)}$, thus $c(x) > 0$ and $x \in \mathscr{G}(B_{c(x)-1}) \subseteq \mathscr{G}(B)$. It follows that B is maximal $\mathscr{G}$-discrete. $\dashv$

Given $f : X \to X$, $B \subseteq X$, and $x \in X$, the *distance from x to B* is

$$d_B(x) = \begin{cases} n & \text{if } n \text{ is the least natural number such that } f^{-n}(x) \in B, \\ \infty & \text{if no such natural exists.} \end{cases}$$

The set B has *bounded gaps* if $\sup_{x \in X} d_B(x) < \infty$, and $B \subseteq X$ is *doubly recurrent* if

$$\forall x \in B \exists m < 0 < n \ (f^m(x), f^n(x) \in B).$$

Note that any set with bounded gaps is doubly recurrent. Associated with any doubly recurrent Borel set is the *induced automorphism of B*, given by $f_B(x) = f^{n(x)}(x)$, where $n(x) > 0$ is least such that $f^{n(x)}(x) \in B$.

Proposition 3.2. *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel automorphism. Then there is a decreasing, vanishing sequence of Borel sets with bounded gaps.*

Proof. Set $B_0 = X$, and suppose that B_n is a Borel set with bounded gaps. Then B_n is recurrent, so f induces an automorphism of B_n. Let B_{n+1} be any Borel maximal $\mathscr{G}_{f_{B_n}}$-discrete set. Note that $d_{B_n}(x) < 3^n$ for all $x \in X$, and each B_n is $\mathscr{G}^{<2^n}$-discrete. It follows that while $\bigcap_n B_n$ need not be empty, this intersection is a *partial transversal*, i.e., it intersects each orbit of f in at most one point.

It only remains to show the lemma when f is smooth. Set

$$S_n = \{k \cdot 2^n : k \neq 0\},$$

fix a Borel transversal $B \subseteq X$, and observe that

$$B_n = \bigcup_{k \in S_n} f^k(B)$$

is as desired. $\dashv$

The following fact completes the description of order-preserving embeddability:

Theorem 3.3 (Miller-Rosendal). *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel automorphism. Then f order-preservingly embeds into σ.*

Proof. We may assume without loss of generality that $X = \mathscr{C}$. By Proposition 18, there is a decreasing, vanishing sequence of Borel complete sections B_n such that

$$\forall x \in X\ (d_{B_n}(x) < 2^n).$$

Let $f_n(x) = f^{-d_{B_n}(x)}(x)$, and define $\varphi_n : \mathscr{C} \to 2^{n \cdot 2^n}$ by

$$\varphi_n(x) = \bigoplus_{m<2^n} f^m \circ f_n(x)|n,$$

where $\oplus$ denotes concatenation. Let $b_2(n) = \sigma^n(0^\infty)$ be the base 2 representation of n, noting that

$$\forall m, n \in \mathbb{N}\ (m \leq n \Leftrightarrow b_2(m) \leq_0 b_2(n)),$$

and define $\psi_n : X \to 2^{n+1}$ by

$$\psi_n(x) = b_2(d_{B_{n+1}}(f_n(x)))|(n+1) = b_2(d_{B_{n+1}}(x) - d_{B_n}(x))|(n+1).$$

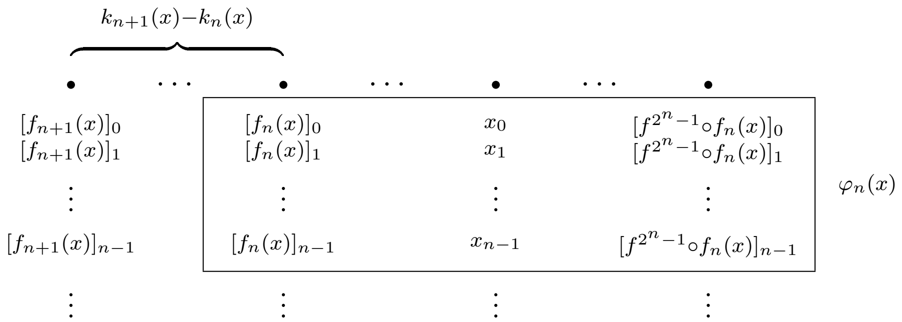

Figure 2.3: φ_n approximates $[x]_f$ and ψ_n codes the distance between $f_n(x), f_{n+1}(x)$.

We claim that

$$\pi(x) = \bigoplus_{n \in \mathbb{N}} \varphi_n(x) \oplus \psi_n(x)$$

is the desired reduction. As the range of π is clearly contained in $\mathscr{C}_0$, it is enough to show that

$$\forall x, y \in X\ (x \leq_f y \Leftrightarrow \pi(x) \leq_0 \pi(y)).$$

To see ($\Rightarrow$), it is enough to check that if $x <_f y$, then $\pi(x) <_0 \pi(y)$. Let $n \in \mathbb{N}$ be maximal such that $f_n(x) \neq f_n(y)$, and note that

$$d_{B_{n+1}}(f_n(x)) < d_{B_{n+1}}(f_n(y)).$$

As $d_{B_{n+1}}(f_n(x)), d_{B_{n+1}}(f_n(y)) < 2^{n+1}$, it follows that $\psi_n(x) <_0 \psi_n(y)$. As

$$\forall m > n\ (\varphi_m(x) \oplus \psi_m(x) = \varphi_m(y) \oplus \psi_m(y)),$$

it follows that $\pi(x) <_0 \pi(y)$.

To see ($\Leftarrow$), it is enough to show that $\pi(x) \leq_0 \pi(y) \Rightarrow x E_t(f) y$, since then $x \leq_0 y$ by ($\Rightarrow$). So suppose that $\pi(x) \leq_0 \pi(y)$, and fix $n \in \mathbb{N}$ sufficiently large that

$$\forall m \geq n\ (\varphi_m(x) \oplus \psi_m(x) = \varphi_m(y) \oplus \psi_m(y)).$$

Set $k = d_{B_n}(x) - d_{B_n}(y)$, noting that $d_{B_m}(x) = d_{B_m}(y) + k$ for all $m \geq n$. Identifying $\varphi_m(x), \varphi_m(y)$ with the corresponding elements of $(2^m)^{2^m}$, it follows that

$$\begin{aligned} x|m &= (\varphi_m(x))_{d_{B_m}(x)} \\ &= (\varphi_m(y))_{d_{B_m}(y)+k} \\ &= f^{d_{B_m}(y)+k} \circ f_m(y)|m \\ &= f^k(y)|m, \end{aligned}$$

thus $x = f^k(y)$, so $x E_t(f) y$. ⊣

Corollary 3.4. *Every aperiodic Borel automorphism of a Polish space can be order-preservingly Borel embedded into every non-smooth Borel automorphism of a Polish space. Thus, two aperiodic Borel automorphisms of a Polish space are order-preservingly bi-embeddable if and only if they are both smooth or both non-smooth.*

Proof. This follows directly from Theorems 2.4 and 3.3. ⊣

Suppose that X and Y are Polish spaces and $f : X \to X$ and $g : Y \to Y$ are Borel automorphisms. A set $A \subseteq X$ is a *complete section for f* if A intersects every orbit of f. The automorphisms f and g are *descriptive Kakutani equivalent* if there are Borel complete sections A, B for f, g and a Borel isomorphism $\pi : A \to B$ which is a reduction of $\leq_f$ to $\leq_g$. In Nadkarni [61], it is asked if all non-smooth Borel automorphisms of finite rank are descriptive Kakutani equivalent. In fact, we have the following:

Theorem 3.5 (Miller-Rosendal). *All non-smooth aperiodic Borel automorphisms of a Polish space are descriptive Kakutani equivalent. Moreover, if X and Y are Polish spaces and $f : X \to X$ and $g : Y \to Y$ are non-smooth aperiodic Borel automorphisms, then there are invariant Borel sets $A \subseteq X$ and $B \subseteq Y$ such that $f|A, g|(Y \setminus B)$ are order-preservingly Borel embeddable onto complete sections of $g|B, f|(X \setminus A)$.*

Proof. Fix order-preserving Borel embeddings φ, ψ of f, g into g, f. We will proceed via a standard Schröder-Bernstein argument, albeit with respect to the maps that φ, ψ induce on the quotients $X/E_t(f), Y/E_t(g)$. Set $A_0 = X \setminus [\psi(Y)]_f$, and recursively define

$$B_n = [\varphi(A_n)]_g \text{ and } A_{n+1} = [\psi(B_n)]_f.$$

Setting $A = \bigcup_n A_n$ and $B = \bigcup_n B_n$, it follows that φ is an order-preserving Borel embedding of $f|A$ onto a complete section of $g|B$.

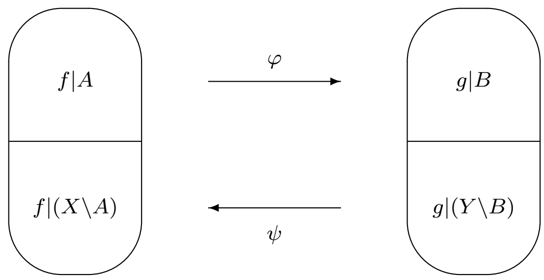

Figure 2.4: A witness to the descriptive Kakutani equivalence of f and g.

To see that ψ is a order-preserving Borel embedding of $g|(Y \setminus B)$ onto a complete section of $f|(X \setminus A)$, simply observe that

$$\begin{aligned}
[\psi(Y \setminus B)]_f &= [\psi(Y) \setminus \psi(B)]_f \\
&= \left[\psi(Y) \setminus \bigcup_{n\in\mathbb{N}} \psi(B_n)\right]_f \\
&\supseteq [\psi(Y)]_f \setminus \bigcup_{n\in\mathbb{N}} [\psi(B_n)]_f \\
&= (X \setminus A_0) \setminus \bigcup_{n>0} A_n \\
&= X \setminus A.
\end{aligned}$$

⊣

If $\mathbb{R}$ acts freely on X, then the usual ordering of the reals can be pushed through the action to orderings of each orbit. Two actions of the reals are *trajectory equivalent* if there is a Borel isomorphism of their underlying spaces which preserves these induced orderings.

Theorem 3.6 (Rosendal). *All non-smooth Borel free actions of* $\mathbb{R}$ *are trajectory equivalent.*

Proof. By a theorem of Wagh (see Nadkarni [61]), any Borel free action of $\mathbb{R}$ has a Borel complete section which is discrete with respect to the induced ordering. Clearly such a section can be modified so as to ensure that its intersection with each class is of type $\mathbb{Z}$, so that there is a non-smooth aperiodic Borel automorphism of the complete section which induces the same ordering as the action of $\mathbb{R}$. Find such automorphisms corresponding to each of the actions, apply Theorem 3.5 to obtain a descriptive Kakutani equivalence of these two automorphisms, and note that any such map can easily be extended to a trajectory equivalence. ⊣

In Dougherty-Jackson-Kechris [24], the analog of Theorem 3.5 is shown for the weaker notion of *Borel embeddability*, in which the embedding is not required to be order-preserving. Actually, in that case something even stronger holds:

Proposition 3.7. *Suppose that f, g are aperiodic non-smooth Borel automorphisms. Then at least one of f, g Borel embeds onto a complete section of the other.*

Proof. By reversing the roles of f, g if necessary, we may assume that f admits at least as many invariant, ergodic probability measures as g. By Dougherty-Jackson-Kechris [24], there is a Borel complete section $B \subseteq Y$ such that $E_t(f), E_t(g)|B$ admit the same number of invariant, ergodic probability measures. But then $E_t(f)$ is Borel isomorphic to $E_t(g)|B$, again by Dougherty-Jackson-Kechris [24], and this isomorphism is a Borel embedding of $E_t(f)$ onto a complete section of $E_t(g)$. ⊣

The analogous property for order-preserving Borel embeddability is false:

Proposition 3.8. *There is a σ-invariant Borel set $B \subseteq \mathscr{C}$ such that neither of*

$$\sigma|B, \sigma|(X \setminus B)$$

is order-preservingly Borel embeddable onto a complete section of the other.

Proof. Let μ be the usual product measure on $\mathscr{C}$, and note that since μ is the unique invariant ergodic probability measure for σ, then exactly one of $\sigma|B, \sigma|(X \setminus B)$ has an invariant ergodic finite measure. We will arrange things so that $\sigma|B$ has such a measure. Note that this guarantees that $\sigma|(X \setminus B)$ does not Borel embed onto a complete section of $\sigma|B$, since $\mu|B$ could be pulled back through any such embedding.

For each $n \geq 1$, fix a maximal $\sigma^{<n \cdot 3^n}(\sigma)$-discrete Borel set $A_n \subseteq X$ and find $i < 3^n$ such that

$$A'_n = \bigcup_{in \leq j < (i+1)n} \sigma^j(A_n)$$

is of measure $\leq 1/3^n$. Now set $A = X \setminus \bigcup_{n \geq 1} A'_n$ and $B = [A]_\sigma$, noting that $\mu(A) \geq 1/2$, thus B is of full measure.

Suppose, towards a contradiction, that $\pi : B \to X \setminus B$ is an order-preserving Borel embedding of $\sigma|B$ onto a complete section of $\sigma|(X \setminus B)$. Then $A' = A \cup \pi(A)$ intersects every orbit $[x]_\sigma$ in a set with *large gaps*, i.e.,

$$\forall x \in X \, \forall m \in \mathbb{N} \, \exists n \in \mathbb{N} \; (\sigma^n(x), \sigma^{n+1}(x), \ldots, \sigma^{n+m}(x) \notin A').$$

As A' is a complete section for σ, there is some $s \in 2^{<\mathbb{N}}$ such that $\mathscr{N}_s \setminus A'$ is meager. It follows that

$$\forall^* x \in X \, \exists n \in \mathbb{N} \, \forall k \in \mathbb{N} \; \left(k \equiv n \pmod{2^{|s|}} \Rightarrow \sigma^k(x) \in A'\right),$$

which contradicts the fact that A' intersects each orbit in a set with large gaps. $\dashv$

In fact, there are large collections of Borel automorphisms whose induced equivalence relations are Borel isomorphic, but for which no automorphism in the collection can be order-preservingly Borel embedded onto a complete section of any of the others. One reason for this is that the sorts of complications which arise in the measure-theoretic version of order-preserving Borel embeddability come into play here. Suppose $f : X \to X$ and $g : Y \to Y$ are Borel automorphisms with invariant, ergodic probability measures μ and ν. Then (f, μ) and (g, ν) are *Kakutani equivalent* if there are recurrent Borel sets $A \subseteq X$ and $B \subseteq Y$ of positive measure and a Borel isomorphism π of $(f_A, \mu|A)$ with $(g_B, \nu|B)$. It follows from the arguments of Ornstein-Rudolph-Weiss [63] that there are large collections of such pairs which are pairwise non-Kakutani equivalent. So it is enough to note the following fact:

Proposition 3.9. *Suppose that f, g are Borel automorphisms with unique invariant (ergodic) probability measures μ, ν. If $(f, \mu), (g, \nu)$ are not Kakutani equivalent, then neither of f, g order-preservingly Borel embeds onto a complete section of the other.*

Proof. Simply observe that by unique ergodicity, any order-preserving Borel embedding of f onto a complete section of g is measure-preserving, and therefore provides a witness to the Kakutani equivalence of (f, μ) and (g, ν). $\dashv$

We will close this section by answering a related question of Louveau. Suppose that X and Y are Polish spaces and $f : X \to X$ and $g : Y \to Y$ are Borel automorphisms. An *embedding* of f into g is a Borel injection $\pi : X \to Y$ such that $\pi \circ f = g \circ \pi$. Equivalently, an embedding is a Borel isomorphism of (X, f) with $(B, g|B)$, for some g-invariant Borel set $B \subseteq Y$. It follows from Clemens [17] (and the generalization provided by Gao [41]) that Borel embeddability of Borel automorphisms is far more complicated than order-preserving Borel embeddability. However, their results do leave open the possibility of an analog of Theorem 2.4 for embeddability, i.e., the existence of a minimal non-smooth Borel automorphism, and Louveau asked if such an automorphism exists. The answer is no:

Theorem 3.10. *Suppose that $\mathscr{A}$ is a collection of Borel automorphisms of a Polish space, and for every non-smooth Borel automorphism $f : X \to X$ of a Polish space X, there is an automorphism in $\mathscr{A}$ which Borel embeds into f. Then $|\mathscr{A}| = \mathfrak{c}$.*

Proof. To show this, we will simply note that the construction of Eigen-Hajian-Weiss [31] does a bit more than the authors intended. Fix a set $\{x_\alpha\}_{\alpha \in \mathscr{C}} \subseteq \mathscr{C}_0$ which is *almost disjoint*, i.e.,

$$\forall \alpha \neq \beta \ \ (|\operatorname{supp}(x_\alpha) \cap \operatorname{supp}(x_\beta)| < \aleph_0),$$

define $W_\alpha = \{x \in \mathscr{C} : \operatorname{supp}(x) \subseteq \operatorname{supp}(x_\alpha)\}$, set

$$X_\alpha = [W_\alpha]_{E_0} = \big\{x \in \mathscr{C} : |\operatorname{supp}(x) \setminus \operatorname{supp}(x_\alpha)| < \aleph_0\big\},$$

and define $X'_\alpha = \bigcup_{\alpha \neq \beta} X_\beta$. The main observation is the following:

Lemma 3.11 (essentially Eigen-Hajian-Weiss). *Suppose that $\alpha \in \mathscr{C}$ and $B \subseteq X_\alpha$ is a σ-invariant Borel set for which $\sigma|B$ is non-smooth. Then $\sigma|B$ does not Borel embed into $\sigma|X'_\alpha$.*

Proof. Suppose that $B \subseteq X_\alpha$ is a σ-invariant Borel set and π is a Borel embedding of $\sigma|B$ into $\sigma|X'_\alpha$. Define

$$W'_\alpha = \{x \in \pi(B) : \operatorname{supp}(x) \cap \operatorname{supp}(x_\alpha) = \emptyset\},$$

and note that

$$\forall x \in \pi(B) \ \ (|\operatorname{supp}(x) \cap \operatorname{supp}(x_\alpha)| < \aleph_0),$$

thus $\pi(B) = [W'_\alpha]_{E_0}$.

The main point here is that for all $n \in \mathbb{N}$,

$$W_\alpha \cap \sigma^n(W_\alpha) = \emptyset \text{ or } W'_\alpha \cap \sigma^n(W'_\alpha) = \emptyset.$$

To see this, let k be the smallest digit on which the base 2 representation of n is non-zero. If $k \in \operatorname{supp}(x_\alpha)$ then $W'_\alpha \cap \sigma^n(W'_\alpha) = \emptyset$, and if $k \notin \operatorname{supp}(x_\alpha)$ then $W_\alpha \cap \sigma^n(W_\alpha) = \emptyset$.

Setting $W''_\alpha = \pi^{-1}(W'_\alpha)$, it follows that $\{\sigma^i(W_\alpha) \cap \sigma^j(W''_\alpha)\}_{i,j\in\mathbb{Z}}$ is a countable family of Borel sets each of which intersects every $E_0|B$-class in at most one point, and which together cover B. It follows that $\sigma|B$ is smooth. ⊣

In particular, any non-smooth Borel automorphism embeds into at most one automorphism of the form $\sigma|X_\alpha$, and the theorem follows. ⊣

4 Betweenness-preserving embeddability of $\mathscr{L}_0$

Suppose that X is a Polish space. A graph $\mathscr{L} \subseteq X^2$ is a *forest of lines* if every connected component of $\mathscr{L}$ is a tree of vertex degree two, and such a forest is *directable* if there is a Borel automorphism $f : X \to X$ such that $\mathscr{L} = \mathscr{L}_f$, where $\mathscr{L}_f$ denotes the union of the graphs of f and f^{-1}. As pointed out by Scot Adams [1], not all Borel forests of lines are directable.

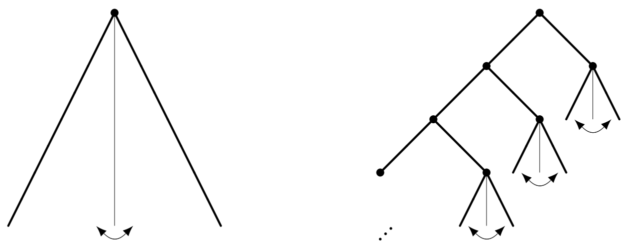

Figure 2.5: The action of i, j on $\mathscr{C}_0$.

One way of seeing this is as follows: for $x \in \mathscr{C}$, let $\overline{x}_n = 1 - x_n$, set

$$i(x) = \overline{x} \text{ and } j(x) = \sigma(\overline{x}),$$

and put $\mathscr{L}_0 = \mathrm{graph}(i) \cup \mathrm{graph}(j)$. Note that $j(0^n 1 x) = 0^n 1 \overline{x}$.

Proposition 4.1. *$\mathscr{L}_0|\mathscr{C}_0$ is an undirectable Borel forest of lines.*

Proof. First, we will show that $\mathscr{L}_0|\mathscr{C}_0$ is a forest of lines. Clearly $i|\mathscr{C}_0$ and $j|\mathscr{C}_0$ are fixed-point free. As the odometer is also fixed-point free, it follows that

$i(x) \neq \sigma \circ i(x) = j(x)$, thus x, $i(x)$, and $j(x)$ are always distinct. It follows that $\mathscr{L}_0|\mathscr{C}_0$ is a graph of vertex degree 2. As $j \circ i(x) = \sigma(x)$ and σ has no finite orbits, each connected component of $\mathscr{L}_0|\mathscr{C}_0$ must be infinite. Since the only infinite, connected graph of vertex degree 2 is a line, it follows that $\mathscr{L}_0|\mathscr{C}_0$ is a forest of lines.

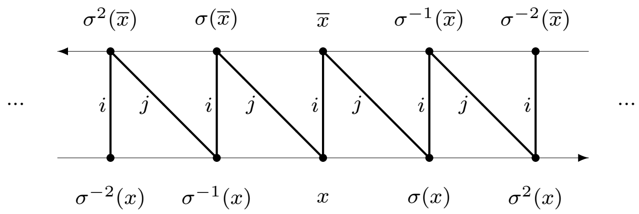

Figure 2.6: The components of $\mathscr{L}_0$ weave together pairs of E_0-classes.

It remains to check that $\mathscr{L}_0|\mathscr{C}_0$ is undirectable. Suppose, towards a contradiction, that there is a Borel automorphism $f : \mathscr{C}_0 \to \mathscr{C}_0$ with $\mathscr{L}_f = \mathscr{L}_0|\mathscr{C}_0$, and observe that

$$B = \{x \in \mathscr{C}_0 : f(x) = i(x)\}$$

is an E_0-invariant Borel set which, together with $\overline{B}$, partitions $\mathscr{C}_0$. As these two sets are either both meager or both comeager, this is a contradiction. $\dashv$

Suppose that X and Y are Polish spaces and $\mathscr{L} \subseteq X^2$ and $\mathscr{M} \subseteq Y^2$ are Borel forests of lines. An *$\mathscr{L}$-path from x to x'* is a sequence $x = x_0, x_1, \ldots, x_n = x'$ of distinct points with $(x_i, x_{i+1}) \in \mathscr{L}$. The *notion of betweenness induced by $\mathscr{L}$* is given by

$$xB_{\mathscr{L}}(y, z) \Leftrightarrow \exists \text{an } \mathscr{L}\text{-path from } y \text{ to } z \text{ which goes through } x.$$

A *betweenness-preserving embedding of $\mathscr{L}$ into $\mathscr{M}$* is an injection $\pi : X \to Y$ which is simultaneously a reduction of $E_{\mathscr{L}}$ into $E_{\mathscr{M}}$ and of $B_{\mathscr{L}}$ into $B_{\mathscr{M}}$.

Proposition 4.2. *Suppose that X and Y are sets and $\mathscr{L}$ and $\mathscr{M}$ are forests of lines on X and Y. Then every reduction of $B_{\mathscr{L}}$ into $B_{\mathscr{M}}$ is a betweenness-preserving embedding of $\mathscr{L}$ into $\mathscr{M}$.*

Proof. Suppose π is a reduction of $B_{\mathscr{L}}$ into $B_{\mathscr{M}}$. To see that π is injective, simply note that if $\pi(x) = \pi(x')$ then $\pi(x)B_{\mathscr{M}}(\pi(x'), \pi(x'))$, thus $xB_{\mathscr{L}}(x', x')$, so $x = x'$. To see that π is a reduction of $E_{\mathscr{L}}$ into $E_{\mathscr{M}}$, simply note that for all $x, x' \in X$,

$$\begin{aligned} xE_{\mathscr{L}}x' &\Leftrightarrow xB_{\mathscr{L}}(x, x') \\ &\Leftrightarrow \pi(x)B_{\mathscr{M}}(\pi(x), \pi(x')) \\ &\Leftrightarrow \pi(x)E_{\mathscr{M}}\pi(x'). \end{aligned}$$

$\dashv$

The primary goal of this section is to show the following analog of Theorem 2.4:

Theorem 4.3. *Suppose that X is a Polish space and $\mathscr{L}$ is a Borel forest of lines on X. Then exactly one of the following holds:*

1. *$\mathscr{L}$ is directable.*
2. *$\mathscr{L}_0$ betweenness-preservingly Borel embeds into $\mathscr{L}$.*

Already (1) $\Rightarrow \neg$(2) follows from Proposition 4.1, as any betweenness-preserving embedding of $\mathscr{L}_0$ into a directable line would give a way of directing $\mathscr{L}_0$.

As in §2, we must develop several tools before getting to the main construction. We will begin with a lemma which will be used in the spirit of Weiss [79], and removes the need to play games:

Proposition 4.4. *Suppose that X is a Polish space, $\mathscr{B}$ is a countable family of Borel subsets of X, and $\mathscr{F}$ is a countable family of finite-to-1 Borel partial functions on X. Then there is a zero-dimensional Polish topology on X, finer than the one with which X was originally endowed but compatible with the underlying Borel structure of X, in which each element of $\mathscr{B}$ is clopen and each element of $\mathscr{F}$ is a continuous open-and-closed map.*

Proof. Let $<$ be a Borel linear ordering of X, and for each $f \in \mathscr{F}$ define

$$X_{(f,n)} = \{x \in \mathrm{dom}(f) : x \text{ is the } n^{\text{th}} \text{ element of } f^{-1}(\{f(x)\})\}.$$

Note that the restriction of f to $X_{(f,n)}$ is injective. Let $\mathscr{U}_0$ be a countable basis for the topology with which X was originally endowed, and let $\mathscr{A}_0$ be the smallest algebra of subsets of X which includes each $X_{(f,n)}$, each element of $\mathscr{B} \cup \mathscr{U}_0$, and is closed under images and pre-images under elements of $\mathscr{F}$. Noting that every element of $\mathscr{A}_0$ is Borel, it follows from Exercise 13.5 of [51] that there is a Polish topology τ_0, compatible with the underlying Borel structure of X, in which each element of $\mathscr{A}_0$ is clopen.

Now suppose we have constructed an algebra $\mathscr{A}_n$ of subsets of X and a Polish topology τ_n, compatible with the underlying Borel structure of X, in which each element of $\mathscr{A}_n$ is clopen. Let $\mathscr{U}_{n+1}$ be a countable open basis for τ_n, and let $\mathscr{A}_{n+1}$ be the smallest algebra of subsets of X which includes each element of $\mathscr{A}_n \cup \mathscr{U}_{n+1}$ and is closed under images and pre-images under elements of $\mathscr{F}$. Again, every element of $\mathscr{A}_{n+1}$ is Borel, so by appealing to Exercise 13.5 of [51], we can find a Polish topology τ_{n+1}, compatible with the underlying Borel structure of X, in which each element of $\mathscr{A}_{n+1}$ is clopen.

Letting τ be the topology generated by the τ_n's, it follows from Lemma 13.3 of [51] that τ is Polish, compatible with the underlying Borel structure of X, and has a clopen basis given by $\mathscr{A} = \bigcup_n \mathscr{A}_n$. As $\mathscr{B} \subseteq \mathscr{A}_0 \subseteq \mathscr{A}$, it is clear that each element of $\mathscr{B}$ is τ-clopen. Now suppose $f \in \mathscr{F}$. To see that f is τ-continuous, suppose that U is τ-open, find a countable collection of sets A_n in $\mathscr{A}$ such that $U = \bigcup_n A_n$, and note that $f^{-1}(U) = \bigcup_n f^{-1}(A_n)$, which is clearly τ-open. To see that f sends τ-open sets to τ-open sets, suppose that U is τ-open, find a countable collection of sets A_n in $\mathscr{A}$ such that $U = \bigcup_n A_n$, and note that $f(U) = \bigcup_n f(A_n)$, which is clearly τ-open. To see that f sends τ-closed sets to τ-closed sets, suppose C is τ-closed, find a collection of sets A_n in $\mathscr{A}$ with $C = \bigcap_n A_n$, and note that

$$f(C) = \bigcup_m f\left(C \cap X_{(f,m)}\right) = \bigcup_m f\left(\bigcap_n A_n \cap X_{(f,m)}\right).$$

It remains to check that if $z \in \overline{f(C)}$, then $z \in f(C)$. Find a sequence of points $x_i \in \bigcap_n A_n \cap X_{(f,m_i)}$ such that $f(x_i) \to z$. As $f(X)$ is clopen, it follows that $z \in f(X)$, thus we can find $m \in \mathbb{N}$ such that $z \in f(X_{(f,m)})$. Note that since this set is τ-open, it must be that m_i is eventually constant with value m. It follows that all but finitely many of the x_i's are in

$$\bigcap_n f\left(A_n \cap X_{(f,m)}\right),$$

and as this latter set is τ-closed, it follows that $z \in f(C)$. ⊣

Remark 4.5. If we drop the requirement that f is a closed map, then finite-to-one can be weakened to countable-to-one in Proposition 4.4. However, the analog of Proposition 4.4 for any countable-to-one Borel function for which uncountably many points have infinite pre-images is false.

Suppose $\mathscr{L}$ is a Borel forest of lines. Given any set B, we will use $\mathscr{L}_B$ to denote the set of pairs $(x, y) \in B^2$ which are the endpoints of an $\mathscr{L}$-path which contains no points of B, with the exception of x and y. Note that every component of $\mathscr{L}_B$ is a line exactly when B is *recurrent* with respect to $\mathscr{L}$, i.e., when any $\mathscr{L}$-path through $[B]_{\mathscr{L}}$ can be extended to an $\mathscr{L}$-path whose endpoints lie in B, where $[B]_{\mathscr{L}} = [B]_{E_{\mathscr{L}}}$ is the *$E_{\mathscr{L}}$-saturation of B*. Even when B is not recurrent, I will say that $\mathscr{L}_B$ is *directable* if the restriction of $\mathscr{L}_B$ to the union of its components which are lines is directable. Note that in this case, $\mathscr{L}_B$ is induced by a partial Borel injection.

Associated with $\mathscr{L}$ is the σ-ideal $\mathscr{I}$ of Borel sets B for which $\mathscr{L}_B$ is directable. Next we will introduce a notion which makes working with this σ-ideal a bit easier. A sequence $\mathfrak{F} = \langle f_n \rangle_{n\in\mathbb{Z}}$ of Borel functions on X is a *local director* for $\mathscr{L}$ if for each $x \in X$ the $f_k(x)$'s are distinct and $(f_k(x), f_{k+1}(x)) \in \mathscr{L}$.

Proposition 4.6. *Every Borel forest of lines admits a local director.*

Proof. Set $f_0 = \mathrm{id}$. By Theorem 18.10 of [51], there are Borel functions f_{-1} and f_1 whose graphs partition $\mathscr{L}$. Recursively define $\mathfrak{F} = \langle f_n \rangle_{n\in\mathbb{Z}}$ by letting $f_{\pm(n+2)}(x)$ be the unique $\mathscr{L}$-neighbor of $f_{\pm(n+1)}(x)$ other than $f_{\pm n}(x)$. ⊣

Associated with a local director $\mathfrak{F} = \langle f_n \rangle_{n\in\mathbb{Z}}$ is an *induced equivalence relation*,

$$xE_{\mathfrak{F}}y \Leftrightarrow \exists m \forall n\ (f_n(x) = f_{m+n}(y)),$$

as well as an *induced partial automorphism* $f_{\mathfrak{F}}(x) = f_{n(x)}(x)$, where $n(x) > 0$ is least such that $xE_{\mathfrak{F}}f_{n(x)}(x)$. Note that every $E_{\mathscr{L}}$-class consists of at most 2 $E_{\mathfrak{F}}$-classes, and

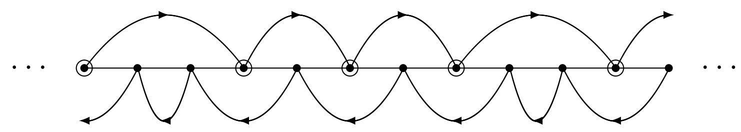

Figure 2.7: The automorphism induced by a local director.

that $E_{\mathfrak{F}} = E_t(f_{\mathfrak{F}})$. Given Borel equivalence relations $E \subseteq F$ such that each F-class consists of at most two E-classes, F/E is *smooth* if there is an E-invariant Borel set B such that B/E is a transversal of F/E.

Proposition 4.7. *Suppose that X is a Polish space, $\mathscr{L}$ is a Borel forest of lines on X, $\mathfrak{F}$ is a local director for $\mathscr{L}$, and $B \subseteq X$ is Borel. Then $\mathscr{L}_B$ is directable $\Leftrightarrow$ $(E_{\mathscr{L}}|B)/(E_{\mathfrak{F}}|B)$ is smooth.*

Proof. To see ($\Rightarrow$), suppose that f generates $\mathscr{L}_B$ and observe that

$$A = \{[x]_{E_{\mathfrak{F}}} : \exists n > 0\ (f(x) = f_n(x))\}$$

is a transversal of $E_{\mathscr{L}}/E_{\mathfrak{F}}$. To see ($\Leftarrow$), suppose that $A \subseteq B$ is an $f_{\mathfrak{F}}$-invariant Borel set which induces a transversal of $(E_{\mathscr{L}}|B)/(E_{\mathfrak{F}}|B)$, set

$$f(x) = \begin{cases} f_n(x) & \text{if } x \in A \text{ and } n > 0 \text{ is least such that } f_n(x) \in B, \\ f_{-n}(x) & \text{if } x \notin A \text{ and } n > 0 \text{ is least such that } f_{-n}(x) \in B, \end{cases}$$

and note that f is a partial Borel injection from B which generates $\mathscr{L}_B$. $\dashv$

Before getting to the main construction, it will be important to make one final observation which does not have a counterpart in §2. The need for this arises out of the fact that, while $\mathscr{L}_0$ is a very natural example of an undirectable line, it is of index two *above* E_0, and the sorts of arguments we use require objects which live within E_0.

Let $\mathscr{C}_0'$ be the set of non-eventually 0 points of $\mathscr{C}$, define

$$i'(x) = \overline{x_0}x_1x_2\ldots \text{ and } j'(0^n1x) = 0^n1\overline{x_0}x_1x_2\ldots,$$

and set $\mathscr{L}_0' = \text{graph}(i') \cup \text{graph}(j')$.

Proposition 4.8. *There is an isometry $\pi : \mathscr{C} \to \mathscr{C}$ which sends $(\mathscr{C}_0, E_{\mathscr{L}_0}, \mathscr{L}_0)$ to $(\mathscr{C}_0', E_0|\mathscr{C}_0', \mathscr{L}_0')$.*

Proof. Put $R_0 = \{1\}$ and for each $n \in \mathbb{N}$, define $R_{n+1} \subseteq 2^{n+2}$ by

$$R_{n+1} = \{x \in 2^{n+2} : x_n \neq x_{n+1}\}.$$

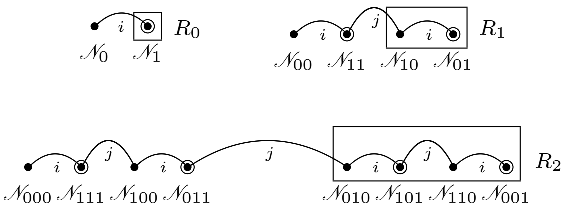

Figure 2.8: The points picked out by the first three R_n's.

It is clear that the function

$$\pi(x) = \langle \chi_{R_n}(x_0 x_1 \dots x_n) \rangle_{n \in \mathbb{N}}$$

is an isometry. To see that $\pi(\mathscr{C}_0) = \mathscr{C}_0'$, note that

$$\begin{aligned} x \in \mathscr{C}_0 &\Leftrightarrow \exists^\infty n \in \mathbb{N}\ (x_n \neq x_{n+1}) \\ &\Leftrightarrow \exists^\infty n \in \mathbb{N}\ (\pi_n(x) = 1) \\ &\Leftrightarrow \pi(x) \in \mathscr{C}_0'. \end{aligned}$$

We claim that $E_{\mathscr{L}_0'} = E_0|\mathscr{C}_0'$. As $E_{\mathscr{L}_0'} \subseteq E_0$ is clear, it is enough to check that if $x, y \in \mathscr{C}_0'$ are E_0-equivalent, then x and y are $\mathscr{L}_0'$-connected. For this, it is enough to check that for all $n \in \mathbb{N}$ and $z \in \mathscr{C}_0'$,

$$\{0^{n+1}z, i'(0^{n+1}z), j' \circ i'(0^{n+1}z), \dots, i' \circ (j' \circ i')^{2^n - 1}(0^{n+1}z)\} = \{sz\}_{s \in 2^{n+1}}. \qquad (\dagger)$$

We will simultaneously show $(\dagger)$ and

$$\forall z \in \mathscr{C}_0' \ \left(i' \circ (j' \circ i')^{2^n - 1}(0^{n+1}z) = 0^n 1 z\right), \qquad (\ddagger)$$

by induction on $n \in \mathbb{N}$. The case $n = 0$ is trivial, so suppose that (†) and (‡) have been established strictly below n. Applying (‡) at $n-1$ to $0z$ and $1z$ gives

$$i' \circ (j' \circ i')^{2^{n-1}-1}(0^n 0z) = 0^{n-1}10z \text{ and } i' \circ (j' \circ i')^{2^{n-1}-1}(0^n 1z) = 0^{n-1}11z.$$

By applying $i' \circ (j' \circ i')^{2^{n-1}-1}$ to each side of the latter equation, we obtain

$$i' \circ (j' \circ i')^{2^{n-1}-1}(0^{n-1}11z) = 0^n 1z.$$

Noting that $j'(0^{n-1}10z) = 0^{n-1}11z$, it follows that

$$\begin{aligned} i' \circ (j' \circ i')^{2^n-1}(0^{n+1}z) &= i' \circ (j' \circ i')^{2^{n-1}-1} \circ j' \circ i' \circ (j' \circ i')^{2^{n-1}-1}(0^n 0z) \\ &= i' \circ (j' \circ i')^{2^{n-1}-1} \circ j'(0^{n-1}10z) \\ &= i' \circ (j' \circ i')^{2^{n-1}-1}(0^{n-1}11z) \\ &= 0^n 1z, \end{aligned}$$

thus (‡) holds at n and

$$\{0^{n+1}z, i'(0^{n+1}z), j' \circ i'(0^{n+1}z), \ldots, i' \circ (j' \circ i')^{2^n-1}(0^{n+1}z)\}$$

is of cardinality 2^{n+1}, so (†) holds at n as well.

Thus, to show that π sends $E_{\mathscr{L}_0}$ to $E_0|\mathscr{C}_0'$, it is enough to show that π sends $\mathscr{L}_0$ to $\mathscr{L}_0'$. Clearly $\mathscr{L}_0'$ is of vertex degree 2, and since each E_0-class is infinite, it follows that $\mathscr{L}_0'$ is a Borel forest of lines. Thus, we need only show that π sends $\mathscr{L}_0$ into $\mathscr{L}_0'$. Noting that $\pi(x)$ and $\pi(\overline{x})$ agree off of their first digit, it follows that π sends graph(i) into graph(i'). As π fixes the first $n+1$ digits of any sequence which begins with $0^n 1$, it follows that $\pi(0^n 1x)$ and $\pi(0^n 1\overline{x})$ agree off of their $(n+1)^{\text{st}}$ digit, thus π sends graph(j) into graph(j'). ⊣

From the point of view of the construction to come, it is really $\mathscr{L}_0'$ that plays the role of σ. The following fact completes the proof of Theorem 4.3:

Proposition 4.9. *Suppose X is a Polish space and $\mathscr{L}$ is an undirectable Borel forest of lines on X. Then $\mathscr{L}_0$ continuously betweenness-preservingly embeds into $\mathscr{L}$.*

Proof. By Lemma 4.6, there is a local director $\mathfrak{F} = \langle f_n \rangle_{n \in \mathbb{N}}$ of $\mathscr{L}$. By throwing away an $\mathscr{L}$-invariant Borel set on which $\mathscr{L}$ is directable, I can assume that $f_{\mathfrak{F}}$ is a Borel automorphism. By Lemma 4.4, there is a Polish topology τ on X, finer than that with which X was originally endowed but compatible with the underlying Borel structure of X, in which each $\{x : x E_{\mathfrak{F}} f_k(x)\}$ is clopen and each $f_k^{\pm 1}$ sends closed sets to closed sets. Let d be a Polish metric which is compatible with this topology.

Next, we will recursively choose natural numbers k_n and $\mathscr{I}$-positive, τ-clopen sets B_n which satisfy the following additional properties:

1. $\forall x \in B_{n+1}\ (x, f_{k_n}(x) \in B_n$ and $(x, f_{k_n}(x)) \notin E_{\mathfrak{F}})$, and

2. $f_{-7k_n}(B_{n+1}), \ldots, f_{7k_n}(B_{n+1})$ are pairwise disjoint and of diameter $< 1/(n+1)$.

Note that (1) and (2) together imply that $k_{n+1} > 7k_n$, thus

$$\begin{aligned} \sum_{i \leq n} k_i &< \sum_{i \leq n} k_{n+1}/7^{i+1} \\ &< k_{n+1} \sum_{i > 0} 1/7^i \\ &= k_{n+1}/6. \end{aligned}$$

The recursion begins by setting $B_0 = X$. Now suppose that we have built $\{B_m\}_{m \leq n}$ and $\{k_m\}_{m < n}$. We claim that for some natural number $k > 0$, the τ-clopen set

$$A_k = \left\{ x : x, f_k(x) \in B_n \text{ and } (x, f_k(x)) \notin E_{\mathfrak{F}} \right\}$$

is $\mathscr{I}$-positive. To see this, it is enough to check that the set

$$B = B_n \setminus \bigcup_{k > 0} A_k$$

is $\mathscr{I}$-null. Since

$$\forall x \in B \forall k > 0\ (f_k(x) \in B \Rightarrow x E_{\mathfrak{F}} f_k(x)),$$

it follows that B cannot intersect any $E_{\mathscr{L}}$-class C in an $(E_{\mathfrak{F}} | C)$-complete section which is recurrent for f, thus $B \in \mathscr{I}$.

So fix k_n with $A_{k_n} \notin \mathscr{I}$, and note that for each $x \in A_{k_n}$, it follows from the τ-continuity of the f_i's that there is a τ-clopen neighborhood U of x, such that

$$f_{-7k_n}(U), \dots, f_{7k_n}(U) \text{ are pairwise disjoint and of diameter } < 1/(n+1).$$

It follows that we can partition A_{k_n} into countably many such sets. Let B_{n+1} be any $\mathscr{I}$-positive set from such a partition.

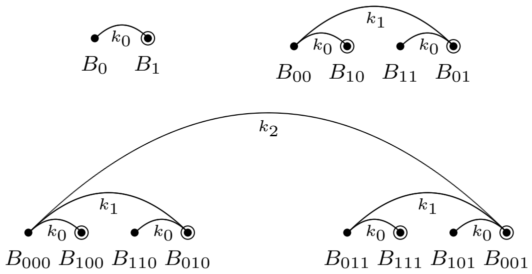

Figure 2.9: The first three stages of the construction of $\pi : \mathscr{C}_0' \hookrightarrow X$.

Set $B_\emptyset = X$, and for $s \in 2^{n+1}$, define

$$B_s = f_{k_0}^{s_0} \circ f_{k_1}^{s_1} \circ \cdots \circ f_{k_n}^{s_n}\big(B_{n+1}\big).$$

Then, for each $x \in 2^{\mathbb{N}}$, $\langle B_{x|n} \rangle_{n\in\mathbb{N}}$ is a decreasing sequence of τ-closed sets of vanishing diameter. It follows that $\bigcap_{n\in\mathbb{N}} B_{x|n}$ consists of a unique element $\pi(x)$.

It is clear that $\pi : \mathscr{C}_0' \to X$ is continuous, so it only remains to check that π is an isomorphism of $\mathscr{L}_0'$ with $\mathscr{L}_{\pi(\mathscr{C}_0')}$. As $\mathscr{L}_0', \mathscr{L}$ are forests of lines, it is enough to check the following:

1. $\forall x, y \in \mathscr{C}_0'$ $(\pi(x) E_\ell \pi(y) \Rightarrow x E_0 y)$.

2. For all $x \in \mathscr{C}_0'$ and $n \in \mathbb{N}$, both

$$(\pi(0x), \pi(1x)) \text{ and } (\pi(0^n10x), \pi(0^n11x))$$

are in $\mathscr{L}_{\pi(\mathscr{C}_0')}$.

Given $x, y \in X$, set $d_{\mathscr{L}}(x, y) = n$ if there is an $\mathscr{L}$-path from x to y with n edges, and set $d_{\mathscr{L}}(x, y) = \infty$ if $(x, y) \notin E_{\mathscr{L}}$. I claim that for $x, y \in \mathscr{C}_0'$,

$$d_{\mathscr{L}}(\pi(x), \pi(y)) \leq k_n \Leftrightarrow \forall m > n\ (x_m = y_m).$$

To see ($\Rightarrow$), it is enough to check that if $x_{n+1} \neq y_{n+1}$, then

$$d_{\mathscr{L}}(\pi(x), \pi(y)) > k_n.$$

By reversing the roles of x and y if necessary, we may assume $x_{n+1} = 0$, and then

$$\begin{aligned}
d_{\mathscr{L}}(\pi(x), \pi(y)) &\geq d_{\mathscr{L}}(f^{x_0}_{k_0} \circ \cdots \circ f^{x_{n+1}}_{k_{n+1}}(B_{n+2}), f^{y_0}_{k_0} \circ \cdots \circ f^{y_{n+1}}_{k_{n+1}}(B_{n+2})) \\
&= d_{\mathscr{L}}(f^{x_0}_{k_0} \circ \cdots \circ f^{x_n}_{k_n}(B_{n+2}), f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n} \circ f_{k_{n+1}}(B_{n+2})) \\
&\geq k_{n+1} - 2\sum_{i \leq n} k_i \\
&> k_{n+1} - k_{n+1}/3 \\
&> k_n.
\end{aligned}$$

To see ($\Leftarrow$), suppose that $\forall m > n\ (x_m = y_m)$, and note that for $m \geq n$, the restriction of

$$f' = \left(f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}\right) \circ \left(f^{x_n}_{k_n} \circ \cdots \circ f^{x_0}_{k_0}\right)$$

to $B' = f^{x_0}_{k_0} \circ \cdots \circ f^{x_m}_{k_m}(B_{m+1})$ is injective. This is because any two distinct points of B' are of $d_{\mathscr{L}}$-distance at least $k_{m+1} - 2\sum_{i \leq m} k_m$ apart, thus their images under f' are of $d_{\mathscr{L}}$-distance at least $k_{m+1} - 6\sum_{i \leq m} k_m > 0$ apart. It follows that

$$\begin{aligned}
\{\pi(y)\} &= \bigcap_{m \geq n} f^{y_0}_{k_0} \circ \cdots \circ f^{y_m}_{k_m}(B_{m+1}) \\
&= \bigcap_{m \geq n} \left(f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}\right) \circ \left(f^{x_n}_{k_n} \circ \cdots \circ f^{x_0}_{k_0}\right) \circ \left(f^{x_0}_{k_0} \circ \cdots \circ f^{x_m}_{k_m}(B_{m+1})\right) \\
&= \left(f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}\right) \circ \left(f^{x_n}_{k_n} \circ \cdots \circ f^{x_0}_{k_0}\right) \left(\bigcap_{m \geq n} f^{x_0}_{k_0} \circ \cdots \circ f^{x_m}_{k_m}(B_{m+1})\right) \\
&= \left\{\left(f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}\right) \circ \left(f^{x_n}_{k_n} \circ \cdots \circ f^{x_0}_{k_0}\right) \circ \pi(x)\right\}.
\end{aligned}$$

Setting $z = \left(f^{x_n}_{k_n} \circ \cdots \circ f^{x_0}_{k_0}\right) \circ \pi(x)$, it follows that

$$\pi(x) = f^{x_0}_{k_0} \circ \cdots \circ f^{x_n}_{k_n}(z) \text{ and } \pi(y) = f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}(z).$$

As $z \in B_{n+1}$, it follows that for all $s \in 2^{n+1}$,

$$f^{s_0}_{k_0} \circ \cdots \circ f^{s_n}_{k_n}(z) = f_{\sum_\ell (-1)^\ell k_{i^s_\ell}}(z),$$

where $\langle i^s_\ell \rangle$ is the decreasing enumeration of $\mathrm{supp}(s)$. Noting that

$$0 \leq \sum_\ell (-1)^\ell k_{i^s_\ell} \leq k_n,$$

it follows that $\pi(x), \pi(y)$ are of the form $f_k(z)$, with $0 \leq k \leq k_n$, thus

$$d_{\mathscr{L}}(\pi(x), \pi(y)) \leq k_n.$$

It is also important to note that

$$d_{\mathscr{L}}(\pi(x), \pi(y)) < k_{n+1} - 2k_n \Rightarrow d_{\mathscr{L}}(\pi(x), \pi(y)) \leq k_n.$$

To see this suppose, towards a contradiction, that

$$k_n < d_{\mathscr{L}}(\pi(x), \pi(y)) < k_{n+1} - 2k_n.$$

Then $x_{n+1} \neq y_{n+1}$ and $\forall m > n+1\ x_m = y_m$, so it follows that

$$\pi(y) = \left(f^{y_0}_{k_0} \circ \cdots \circ f^{y_n}_{k_n}\right) \circ f_{k_{n+1}} \circ \left(f^{x_0}_{k_0} \circ \cdots \circ f^{x_n}_{k_n}\right) \circ \pi(x).$$

But then, as before,

$$d_{\mathscr{L}}(\pi(x), \pi(y)) \geq k_{n+1} - 2 \max_{s \in 2^{n+1}} \sum_\ell (-1)^\ell k_{i^s_\ell} \geq k_{n+1} - 2k_n,$$

and this is the desired contradiction.

It follows that each pair of the form $(ix, \bar{\imath}x)$ is in $\mathscr{L}_{\pi(\mathscr{C}'_0)}$, as these two points are of minimal distance apart. It also follows that, for all $x \in \mathscr{C}'_0$ and $n \in \mathbb{N}$,

$$\pi(0^n 11x) = f_{k_n} \circ f_{k_{n+1}} \circ f_{k_n} \circ \pi(0^n 10x) = f_{2k_n - k_{n+1}}(0^n 11x).$$

Thus, to see that $(\pi(0^n 10x), \pi(0^n 11x)) \in \mathscr{L}_{\pi(\mathscr{C}'_0)}$, it is enough to check that if

$$k = d_{\mathscr{L}}(\pi(0^n 10x), \pi(y))$$

is less than $k_{n+1} - 2k_n$ (thus at most k_n), then $f_k \circ \pi(0^n 10x) = \pi(y)$, as it then follows that $k = k_{n+1} - 2k_n$ is the minimal natural number such that

$$f_{-k} \circ \pi(0^n 10x) \in \pi(\mathscr{C}'_0).$$

To see this, note that

$$\pi(y) = f_{k_0}^{y_0} \circ \cdots \circ f_{k_{n-1}}^{y_{n-1}} \circ f_{k_n} \circ \pi(0^n 10x) = f_k \circ \pi(0^n 10x),$$

for some $k > 0$, since $k_n > 7k_{n-1}$. ⊣

We will close this section with some applications of Theorem 4.3 to the study of Borel marriage problems. Suppose that X is a Polish space, $M, W \subseteq X$ are Borel sets which partition X, and $\mathscr{G}$ is a Borel graph for which M, W are discrete, or equivalently, for which M, W witness that $\mathscr{G}$ is *bipartite*. Intuitively, M should be thought of as a set of men, W should be thought of as a set of women, and $\mathscr{G}$ is the set of pairs (m, w) of men and women who are willing to marry one another. A solution to the *marriage problem associated with* $(\mathscr{G}, M, W)$ is a bijection $f : M \to W$ whose graph is contained in $\mathscr{G}$.

Now suppose that $\mathscr{G}$ is of vertex degree two. Then $\mathscr{G}$ clearly has a solvable marriage problem: fix a transversal $B \subseteq M$ of $E_{\mathscr{G}}$, let $f : B \to W$ be a function whose graph is contained in $\mathscr{G}$, and observe that f has a unique extension to a solution to the marriage problem for $(\mathscr{G}, M, W)$. On the other hand, Laczkovich [59] has shown that there is a Borel graph $\mathscr{G}$ of vertex degree 2 whose marriage problem admits no Borel solution.

Note that if $E_{\mathscr{G}}$ is finite, thus smooth, then the strategy of the previous paragraph can be used to produce a Borel solution to the associated marriage problem. So from the point of view of understanding the Borel marriage problem, I might as well restrict my attention to the case that $\mathscr{G} = \mathscr{L}$ is a Borel forest of lines.

Proposition 4.10. *Suppose that X is a Polish space, $\mathscr{L}$ is a Borel forest of lines on X, M, W partition X into Borel $\mathscr{L}$-discrete sets. Then $(\mathscr{L}, M, W)$ has a solvable Borel marriage problem $\Leftrightarrow$ $\mathscr{L}$ is directable.*

Proof. To see $(\Rightarrow)$, suppose that f is a solution to the Borel marriage problem for $\mathscr{L}$, extend f to a Borel automorphism $g : X \to X$ by letting $g(w)$ be the unique $\mathscr{L}$-neighbor of w other than $f^{-1}(w)$ for $w \in W$, and observe that g induces $\mathscr{L}$. To see $(\Leftarrow)$, suppose that f is a Borel automorphism inducing $\mathscr{L}$, and note that $f|M$ is a solution to the Borel marriage problem of $\mathscr{G}$. ⊣

This immediately yields an alternative solution to the problem of Laczkovich [59]:

Theorem 4.11. *There is a Borel graph of vertex degree 2 on a Polish space and a partition of the underlying space into discrete Borel sets such that the associated Borel marriage problem is unsolvable.*

Proof. The idea is to combine Propositions 4.1 and 4.10. There are several ways of doing this. One way is to find a Borel maximal $(\mathscr{L}_0|\mathscr{C}_0)$-discrete set M, put $W = i(M)$, and set $\mathscr{G} = (\mathscr{L}_0)_{M \cup W}$. Another is to set $M = \mathscr{C}_0$ and $W = \mathscr{L}_0|\mathscr{C}_0$, and put $(x, (y, z)) \in \mathscr{G}$ whenever $x \in \{y, z\}$. ⊣

Laczkovich [59] actually shows that for his example, there is not even a solution to the Lebesgue-measurable marriage problem. Of course the same thing is true for the modifications of $\mathscr{L}_0$ mentioned above, as can be seen by repeating the proof of Proposition 4.1 with (the appropriate modification of) Lebesgue measure in place of Baire category (as it stands, the proof there shows that there is no solution to the Baire-measurable marriage problem). More generally, we have the following:

Theorem 4.12. *Suppose that X is a Polish space, $\mathscr{G} \subseteq X^2$ is a bipartite Borel graph of vertex degree two, and $M, W \subseteq X$ are Borel sets which partition X. If $(\mathscr{G}, M, W)$ has a solvable universally measurable marriage problem, then $(\mathscr{G}, M, W)$ has a solvable Borel marriage problem.*

Proof. Suppose, towards a contradiction, that there is a solution to the universally measurable marriage problem for $(\mathscr{G}, M, W)$, but that the Borel marriage problem is unsolvable. Then there is a universally measurable function f which induces $\mathscr{G}$ and a betweenness-preserving Borel embedding of $\mathscr{L}_0$ into $\mathscr{G}$, and by pulling back f through this embedding, one obtains a universally measurable function g which induces $\mathscr{L}_0$. But there is no such function. ⊣

From this, one can conclude that whenever a marriage problem admits a sufficiently definable solution, then it admits a Borel solution. For instance, under projective determinacy it follows that any Borel marriage problem with a projective solution has a Borel solution.

We will close this section with one final application of Theorem 4.3. Suppose that $(\mathscr{L}, X, Y)$ is a Borel marriage problem, where $\mathscr{L}$ is a Borel forest of lines on $X \cup Y$. Set $S = \mathscr{L} \cap (X \times Y)$, and define $h, v : S \to S$ by

$$h(x, y) = (x', y) \text{ and } v(x, y) = (x, y'),$$

where x', y' are the unique $\mathscr{L}$-neighbors of y, x other than x, y. Let $\mathbb{G} = \langle h, v \rangle$ be the group generated by h, v, and let $\mathbb{G}_0 = \langle hv \rangle$ be the group generated by hv. In Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [58], it is shown that if $(\mathscr{L}, X, Y)$ has a Borel solution, then there is no $\mathbb{G}$-quasi-invariant probability measure which is $\mathbb{G}_0$-ergodic, and it is asked if the converse holds.

We can now give their question an affirmative answer. Define $\mathscr{L}' \subseteq S^2$ by

$$\mathscr{L}' = \mathrm{graph}(h) \cup \mathrm{graph}(v),$$

noting that $\mathscr{L}'$ is a Borel forest of lines on S. As in Proposition 4.10, it is straightforward to check that the existence of a Borel solution to $(\mathscr{L}, X, Y)$ is equivalent to the directability of $\mathscr{L}'$. Now note that the function $f_1 = v$ has a unique extension to a local director $\mathfrak{F}'$ of $\mathscr{L}'$, and the corresponding function $f'_{\mathfrak{F}}$ is simply hv. It follows that if μ is a probability measure on S, then

$$\mu \text{ is } \mathbb{G}\text{-quasi-invariant, } \mathbb{G}_0\text{-ergodic} \Leftrightarrow \mu \text{ is } E_{\mathscr{L}'}\text{-quasi-invariant, } E_{\mathfrak{F}'}\text{-ergodic.}$$

So it is certainly enough to show the following more general fact:

Theorem 4.13. *Suppose X is a Polish space, $\mathscr{L}$ is a Borel forest of lines on X, and $\mathfrak{F} = \langle f_n \rangle_{n \in \mathbb{Z}}$ is a local director for $\mathscr{L}$ such that*

$$\forall x \in X \ ([x]_{E_{\mathfrak{F}}} \subsetneq [x]_{E_{\mathscr{L}}}).$$

Then exactly one of the following holds:

1. *$\mathscr{L}$ is directable.*

2. *There is an $E_{\mathscr{L}}$-quasi-invariant, $E_{\mathfrak{F}}$-ergodic probability measure.*

Proof. We will begin with (1) $\Rightarrow \neg$(2). Suppose that $g : X \to X$ is a Borel automorphism which induces $\mathscr{L}$, and observe that

$$B = \{x \in X : g(x) = f_1(x)\}$$

is a Borel $E_{\mathfrak{F}}$-invariant $E_{\mathscr{L}}$-complete, co-complete section. It follows that there are no $E_{\mathscr{L}}$-quasi-invariant, $E_{\mathfrak{F}}$-ergodic probability measures.

It remains to show $\neg$(1) $\Rightarrow$ (2). Suppose that $\mathscr{L}$ is not directable, and apply Theorem 4.3 to find a betweenness-preserving Borel embedding π of $\mathscr{L}_0$ into $\mathscr{L}$. Let μ be the Lebesgue measure on $\mathscr{C}$, and define ν on X by

$$\nu(B) = \sum_{n \in \mathbb{Z}} \mu(\pi^{-1} \circ f_{\mathfrak{F}}^n(B))/2^{n+1}.$$

It is easily verified that ν is $E_{\mathfrak{F}}$-quasi-invariant and $E_{\mathfrak{F}}$-ergodic. Suppose, towards a contradiction, that ν is not $E_{\mathscr{L}}$-quasi-invariant, and find a Borel set $B \subseteq X$ with

$$\nu(B) = 0 \text{ and } \nu([B]_{E_{\mathscr{L}}}) > 0.$$

By the definition of ν, we may assume that B if $E_{\mathfrak{F}}$-invariant. By throwing away a null $E_{\mathscr{L}}$-invariant Borel set, we may assume that B intersects exactly one $E_{\mathfrak{F}}$-class within every $E_{\mathscr{L}}$-class within $[B]_{E_{\mathscr{L}}}$. It follows that $\mathscr{L}|[B]_{E_{\mathscr{L}}}$ is directable. Setting

$$A = \pi^{-1}(B),$$

it follows that $\mathscr{L}_0|[A]_{E_0}$ is also directable. As $\mu([A]_{E_0}) > 0$, this contradicts the fact that $\mathscr{L}_0$ is not directable on any Borel set of positive measure. $\dashv$

5 More on betweenness and directability

In this section, we will provide strengthenings of the following:

Proposition 5.1. *Suppose that X and Y are Polish spaces and $\mathscr{L}$ and $\mathscr{L}'$ are Borel forests of lines on X and Y.*

1. *If $\mathscr{L}$ is directable and $\mathscr{L}'$ is undirectable, then $\mathscr{L}$ betweenness-preservingly embeds into $\mathscr{L}'$.*

2. If $\mathscr{L}'$ is non-smooth, then there is a Borel set on which $\mathscr{L}'$ is non-smooth and directable.

Proof. Of course it is enough to show (2) when $\mathscr{L}'$ is undirectable. Let $\mathscr{L}_\sigma$ be the Borel forest of lines which is induced by the odometer. It is easily verified that the function $\pi : \mathscr{C} \to \mathscr{C}$, given by $\pi(x) = x_0 0 x_1 0 \ldots$, is a betweenness-preserving embedding of $\mathscr{L}_\sigma$ into $\mathscr{L}_0$. By Theorems 3.3 and 4.3, there are betweenness-preserving Borel embeddings π', π'' of $\mathscr{L}$ into $\mathscr{L}_\sigma$, $\mathscr{L}_0$ into $\mathscr{L}'$.

To see (1), simply note that $\pi'' \circ \pi \circ \pi'$ is a betweenness-preserving embedding of $\mathscr{L}$ into $\mathscr{L}'$. To see (2), observe that $\pi'' \circ \pi(\mathscr{C})$ is a Borel set on which $\mathscr{L}'$ is directable but non-smooth. $\dashv$

It will be convenient to have an alternative description of $\mathscr{L}_0'$ in terms of linear orderings of 2^n. For $x, y \in 2^{n+1}$, put

$$x \leq_0' y \Leftrightarrow x = y \text{ or } x_n < y_n \text{ or } (x_n = y_n \text{ and } (x|n \leq_0' y|n \Leftrightarrow x_n = 0)).$$

Letting $n(x,y) \leq n$ be greatest such that $x_{n(x,y)} \neq y_{n(x,y)}$, it is easily verified that

$$x \leq_0' y \Leftrightarrow \left[x = y \text{ or } \left(x_{n(x,y)} < y_{n(x,y)} \Leftrightarrow \sum_{n(x,y) < i \leq n} x_i \equiv 0 \pmod 2 \right) \right].$$

Proposition 5.2. *Suppose $x, y, z \in \mathscr{C}_0'$ and $x E_0 y E_0 z$. The following are equivalent:*

1. y is $\mathscr{L}_0'$-between x, z.

2. $\forall^\infty n \in \mathbb{N}$ ($y|n$ is $\leq_0'$-between $x|n, z|n$).

Proof. Clearly we may assume that x, y, z are not all equal. Let $n(x,y,z)$ be the greatest digit on which x, y, z do not all agree. Then y is $\mathscr{L}_0'$-between x, z exactly when $y|n$ appears between $x|n, z|n$ in the sequence

$$0^n, i'(0^n), j' \circ i'(0^n), \ldots, i' \circ (j' \circ i')^{2^{n-1}-1}(0^n),$$

for every $n > n(x,y,z)$. Thus, it is enough to check that for each n,

$$0^{n+1}, i'(0^{n+1}), j' \circ i'(0^{n+1}), \ldots, i' \circ (j' \circ i')^{2^n - 1}(0^{n+1})$$

is $\leq_0'$-increasing. This follows from the obvious induction, the fact that

$$\begin{aligned} i' \circ (j' \circ i')^{2^{n-1}-1}(0^{n+1}) &= 0^{n-1}10 \leq_0' 0^{n-1}11 \\ &= (j' \circ i')^{2^{n-1}}(0^{n+1}), \end{aligned}$$

and the proof of Proposition 4.8. ⊣

The following fact is the analog of Theorem 3.3:

Theorem 5.3 (Miller-Rosendal). *Suppose that X is a Polish space and $\mathscr{L}$ is a Borel forest of lines on X. Then $\mathscr{L}$ betweenness-preservingly embeds into $\mathscr{L}_0$.*

Proof. It is enough to show that every Borel forest of lines $\mathscr{L}$ on $X = \mathscr{C}$ betweenness-preservingly Borel embeds into $\mathscr{L}_0'$. Let $\mathfrak{F} = \langle f_n \rangle_{n \in \mathbb{Z}}$ be a local director for $\mathscr{L}$. Given a recurrent set $B \subseteq X$, put

$$d_B^-(x) = \min\{n \in \mathbb{N} : f_{-n}(x) \in B\} \text{ and } d_B^+(x) = \min\{n \in \mathbb{N} : f_n(x) \in B\}.$$

Fix a decreasing, vanishing sequence of Borel complete sections $B_n \subseteq X$ such that

$$\forall n \in \mathbb{N} \, \forall x \in X \, \left(d_{B_n}^-(x), d_{B_n}^+(x) < 2^{n-1}\right).$$

Fix a Borel linear ordering $\preceq$ of X, define $g_n : X \to X$ by

$$g_n(x) = \min_{\preceq}(f_{-d_{B_n}^-(x)}(x), f_{d_{B_n}^+(x)}(x)),$$

put

$$j_n(x) = \text{the unique } j \in \mathbb{Z} \text{ such that } f_j \circ g_n(x) = x,$$

and set $k_n(x) = j_n(x) + 2^{n-1}$, noting that $0 \leq k_n(x) < 2^n$.

Now define $\varphi_n : X \to 2^{n \cdot 2^n}$ by

$$\varphi_n(x) = \bigoplus_{-2^{n-1} \leq m < 2^{n-1}} f_m \circ g_n(x) | n,$$

define $\psi_n : X \to 2^{n+1}$ by $\psi_n(x) = b_{n+1}(k_{n+1} \circ g_n(x))$, where

$$b_n(m) = \text{ the } m^{\text{th}} \text{ element of } \leq_0' | 2^n,$$

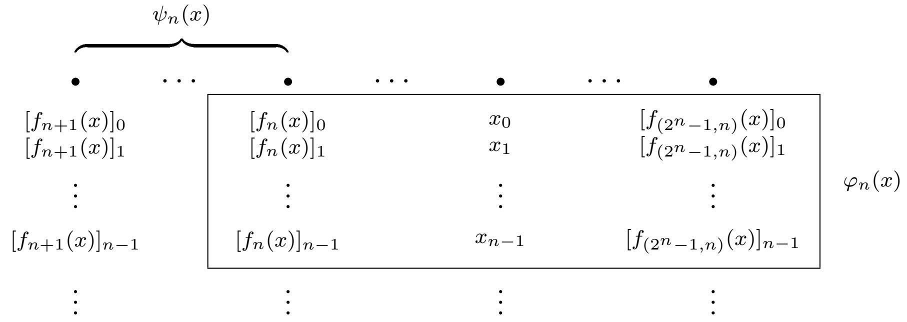

Figure 2.10: φ_n approximates $[x]_f$ and ψ_n specifies how to put the approximations together.

and define $\xi_n : X \to \{0,1\}$ by

$$\xi_n(x) = \begin{cases} 0 & \text{if } g_n(x) E_{\mathfrak{F}} g_{n+1}(x), \\ 1 & \text{otherwise.} \end{cases}$$

We claim that

$$\pi(x) = \bigoplus_{n \in \mathbb{N}} \langle \xi_n(x) \rangle \oplus \varphi_n(x) \oplus \varphi_n(x) \oplus \psi_n(x) \oplus \psi_n(x)$$

is the desired reduction, or equivalently, that

$$\forall x, y, z \in X \ \big(y \text{ is } \mathscr{L}\text{-between } x, z \Leftrightarrow \pi(y) \text{ is } \mathscr{L}_0'\text{-between } \pi(x), \pi(z)\big).$$

For each $n \in \mathbb{N}$, let $n' \in \mathbb{N}$ be the unique natural number such that for all $x \in X$,

$$\pi(x)|n' = \bigoplus_{m<n} \langle \xi_n(x) \rangle \oplus \varphi_n(x) \oplus \varphi_n(x) \oplus \psi_n(x) \oplus \psi_n(x).$$

Also, define

$$x E_n y \Leftrightarrow g_n(x) = g_n(y) \text{ and } x <_n y \Leftrightarrow \big(x E_n y \text{ and } k_n(x) < k_n(y)\big),$$

noting that

$$\begin{aligned} x E_{\mathscr{L}} y &\Leftrightarrow \exists n \in \mathbb{N} \ (x E_n y) \\ &\Leftrightarrow \forall^\infty n \in \mathbb{N} \ (x E_n y), \end{aligned}$$

for all $x, y \in X$. We will show the following:

1. $\forall n \in \mathbb{N}\, \forall x, y \in X\ \big(x <_n y \Rightarrow \pi(x)|n' <_0' \pi(y)|n'\big)$.

2. $\forall x, y \in X\ \big(\pi(x) E_0 \pi(y) \Rightarrow x E_{\mathscr{L}} y\big)$.

To see that this is sufficient, note that (1) and (2) together imply that

$$\forall n \in \mathbb{N}\, \forall x, y \in E_n\ \big(x \leq_n y \Leftrightarrow \pi(x)|n' \leq_0' \pi(y)|n'\big).$$

As the range of π is clearly contained in $\mathscr{C}_0'$, it follows that for all $x, y, z \in X$,

$$\begin{aligned} y \text{ is } \mathscr{L}\text{-between } x, z &\Leftrightarrow \forall^\infty n \in \mathbb{N}\ (y \text{ is } \leq_n\text{-between } x, z) \\ &\Leftrightarrow \pi(x) E_0 \pi(y) E_0 \pi(z) \text{ and} \\ &\ \forall^\infty n \in \mathbb{N}\ (\pi(y)|n' \text{ is } \leq_0'\text{-between } \pi(x)|n', \pi(z)|n') \\ &\Leftrightarrow \pi(x) E_0 \pi(y) E_0 \pi(z) \text{ and} \\ &\ \forall^\infty n \in \mathbb{N}\ (\pi(y)|n \text{ is } \leq_0'\text{-between } \pi(x)|n, \pi(z)|n) \\ &\Leftrightarrow \pi(y) \text{ is } \mathscr{L}_0'\text{-between } \pi(x), \pi(z). \end{aligned}$$

To see (1), suppose $x \neq y$ lie in the same connected component of $\mathscr{L}$ and set

$$n(x, y) = \max\{n \in \mathbb{N} : g_n(x) \neq g_n(y)\}.$$

By reversing the roles of x, y if necessary, we may assume that $x <_{n(x,y)+1} y$. As

$$k_{n(x,y)+1}(x) < k_{n(x,y)+1}(y) \Rightarrow k_{n(x,y)+1} \circ g_{n(x,y)}(x) < k_{n(x,y)+1} \circ g_{n(x,y)}(y),$$

it follows that $\psi_{n(x,y)}(x) <_0' \psi_{n(x,y)}(y)$, thus

$$\pi(x)|(n(x, y) + 1)' <_0' \pi(y)|(n(x, y) + 1)'.$$

Now note that if $m \geq n$ and $x <_n y$, then

$$x <_{n+1} y \Leftrightarrow \xi_n(x) = 0.$$

Similarly, if $m \geq n$ and $\pi(x)|n' <_0' \pi(y)|n'$, then

$$\pi(x)|(n + 1)' <_0' \pi(y)|(n + 1)' \Leftrightarrow \xi_n(x) = 0.$$

A simple inductive argument now gives (1).

To see (2), suppose that $\pi(x) E_0 \pi(y)$, fix $n \in \mathbb{N}$ sufficiently large that

$$\forall m \geq n \ \big(\langle \xi_m(x) \rangle \oplus \varphi_m(x) \oplus \psi_m(x) = \langle \xi_m(y) \rangle \oplus \varphi_m(y) \oplus \psi_m(y)\big),$$

and set $\ell_m = k_m(x) - k_m(y)$, noting that for all $m \geq n$,

$$\ell_m = (-1)^{\sum_{n \leq i < m} \xi_i(x)} \ell_n.$$

Identifying $\varphi_m(x), \varphi_m(y)$ with the corresponding elements of $(2^m)^{2^m}$, it follows that

$$\begin{aligned}
x|m &= f_{k_m(x)} \circ g_m(x)|m \\
&= (\varphi_m(x))_{k_m(x)} \\
&= (\varphi_m(y))_{k_m(y)+\ell_m} \\
&= f_{k_m(y)+\ell_m} \circ g_m(y)|m \\
&= f_{k_m(y)+(-1)^{n \leq i < m}\xi_i(x)} \circ g_m(y)|m \\
&= \begin{cases} f_{k_m(y)+\ell_n} \circ g_m(y)|m & \text{if } g_m(y) E_{\mathfrak{F}} g_n(y), \\ f_{k_m(y)-\ell_n} \circ g_m(y)|m & \text{otherwise,} \end{cases} \\
&= f_{k_n(y)+\ell_n} \circ g_n(y)|m,
\end{aligned}$$

thus $x = f_{k_n(y)+\ell_n} \circ g_n(y)$, so $x E_{\mathscr{L}} y$. $\dashv$

There is an analog of Kakutani equivalence in this setting. Two Borel forests of lines $\mathscr{L}$ and $\mathscr{M}$ are *Kakutani equivalent* if there are Borel complete sections A and B such that $\mathscr{L}_A \cong \mathscr{M}_B$. A proof identical to that given in §3 shows that betweenness-preserving Borel bi-embeddability of lines implies Kakutani equivalence. In particular, this gives our strengthening of the first half of Proposition 5.1:

Theorem 5.4. *Up to Kakutani equivalence, there are exactly three Borel forests of lines on Polish spaces. In order of betweenness-preserving Borel embeddability, these are: those which are induced by smooth Borel automorphisms, those which are induced by non-smooth Borel automorphisms, and those which are not induced by Borel automorphisms at all.*

Proof. Simply combine Theorems 4.3 and 5.3 with Proposition 5.1. $\dashv$

Various issues surrounding the proof of Theorem 5.3 lead to another natural forest of lines, which I will mention briefly here. Put $\mathscr{C}_0'' = \mathscr{C}^2 \setminus E_0$, define

$$i''(x, y) = (y, x) \text{ and } j''(x, y) = (\sigma^{-1}(y), \sigma(x)),$$

and set $\mathscr{L}_0'' = \operatorname{graph}(i'') \cup \operatorname{graph}(j'')$.

Proposition 5.5. $\mathscr{L}_0'' | \mathscr{C}_0'' \cong_B \mathscr{L}_0 | \mathscr{C}_0 \times \Delta(\mathscr{C})$.

Proof. Let $+$ denote "addition with right carry" on $\mathscr{C}$. Clearly $(\mathscr{C}, +)$ is an abelian group with identity 0^∞. We will use $\cdot$ to denote multiplication,

$$x \cdot y = \sum_{y_n = 1} 0^n x_0 x_1 \ldots,$$

and we will use 1 to denote 10^∞, the multiplicative identity.

One should note that i, j have simple representations in terms of this notation:

$$i(x) = \overline{x} = -x - 1 \text{ and } j(x) = \sigma \circ i(x) = -x.$$

Define

$$\mathscr{C}_\alpha = \mathscr{C} \setminus [\alpha_1 \alpha_2 \ldots]_\sigma,$$

where $[x]_\sigma$ denotes the orbit of x under σ, define i_α, j_α on $\mathscr{C}_\alpha$ by

$$i_\alpha(x) = \alpha - x - 1 \text{ and } j_\alpha(x) = \alpha - x,$$

and put $\mathscr{L}_\alpha = \operatorname{graph}(i_\alpha) \cup \operatorname{graph}(j_\alpha)$. Letting

$$\mathscr{L} = \{((\alpha, x), (\alpha, y)) : (x, y) \in \mathscr{L}_\alpha\},$$

it follows that the map $(x, y) \mapsto (x + y, x)$ is an isomorphism of $\mathscr{L}_0''$ with $\mathscr{L}$. So it only remains to construct an isomorphism of $\mathscr{L}_0 \times \Delta(\mathscr{C})$ with $\mathscr{L}$, and for this it is enough to provide a uniform-in-α collection of isomorphisms from $\mathscr{L}_0$ to $\mathscr{L}_\alpha$. Fix $\alpha \in \mathscr{C}$, define

$$\pi_\alpha(x) = (-1)^{\alpha_0} x + \alpha_1 \alpha_2 \ldots,$$

and note that

$$\begin{aligned}
\pi_\alpha(\mathrm{graph}(i)) &= \{((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots, (-1)^{\alpha_0}(-x-1) + \alpha_1\alpha_2\ldots)\}_{x\in\mathscr{C}_0} \\
&= \{((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots, \\
&\qquad -((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots) + (-1)^{\overline{\alpha_0}} + 0\alpha_1\alpha_2\ldots)\big\}_{x\in\mathscr{C}_0} \\
&= \big\{(x, -x + (-1)^{\overline{\alpha_0}} + 0\alpha_1\alpha_2\ldots)\big\}_{x\in\mathscr{C}_\alpha} \\
&= \begin{cases} \mathrm{graph}(i_\alpha) & \text{if } \alpha_0 = 0, \\ \mathrm{graph}(j_\alpha) & \text{if } \alpha_0 = 1, \end{cases}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\pi_\alpha(\mathrm{graph}(j)) &= \{((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots, (-1)^{\alpha_0}(-x) + \alpha_1\alpha_2\ldots)\}_{x\in\mathscr{C}_0} \\
&= \{((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots, \\
&\qquad -((-1)^{\alpha_0}x + \alpha_1\alpha_2\ldots) + 0\alpha_1\alpha_2\ldots)\}_{x\in\mathscr{C}_0} \\
&= \{(x, -x + 0\alpha_1\alpha_2\ldots)\}_{x\in\mathscr{C}_\alpha} \\
&= \begin{cases} \mathrm{graph}(j_\alpha) & \text{if } \alpha_0 = 0, \\ \mathrm{graph}(i_\alpha) & \text{if } \alpha_0 = 1, \end{cases}
\end{aligned}$$

thus π_α is an isomorphism of $\mathscr{L}_0$ with $\mathscr{L}_\alpha$, and it follows that

$$\pi(x, y) = (\pi_{x+y}^{-1}(x), x + y)$$

is an isomorphism of $\mathscr{L}_0''$ with $\mathscr{L}_0 \times \Delta(\mathscr{C})$. $\dashv$

Remark 5.6. In the above proof, it was necessary for some of the π_α's to send (i, j) to (j_α, i_α). In fact, there is a Borel isomorphism sending (i_α, j_α) to (i_β, j_β) exactly when $\alpha_0 = \beta_0$. To see this, it is enough to show that there is no Borel isomorphism of (i, j) with (i_1, j_1). Suppose, towards a contradiction, that π is such a map. As $j_\alpha \circ i_\alpha = j_\beta \circ i_\beta = \sigma$, it follows that π must carry σ to σ, i.e., π and σ must commute. But then, after throwing away an invariant Borel meager set, $\pi(x) = x + \gamma$ for some fixed $\gamma \in \mathscr{C}$, and no such map can carry i to any i_α with $\alpha_0 = 1$.

Next, we will turn to a strengthening of the second half of Proposition 5.1. Given countable Borel equivalence relations $E_1 \subseteq E_2$ on X, we will say that E_2/E_1 is *smooth* if the underlying space X can be covered with countably many E_1-invariant Borel set B_n which are *partial transversals of* E_2/E_1, i.e., which intersect at most 1 E_1-class within every E_2-class. Note that when E_2 is *countably generated over* E_1, i.e., when there are countably many E_1-invariant Borel functions f_n which together with E_1 generate E_2, then the smoothness of E_2/E_1 is equivalent to the existence of an E_1-invariant Borel set B which gives a transversal of E_2/E_1. In the same way that one may associate a σ-ideal $\mathscr{I}(E) = \{B : E|B \text{ is smooth}\}$ with every Borel equivalence relation, one may associate a σ-ideal

$$\mathscr{I}(E_2/E_1) = \{B : (E_2|B)/(E_1|B) \text{ is smooth}\}$$

with every pair of countable Borel equivalence relations $E_1 \subseteq E_2$.

Theorem 5.7. *Suppose that X is a Polish space, $E_1 \subseteq E_2$ are countable Borel equivalence relations on X, $[E_2 : E_1] < \aleph_0$, and E_2/E_1 is non-smooth. Then $\mathscr{I}(E_2/E_1) \not\subseteq \mathscr{I}(E)$, for any non-smooth Borel equivalence relation E.*

Proof. Let $\mathscr{J}(E_2/E_1)$ be the σ-ideal of Borel sets B for which $(E_2|B)/(E_1|B)$ admits a Borel transversal. We will show the apparently weaker fact that

$$\mathscr{J}(E_2/E_1) \not\subseteq \mathscr{I}(E),$$

for any non-smooth Borel equivalence relation E. To see that this is enough, assume without loss of generality that $[E_2 : E_1] = n$, find a sequence of Borel functions f_i, with $f_0 = \text{id}$, such that

$$\forall x \left([x]_{E_2} = \bigcup_{i<n} [f_i(x)]_{E_1} \right),$$

and put $xE_1'y \Leftrightarrow \forall i < n \ (f_i(x)E_1f_i(y))$. Clearly $\mathscr{J}(E_2/E_1') \subseteq \mathscr{I}(E_2/E_1)$, so if $\mathscr{J}(E_2/E_1') \not\subseteq \mathscr{I}(E)$, then $\mathscr{I}(E_2/E_1) \not\subseteq \mathscr{I}(E)$ as well.

Now suppose, towards a contradiction, that E is a Borel equivalence relation with the property that whenever $E|B$ is non-smooth, $(E_2|B)/(E_1|B)$ has no Borel transversal. By [44], there is a Borel set B such that $E|B$ is non-smooth and hyperfinite. By replacing E with $E|B$, I may assume that E is aperiodic and hyperfinite. I will perform the Glimm-Effros style embedding of E_0 into E from [79], along the way ensuring that the image of $\mathscr{C}_0$ intersects at most one E_1-class within every E_2-class, thus lies within $\mathscr{J}(E_2/E_1) \setminus \mathscr{I}(E)$, the desired contradiction.

Let f be a Borel automorphism which generates E, put $f_n = f^n$, fix a countable collection of Borel automorphisms g_n, the union of whose graphs is E_2, let τ be a zero-dimensional Polish topology on X, finer than that with which X was originally endowed but compatible with the underlying Borel structure of X, in which each $\{x : f_m(x) E_i f_n(x)\}$ is clopen, the support of any composition of the f's and g_n's is clopen, and f and the g_n's are homeomorphisms, and let d be a Polish metric compatible with τ.

I will recursively choose natural numbers k_n and $\mathscr{I}(E)$-positive, τ-clopen sets B_n which satisfy several additional properties. Letting $f_s = f_{k_0}^{s_0} \circ \cdots \circ f_{k_n}^{s_n}$, for $s \in 2^{n+1}$, and

$$\mathscr{H}_n = \{\mathrm{id}\} \cup \{f_i\}_{|i| \leq n} \cup \{g_i\}_{i \leq n},$$

these properties are as follows:

1. $\forall x \in B_{n+1} \left(x, f_{k_n}(x) \in B_n \text{ and } \forall s, t \in 2^n \forall h \in \mathscr{H}_n \ (f_s(x) \neq h \circ f_t \circ f_{k_n}(x))\right)$,
2. $\forall x \in B_{n+1} \forall s, t \in 2^{n+1} \left((f_s(x), f_t(x)) \notin E_2 \setminus E_1\right)$, and
3. $\forall s \in 2^{n+1} \left(f_s(B_{n+1}) \text{ is of diameter } < 1/(n+1)\right)$.

The recursion begins by setting $B_0 = X$. Now suppose I have built $\{B_m\}_{m \leq n}$ and $\{k_m\}_{m<n}$. I claim that for some natural number k, the τ-clopen set A_k of all x such that

1′. $\left(x, f_k(x) \in B_n \text{ and } \forall s, t \in 2^n \forall h \in \mathscr{H}_n \ (f_s(x) \neq h \circ f_t \circ f_k(x)\right)$, and

2′. $\forall s, t \in 2^n \left((f_s(x), f_t \circ f_k(x)) \notin E_2 \setminus E_1\right)$,

is $\mathscr{I}(E)$-positive. To see this, it suffices to check that $B = B_n \setminus \bigcup [A_k]_E$ is $\mathscr{I}(E)$-null. It follows from the definition of B that

$$\forall x \in B \forall^\infty y \in B \cap [x]_E \exists s, t \in 2^n \left((f_s(x), f_t(y)) \in E_2 \setminus E_1\right). \tag{$*$}$$

I will show that this implies that B intersects every E-class C in only finitely many points, which of course implies that $B \in \mathscr{I}(E)$.

Let $\langle C_i \rangle$ be an enumeration of C/E_1, let $\langle x_i \rangle$ be an enumeration of $B \cap C$, set

$$T_i = \{j \in \mathbb{N} : \exists s \in 2^n \ (f_s(x_i) \in C_j)\},$$

and define a finite equivalence relation F on $\mathbb{N}$, by putting

$$iFj \Leftrightarrow C_i, C_j \text{ lie within the same } E_2\text{-class.}$$

Clearly $\langle T_i \rangle$ is a sequence of partial transversals of F, and it follows from $(*)$ that the union along any infinite subsequence is not a partial transversal. So it only remains to note the following:

Lemma 5.8. *Suppose that F is a finite equivalence relation on $\mathbb{N}$ and $\langle T_i \rangle_{i \in \mathbb{N}}$ is a sequence of partial transversals of F of bounded cardinality. Then there is an infinite set $N \subseteq \mathbb{N}$ such that $\bigcup_{i \in N} T_i$ is a partial transversal of F.*

Proof. By induction on $k = \max_i |S_i|$. The case $k = 1$ is trivial, so suppose that we have shown the lemma below k. Let $\langle D_i \rangle$ be an enumeration of the equivalence classes of F, put $D'_i = \bigcup_{j \leq i} D_j$, and note that if

$$\forall i \forall^\infty j \ (T_j \cap D'_i = \emptyset),$$

then N can be easily built: set $i_0 = 0$, and given i_n, fix i sufficiently large that $T_{i_0}, \ldots, T_{i_n} \subseteq D'_i$ and choose i_{n+1} such that $T_{i_{n+1}} \cap D'_i = \emptyset$. Clearly $N = \{i_n\}_{n \in \mathbb{N}}$ is as desired.

So we may assume that for some i, there are infinitely many j with $T_j \cap D'_i \neq \emptyset$. By passing to an infinite subsequence, I may assume that this is true for all j, and moreover, that this intersection is independent of j (here is where we use that F is finite). Now it follows from the induction hypothesis that by passing to an infinite subsequence once more, I may assume that $\bigcup S_j \setminus D'_i$ is a partial transversal of F, thus so too is $\bigcup S_j$. $\dashv$

So fix such a $k = k_n$, and note that, by the τ-continuity of the f_i's and g_i's, each $x \in A_{k_n}$ has a τ-clopen neighborhood U such that $f_s(U)$ is of diameter $< 1/(n+1)$, for each $s \in 2^n$. It follows that there is a partition of A_{k_n} into countably many such sets. Let B_{n+1} be any $\mathscr{I}(E)$-positive set from such a partition.

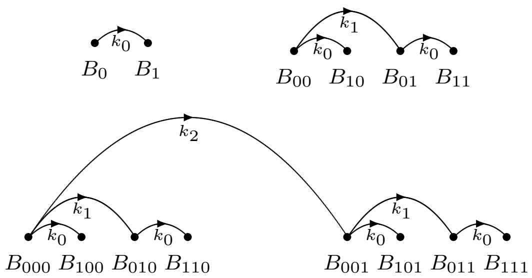

Figure 2.11: The first three stages of the construction of $\pi : \mathscr{C}_0 \hookrightarrow X$.

Set $B_\emptyset = X$, and for $s \in 2^{n+1}$, define

$$B_s = f_s(B_{n+1}) = f_{k_0}^{s_0} \circ f_{k_1}^{s_1} \circ \cdots \circ f_{k_n}^{s_n}(B_{n+1}).$$

Then, for each $x \in 2^{\mathbb{N}}$, $\{B_{x|n}\}$ is a decreasing sequence of τ-clopen sets of vanishing diameter, and it follows that $\bigcap B_{x|n}$ consists of a unique element $\pi(x)$.

It is clear that $\pi : \mathscr{C}_0 \to X$ is continuous, so it only remains to check that π is an embedding of $E_0|\mathscr{C}_0$ into E, the image of which consists of at most one E_1-class within any E_2-class. For this, it is enough to check that

(a) $\forall x, y\ ((x, y) \notin E_0 \Rightarrow (\pi(x), \pi(y)) \notin E \cup E_2)$, and

(b) $\forall x \forall s, t\ (\pi(sx), \pi(tx)) \in E \setminus (E_2 \setminus E_1)$.

To see (a), simply note that if $x_n \neq y_n$, then $\pi(x) \neq h \circ \pi(y)$ for any $h \in \mathscr{H}_n$, by (1). Thus, if x, y disagree on infinitely many coordinates, then $\pi(x) \neq h \circ \pi(y)$ for any $h \in \bigcup \mathscr{H}_n$, and then (a) follows from the fact that $E \cup E_2 = \bigcup_{h \in \bigcup \mathscr{H}_n} \operatorname{graph}(h)$.

To see (b), simply note that

$$\begin{aligned}
\{\pi(sx)\} &= \bigcap_{m>n} f_{k_0}^{s_0} \circ \cdots \circ f_{k_n}^{s_n} \circ f_{k_{n+1}}^{x_0} \circ \cdots \circ f_{k_m}^{x_{m-n-1}}(B_{m+1}) \\
&= f_{k_0}^{s_0} \circ \cdots \circ f_{k_n}^{s_n} \left(\bigcap_{m>n} f_{k_{n+1}}^{x_0} \circ \cdots \circ f_{k_m}^{x_{m-n-1}}(B_{m+1}) \right) \\
&= \left\{ f_{k_0}^{s_0} \circ \cdots \circ f_{k_n}^{s_n} \circ \pi(0^{n+1}x) \right\},
\end{aligned}$$

and similarly $\pi(tx) = f_{k_0}^{t_0} \circ \cdots \circ f_{k_n}^{t_n} \circ \pi(0^{n+1}x)$, thus $(\pi(sx), \pi(tx)) \in E \setminus (E_2 \setminus E_1)$. $\dashv$

Corollary 5.9. *Suppose that X is a Polish space and $\mathscr{L}$ is an undirectable Borel forest of lines on X. Then the directability σ-ideal of $\mathscr{L}$ is not the smoothness σ-ideal of any Borel equivalence relation on X.*

Proof. Let $\mathfrak{F}$ be a local director for $\mathscr{L}$, recall that the directability σ-ideal induced by $\mathscr{L}$ is exactly $\mathscr{I}(E_{\mathscr{L}}/E_{\mathfrak{F}})$, and apply Theorem 5.7. $\dashv$

6 Ergodic equivalence relations on quotients

Suppose that $F_1, \ldots, F_n$ are countable Borel equivalence relations on Polish spaces $X_1, \ldots, X_n$. Although the *$\prod_i F_i$-saturation of $B \subseteq \prod_i X_i$* is given by

$$[B]_{\prod_i F_i} = \{([x_1]_{F_1}, \ldots, [x_n]_{F_n}) : (x_1, \ldots, x_n) \in B\},$$

we will also use this notation to denote the corresponding subset of $\prod_i X_i / \prod_i F_i$. The *quotient Borel structure on $\prod_i X_i/F_i$* is given by

$$\mathscr{B}\Big(\prod_i X_i/F_i\Big) = \Big\{[B]_{\prod_i F_i} : B \subseteq \prod_i X_i \text{ is Borel}\Big\}.$$

When each F_i is smooth, the space $\prod_i X_i/F_i$ is standard Borel. On the other hand, as soon as one of the F_i's is non-smooth, the quotient Borel structure is neither countably generated nor is it generated by products of Borel rectangles.

A function $f : X_1/F_1 \to X_2/F_2$ is *Borel* if its graph,

$$\operatorname{graph}(f) = \{([x]_{F_1}, [y]_{F_2}) \in X_1 \times X_2/F_1 \times F_2 : f([x]_{F_1}) = [y]_{F_2}\}$$

is Borel. It follows from the Lusin-Novikov uniformization theorem (see 18.10 of Kechris [51]) that a countable-to-one function $f : X_1/F_1 \to X_2/F_2$ is Borel exactly when it has a *Borel lifting*, i.e., a Borel function $\tilde{f} : X_1 \to X_2$ such that $\operatorname{graph}(f) = [\operatorname{graph}(\tilde{f})]_{F_1 \times F_2}$, or equivalently, such that

$$\forall x \in X_1 \ \big(\tilde{f}(x) \in f([x]_{E_1})\big).$$

In fact, there is useful strengthening of this:

Proposition 6.1. *Suppose that X_1, X_2 are Polish spaces, F_1, F_2 are countable Borel equivalence relations on X_1, X_2, F_2 is aperiodic, and $f : X_1/F_1 \to X_2/F_2$ is a countable-to-one Borel function. Then f has a finite-to-one Borel lifting.*

Proof. By the Lusin-Novikov uniformization theorem (see 18.10 of Kechris [51]), there is a countable family of Borel partial injections $f_n : X_1 \to X_2$ whose domains partition X_1 with

$$\forall n \in \mathbb{N} \forall x \in \mathrm{dom}(f_n) \ \big(f_n(x) \in f([x]_{F_1})\big).$$

Let $n(x)$ be the unique natural number such that $x \in \mathrm{dom}(f_{n(x)})$.

Lemma 6.2. *Suppose that X is a Polish space and F is an aperiodic countable Borel equivalence relation on X. Then there is a partition of X into Borel complete sections C_n and a finite-to-one Borel function $\varphi : X \to X$ such that $\forall n \in \mathbb{N}$ $(\varphi(C_n) \subseteq C_{n+1})$ and $\forall x \in X$ $(\varphi(x) \in [x]_F)$.*

Proof. Note that it is enough to prove the lemma off of an F-invariant Borel set on which F is smooth, for it is clear how to proceed in the smooth case. By Feldman-Moore [36], we can find Borel involutions $i_n : X \to X$ such that

$$\forall n \in \mathbb{N} \left(F = \bigcup_{m \geq n} \mathrm{graph}(i_m) \right).$$

Now we will recursively define several sets and involutions. Put $A_0 = X$, and given A_n, define

$$B^n_m = (\mathrm{supp}(i_{m+n}) \cap A_n \cap i_{m+n}(A_n)) \setminus \bigcup_{\ell < m} B^n_\ell \cup i_{m+n}(B^n_\ell),$$

where $\mathrm{supp}(f) = \{x \in X : f(x) \neq x\}$ denotes the *support of f*. Set $B_n = \bigcup_{m \in \mathbb{N}} B^n_m$, and for each $x \in B_n$, let $m_n(x)$ denote the unique natural number such that $x \in B^n_{m_n(x)}$. Now define $j_n : B_n \to B_n$ by

$$j_n(x) = i_{m_n(x)+n}(x).$$

As each B^n_m is i_{m+n}-invariant, it follows that j_n is an involution. As $A_n \setminus B_n$ is a partial transversal for F, we may assume that $A_n = B_n$. Let C_n be a Borel transversal of the orbit equivalence relation induced by j_n, and put $A_{n+1} = j_n(C_n)$.

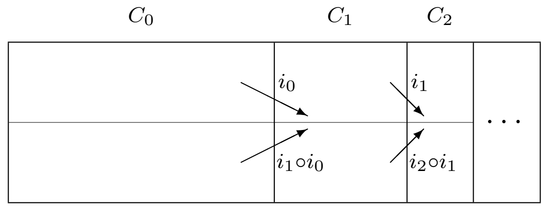

Figure 2.12: The construction of $\langle C_n \rangle_{n\in\mathbb{N}}$ and φ.

Noting that $X \setminus \bigcup_{n\in\mathbb{N}} C_n$ is a partial transversal of F, we may assume that $\langle C_n \rangle_{n\in\mathbb{N}}$ is a partition of X. Now define $\varphi_n : C_n \to C_{n+1}$ by

$$\varphi(x) = \begin{cases} i_n(x) & \text{if } x \in C_n \text{ and } i_n(x) \in C_{n+1}, \\ i_{n+1} \circ i_n(x) & \text{otherwise.} \end{cases}$$

It is clear that $\varphi : X \to X \setminus C_0$ is exactly two-to-one. ⊣

Now fix C_n and $\varphi : X_2 \to X_2$ as in Lemma 6.2, and define $\tilde{f} : X_1 \to X_2$ by

$$\tilde{f}(x) = \varphi^{n(x)} \circ f_{n(x)}(x).$$

As φ is finite-to-one, so too is $\tilde{f}$. ⊣

There is a connection between the notions we have mentioned thus far and the ergodic-theoretic study of normalizers of full groups. Suppose that E is a countable Borel equivalence relation on a Polish space X. The *full group of* E is

$$[E] = \{f : X \to X \text{ Borel} : \text{graph}(f) \subseteq E\}.$$

The *normalizer of* $[E]$ is the group $N[E]$ of Borel automorphisms f such that $f \circ [E] \circ f^{-1} = [E]$, or equivalently, such that $\forall x, y \in X\ (xEy \Leftrightarrow f(x)Ef(y))$.

It is clear that any element of $N[E]$ gives rise to a Borel automorphism of X/E, and moreover, that the corresponding map from $N[E]/E$ to the Borel automorphisms of X/E is injective. Next we will see that in the ergodic-theoretic context, this is actually a bijection. We will use E_G denote the *orbit equivalence relation* on X induced by an action of the group G on X/E.

Proposition 6.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $f : X/E \to X/E$ is a Borel automorphism, and μ is an $E_{\langle f \rangle}$-invariant probability measure on X. Then there exists $\tilde{f} \in N[E]$ such that $\forall^*_\mu x \in X \ (\tilde{f}(x) \in f([x]_E))$.*

Proof. By the Lusin-Novikov uniformization theorem (see 18.10 of [51]), there is a sequence of partial Borel injections $f_n : X \to X$ such that

$$\operatorname{graph}(f) = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(f_n).$$

Recursively define $A_n = \operatorname{dom}(f_n) \setminus \bigcup_{m<n} A_m \cup f_n(A_m)$, set $A = \bigcup_{n \in \mathbb{N}} A_n$, and for $x \in A$, let $n(x)$ be the unique natural number such that $x \in A_{n(x)}$. Now define

$$\tilde{f}(x) = f_{n(x)}(x),$$

and note that for each $x \in X$, either $[x]_E \subseteq \operatorname{dom}(\tilde{f})$ or $f([x]_E) \subseteq \operatorname{rng}(\tilde{f})$. As $X \setminus \operatorname{dom}(\tilde{f})$ and $X \setminus \operatorname{rng}(\tilde{f})$ are both null by the invariance of μ, the proposition follows. $\dashv$

Remark 6.4. If E and $E_{\langle f \rangle}$ have the same set of invariant (ergodic) probability measures, then a combination of the above proof with a compressibility argument shows that f has a lifting in $N[E]$.

Suppose E is a countable Borel equivalence relation on X/F. The *E-saturation* of $B \subseteq X/F$ is

$$[B]_E = \{x \in X/F : \exists y \in B \ (xEy)\}.$$

A *complete section* for E is a set $B \subseteq X/F$ such that $[B]_E = X/F$. A *co-complete section* for E is the complement of a complete section. The equivalence relation E is *ergodic* if it admits no Borel complete section which is also co-complete. The following fact provides plenty of examples of ergodic equivalence relations:

Proposition 6.5. *Suppose that $F \subseteq E$ are Borel equivalence relations on a Polish space X and there is an F-ergodic, E-quasi-invariant probability measure μ on X. Then E/F is ergodic.*

Proof. To see this, suppose that B is a Borel complete section for E/F. It follows that the corresponding Borel set $\tilde{B} \subseteq X$ is an F-invariant Borel complete section for E. Then $\mu(\tilde{B}) = 1$ by F-ergodicity of μ, thus $\mu(X \setminus \tilde{B}) = 0$. It then follows from the E-quasi-invariance of μ that $X \setminus \tilde{B}$ is not a complete section for E, thus B is not a co-complete section for E/F. $\dashv$

7 Rigidity for ergodic actions

Suppose that G is a countable group. Put $X_0(G) = G^{\mathbb{N}}$, define

$$\vec{g} E_0(G) \vec{h} \Leftrightarrow \exists N \in \mathbb{N} \forall n \geq N \ \big(\vec{g}_n = \vec{h}_n\big),$$

and let G act on $X_0(G)$ via $g \cdot \vec{g} = \langle g\vec{g}_0, g\vec{g}_1, \ldots \rangle$. Of course, the action of G on $X_0(G)$ is smooth.

Proposition 7.1. *The induced action of G on $X_0(G)/E_0(G)$ is ergodic.*

Proof. Note that both $E_0(G)$ and E_G are generic and generically ergodic, and repeat the proof of Proposition 6.5 with category in place of measure. $\dashv$

For the sake of the arguments to come, it will be convenient to work with a slight modification of the action of G on $X_0(G)/E_0(G)$. Define $F_0(G) \subseteq E_0(G)$ by

$$\vec{g} F_0(G) \vec{h} \Leftrightarrow \exists N \in \mathbb{N} \ \big(\vec{g}_0 \cdots \vec{g}_N = \vec{h}_0 \ldots \vec{h}_N \text{ and } \forall n > N \ \big(\vec{g}_n = \vec{h}_n\big)\big),$$

and let G act on $X_0(G)$ via $g \cdot \langle \vec{g}_0, \vec{g}_1, \ldots \rangle = \langle g\vec{g}_0, \vec{g}_1, \vec{g}_2, \ldots \rangle$. Note that the equivalence relation which is generated by the induced action of G on $X_0(G)/F_0(G)$ is $E_0(G)/F_0(G)$. It is easily verified that the function $\pi : X_0(G) \to X_0(G)$ which is given by

$$\pi_n(\vec{g}) = \begin{cases} \vec{g}_0 & \text{if } n = 0, \\ \vec{g}_{n-1}^{-1}\vec{g}_n & \text{otherwise,} \end{cases}$$

is an isometry which simultaneously carries $E_0(G)$ to $F_0(G)$ and the old action of G to the new action of G. The following fact shows that the action of G on $X_0(G)/F_0(G)$ (as well as the action of G on $X_0(G)/E_0(G)$) is the minimal free ergodic G-action:

Theorem 7.2. *Suppose that X is a standard Borel space, F is a countable Borel equivalence relation on X, and G is a countable group which acts freely and in a Borel fashion on X/F. Then exactly one of the following holds:*

1. *The action of G on X/F is not ergodic.*
2. *There is a continuous embedding of $X_0(G)$ into X which induces a G-action embedding of $X_0(G)/F_0(G)$ into X/F.*

Proof. Proposition 7.1 essentially shows (1) $\Rightarrow \neg$(2), so it is enough to show $\neg$(1) $\Rightarrow$ (2). We claim that for this, it is enough to show that if the action of G on X/F is ergodic, then there is a continuous injection $\pi : X \to X_0(G)$ such that

(a) $\forall (\vec{g}, \vec{h}) \in E_0(G) \forall g \in G \ \big(g \cdot \vec{g} F_0(G) \vec{h} \Leftrightarrow g \cdot [\pi(\vec{g})]_F = [\pi(\vec{h})]_F\big)$.

(b) $\forall (\vec{g}, \vec{h}) \notin E_0(G) \ \big((\pi(\vec{g}), \pi(\vec{h})) \notin F\big)$.

To see this really is sufficient, set $E = E_G$ and suppose, towards a contradiction, that there exists $(\vec{g}, \vec{h}) \notin E_0(G)$ such that $\pi(\vec{g}) E \pi(\vec{h})$. Then there exists $g \in G$ such that $g \cdot [\pi(\vec{g})]_F = [\pi(\vec{h})]_F$. Now it follows from (a) that $\pi(g \cdot \vec{g}) \in g \cdot [\pi(\vec{g})]_F$, thus $(g \cdot \vec{g}, \vec{h}) \notin E_0(G)$ but $(g \cdot \vec{g}, \vec{h}) \in F$, which contradicts (b). Thus we have shown that (a) and (b) together imply

(b′) $\forall (\vec{g}, \vec{h}) \notin E_0(G) \ \big((\pi(\vec{g}), \pi(\vec{h})) \notin E\big)$.

Clearly (a) and (b′) imply that π is simultaneously an embedding of $F_0(G)$ into F and of $E_0(G)$ into E, and it then follows that the induced map from $X_0(G)/F_0(G)$ to X/F is a G-action embedding.

So suppose that the action of G on X/F is ergodic. Let $\mathscr{I}$ be the σ-ideal of Borel sets $B \subseteq X$ such that $(E|B)/(F|B)$ is not ergodic. By Feldman-Moore [36], there are Borel automorphisms $f_n : X \to X$, with $f_0 = \mathrm{id}$, such that $F = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$, and by Proposition 6.1 there are finite-to-one Borel functions $f_g : X \to X$ such that

$$\forall g \in G \forall x \in X \ \big(f_g(x) \in g \cdot [x]_F\big).$$

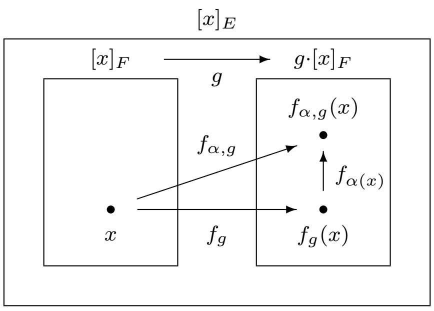

Figure 2.13: The action of $f_{\alpha,g}$ on $[x]_E$.

By Proposition 4.4, there is a Polish metric d on X which is compatible with the underlying Borel structure of X and in which each f_n and f_g is a continuous open-and-closed map.

We will associate with every Borel partial function $\alpha \colon X \to \mathbb{N}$ and $g \in G$ a lifting $f_{\alpha,g} : X \to X$ of $g|\operatorname{dom}(\alpha)$,

$$f_{\alpha,g}(x) = \begin{cases} f_{\alpha(x)} \circ f_g(x) & \text{if } g \neq 1_G \text{ and } x \in \operatorname{dom}(\alpha), \\ x & \text{otherwise.} \end{cases}$$

Let $\mathbf{\Pi}$ be the set of sequences $\langle \vec{\alpha}_{(g,n)} \colon X \to \mathbb{N} \rangle_{(g,n) \in G^+ \times \mathbb{N}}$ of Borel partial functions such that

$$\forall^\infty (g,n) \in G^+ \times \mathbb{N} \left(\alpha_{(g,n)} = \emptyset\right), \tag{$\dagger$}$$

where $G^+ = G \setminus \{1_G\}$. Associated with each $\vec{\alpha} \in \mathbf{\Pi}$, $\vec{g} \in X_0(G)$, and $S \subseteq \mathbb{N}$ is a function $f_{\vec{\alpha},\vec{g},S} : X \to X$ given by

$$f_{\vec{\alpha},\vec{g},S} = f_{\vec{\alpha}_{(\vec{g}_{n_0},n_0)},\vec{g}_{n_0}} \circ f_{\vec{\alpha}_{(\vec{g}_{n_1},n_1)},\vec{g}_{n_1}} \circ \cdots \circ f_{\vec{\alpha}_{(\vec{g}_{n_k},n_k)},\vec{g}_{n_k}} \circ \cdots ,$$

where $n_0, n_1, \ldots$ is the increasing enumeration of S. Note that this makes sense even when S is infinite, as ($\dagger$) ensures that only finitely many of the functions in this composition are non-trivial. Associated with each finite family $\mathscr{F} \subseteq G^+ \times \mathbb{N}$ is the set

$$X_{\mathscr{F}} = \left\{\vec{g} \in X_0(G) : \forall n \in \mathbb{N} \left(\vec{g}_n = 1_G \text{ or } (\vec{g}_n, n) \in \mathscr{F}\right)\right\}.$$

Given $\mathscr{F} \subseteq G^+ \times \mathbb{N}$, $B \subseteq X$, and $\epsilon > 0$, we say that $\vec{\alpha} \in \mathbf{\Pi}$ is $(\mathscr{F}, B, \epsilon)$-*good* if:

1. $\mathscr{F}$ is finite and $\forall (g,n) \notin \mathscr{F} \ \big(\alpha_{(g,n)} = \emptyset\big)$.

2. B is clopen and $\mathscr{I}$-positive.

3. $\forall \vec{g} \in X_{\mathscr{F}} \ \big(\operatorname{diam}\big(f_{\vec{\alpha},\vec{g},[0,\infty)}(B)\big) < \epsilon\big)$.

4. $\forall \vec{g}, \vec{h} \in X_{\mathscr{F}} \forall n \in \mathbb{N} \ \big(\vec{g}_n \neq \vec{h}_n \Rightarrow \forall i \leq n \ \big(f_{\vec{\alpha},\vec{g},[0,\infty)}(B) \cap f_i \circ f_{\vec{\alpha},\vec{h},[0,\infty)}(B) = \emptyset\big)\big)$.

For $\vec{g} \in X_{\mathscr{F}}$ and $n \in \mathbb{N}$, we will use $\vec{g}^{(n)}$ to denote the element of $X_{\mathscr{F}}$ which results from replacing each of the first n entries of $\vec{g}$ with 1_G. Note that for all $\vec{g} \in X_{\mathscr{F}}$,

$$f_{\vec{\alpha},\vec{g},[n,\infty)} = f_{\vec{\alpha},\vec{g}^{(n)},[n,\infty)} = f_{\vec{\alpha},\vec{g}^{(n)},[0,\infty)}.$$

Setting $X^{(n)}_{\mathscr{F}} = \big\{\vec{g}^{(n)} : \vec{g} \in X_{\mathscr{F}}\big\}$, it follows from (4) that

$$\forall \vec{g}, \vec{h} \in X^{(n)}_{\mathscr{F}} \ \big(\vec{g} \neq \vec{h} \Rightarrow f_{\vec{\alpha},\vec{g},[n,\infty)}(B) \cap f_{\vec{\alpha},\vec{h},[n,\infty)}(B) = \emptyset\big).$$

Now suppose that $\vec{\alpha} \in \mathbf{\Pi}$ is $(\mathscr{F}, B, \epsilon)$-good and $\vec{\alpha}' \in \mathbf{\Pi}$ is $(\mathscr{F}', B', \epsilon')$-good. Then $(\vec{\alpha}', \mathscr{F}', B', \epsilon')$ *extends* $(\vec{\alpha}, \mathscr{F}, B, \epsilon)$ if the following conditions are satisfied:

1. $\mathscr{F} \subseteq \mathscr{F}'$.

2. $B' \subseteq B$.

3. $\forall \vec{g} \in X_0(G) \forall (g,n) \in \mathscr{F} \ \big(\vec{\alpha}'_{(g,n)} = \vec{\alpha}_{(g,n)}\big)$.

4. $\forall \vec{g} \in X_0(G) \, \forall (g,n) \notin \mathscr{F} \ \big(f_{\vec{\alpha}'_{(g,n)},g} \circ f_{\vec{\alpha},\vec{g},(n,\infty)}(B') \subseteq f_{\vec{\alpha},\vec{g},(n,\infty)}(B)\big)$.

Note that if $(\vec{\alpha}', \mathscr{F}', B', \epsilon')$ extends $(\vec{\alpha}, \mathscr{F}, B, \epsilon)$, then

$$\forall \vec{g} \in X_{\mathscr{F}} \ \big(f_{\vec{\alpha}',\vec{g},[0,\infty)} = f_{\vec{\alpha},\vec{g},[0,\infty)}\big) \text{ and } \forall \vec{g} \in X_0(G) \ \big(f_{\vec{\alpha}',\vec{g},[0,\infty)}(B') \subseteq f_{\vec{\alpha},\vec{g},[0,\infty)}(B)\big).$$

The rest of the proof will be quite similar to the proofs of the previous sections, as soon as we establish the following lemma:

Lemma 7.3. *Suppose that $\mathscr{F} \subseteq \mathscr{F}'$ are finite subsets of $G^+ \times \mathbb{N}$, $\epsilon' > 0$, and $\vec{\alpha}$ is $(\mathscr{F}, B, \epsilon)$-good. Then there exists $B' \subseteq B$ and an $(\mathscr{F}', B', \epsilon')$-good $\vec{\alpha}' \in \mathscr{P}$ such that $(\vec{\alpha}', \mathscr{F}', B', \epsilon')$ extends $(\vec{\alpha}, \mathscr{F}, B, \epsilon)$.*

Proof. By the obvious induction, we may assume that $\mathscr{F}' = \mathscr{F} \cup \{(g,n)\}$, where $(g,n) \notin \mathscr{F}$. Fix an enumeration $\vec{g}^0, \ldots, \vec{g}^{M-1}$ of $X_{\mathscr{F}}^{(n+1)}$, and set

$$X_m = X_{\mathscr{F}} \cup \left\{\vec{g} \in X_{\mathscr{F}'} : \exists \ell < m \left(\vec{g}^{(n+1)} = (\vec{g}^\ell)^{(n+1)}\right)\right\}.$$

I will recursively define $\mathscr{I}$-positive Borel sets $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_M$ and $\vec{\alpha}^0 \subseteq \vec{\alpha}^1 \subseteq \cdots \subseteq \vec{\alpha}^M \in \mathbf{\Pi}$, beginning with $B_0 = B$ and $\vec{\alpha}^0 = \vec{\alpha}$. Suppose now that we have already defined B_m and $\vec{\alpha}^m$, for some $m < M$. For each $k \in \mathbb{N}$, define

$$\vec{\alpha}^{(m,k)}_{(g',n')}(x) = \begin{cases} k & \text{if } x \in f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(B_m) \text{ and } (g',n') = (g,n), \\ \vec{\alpha}^m_{(g',n')}(x) & \text{otherwise,} \end{cases}$$

put $\ell = \max\{k \in \mathbb{N} : \exists g \in G\ ((g,k) \in \mathscr{F}')\}$, and let A_k be the set of all $x \in B_m$ such that:

(a) $(f_k \circ f_g) \circ f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(x) \in f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(B_m)$, and

(b) $\forall i \leq \ell\, \forall \vec{g} \in X_m \forall \vec{h} \in X_{m+1} \setminus X_m \left(f_{\vec{\alpha}^{(m,k)},\vec{g},[0,\infty)}(x) \neq f_i \circ f_{\vec{\alpha}^{(m,k)},\vec{h},[0,\infty)}(x)\right)$.

I claim that $A = B_m \setminus \bigcup_{k \in \mathbb{N}} A_k$ is $\mathscr{I}$-null. To see this, suppose that $x \in A$ and set

$$y = f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(x).$$

Note that if $\vec{g} \in X_m$, then

$$\vec{g}_n = g \Rightarrow f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(B_m) \cap f_{\vec{\alpha}^m,\vec{g},(n,\infty)}(B_m) = \emptyset,$$

thus $f_{\vec{\alpha}^{(m,k)},\vec{g},[0,\infty)} = f_{\vec{\alpha}^m,\vec{g},[0,\infty)}$. It follows that if $k \in \mathbb{N}$ and

$$(f_k \circ f_g)(y) \in f_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(B_m),$$

then there exists $i \leq \ell$, $\vec{g} \in X_m$, and $\vec{h} \in X_{m+1} \setminus X_m$ such that

$$f_{\vec{\alpha}^m,\vec{g},[0,\infty)}(x) = f_i \circ f_{\vec{\alpha}^m,\vec{h},[0,n)} \circ (f_k \circ f_g)(y),$$

thus

$$(f_k \circ f_g)(y) \in f^{-1}_{\vec{\alpha}^m,\vec{h},[0,n)} \circ f_i^{-1} \circ f_{\vec{\alpha}^m,\vec{g},[0,\infty)} \circ f^{-1}_{\vec{\alpha}^m,\vec{g}^m,(n,\infty)}(y).$$

As there are only finitely many possibilities for $i, \vec{g}, \vec{h}$ and all of the functions involved are finite-to-one, it follows that there are only finitely many possible values of $(f_k \circ f_g)(y)$ which are in $f_{\vec{\alpha}^m, \vec{g}^m, (n,\infty)}(B_m)$. That is,

$$f_{\vec{\alpha}^m, \vec{g}^m, (n,\infty)}(B_m) \cap \left(g \prod_{j>n} \vec{g}_j^m \right) \cdot [x]_F$$

is finite. It follows that A can be partitioned into a pair of Borel sets A_0, A_1 such that:

1. $f_{\vec{\alpha}^m, \vec{g}^m, (n,\infty)}(B_m)$ intersects every $F|A_0$-class in a finite non-zero set.

2. $f_{\vec{\alpha}^m, \vec{g}^m, (n,\infty)}(B_m)$ entirely misses at least one F-class and has non-zero intersection with at least one F-class within each $E|A_1$-class.

Clearly $(E|A_1)/(F|A_1)$ is not ergodic. It follows that $(E|A_0)/(F|A_0)$ is also not ergodic, since $F|A_0$ is smooth. Thus $A \in \mathscr{I}$.

It now follows that there exists $k \in \mathbb{N}$ such that A_k is $\mathscr{I}$-positive. Put $\vec{\alpha}^{k+1} = \vec{\alpha}^{m,k}$, and note that since A_k is open, it can be thinned down to an $\mathscr{I}$-positive clopen set B_{m+1} with

$$\forall \vec{g} \in X_{m+1} \left(\mathrm{diam}\left(f_{\vec{\alpha}^{k+1}, \vec{g}, [0,\infty)}(B_{m+1}) \right) < \epsilon' \right).$$

Setting $B' = B_M$ and $\vec{\alpha}' = \vec{\alpha}^M$, it follows that $\vec{\alpha}'$ is $(\mathscr{F}', B', \epsilon')$-good and that $(\vec{\alpha}', \mathscr{F}', B', \epsilon')$ extends $(\vec{\alpha}, \mathscr{F}, B, \epsilon)$. ⊣

Now fix an increasing, exhaustive sequence $\mathscr{F}_n$ of finite subsets of $G^+ \times \mathbb{N}$ and a sequence of positive real numbers $\epsilon_n \to 0$. Put $B_0 = X$, let $\vec{\alpha}^0$ be the trivial element of $\mathbf{\Pi}$, and repeatedly apply Lemma 7.3 so as to produce a decreasing sequence of $\mathscr{I}$-positive clopen sets B_n and an increasing sequence $\vec{\alpha}^n \in \mathbf{\Pi}$ such that each $\vec{\alpha}^{n+1}$ is $(\mathscr{F}_{n+1}, B_{n+1}, \epsilon_{n+1})$-good and each $(\vec{\alpha}^{n+1}, \mathscr{F}_{n+1}, B_{n+1}, \epsilon_{n+1})$ extends $(\vec{\alpha}, \mathscr{F}_n, B_n, \epsilon_n)$. It follows from conditions (2) and (3) of the definition of goodness that each $\langle f_{\vec{\alpha}^n, \vec{g}, [0,\infty)}(B_n) \rangle_{n \in \mathbb{N}}$ is a decreasing sequence of closed sets with vanishing diameter. Thus

$$B_{\vec{g}} = \bigcap_{n \in \mathbb{N}} f_{\vec{\alpha}^n, \vec{g}, [0,\infty)}(B_n)$$

consists of a single point, which we will denote by $\pi(\vec{g})$. It follows from condition (3) of the definition of goodness that the function $\pi : X_0(G) \to X$ is continuous, and it follows from condition (4) of the definition of goodness that π is injective.

Now observe that for $\vec{g} \in X_0(G)$, $m \in \mathbb{N}$, and $\ell \in \mathbb{N}$ sufficiently large that $\forall i < m \ ((\vec{g}_i, i) \in \mathscr{F}_\ell)$,

$$\begin{aligned} B_{\vec{g}^{(m)}} &= \bigcap_{n \in \mathbb{N}} f_{\vec{\alpha}^n, \vec{g}^{(m)}, [0,\infty)}(B_n) \\ &= \bigcap_{n \in \mathbb{N}} f_{\vec{\alpha}^n, \vec{g}, [m,\infty)}(B_n) \\ &\subseteq \bigcap_{n \in \mathbb{N}} f^{-1}_{\vec{\alpha}^n, \vec{g}, [0,m)}\big(f_{\vec{\alpha}^n, \vec{g}, [0,m)} \circ f_{\vec{\alpha}^n, \vec{g}, [m,\infty)}(B_n)\big) \\ &= f^{-1}_{\vec{\alpha}^\ell, \vec{g}, [0,m)}\Big(\bigcap_{n \in \mathbb{N}} f_{\vec{\alpha}^n, \vec{g}, [0,\infty)}(B_n)\Big) \\ &= f^{-1}_{\vec{\alpha}^\ell, \vec{g}, [0,m)}\big(B_{\vec{g}}\big), \end{aligned}$$

thus $\pi(\vec{g}) = f_{\vec{\alpha}^\ell, \vec{g}, [0,m)} \circ \pi(\vec{g}^{(m)})$. Now suppose that $\vec{g} E_0(G) \vec{h}$, fix $\ell, m \in \mathbb{N}$ sufficiently large that

$$\vec{g}^{(m)} = \vec{h}^{(m)} \text{ and } \forall i < m \ ((\vec{g}_i, i), (\vec{h}_i, i) \in \mathscr{F}_\ell),$$

and observe that

$$[\pi(g \cdot \vec{g})]_F = \big[f_{\vec{\alpha}^\ell, g \cdot \vec{g}, [0,m)} \circ \pi\big(\vec{g}^{(m)}\big)\big]_F = g\vec{g}_0 \ldots \vec{g}_{m-1} \cdot \big[\pi\big(\vec{g}^{(m)}\big)\big]_F = g \cdot [\pi(\vec{g})]_F$$

and

$$[\pi(\vec{h})]_F = \big[f_{\vec{\alpha}^\ell, \vec{h}, [0,m)} \circ \pi\big(\vec{h}^{(m)}\big)\big]_F = \vec{h}_0 \ldots \vec{h}_{m-1} \cdot \big[\pi\big(\vec{h}^{(m)}\big)\big]_F,$$

thus

$$\begin{aligned} \pi(g \cdot \vec{g}) F \pi(\vec{h}) &\Leftrightarrow g \cdot \pi(\vec{g}) F \pi(\vec{h}) \\ &\Leftrightarrow g\vec{g}_0 \ldots \vec{g}_{m-1} = \vec{h}_0 \ldots \vec{h}_{m-1} \\ &\Leftrightarrow g \cdot \vec{g} F_0(G) \vec{h}, \end{aligned}$$

thus (a) holds.

It only remains to check (b). It is enough to show that if $\vec{g}, \vec{h} \in X_0(G)$ and $\vec{g}_n \neq \vec{h}_n$, then

$$\forall i \leq n \ \big(\pi(\vec{g}) \neq f_i \circ \pi(\vec{h})\big). \tag{‡}$$

Fix $\ell \in \mathbb{N}$ sufficiently large that $\forall i < m$ $((\vec{g}_i, i), (\vec{h}_i, i) \in \mathscr{F}_\ell)$, and note that by condition (4) of the definition of goodness,

$$\forall i \leq n \left(f_{\vec{\alpha}^{(\ell)}, \vec{g}, [0,\infty)}(B_\ell) \cap f_i \circ f_{\vec{\alpha}^{(\ell)}, \vec{h}, [0,\infty)}(B_\ell) = \emptyset \right),$$

and (‡) follows. $\dashv$

As a corollary, we see that Proposition 6.5 is the only route to ergodic free actions:

Theorem 7.4. *Suppose that X is a standard Borel space, F is a countable Borel equivalence relation on X, and G is a countable group which acts freely and in a Borel fashion on X/F. Then the following are equivalent:*

1. *The action of G on X is ergodic.*
2. *There is an E-ergodic, E_G-quasi-invariant probability measure on X.*

Proof. By Proposition 6.5, it is enough to show (1) $\Rightarrow$ (2). Let $\pi : X_0(G) \to X$ be the embedding of Theorem 7.2, let μ be a strictly positive probability measure on $\mathscr{P}(G)$, let ν be the associated product measure on $X_0(G)$, and fix Borel automorphisms $f_n : X \to X$ such that

$$F = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n).$$

It is easily verified that $\nu(B) = \sum_{n \in \mathbb{N}} \mu(\pi^{-1} \circ f_n(B))/2^{n+1}$ is as desired. $\dashv$

Now suppose that $L = \{R_i\}_{i \in I}$ is a countable relational language and $\mathscr{K}$ is a class of countable L-structures. An assignment $C \mapsto K_C$ of L-structures to the classes of E/F is *Borel* if

$$\forall i \in I \left(\left\{ ([x_1]_F, \ldots, [x_n]_F) \in C^n : C \in E/F \text{ and } R_i^{K_C}([x_1]_F, \ldots, [x_n]_F) \right\} \text{ is Borel} \right),$$

and such an assignment is a *$\mathscr{K}$-structuring* if each K_C is isomorphic to some L-structure in $\mathscr{K}$.

Theorem 7.5. *Suppose that X is a Polish space, $F \subseteq E$ are countable Borel equivalence relations on X, E/F is generated by a free ergodic action of G, $L = \{R_i\}_{i \in I}$*

is a countable relational language, and $\mathscr{K}$ is a set of L-structures on G. Then the following are equivalent:

1. *$\mathscr{K}$ contains a right-invariant L-structure.*

2. *There is a Borel $\mathscr{K}$-structuring of E/F.*

Proof. To see (1) $\Rightarrow$ (2), note that the right-invariance of $K_G \in \mathscr{K}$ ensures that

$$R_i^C = \left\{([x_1]_F, \ldots, [x_n]_F) \in C^n : \exists x \in C \exists (g_1, \ldots, g_n) \in R_i^G \forall 1 \leq i \leq n \ (g_i \cdot x = x_i)\right\}$$

defines the desired assignment of L-structures.

To see (2) $\Rightarrow$ (1), it follows from Theorem 7.2 that it is enough to show that every Borel $\mathscr{K}$-structuring $C \mapsto K_C$ of $E_0(G)/F_0(G)$ gives rise to a right-invariant L-structure $K_G \in \mathscr{K}$. For each $i \in I$ and $g_1, \ldots, g_{n_i} \in G$, note that

$$X^i_{g_1,\ldots,g_{n_i}} = \left\{x \in X_0(G) : R_i^{[x]_{E_0(G)}}(g_1 \cdot [x]_{F_0(G)}, \ldots, g_{n_i} \cdot [x]_{F_0(G)})\right\}$$

is $F_0(G)$-invariant, and therefore either meager or comeager. Setting

$$S = \{(i, g_1, \ldots, g_{n_i}) : X^i_{g_1,\ldots,g_{n_i}} \text{ is comeager}\},$$

it follows that there is a comeager $E_0(G)$-invariant Borel set B contained in

$$\bigcap_{(i,g_1,\ldots,g_{n_i}) \in S} X^i_{g_1,\ldots,g_{n_i}} \cap \bigcap_{(i,g_1,\ldots,g_{n_i}) \notin S} X \setminus X^i_{g_1,\ldots,g_{n_i}}.$$

Fix $x \in B$, set $C = [x]_{E_0(G)}$, and note that

$$R_i^G = \left\{(g_1, \ldots, g_{n_i}) \in G^n : (g_1 \cdot [x]_F, \ldots, g_{n_i} \cdot [x]_F) \in R_i^{\mathscr{K}_C}\right\}$$

defines the desired L-structure. $\dashv$

It is worth noting that the definability constraint on the structuring can be weakened under appropriate determinacy hypothesis. For instance, under projective determinacy Borel can be weakened to projective.

Theorem 7.6. *Suppose that X is a Polish space, $F \subseteq E$ are countable Borel equivalence relations on X, and E/F is ergodic. Then there is at most one group which freely acts in a Borel fashion on X/F so as to generate E/F.*

Proof. Suppose that G and H are groups which both act freely on X/F so as to generate E/F. It then follows from Theorem 7.5 that there is a transitive free action of H on G and a function $\pi : H \to G$ such that $h \cdot g = \pi(h)g$. Clearly any such π is an isomorphism of H and G. $\dashv$

Theorem 7.5 also gives a recipe for building equivalence relations which are difficult to distinguish:

Proposition 7.7. *Suppose that X_1 and X_2 are Polish spaces, $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ are countable Borel equivalence relations on X_1 and X_2, and E_1/F_1 and E_2/F_2 are generated by an ergodic action of the same group. Then E_1/F_1 and E_2/F_2 admit the same definable structurings.*

Finally, Theorem 7.5 makes it easy to see that various sorts of structurability behave much differently for ergodic equivalence relations than for equivalence relations on Polish spaces. We will just mention one such example here:

Proposition 7.8. *Suppose that X is a standard Borel space, $F \subseteq E$ are countable Borel equivalence relations on X, and E/F is ergodic. Then the following are mutually exclusive:*

1. *E/F is generated by the action of an aperiodic Borel automorphism of X/F.*
2. *E/F is the increasing union of finite Borel subequivalence relations on X/F.*

Proof. Suppose, towards a contradiction, that both (1) and (2) hold. Then E/F is induced by a free Borel action of $\mathbb{Z}$, and (2) coupled with Theorem 7.5 then imply that $\mathbb{Z}$ is locally finite, a contradiction. $\dashv$

8 Ergodic hyperfinite actions

Suppose that X is a Polish space and $F \subseteq E$ are countable Borel equivalence relations on X. The equivalence relation E/F is *hyperfinite* if E is hyperfinite. A Borel action of a group G on X/F is *hyperfinite* if the induced equivalence relation on

X/F is hyperfinite. It is not hard to see that the equivalence relation $E_0(G)/F_0(G)$ of §7 is hyperfinite.

Theorem 8.1. *Suppose that X is a Polish space, F is a countable Borel equivalence relation on X, and G acts freely and hyperfinitely on X/F. Then there is a Borel G-action embedding of X/F into $X_0(G)/F_0(G)$.*

Proof. We can assume that $X = X_0(G)$. Let $f : X_0(G) \to X_0(G)$ be a Borel automorphism which induces $E = E_G$, and fix a decreasing sequence of Borel sets B_n such that

$$\forall n \in \mathbb{N} \left(X = \bigcup_{k<2^n} f^k(B_n) \right),$$

and whose intersection is a transversal for the restriction of E to its periodic part. Put

$$k_n(x) = \min \left\{ k \in \mathbb{N} : f^{-k}(x) \in B_n \right\},$$

define $\varphi_n : X \to G^{n \cdot 2^n}$ by

$$\varphi_n(x) = \bigoplus_{k<2^n} f^{k-k_n(x)}(x)|n,$$

where $\oplus$ denotes concatenation. Fix $g_0 \in G \setminus \{1\}$, put $g_1 = 1$, define $\psi_{n+1} : X \to G^{n+1}$ by

$$\psi_{n+1}(x) = \langle g_{b_0}, g_{b_1}, \ldots, g_{b_n} \rangle,$$

where $b_0 b_1 \ldots b_n$ is the base 2 representation of $k_{n+1}(x) - k_n(x)$, and let $g_{n+1}(x)$ be the unique element of G such that

$$\left(\prod \varphi_{n+1}(x) \right) \left(\prod \psi_{n+1}(x) \right) g_{n+1}(x) \cdot [f_{n+1}(x)]_F = [f_n(x)]_F,$$

where $\prod$ denotes the product of the elements along the sequence. Now define $\pi : X \to X_0(G)$ by

$$\pi(x) = \bigoplus_{n>0} \varphi_n(x) \oplus \psi_n(x) \oplus \langle g_n(x) \rangle.$$

Lemma 8.2. *If xEy, then $\pi(x)E_0(G)\pi(y)$ and $g \cdot [x]_F = [y]_F \Leftrightarrow g \cdot \pi(x)F_0(G)\pi(y)$.*

Proof. Set $f_n(x) = f^{-k_n(x)}(x)$. For the first assertion, fix $N \in \mathbb{N}$ with $f_N(x) = f_N(y)$ and note that for $n \geq N$,

$$\varphi_{n+1}(x) \oplus \psi_{n+1}(x) \oplus \langle g_{n+1}(x) \rangle = \varphi_{n+1}(y) \oplus \psi_{n+1}(y) \oplus \langle g_{n+1}(y) \rangle ,$$

thus $\pi(x) E_0(G) \pi(y)$. For the second assertion, define

$$g_z = \prod_{n \leq N} \Big(\prod \varphi_{n+1}(z) \Big) \Big(\prod \psi_{n+1}(z) \Big) g_{n+1}(z),$$

for $z \in X$, and note that

$$g \cdot \pi(x) F_0(G) \pi(y) \Leftrightarrow g g_x = g_y.$$

Now it follows from the definition of g_{n+1} that for all $z \in X$,

$$g_z \cdot [f_{N+1}(z)]_F = [z]_F,$$

and since $f_{N+1}(x) = f_{N+1}(y)$ and the action of G is free, it follows that

$$\begin{aligned} g \cdot [x]_F = [y]_F \quad &\Leftrightarrow \quad g g_x \cdot [f_{N+1}(x)]_F = g_y \cdot [f_{N+1}(y)]_F \\ &\Leftrightarrow \quad g g_x = g_y \\ &\Leftrightarrow \quad g \cdot \pi(x) F_0(G) \pi(y), \end{aligned}$$

and the lemma follows. $\dashv$

Lemma 8.3. *θ is injective, and moreover, if $\pi(x) E_0(G) \pi(y)$ then $x E y$.*

Proof. Suppose that $\pi(x) E_0(G) \pi(y)$ and fix $N \in \mathbb{N}$ sufficiently large that for all $n \geq N$,

$$\varphi_{n+1}(x) \oplus \psi_{n+1}(x) \oplus \langle g_{n+1}(x) \rangle = \varphi_{n+1}(y) \oplus \psi_{n+1}(y) \oplus \langle g_{n+1}(y) \rangle .$$

It follows that $k_n(x) - k_N(x) = k_n(y) - k_N(y)$ for all $n > N$, thus

$$x|n = f^{k_n(x)-k_n(y)}(y)|n = f^{k_N(x)-k_N(y)}(y)|n,$$

thus $x = f^{k_N(x)-k_N(y)}(y)$, so $x E y$. Also, if $x \neq y$ then $k_N(x) \neq k_N(y)$, thus π is injective. $\dashv$

By Lemma 8.2, $xFy \Rightarrow xEy \Rightarrow \pi(x)F_0(G)\pi(y)$, and by Lemma 8.3,

$$\pi(x)F_0(G)\pi(y) \Rightarrow \pi(x)E_0(G)\pi(y) \Rightarrow xEy \Rightarrow xFy,$$

thus π is simultaneously an embedding of F into $F_0(G)$ and an embedding of E into $E_0(G)$. It follows from Lemma 8.2 that π induces an embedding of G-actions. $\dashv$

Putting together this result with the results of the previous section, we now have:

Theorem 8.4. *Suppose that G is a countable group. Then any two free ergodic hyperfinite actions of G on quotients of Polish spaces by countable Borel equivalence relations are Borel isomorphic.*

Proof. As in the proof of Theorem 3.5, this follows from a Schröder-Bernstein argument. $\dashv$

Corollary 8.5. *Suppose that G is a finite group. Then any two free ergodic actions of G on quotients of Polish spaces by hyperfinite equivalence relations are Borel isomorphic.*

Proof. By Proposition 1.3 of Jackson-Kechris-Louveau [48], every Borel equivalence relation with a finite index hyperfinite subequivalence relation is itself hyperfinite, and the corollary follows. $\dashv$

Chapter 3

Measures and graphings

1 Introduction

The primary motivation underlying this chapter is the desire to understand the sort of information that can be discerned about countable Borel equivalence relations from probability measures and graphings which live on them.

Suppose that X is a Polish space and $f : X \to X$ is a Borel automorphism. A probability measure μ on X is *f-invariant* if

$$\forall B \subseteq X \text{ Borel } (\mu(B) = \mu(f^{-1}(B))).$$

A probability measure μ on X is *f-quasi-invariant* if

$$\forall B \subseteq X \text{ Borel } (\mu(B) = 0 \Leftrightarrow \mu(f^{-1}(B)) = 0).$$

The *full group* of a countable Borel equivalence relation E on X is the group $[E]$ of Borel automorphisms $f : X \to X$ such that

$$\forall x \in X \ (xEf(x)),$$

and a set $B \subseteq X$ is *E-invariant* if it is equal to its E-saturation,

$$[B]_E = \{x \in X : \exists y \in B \ (xEy)\}.$$

The probability measure μ is *E-(quasi-)invariant* if it is (quasi-)invariant with respect to every element of $[E]$, and the probability measure μ is *E-ergodic* if every E-invariant Borel set is null or conull.

The following theorem shows that we can get our hands on all such measures:

Theorem (Farrell-Varadarajan). *Suppose X is a Polish space and E is a countable Borel equivalence relation on X which admits an invariant probability measure. Then the set $\mathscr{E}\mathscr{I}(E)$ of E-invariant, E-ergodic probability measures is non-empty and Borel, and there is a (surjective) Borel function $\pi : X \to \mathscr{E}\mathscr{I}(E)$ such that*

1. *If μ is E-invariant, then $\mu = \int \pi(x)\ d\mu(x)$.*

2. *If μ is also E-ergodic, then $\pi(x) = \mu$ for μ-almost every $x \in X$.*

Of course, this says nothing when E admits no invariant measures. Nadkarni [62] filled this void by showing that the inexistence of invariant measures always has a very specific sort of witness. The *full semigroup of E* is the group $[\![E]\!]$ of partial Borel injections $f : X \to X$ such that $xEf(x)$ whenever $x \in \mathrm{dom}(f)$. A *complete section* for E is a set $B \subseteq X$ which intersects every class of E. A *compression* of E is a partial Borel injection $f \in [\![E]\!]$ with full domain and co-complete range. Intuitively, a compression can be thought of as a uniform witness to the fact that each class of E is Dedekind infinite.

Theorem (Nadkarni). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then E has no invariant probability measure exactly when E is compressible.*

By making use of Nadkarni's theorem, one can also obtain an analog of Tarski's theorem on the existence of finitely additive invariant probability measures. An equivalence relation E is *paradoxical* if there are partial injections $f, g \in [\![E]\!]$ with full domains, whose ranges partition X.

Theorem (Becker-Kechris). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then E has no invariant probability measure exactly when E is paradoxical.*

Together, these results give rise to a powerful tool for studying equivalence relations, for they imply that one can prove things about equivalence relations by combining sufficiently uniform measure-theoretic arguments with arguments that presume the existence of a compression or a paradoxical decomposition. A notable such application appears in Dougherty-Jackson-Kechris [24], where hyperfinite equivalence relations are classified up to Borel isomorphism.

Unfortunately, most Borel probability measures are not E-invariant, and these theorems say nothing about such measures. The theorem of Farrell-Varadarajan has been generalized to quasi-invariant measures, first by Kifer-Pirogov [57] in the hyperfinite case (the proof of their result was later simplified by Schmidt [71]), and then by Ditzen [23] in the general case. On the other hand, the results of Nadkarni [61] and Becker-Kechris [6] have not been so generalized.

In §2, we present several facts about quasi-invariant measures. We begin by showing that every probability measure is E-quasi-invariant on a complete section, so that the study of probability measures essentially reduces to the study of those which are quasi-invariant. (Actually, this result is morally due to Woodin, who showed the analogous fact for Baire category.) Here we also introduce the well-known way of associating with each quasi-invariant measure a Borel function $D : E \to \mathbb{R}^+$, which describes a relative notion of mass between E-related points of X. That is, when xEy we think of x as being $D(x, y)$ times more massive than y. This intuition is solidified by the fact that if $f \in [E]$ and $B \subseteq X$ is Borel, then

$$\mu(f^{-1}(B)) = \int_B D(f^{-1}(x), x) \; d\mu(x).$$

In particular, E is μ-invariant if the associated function $D : E \to \mathbb{R}^+$ has constant value 1. We describe how a simple proof of the analog of the Lebesgue density theorem for Polish ultrametric spaces can be used to calculate D from μ. We then isolate a σ-ideal (which agrees with usual σ-ideal of smooth sets when $D = 1$), which plays a fundamental role in our work, and describe some of its properties.

In §3, we prove several selection theorems which allow us to build finite Borel subequivalence relations whose classes satisfy a wide range of properties. We also note various barriers to further strengthenings of these theorems. Coincidentally, one of these barriers gives rise to a quite simple answer to a question of Nadkarni [62] and its subsequent modification in Eigen-Hajian-Nadkarni [30] (where the original question was answered in a different manner) regarding a potential alternative characterization of compressibility.

In §4 and §5, we embark upon the program of providing analogs and generalizations of the theorems of Farrell-Varadarajan, Nadkarni, and Becker-Kechris for probability measures with a given $D : E \to \mathbb{R}^+$. The generalization of the theorem of Farrell-Varadarajan to this context has already been achieved, first by Kifer-Pirogov [57] for hyperfinite equivalence relations, and then in general by Ditzen [23]. The proof of Kifer-Pirogov [57] was quite complex, and their theorem was given a much simpler proof, using only the Hurewicz ergodic theorem, by Schmidt [71]. Ditzen's proof uses quite a bit of ergodic theory beyond this, however. While analogs of the theorems of Nadkarni and Becker-Kechris did not exist before now, it is worth noting that Nadkarni has found a new proof of the hyperfinite case of his theorem which uses little more than Srivatsa's descriptive strengthening of the Birkhoff ergodic theorem (the invariant special case of the Hurewicz ergodic theorem).

The main result of §4 is a slight weakening of the Hurewicz ergodic theorem which holds for all countable Borel equivalence relations. We prove this theorem using nothing more than the selection results of §3. We then describe how this can be used to give two new proofs of Ditzen's [23] theorem. The second of these proofs is really a reduction of Ditzen's [23] theorem to that of Kifer-Pirogov [57]. The source of this reduction is the following fact, which we again prove with our bare hands:

Theorem. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. Then there is a hyperfinite equivalence relation $F \subseteq E$ such the the set of D-invariant probability measures and the set of $D|F$-invariant probability measures coincide.*

In §5, we give the analog of the theorems of Nadkarni [61] and Becker-Kechris [6] in the quasi-invariant setting. One of the main difficulties here is that the function $D : E \to \mathbb{R}^+$ can impose limitations on the existence of invariant functions in $[\![E]\!]$ so severe that it may be impossible to find a compression, even when no D-invariant probability measures exist. This is similar to the situation that arises with a countable group Γ of Borel automorphisms of X. Even if there are no Γ-invariant probability measures, it is clearly impossible for any element of Γ to be a compression, for every element of Γ has full range. The solution is to search for compressions in the full semigroup of the *orbit equivalence relation associated with* Γ,

$$xE_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y).$$

This allows us to break up sets and map pieces of one set to pieces of another via different elements of Γ. We handle the new limitations imposed by D by going one step further, and work with an enlarged full group $[\![D]\!]$ in which we can break up points and map pieces of one point to pieces of another via different elements of Γ. This leads to natural analogs of the notions of aperiodicity, compressibility, and paradoxicality. We show that the natural analogs of the theorems of Nadkarni and Becker-Kechris go through with respect to these notions (whose definitions we shall suppress until §5):

Theorem. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E is D-aperiodic. Then exactly one of the following holds:*

1. *E admits a D-invariant probability measure.*

2. *E is D-compressible.*

3. *E is D-paradoxical.*

As an application of our results, we also provide compressibility-like criteria for the existence of an invariant probability measure for a countable-to-one function, answering a question of Nadkarni. We close this section with a result (joint with Kechris) on the incompatibility between measure and category in this setting, which generalizes a result of Wright [80].

In §6, we turn our attention to graphings. We focus on results concerning *ends of graphs*. All of our results here generalize work of Paulin [65], which itself generalizes work of Adams [1]. Again, the main tools of this section are the selection results of §3. With few exceptions, our results do not depend at all on the presence of D-invariant measures. For those which do, we still give descriptive results by showing them off of a D-compressible set.

We begin by showing that every aperiodic countable Borel equivalence relation admits a single-ended locally finite graphing. We then show the following:

Theorem. *Suppose X is a Polish space and $\mathscr{G}$ is a locally countable Borel graph on X, each of whose connected components has exactly two ends. Then $E_{\mathscr{G}}$ is hyperfinite.*

We also show the following, which was independently observed by Blanc [15]:

Theorem. *Suppose that X is a Polish space and $\mathscr{G}$ is a locally countable Borel graph on X. Then off of an E-invariant Borel set on which E is smooth, every component of $\mathscr{G}$ has 0, 1, 2, or infinitely many ends.*

In order to strengthen this theorem, we then work with cocycles. Given a Borel graph $\mathscr{G}$ and a Borel cocycle $D : E_{\mathscr{G}} \to \mathbb{R}^+$, we study a natural subset of the ends of $\mathscr{G}$ which we term the *D-ends*. When $D = 1$, these are exactly the usual ends of $\mathscr{G}$. Here we show the following:

Theorem. *Suppose that X is a Polish space, $\mathscr{G}$ is a locally countable Borel graph on X, and $D : E_{\mathscr{G}} \to \mathbb{R}^+$ is a Borel cocycle. Then off of a D-negligible E-invariant Borel set, every component of $\mathscr{G}$ has 0, 1, 2, or perfectly many D-ends.*

We close the section with a generalization of the Poincaré recurrence lemma:

Theorem. *Suppose that X is a Polish space, $\mathscr{G}$ is a locally countable Borel graph on X, $D : E_{\mathscr{G}} \to \mathbb{R}^+$ is a Borel cocycle, and B is a Borel complete section for $E_{\mathscr{G}}$. Then off of a D-negligible E-invariant Borel set, B is dense in the D-ends of $\mathscr{G}$.*

2 Quasi-invariant measures

Suppose that X is a Polish space. By a *measure on X*, we mean a countably additive extended real-valued function μ on the Borel subsets of X such that $\mu(\emptyset) = 0$.

By a *probability measure on* X, we mean a measure μ on X such that $\mu(X) = 1$. The set of all probability measures on X is denoted by $P(X)$.

Now suppose that E is a countable Borel equivalence relation on X. The *E-saturation* of a set $B \subseteq X$ is given by

$$[B]_E = \{x \in X : \exists y \in B\ (xEy)\}.$$

The set $B \subseteq X$ is *E-invariant* if $B = [B]_E$, and the set $B \subseteq X$ is a *complete section for* E if $[B]_E = X$. By the Lusin-Novikov Uniformization Theorem (see Theorem 18.10 of Kechris [51]), every countable Borel equivalence relation is the union of countably many Borel graphs. It follows that the E-saturation of every Borel set is also Borel.

A probability measure μ is *E-quasi-invariant* if the saturation of every null set is null. There is a substantial sense in which every probability measure on X is nearly E-quasi-invariant. This is essentially due to Woodin, who showed the analogous fact for Baire category:

Proposition 2.1 (essentially Woodin). *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and μ is a probability measure on X. Then there is a conull Borel E-complete section $B \subseteq X$ such that $\mu|B$ is $(E|B)$-quasi-invariant.*

Proof. We may assume, without loss of generality, that $X = \mathbb{R}$. Fix a sequence of open intervals $\mathscr{U}_n$ which form a basis for the usual topology on $\mathbb{R}$. Also, fix a sequence of Borel automorphisms $f_n : X \to X$ such that

$$E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(f_n).$$

For each pair of natural numbers (m, n) for which it is possible, fix a Borel set $B_{mn} \subseteq \mathscr{U}_m$ such that $\mu(B_{mn}) \geq \mu(\mathscr{U}_m)/2$ and $\mu(f_n^{-1}(B_{mn})) = 0$. Now define

$$A = X \setminus \bigcup_{m,n} f_n^{-1}(B_{mn}).$$

Lemma 2.2. *$\mu|A$ is $(E|A)$-quasi-invariant.*

Proof. Suppose, towards a contradiction, that there is a Borel null set $A' \subseteq A$ such that $[A']_{E|A}$ is non-null. Set $A_n = A \cap f_n^{-1}(A)$, and note that

$$[A']_{E|A} = \bigcup_{n \in \mathbb{N}} f_n(A' \cap A_n).$$

In particular, it follows that there exists $n \in \mathbb{N}$ such that

$$\mu(f_n(A' \cap A_n)) > 0.$$

By the Lebesgue density theorem, there exists $m \in \mathbb{N}$ such that

$$\mu(f_n(A' \cap A_n) \cap \mathscr{U}_m) > \mu(\mathscr{U}_m)/2.$$

It follows that B_{mn} exists, and since $\mu(B_{mn}) \geq \mu(\mathscr{U}_m)/2$, we have that

$$f_n(A' \cap A_n) \cap B_{mn} \neq \emptyset.$$

It then follows that $A \cap f_n^{-1}(B_{mn}) \neq \emptyset$, a contradiction. $\dashv$

It now follows that the set

$$B = A \cup (X \setminus [A]_E)$$

is a conull Borel complete section for E and $\mu|B$ is $(E|B)$-quasi-invariant. $\dashv$

It was Kechris who pointed out that Woodin's argument can be used here. As our original proof is a bit more in the spirit of the arguments to come (and avoids the need for the Lebesgue density theorem), it seems worthwhile to reproduce it here:

Alternative Proof (Miller). A topological space is *zero-dimensional* if it has a clopen basis. By change of topology results (see §13 of Kechris [51]), there is a zero-dimensional Polish topology on X, compatible with its underlying Borel structure. An *ultrametric* on X is a metric d which satisfies the *strong triangle inequality*,

$$\forall x, y, z \in X \ \big(d(x, z) \leq \max(d(x, y), d(y, z))\big).$$

It follows that there is a complete ultrametric d on X which is compatible with its underlying Borel structure, as any zero-dimensional Polish space admits such an ultrametric. We will use $\mathscr{B}(x, \epsilon)$ to denote the ball around x of radius ϵ. The requirement that d is an ultrametric ensures that

$$\forall \delta \leq \epsilon \, \forall x, y \in X \, \big(\mathscr{B}(x, \delta) \cap \mathscr{B}(y, \epsilon) = \emptyset \text{ or } \mathscr{B}(x, \delta) \subseteq \mathscr{B}(y, \epsilon)\big).$$

For each $f \in [E]$ and $\epsilon > 0$, set

$$\rho_{f,\epsilon}(x) = \frac{\mu(f(\mathscr{B}(x, \epsilon)))}{\mu(\mathscr{B}(x, \epsilon))}.$$

Note that $\rho_{f,\epsilon}$ is well-defined off of a (countable) union of null open balls, and therefore has conull domain. As $\mathrm{dom}(\rho_{f,\epsilon})$ decreases as $\epsilon \to 0$, it follows that

$$\bigcap_{\epsilon > 0} \mathrm{dom}(\rho_{f,\epsilon}) = \bigcap_{n > 0} \mathrm{dom}(\rho_{f,1/n}),$$

thus for all $f \in [E]$, almost every point of X is in the domain of every $\rho_{f,\epsilon}$. Set

$$A_f = \left\{ x \in \bigcap_{\epsilon > 0} \mathrm{dom}(\rho_{f,\epsilon}) : \limsup_{\epsilon \to 0} \rho_{f,\epsilon}(x) < \infty \right\}.$$

Lemma 2.3. *For all $f \in [E]$, the set A_f is conull.*

Proof. The measure μ is *regular* if

$$\forall B \subseteq X \text{ Borel } \big(\mu(B) = \inf_{B \subseteq U \text{ open}} \mu(U)\big),$$

and the measure μ is *tight* if

$$\forall B \subseteq X \text{ Borel } \big(\mu(B) = \inf_{B \supseteq K \text{ compact}} \mu(K)\big).$$

Every probability measure on a Polish space is regular and tight (see, for example, §17 of Kechris [51]).

Now suppose, towards a contradiction, that $X \setminus A_f$ is of positive measure. It follows from the tightness of μ that there is a compact set $K \subseteq X \setminus A_f$ of positive measure which is contained in the domain of every $\rho_{f,\epsilon}$, for $\epsilon > 0$. We will show that $n\mu(K) \leq \mu(f(K))$ for all $n \in \mathbb{N}$, contradicting the fact that $\mu(f(K)) \leq 1$.

It follows from the regularity of the probability measure $B \mapsto \mu(f(B))$ that there is an open set $\mathscr{U} \supseteq K$ such that

$$\mu(f(\mathscr{U})) \leq \mu(f(K)) + \mu(K).$$

For each $x \in K$, fix $\epsilon_x > 0$ such that $\mathscr{B}(x, \epsilon_x) \subseteq \mathscr{U}$ and $\rho_{f,\epsilon_x}(x) \geq n + 1$. As K is compact, we can find a finite family of points $x_i \in K$ such that the set

$$\mathscr{V} = \bigcup_{i<k} \mathscr{B}(x_i, \epsilon_{x_i})$$

contains $\mathscr{K}$. As (X, d) is an ultrametric space, after throwing out some of the x_i's we may assume that the balls $\mathscr{B}(x_i, \epsilon_{x_i})$ partition $\mathscr{V}$. As $\mathscr{V} \subseteq \mathscr{U}$, it follows that

$$\begin{aligned}
(n+1)\mu(\mathscr{V}) &= \sum_{i<k} (n+1)\mu(\mathscr{B}(x_i, \epsilon_{x_i})) \\
&\leq \sum_{i<k} \mu(f(\mathscr{B}(x_i, \epsilon_{x_i}))) \\
&= \mu(f(\mathscr{V})) \\
&\leq \mu(f(K)) + \mu(K),
\end{aligned}$$

thus $n\mu(K) \leq (n+1)\mu(\mathscr{V}) - \mu(K) \leq \mu(f(K))$, the desired contradiction. $\dashv$

By Feldman-Moore [36], there are Borel automorphisms $f_n : X \to X$ such that

$$E = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(f_n).$$

It follows from Lemma 2.3 that the set $A = \bigcap_{n \in \mathbb{N}} A_{f_n}$ is conull.

Lemma 2.4. *$\mu|A$ is $E|A$-quasi-invariant.*

Proof. Suppose, towards a contradiction, that $B \subseteq A$ is a null Borel set whose saturation is of positive measure. As

$$[B]_E = \bigcup_{n \in \mathbb{N}} f_n(B),$$

it follows that there exists $n \in \mathbb{N}$ such that $\mu(f_n(B)) > \epsilon$, for some $\epsilon > 0$. Set $f = f_n$. By the tightness of the measure $C \mapsto \mu(f(C))$, there is a compact set $K \subseteq B$ such that $\mu(f(K)) > \epsilon$. It follows from Lemma 2.3 that for $m \in \mathbb{N}$ sufficiently large,

$$\limsup_{\epsilon \to 0} \rho_{f,\epsilon}(x) \leq m.$$

For each $x \in K$, fix $\epsilon_x > 0$ such that $\rho_{f,\epsilon}(x) \leq m$. As K is compact, we can find a finite family of points $x_i \in K$ such that

$$\mathscr{V} = \bigcup_{i<k} \mathscr{B}(x_i, \epsilon_{x_i})$$

contains $\mathscr{K}$. It now follows that

$$\begin{aligned} \mu(f(K)) &\leq \sum_{i<k} \mu(f(\mathscr{B}(x_i, \epsilon_{x_i}))) \\ &\leq \sum_{i<k} m\mu(\mathscr{B}(x_i, \epsilon_{x_i})) \\ &= 0, \end{aligned}$$

the desired contradiction. $\dashv$

It now follows that the set

$$B = A \cup (X \setminus [A]_E)$$

is a conull Borel complete section for E and $\mu|B$ is $(E|B)$-quasi-invariant. $\dashv$

Suppose that $f : X \to X$ is a Borel automorphism. The measure μ is *f-quasi-invariant* if f sends null sets to null sets. We will use $f_*\mu$ to denote the probability measure given by $B \mapsto \mu(f^{-1}(B))$. Note that μ is f-quasi-invariant exactly when μ and $f_*\mu$ have the same null sets. By the Radon-Nikodym Theorem (see Theorem 6.10 of Rudin [67]), there is a Borel function $df_*\mu/d\mu : X \to \mathbb{R}^+$ in $L^1(\mu)$ such that for every $\varphi : X \to \mathbb{R}^+$ in $L^1(\mu)$,

$$\int \varphi(x)\ df_*\mu(x) = \int \varphi(x)(df_*\mu/d\mu)(x)\ d\mu(x),$$

and moreover, the function $df_*\mu/d\mu$ is unique modulo null sets. In Polish ultrametric spaces, these *derivatives* are not difficult to compute:

Proposition 2.5. *Suppose that (X, d) is a Polish ultrametric space, $f : X \to X$ is a Borel automorphism, and μ is an f-quasi-invariant probability measure. Then*

$$(df_*\mu/d\mu)(x) = \lim_{\epsilon \to 0} \left(\frac{f_*\mu(\mathscr{B}(y, \epsilon))}{\mu(\mathscr{B}(y, \epsilon))} \right) \ \mu\text{–a.e.}$$

Proof. The main point is the following version of the Lebesgue density theorem:

Lemma 2.6. *Suppose that (X, d) is a Polish ultrametric space, μ is a probability measure on X, and $\varphi : X \to \mathbb{R}^+$ is (locally) in $L^1(\mu)$. Then*

$$\varphi(x) = \lim_{\epsilon \to 0} \left(\frac{\int_{\mathscr{B}(x,\epsilon)} \varphi(y)\ d\mu(y)}{\mu(\mathscr{B}(x,\epsilon))} \right)\ \mu\text{−a.e.}$$

Proof. By an approximation argument, it is enough to show the lemma for simple functions, thus for characteristic functions. For this, it is enough to show

$$\forall B \subseteq X \text{ Borel}\, \forall^*_\mu x \in B\ \left(\lim_{\epsilon \to 0} \left(\frac{\mu(B \cap \mathscr{B}(x,\epsilon))}{\mu(\mathscr{B}(x,\epsilon))} \right) = 1 \right), \tag{†}$$

as (†) can then be applied to B and $X \setminus B$. Fix a Borel set $B \subseteq X$, and for $x \in B$ and $\epsilon > 0$, put

$$\rho_x(\epsilon) = \frac{\mu(B \cap \mathscr{B}(x,\epsilon))}{\mu(\mathscr{B}(x,\epsilon))}.$$

Note that by replacing X with the conull closed set

$$\{x \in X : \forall \epsilon > 0\ (\mu(\mathscr{B}(x,\epsilon)) > 0)\},$$

we may assume that $\rho_x(\epsilon)$ is defined everywhere. Now suppose, towards a contradiction, that

$$A = \left\{ x \in X : \liminf_{\epsilon \to 0} \rho_x(\epsilon) < 1 \right\}$$

is of positive measure. Then we can find $\delta > 0$ such that

$$A' = \left\{ x \in A : \liminf_{\epsilon \to 0} \rho_x(\epsilon) < 1 - \delta \right\}$$

is of positive measure. By the tightness of μ, there is compact set $K \subseteq A'$ of positive measure. Now suppose $\mathscr{U} \supseteq K$ is open, and for each $x \in K$, choose $\epsilon_x > 0$ such that

$$\mathscr{B}(x, \epsilon_x) \subseteq \mathscr{U} \text{ and } \rho_x(\epsilon_x) < 1 - \delta.$$

As $\{\mathscr{B}(x, \epsilon_x)\}_{x \in K}$ forms an open cover of K, there is a finite subcover $\{\mathscr{B}(x_i, \epsilon_{x_i})\}_{i<n}$. As d is an ultrametric, we may assume this subcover is pairwise disjoint. Setting

$$\mathscr{V} = \bigcup_{i<n} \mathscr{B}(x_i, \epsilon_{x_i}),$$

it follows that

$$\begin{aligned} \frac{\mu(K)}{\mu(\mathscr{U})} &\leq \frac{\mu(K)}{\mu(\mathscr{V})} \\ &= \frac{\sum_{i<n} \mu(K \cap \mathscr{B}(x_i, \epsilon_{x_i}))}{\sum_{i<n} \mu(\mathscr{B}(x_i, \epsilon_{x_i}))} \\ &< 1 - \delta, \end{aligned}$$

thus there is no open set $\mathscr{U} \supseteq K$ such that $\mu(\mathscr{U}) \leq \mu(K)/(1-\delta)$, contradicting the regularity of μ. It follows that

$$\forall^*_\mu x \in B \ \left(\liminf_{\epsilon \to 0} \rho_x(\epsilon) \geq 1 \right),$$

thus (†) holds. $\dashv$

It now follows that for μ-almost all $x \in X$,

$$\begin{aligned} (df_*\mu/d\mu)(x) &= \lim_{\epsilon \to 0} \left(\frac{\int_{\mathscr{B}(x,\epsilon)} (df_*\mu/d\mu)(y) \ d\mu(y)}{\mu(\mathscr{B}(x,\epsilon))} \right) \ \mu\text{−a.e.} \\ &= \lim_{\epsilon \to 0} \left(\frac{f_*\mu(\mathscr{B}(x,\epsilon))}{\mu(\mathscr{B}(x,\epsilon))} \right) \ \mu\text{−a.e.}, \end{aligned}$$

and the proposition follows. $\dashv$

The *full group of* E is the group $[E]$ of Borel automorphisms of X whose graphs are contained in X, or equivalently, the group of Borel automorphisms $f : X \to X$ such that

$$\forall x \in X \ (xEf(x)).$$

Note that μ is E-quasi-invariant exactly when μ is quasi-invariant with respect to every element of $[E]$. A function $D : E \to \mathbb{R}^+$ is a *cocycle* if $D(x,z) = D(x,y)D(y,z)$ whenever $xEyEz$.

Proposition 2.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and μ is an E-quasi-invariant probability measure. Then there is a Borel cocycle $D : E \to \mathbb{R}^+$ such that for every $f \in [E]$ and (locally) integrable $\varphi : X \to \mathbb{R}^+$,*

$$\int \varphi(x) \ df_*\mu(x) = \int \varphi(x) D(f^{-1}(x), x) \ d\mu(x).$$

Proof. By Feldman-Moore [36], there is a countable group $\Gamma \leq [E]$ which *generates* E. That is, E coincides with the *orbit equivalence relation associated with* Γ, which is given by

$$xE_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \ (\gamma \cdot x = y).$$

By the Lusin-Novikov Uniformization Theorem, there is a Borel function $\pi : E \to \Gamma$ such that

$$\forall (x, y) \in E \ (\pi(x, y) \cdot y = x).$$

Define $D : E \to \mathbb{R}^+$ by

$$D(y, x) = (d\pi(x, y)_* \mu / d\mu)(y).$$

Now suppose that $f \in [E]$, for each $\gamma \in \Gamma$ put

$$A_\gamma = \{x \in X : f^{-1}(x) = \gamma^{-1} \cdot x \text{ and } \pi(x, \gamma^{-1} \cdot x) = \gamma\},$$

and observe that

$$\begin{aligned}
\int \varphi(x) \ df_* \mu(x) &= \sum_{\gamma \in \Gamma} \int_{A_\gamma} \varphi(x) \ d\gamma_* \mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{A_\gamma} \varphi(x) (d\gamma_* \mu / d\mu)(x) \ d\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{A_\gamma} \varphi(x) (d\pi(x, \gamma^{-1} \cdot x)_* \mu / d\mu) \ d\mu(x) \\
&= \sum_{\gamma \in \Gamma} \int_{A_\gamma} \varphi(x) D(\gamma^{-1} \cdot x, x) \ d\mu(x) \\
&= \int \varphi(x) D(f^{-1}(x), x) \ d\mu(x).
\end{aligned}$$

Unfortunately, it need not be the case that D is a cocycle. However, it is the case that D is a *cocycle almost everywhere*, in the sense that

$$\forall_\mu^* x \in X \, \forall u, v, w \in [x]_E \ (D(u, w) = D(u, v) D(v, w)).$$

Granting this, D can be turned into the desired cocycle by setting it equal to 1 on the union of the E-classes on whose restriction it fails to be a cocycle.

To see that D really is a cocycle almost everywhere, it is enough to show that for all $\gamma, \delta \in \Gamma$,

$$D(\delta^{-1} \cdot x, x) = D(\delta^{-1} \cdot x, \gamma^{-1} \cdot x) D(\gamma^{-1} \cdot x, x) \ \mu\text{–a.e.},$$

or equivalently, that

$$(d\delta_*\mu/d\mu)(x) = (d(\gamma\delta^{-1})_*\mu/d\mu)(\gamma^{-1} \cdot x) \cdot (d\gamma_*\mu/d\mu)(x) \ \mu\text{–a.e.} \tag{†}$$

Fix a Polish ultrametric d on X which is compatible with the underlying Borel structure of X. Noting that the pullback of d through the action of γ is also a Polish ultrametric on X which is compatible with its underlying Borel structure, it follows from the uniqueness of the derivative and Proposition 2.5 that

$$\begin{aligned}(d(\gamma\delta^{-1})_*\mu/d\mu)(\gamma^{-1} \cdot x) &= \lim_{\epsilon \to 0} \left(\frac{(\gamma\delta^{-1})_*\mu(\mathscr{B}(\gamma^{-1} \cdot x, \epsilon))}{\mu(\mathscr{B}(\gamma^{-1} \cdot x, \epsilon))} \right) \ \mu\text{–a.e.} \\ &= \lim_{\epsilon \to 0} \left(\frac{\delta_*\mu(\gamma \cdot \mathscr{B}(\gamma^{-1} \cdot x, \epsilon))}{\gamma_*\mu(\gamma \cdot \mathscr{B}(\gamma^{-1} \cdot x, \epsilon))} \right) \ \mu\text{–a.e.} \\ &= (d\delta_*\mu/d\gamma_*\mu)(x) \ \mu\text{–a.e.}\end{aligned}$$

The chain rule (for derivatives of measures) now implies that (†) holds. ⊣

Remark 2.8. It is not difficult to see that $D : E \to \mathbb{R}^+$ is unique modulo null sets, in that any other such map must agree with D on an E-invariant set of full measure.

Remark 2.9. Our primary use of Proposition 2.7 will be when φ is the *characteristic function* $\mathbb{1}_B$ of some Borel set $B \subseteq X$, in which case we obtain that

$$\forall f \in [E] \ \left(\mu(f^{-1}(B)) = \int_B D(f^{-1}(x), x) \ d\mu(x) \right).$$

Note that when $D : E \to \mathbb{R}^+$ is the constant cocycle, this just says that the elements of $[E]$ are all measure-preserving. In this case, we say that μ is *E-invariant*.

Remark 2.10. A probability measure μ on X is *D-invariant* if D agrees with the cocycle of Proposition 2.7 almost everywhere. It follows that if Γ generates E, then μ is D-invariant exactly when

$$\forall^*_\mu x \in X \, \forall \gamma \in \Gamma \ \big(D(\gamma^{-1} \cdot x, x) = (d\gamma_*\mu/d\mu)(x) \big).$$

In fact, if $\mathscr{U}$ is a countable open basis for X, then μ is D-invariant exactly when

$$\forall \gamma \in \Gamma \, \forall U \in \mathscr{U} \ \left(\mu(\gamma^{-1}(U)) = \int_U D(\gamma^{-1} \cdot x, x) \ d\mu(x) \right).$$

To see this, observe that this condition ensures that the map $B \mapsto \int_B D(\gamma^{-1} \cdot x, x)$ is a probability measure. It follows from regularity that both this measure and $\gamma_* \mu$ are determined by their values on $\mathscr{U}$, and therefore must be identical.

Remark 2.11. As noted in Kechris-Miller [55], the cocycle D is simply the Radon-Nikodym derivative of the σ-finite measures M_l and M_r on E which are given by

$$M_l(A) = \int |A_x| \, d\mu(x) \text{ and } M_r(A) = \int |A^y| \, d\mu(y),$$

where $A_x = \{y \in A : (x, y) \in A\}$ and $A^y = \{x \in A : (x, y) \in A\}$.

In light of the observations we have made thus far, it is reasonable to view the Borel cocycle $D : E \to \mathbb{R}^+$ associated with a quasi-invariant measure μ as giving a notion of relative mass between E-related points of X. That is, we think of each equivalence class as being a single mass which has been divided into countably many pieces, and $D(x, y)$ as the ratio of the mass of the piece corresponding to x to the mass of the piece corresponding to y.

In fact, we obtain a notion of relative mass between subsets of each E-class. Let

$$[E]^{<\infty} = \{S \subseteq X \text{ finite} : \forall x, y \in S \ (xEy)\},$$

and define an equivalence relation F on $[E]^{<\infty}$ by

$$(S, T) \in F \Leftrightarrow (S \text{ and } T \text{ are contained in the same } E\text{-class}).$$

Given an E-class C, $z \in C$, and $S \subseteq C$, we will use

$$|S|_z = \sum_{x \in S} D(x, z)$$

to denote the *mass of S relative to z*, and we extend D to $\widetilde{D} : F \to \mathbb{R}^+$ by setting

$$\widetilde{D}(S, T) = \frac{|S|_z}{|T|_z} = \frac{\sum_{x \in S} D(x, z)}{\sum_{y \in T} D(y, z)},$$

where $z \in C$. Note that because D is a Borel cocycle, $\widetilde{D}(S, T)$ is independent of the choice of $z \in C$ and is also a Borel cocycle. As no confusion will result, we will use D to refer to both cocycles.

Now suppose that $S \in [E]^{<\infty}$. Although $|S|_z$ can depend on our choice of z, whether or not $|S|_z$ is finite does not. We say that S is *D-finite* if $|S|_z$ is finite, and S is *D-infinite* otherwise. The equivalence relation E is *D-periodic* if all of its classes are D-finite, and E is *D-aperiodic* if all of its classes are D-infinite. A set $B \subseteq X$ is *D-negligible* if it is null with respect to every D-invariant measure, and φ *holds D-almost everywhere* if the set of x for which $\varphi(x)$ fails is D-negligible.

Proposition 2.12. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle.*

1. *If E is D-periodic, then E is smooth.*

2. *If E is smooth, then D-almost every class of E is D-finite.*

Proof. To see (1), note that the classes of any D-periodic equivalence relation each contain a finite, non-empty set of x's such that

$$\forall y \in [x]_E \ (D(y, x) \leq 1).$$

Letting $\preceq$ be a Borel linear ordering of X, it follows that

$$A = \{x \in B : \forall y \in [x]_E \ (D(y, x) < 1 \text{ or } y \preceq x)\}$$

is a Borel transversal of E.

To see (2) suppose, towards a contradiction, that E is smooth, D-aperiodic and μ is D-invariant. Fix a Borel transversal $B \subseteq X$ of E, and build an infinite pairwise disjoint sequence of Borel complete sections $B_n \subseteq X$ such that

$$\forall n \in \mathbb{N} \ (\mu(B_n) > \mu(B)).$$

Then $\sum_{n \in \mathbb{N}} \mu(B_n) = \infty$, a contradiction. $\dashv$

Now suppose that E is a smooth equivalence relation and $D : E \to \mathbb{R}^+$ is a Borel cocycle. It follows from Proposition 2.12 that the D-aperiodic part of E admits no D-invariant probability measures. Moreover, if $B \subseteq X$ is a Borel transversal of the restriction of E to its D-periodic part, then there is a natural correspondence between $P(B)$ and the space of D-invariant probability measures.

Thus, from now on we will focus on D-aperiodic equivalence relations. The *pre-ordering induced by* D is the assignment of pre-orderings to the classes of E which is given by

$$x \leq_D y \Leftrightarrow (xEy \text{ and } D(x,y) \leq 1).$$

Clearly the restriction of $\leq_D$ to any E-class is pre-linear.

Proposition 2.13. *Suppose that X is a Polish space, E is a countable Borel equivalence relation, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E is D-aperiodic. Then*

$$\{x \in X : \leq_D |[x]_E \textit{ is a discrete linear ordering of } [x]_E\}$$

is D-negligible.

Proof. Suppose, towards a contradiction, that

$$\forall x \in X \ (\leq_D |[x]_E \text{ is a discrete linear ordering of } [x]_E),$$

and that there is a D-invariant probability measure μ on X. By Proposition 2.12, we may assume that $\leq_D$ provides a $\mathbb{Z}$-ordering of each class of E. Let $+$ be the successor function for $\leq_D$, and observe that

$$\begin{aligned} \mu(X) &= \mu(X^+) \\ &= \int_X D(x^+, x) \, d\mu(x) \\ &> \mu(X), \end{aligned}$$

a contradiction. ⊣

Note that in the invariant case, the restriction of $\leq_D$ to each class of $E|B$ is a discrete linear order exactly when B is a partial transversal of E, so that the σ-ideal generated by such sets is just the σ-ideal of Borel sets $B \subseteq X$ for which $E|B$ is smooth. It is not difficult, however, to come up with cocycles $D : E \to \mathbb{R}^+$ for which the latter σ-ideal is strictly contained in the former. It is the former σ-ideal which will be of primary importance in the arguments to come.

It is trivial to see that the restriction of $\leq_D$ to each class of E can be a discrete pre-order, even in the presence of a D-invariant probability measure. Nevertheless, there is a density condition that $\leq_D$ must obey off of a D-negligible set. A cocycle D is *dense around* x if

$$\forall \epsilon > 0 \, \exists^\infty y \in [x]_E \ (1 \leq D(x,y) \leq 1 + \epsilon).$$

Proposition 2.14. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. Then there is a Borel set $B \subseteq X$ such that the restriction of $\leq_D$ to each class of $E|B$ is a discrete linear order and D is dense around every point of $X \setminus [B]_E$.*

Proof. Set $A = \{x \in X : D \text{ is not dense around } x\}$, let $x \mapsto \epsilon(x)$ be a Borel assignment of points of $(0,1)$ to points of A such that

$$\forall x \in A \, \forall^\infty y \in [x]_E \ (D(x,y) < 1 \text{ or } D(x,y) > 1 + \epsilon(x)),$$

and for each E-class C, define

$$\epsilon(C) = \sup_{x \in A \cap C} \epsilon(x).$$

Put $xFy \Leftrightarrow D(x,y) = 1$, fix a Borel transversal $A' \subseteq A$ of $F|A$, and observe that

$$B = \{x \in A' : \epsilon(x) > \epsilon([x]_E)/2\}$$

is an $(E|A)$-complete section and $\leq_D$ discretely orders each class of $E|B$. $\dashv$

Of course, Proposition 2.14 cannot be strengthened so as to ensure that

$$\forall \epsilon > 0 \, \exists^\infty y \in [x]_E \ (1 < D(x,y) < 1 + \epsilon),$$

off of a D-negligible set. It is important to note that we also cannot guarantee that

$$\exists^\infty y \in [x]_E \ (D(x,y) = 1),$$

off of a D-negligible set:

Example 2.15. We will use $\mathscr{C} = 2^{\mathbb{N}}$ to denote *Cantor space*, and $\mathscr{N}_s$ to denote the basic clopen subset of $\mathscr{C}$ which consists of sequences that begin with $s \in 2^{<\mathbb{N}}$. Let μ be the probability measure on $2^{\mathbb{N}}$ which satisfies

$$\mu(\mathscr{N}_{s0}) = \frac{\mu(\mathscr{N}_s)}{1 + 2^{1/2^{|s|}}},$$

for all $s \in 2^{<\mathbb{N}}$, and let

$$\sigma(x) = \begin{cases} 0^n 1y & \text{if } x = 1^n 0y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

be the *odometer* on $\mathscr{C}$. Alternatively, one can think of the odometer as "addition by 10^∞ with right carry." Define E_0 on $\mathscr{C}$ by

$$xE_0y \Leftrightarrow \forall^\infty n \in \mathbb{N}\ (x_n = y_n),$$

noting that off of the eventually constant sequences,

$$xE_0y \Leftrightarrow \exists n \in \mathbb{Z}\ (\sigma^n(x) = y).$$

Now note that if $\mathscr{U} \subseteq \mathscr{C}$ is open, then

$$\mu(\mathscr{U})/4 \leq \mu(\sigma^{\pm 1}(\mathscr{U})) \leq 4\mu(\mathscr{U}),$$

from which it easily follows that μ is E_0-quasi-invariant. By Proposition 2.5, μ is invariant with respect to the cocycle $D : E_0 \to \mathbb{R}^+$ which is given by

$$\begin{aligned} D(x,y) &= \lim_{k\to\infty} \left(\frac{\mu(\sigma^n(\mathscr{N}_{y|k}))}{\mu(\mathscr{N}_{y|k})} \right) \\ &= \lim_{k\to\infty} \left(\frac{\mu(\mathscr{N}_{x|k})}{\mu(\mathscr{N}_{y|k})} \right) \\ &= \lim_{k\to\infty} \prod_{i<k} 2^{(x_i-y_i)/2^i} \\ &= 2^{\sum_{i<k}(x_i-y_i)/2^i}, \end{aligned}$$

where $x = \sigma^n(y)$. In particular, if $D(x,y) = 1$ then x and y must be base 2 representations of the same real. As xE_0y, it follows that $x = y$.

3 Maximal finite subequivalence relations

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. A *(partial) finite subequivalence relation* (or *fsr*) of E is a finite Borel equivalence relation F, defined on a Borel set $\mathrm{dom}(F) \subseteq X$, such that $F \subseteq E$. The standard Borel space of finite subsets of X is denoted by $[X]^{<\infty}$, and the standard Borel space of finite, pairwise E-related subsets of X is denote by $[E]^{<\infty}$.

In contrast with the viewpoint of ergodic theory, we will often find it useful to think of ourselves as working within individual E-classes, albeit in a sufficiently uniform manner that the objects we build are Borel. As a result, it will be useful to think of Borel sets $\Phi \subseteq [E]^{<\infty}$ as definable statements about finite subsets of E-classes. Adopting this point of view, it is natural to use $\Phi(S)$ to denote $S \in \Phi$. An fsr $F \subseteq E$ is Φ-*satisfying* if

$$\forall x \in \mathrm{dom}(F)\ (\Phi([x]_F)),$$

and a Φ-satisfying fsr $F \subseteq E$ is Φ-*maximal* if

$$\forall S \in [E]^{<\infty}\ (S \cap \mathrm{dom}(F) = \emptyset \Rightarrow \neg\Phi(S)),$$

or equivalently, if there is no Φ-satisfying fsr $F' \supsetneq F$ such that

$$\forall x \in \mathrm{dom}(F)\ \big([x]_F = [x]_{F'}\big).$$

The following fact was first explicitly isolated in Kechris-Miller [55], although it can also be easily shown via the techniques of Kechris-Solecki-Todorcevic [56]:

Theorem 3.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $\Phi \subseteq [E]^{<\infty}$ is Borel. Then E admits a Φ-maximal fsr.*

Proof. Consider the graph G on $[E]^{<\infty}$ which is given by

$$G = \{(S,T) \in [E]^{<\infty} : S \neq T \text{ and } S \cap T \neq \emptyset\}.$$

We begin by noting that G admits a Borel $\aleph_0$-*coloring*, i.e., a Borel map $c : [E]^{<\infty} \to I$, with I a countable (discrete) set, such that

$$\forall S,T \in [E]^{<\infty}\ \big((S,T) \in G \Rightarrow c(S) \neq c(T)\big).$$

By Feldman-Moore [36], there are Borel involutions $g_n : X \to X$ such that

$$E = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(g_n).$$

Let $<$ be a Borel linear ordering of X, and given $S \in [E]^{<\infty}$, let $\langle x_i^S\rangle_{i<n}$ be the $<$-increasing enumeration of S and let $c(S)$ be the lexicographically least sequence $\langle k_{ij}\rangle_{i,j<n}$ of natural numbers such that

$$\forall i,j < n \left(g_{k_{ij}} \cdot x_i^S = x_j^S\right).$$

Now suppose, towards contradiction, that c is not a coloring. Then we can find $(S,T) \in G$ such that $c(S) = c(T)$. Put $n = |S| = |T|$ and fix $i,j < n$ such that $x_i^S = y_j^S$. Then

$$\begin{aligned} i < j &\Leftrightarrow x_i^S < x_j^S \\ &\Leftrightarrow x_i^S < g_{k_{ij}}(x_i^S) \\ &\Leftrightarrow y_j^S < g_{k_{ij}}(y_j^S) \\ &\Leftrightarrow y_j^S < y_i^S \\ &\Leftrightarrow j < i, \end{aligned}$$

thus $i = j$ and $x_i^S = y_i^S$. It follows that for all $m < n$,

$$\begin{aligned} x_m^S &= g_{k_{im}}(x_i^S) \\ &= g_{k_{im}}(y_i^S) \\ &= y_m^S, \end{aligned}$$

thus $S = T$, contradicting our assumption that $(S,T) \in G$.

Now that we have seen that c is a coloring, recursively define a sequence of fsr's F_n of E by putting xF_ny exactly when

$$\exists S \in [E]^{<\infty} \left(x,y \in S \text{ and } \Phi(S) \text{ and } c(S) = n \text{ and } S \cap \left(\bigcup_{m<n} \mathrm{dom}(F_m)\right) = \emptyset\right).$$

Noting that

$$\forall m \neq n \; (\mathrm{dom}(F_m) \cap \mathrm{dom}(F_n) = \emptyset),$$

it follows that $F = \bigcup_{n \in \mathbb{N}} F_n$ is an fsr of E. It is also clear that the classes of F satisfy Φ. To see that F is Φ-maximal, it remains to check that if $S \in [E]^{<\infty}$ satisfies Φ, then $S \cap \mathrm{dom}(F) \neq \emptyset$. In fact, it follows from the definition of $F_{c(S)}$ that either S forms a class of $F_{c(S)}$ or $S \cap F_n \neq \emptyset$, for some $n < c(S)$. $\dashv$

Theorem 3.1 is a remarkably useful tool in the study of countable Borel equivalence relations. Here is a simple example of its application:

Lemma 3.2. *Suppose that X is a Polish space, E is an aperiodic countable Borel equivalence relation on X, and n is a positive natural number. Then there is a Borel subequivalence relation F of E, all of whose classes are of cardinality n.*

Proof. It is clear how to proceed when E is smooth. In the general case, put

$$\Phi(S) \Leftrightarrow |S| = n,$$

and let $F \subseteq E$ be a Φ-maximal fsr. Clearly $X \setminus \mathrm{dom}(F)$ intersects each E-class in at most $n-1$ points, thus E is smooth off of the set on which F is as desired. $\dashv$

As can be easily seen by considering $\Phi = [E]^{<\infty}$, a Φ-maximal fsr can be properly contained in another Φ-maximal fsr. Thus we are lead to the following question:

Question 3.3. *Are there stronger notions of maximality which satisfy an analog of Theorem 3.1?*

A Φ-maximal fsr $F \subseteq E$ is *strongly Φ-maximal* if

$$\forall x \in \mathrm{dom}(F)\, \forall S \in [E]^{<\infty} \left(S \cap \mathrm{dom}(F) = \emptyset \Rightarrow \neg\Phi([x]_F \cup S)\right),$$

or equivalently, if there is no Φ-satisfying fsr $F' \supsetneq F$ such that

$$\forall x \in \mathrm{dom}(F)\ ([x]_F = [x]_{F'} \cap \mathrm{dom}(F)).$$

Although strong maximality might appear to be a rather innocuous strengthening of maximality, there are simple obstructions to the existence of such fsr's:

Example 3.4. Set $X = \mathbb{N}$, $E = \mathbb{N}^2$, and define

$$\Phi(S) \Leftrightarrow 0 \in S.$$

Clearly E does not admit a strongly Φ-maximal fsr.

Nevertheless, a version of Theorem 3.1 still goes through:

Theorem 3.5. *Suppose that E is a countable Borel equivalence relation and $\Phi \subseteq [E]^{<\infty}$ is Borel. Then there is an E-invariant Borel set $B \subseteq X$ such that*

1. *$E|B$ admits no invariant probability measures.*

2. *$E|(X \setminus B)$ admits a strongly Φ-maximal fsr.*

Proof. Given a Φ-maximal fsr $F \subseteq E$, let Φ_F be the set of $S \in \Phi$ such that

$$S \cap \mathrm{dom}(F) \text{ is a single } F\text{-class which is properly contained in } S,$$

noting that F is strongly Φ-maximal exactly when $\Phi_F = \emptyset$.

Fix a Φ-maximal fsr $F_0 \subseteq E$, and recursively define an increasing sequence of fsr's of E by letting F_{n+1} be the union of F_n with a Φ_{F_n}-maximal fsr F'_n. Define

$$F = \bigcup_{n \in \mathbb{N}} F_n,$$

let $A \subseteq X$ be the aperiodic part of F, and set $B = [A]_E$. It is clear that $F|(X \setminus B)$ is a strongly Φ-maximal fsr of $E|(X \setminus B)$. As the intersection of each F-class with $\mathrm{dom}(F_0)$ is a non-empty finite set, it follows that $F|A$ is a smooth aperiodic subequivalence relation of $E|A$, thus $E|B$ admits no invariant probability measures. $\dashv$

Remark 3.6. By Example 3.4, the analog of Theorem 3.5 for Baire category is false.

There is another direction in which one can improve Theorem 3.1. Given a sequence of Borel sets $\Phi_n \subseteq [E]^{<\infty}$, we say that an fsr $F \subseteq E$ is *simultaneously Φ_n-satisfying* if for each $n \in \mathbb{N}$, every E-class contains an F-class which satisfies Φ_n. Although it is once again straightforward to see that such an fsr need not always exist, we do have the following:

Theorem 3.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, E is D-aperiodic, and $\langle \Phi_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel subsets of $[E]^{<\infty}$ such that each Φ_n contains a subset of every E-class. Then there is a D-co-negligible E-invariant Borel set $B \subseteq X$ such that $E|B$ admits a simultaneously Φ_n-satisfying fsr.*

Proof. We begin by recursively defining Borel sets $\Psi_n \subseteq [E]^{<\infty}$, fsr's $F_n \subseteq E$, and Borel sets $A_n \subseteq X$. Set

$$\Psi_n(S) \Leftrightarrow \exists T \subseteq S \ (\Phi_n(T) \text{ and } \forall m < n \ (D(A_m \cap S, [T]_{F_m}) > 2^n)), \qquad (*)$$

and fix a Ψ_n-maximal fsr $F_n \subseteq E$. Fix a Borel assignment $S \mapsto T(S)$ of witness to $(*)$ to the classes of F_n and define

$$A_n = \bigcup_{S \text{ an } F_n\text{-class}} T(S).$$

For each $m \in \mathbb{N}$, define

$$B_m = A_m \setminus \bigcup_{n > m} [A_n]_{F_m},$$

and set $F = \bigcup_{n \in \mathbb{N}} F_n | B_n$. As the B_m's are pairwise disjoint and F_m-invariant, it follows that $F \subseteq E|B$ is Φ_n-satisfying, where

$$B = \bigcap_{n \in \mathbb{N}} [B_n]_E.$$

It remains to show that $X \setminus B$ is D-negligible. Suppose, towards a contradiction, that there is a D-invariant probability measure μ such that $\mu([B_m]_E) < 1$, and fix $m \in \mathbb{N}$ such that $A = X \setminus [B_m]_E$ is of positive measure. As $[\mathrm{dom}(F_m)]_E$ is co-D-negligible, it follows that $\mu(A \cap \mathrm{dom}(F_m)) > 0$, thus for all $n > m$,

$$\begin{aligned}
\mu(A \cap [A_n]_{F_m}) &= \int_{A \cap \mathrm{dom}(F_m)} D([T([x]_{F_n})]_{F_m}, [x]_{F_n}) \, d\mu(x) \\
&= \int_{A \cap \mathrm{dom}(F_n)} D([T([x]_{F_n})]_{F_m}, A_m \cap [x]_{F_n}) \cdot \\
&\qquad\qquad D(A_m \cap [x]_{F_n}, [x]_{F_n}) \, d\mu(x) \\
&< \int_{A \cap \mathrm{dom}(F_n)} D(A_m \cap [x]_{F_n}, [x]_{F_n})/2^n \, d\mu(x) \\
&= \mu(A \cap A_m \cap \mathrm{dom}(F_n))/2^n \\
&\leq \mu(A \cap A_m)/2^n.
\end{aligned}$$

It follows that

$$\mu(A \cap A_m) > \sum_{n>0} \mu(A \cap [A_n]_{F_m}),$$

thus $\mu(A \cap B_m) > 0$, the desired contradiction. ⊣

Corollary 3.8. *Suppose that X is a Polish space, E is an aperiodic countable Borel equivalence relation on X, μ is an E-invariant probability measure on X, and $\langle \Phi_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel subsets of $[E]^{<\infty}$ such that each Φ_n contains a subset of every E-class. Then there is a conull E-invariant Borel set $B \subseteq X$ such that $E|B$ admits a simultaneously Φ_n-satisfying fsr.*

Although Corollary 3.8 has several applications, the main reason we mention it here is that it points to a significant difference between measure and category:

Example 3.9. For each $n \in \mathbb{N}$, define $\Phi_n \subseteq [E_0]^{<\infty}$ by

$$\Phi_n(S) \Leftrightarrow \exists x \in X \ \left(S = \{\sigma^i(x)\}_{i<n}\right).$$

Letting μ be the usual product measure on $\mathscr{C}$, it follows from Corollary 3.8 that there is a conull E_0-invariant Borel set $B \subseteq X$ such that $E_0|B$ admits a simultaneously Φ_n-satisfying fsr.

On the other hand, every E_0-invariant Borel set $B \subseteq X$ for which $E_0|B$ admits a simultaneously Φ_n-satisfying fsr is necessarily meager! To see this suppose, towards a contradiction, that $B \subseteq X$ is a non-meager E_0-invariant Borel set and $F \subseteq E_0|B$ is a simultaneously Φ_n-satisfying fsr. Define

$$A = \{x \in \mathscr{C} : (x, \sigma(x)) \notin F\},$$

and note that since A is an $E_0|B$-complete section, there exists $s \in 2^{<\mathbb{N}}$ such that B is comeager in $\mathscr{N}_s$. Find a comeager E_0-invariant Borel set $C \subseteq \mathscr{C}$ such that $C \cap \mathscr{N}_s \subseteq B$, and note that since

$$\sigma^{2^{|s|}}(\mathscr{N}_s) = \mathscr{N}_s,$$

it follows that there are comeagerly many E_0-classes which do not contain an F-class which satisfies Φ_n, for any $n > 2^{|s|}$, a contradiction.

As we have already mentioned, the goal of the next several sections is to provide an effective means of describing the set of D-invariant probability measures. A significant part of this project is to understand the circumstances under which there are no D-invariant probability measures. In the invariant case, an answer to this piece of the puzzle was given by Nadkarni [62]. Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. The *full semigroup of* E is the semigroup $[\![E]\!]$ of all partial Borel injections from X into itself whose graphs are contained in E. A map $f \in [\![E]\!]$ is a *compression of* E if it has full domain and the complement of its range is an E-complete section, and E is *compressible* if it admits a compression.

Theorem 3.10 (Nadkarni). *Suppose X is a Polish space and E is an aperiodic countable Borel equivalence relation on X. Then E has no invariant probability measure $\Leftrightarrow$ E is compressible.*

We will eventually provide a new proof of this theorem, as well as a version for D-invariant measures. In the meantime, we will close this section by noting that the idea behind Example 3.9 can be used to answer questions of Nadkarni [62] and Eigen-Hajian-Nadkarni [30] regarding a potential alternative characterization of compressibility.

Suppose $f : X \to X$ is an aperiodic Borel automorphism. A set $B \subseteq X$ is *weakly wandering* if there is an infinite set $S \subseteq \mathbb{N}$ such that $\langle f^n(B) \rangle_{n \in S}$ is pairwise disjoint.

Question 3.11 (Nadkarni). *Suppose that X is a Polish space, $f : X \to X$ is a Borel automorphism, and E is the orbit equivalence relation associated with f. Is E compressible exactly when E admits a weakly wandering Borel complete section?*

It is straightforward to see that if E admits a weakly wandering Borel complete section, then E is compressible. However, Eigen-Hajian-Nadkarni [30] answered Question 3.11 in the negative by constructing a compressible Borel automorphism which admits no weakly wandering Borel complete section. However, their method left open the following possibility:

Question 3.12 (Eigen-Hajian-Nadkarni). *Suppose that X is a Polish space, $f : X \to X$ is a Borel automorphism, and E is the orbit equivalence relation associated* *with f. Is E compressible exactly when there is a countably generated partition $\mathscr{B}$ of X into E-invariant Borel sets such that $E|B$ admits a weakly wandering Borel complete section, for each $B \in \mathscr{B}$?*

Again, the answer to this question is no. Here is a counterexample:

Example 3.13. Let μ be the usual product measure on $\mathscr{C}$, fix a null comeager E_0-invariant Borel set $A \subseteq \mathscr{C}$, and put $f = \sigma|A$. As μ is the unique invariant probability measure for E_0, it follows that there are no invariant probability measures for $E_0|A$, and thus $E_0|A$ is compressible.

Now suppose that $\mathscr{B}$ is a countably generated partition of A into E_0-invariant Borel sets, and note that there is a comeager set $C \in \mathscr{B}$. As in Example 3.9, it follows that if $B \subseteq C$ is a Borel complete section for $E_0|C$, then after throwing out a meager E_0-invariant Borel subset of C, we may assume that there exists an $s \in 2^{<\mathbb{N}}$ such that $\mathscr{N}_s \cap C \subseteq B$. It then follows that no collection of more than $2^{|s|}$ iterates of B under f is pairwise disjoint, thus $f|C$ admits no weakly wandering Borel complete section.

4 Ergodic decomposition

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and $f : X \to \mathbb{R}$ is a Borel function. For each set $S \in [E]^{<\infty}$, fix $z \in [S]_E$ and put

$$I_S(f) = \frac{\sum_{x \in S} f(x) D(x,z)}{\sum_{x \in S} D(x,z)}.$$

Since D is a cocycle, this quantity does not depend on our choice of z. Intuitively, $I_S(f)$ is simply S's best guess at the value of $\int f \; d\mu$. We will use

$$\mu_S(B) = I_S(\mathbb{1}_B)$$

to denote the *density of B within S*.

Proposition 4.1. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, μ is a D-invariant probability* *measure on X, $F \subseteq E$ is a finite Borel equivalence relation, and $f : X \to \mathbb{R}$ is a bounded Borel function. Then*

$$\int f(x)\ d\mu(x) = \int I_{[x]_F}(f)\ d\mu(x).$$

Proof. By breaking up X into countably many F-invariant pieces, we may assume that each F-class is of cardinality $n > 0$. Let $B \subseteq X$ be a Borel transversal for F, let $g : X \to X$ be a Borel automorphism whose associated orbit equivalence relation is F, and observe that

$$\begin{aligned}
\int f(x)\ d\mu(x) &= \sum_{i<n} \int_{g^i(B)} f(x)\ d\mu(x) \\
&= \sum_{i<n} \int_B f \circ g^i(x) D(g^i(x), x)\ d\mu(x) \\
&= \int_B \sum_{y \in [x]_F} f(y) D(y, x)\ d\mu(x) \\
&= \int_B I_{[x]_F}(f) \sum_{y \in [x]_F} D(y, x)\ d\mu(x) \\
&= \sum_{i<n} \int_{g^i(B)} I_{[x]_F}(f)\ d\mu(x) \\
&= \int I_{[x]_F}(f)\ d\mu(x).
\end{aligned}$$

$\dashv$

The following fact is intended as a descriptive version of the Hurewicz ergodic theorem for countable Borel equivalence relations:

Theorem 4.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and $\mathscr{F}$ is a countable family of bounded real-valued Borel functions on X. Then there is a Borel set $B \subseteq X$ such that the restriction of $\leq_D$ to each class of $E|B$ is a discrete linear order, and an increasing sequence of finite Borel subequivalence relations $F_n \subseteq E$ such that for all $x \in X \setminus [B]_E$, all $f \in \mathscr{F}$, and all D-invariant probability measures $\mu \in P(X)$:*

1. *$\langle I_{[x]_{F_n}}(f) \rangle_{n \in \mathbb{N}}$ converges uniformly in x to some $I_x(f) \in \mathbb{R}$.*

2. *$x \mapsto I_x(f)$ is E-invariant.*

3. $\int f(x)\ d\mu(x) = \int I_x(f)\ d\mu(x)$.

Proof. A finite Borel equivalence relation $F \subseteq E$ is *ϵ-approximating for f* if

$$\forall C \in X/E\, \forall S, T \in C/F \ \ (|I_S(f) - I_T(f)| \leq \epsilon)\,.$$

Lemma 4.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, $\epsilon > 0$, and $F \subseteq E$ is a finite Borel equivalence relation which is ϵ-approximating for f. Then there is a Borel set $B \subseteq X$ and a finite Borel equivalence relation $F \subseteq F' \subseteq E$ such that:*

1. *F' is $(3\epsilon/4)$-approximating for $f|(X \setminus [B]_E)$.*

2. *The restriction of $\leq_D$ to each class of $E|B$ is a discrete linear order.*

Proof. For each E-class C, set

$$I_C(f) = \frac{1}{2}\left(\inf_{x \in C} I_{[x]_F}(f) + \sup_{x \in C} I_{[x]_F}(f)\right),$$

define $\Phi \subseteq [E]^{<\infty}$ by

$$\Phi(S) \Leftrightarrow \big(S \text{ is } F\text{-invariant and } |I_S(f) - I_{[S]_E}(f)| \leq \epsilon/4\big)\,,$$

let $F'' \subseteq E$ be a Φ-maximal fsr, and define $F' = F \cup F''$. Setting

$$A = \{x \in X : \exists y, z \in [x]_E \ (|I_{[y]_{F'}}(f) - I_{[z]_{F'}}(f)| > 3\epsilon/4)\},$$

it is clear that $F'|(X \setminus A)$ is $(3\epsilon/4)$-approximating for f.

Now suppose, towards a contradiction, that there is no Borel complete section $B \subseteq A$ for $E|A$ such that the restriction of $\leq_D$ to each class of $E|B$ is a discrete linear order. We will use D_F to denote the cocycle induced by D on X/F.

Sublemma 4.4. *Suppose that X is a Polish space, E is a countable Borel equivalence relation, $F \subseteq E$ is a finite Borel subequivalence relation, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. If E/F admits a $\leq_{D_F}$-linearly discretely ordered Borel complete section, then E admits a $\leq_D$-linearly discretely ordered Borel complete section.*

Proof. By breaking X into countably many E-invariant Borel sets and going down to an F-invariant Borel complete section, we may assume that each class of F is of cardinality $n \in \mathbb{N}$. Fix an F-invariant Borel complete section $A \subseteq X$ for E such that the restriction of $\leq_{D_F}$ to each class of $(E/F)|(B/F)$ is a discrete linear order. As it is clear how to proceed when E/F is smooth, we may assume that

$$\forall x \in A\, \exists y, z \in A\ (D([y]_F, [x]_F), D([x]_F, [z]_F) \geq n).$$

Define $B \subseteq A$ by

$$B = \{x \in A : \forall y \in [x]_F\ (D(x, y) \geq 1)\},$$

noting that B is a complete section for $F|A$ and

$$\forall x \in B\ (1 \leq |[x]_F|_x \leq n).$$

As $D([x]_F, [y]_F) = D([x]_F, x)D(x, y)D(y, [y]_F)$, it follows that

$$D(x, y)/n \leq D([x]_F, [y]_F) \leq nD(x, y),$$

thus for all $x, y \in B$ there exists $x', y' \in B$ such that

$$\forall z \in B\ (x \leq_D z \leq_D y \Rightarrow [x']_F \leq_{D_F} [z]_F \leq_{D_F} [y']_F).$$

As D_F is discrete, the set of all such z must be finite. Define F' on B by

$$xF'y \Leftrightarrow D(x, y) = 1,$$

note that the classes of F' are finite, and let $C \subseteq B$ be a Borel transversal of F'. It is clear that C is an E-complete section and the restriction of $\leq_D$ to each class of $E|C$ is a discrete linear order. ⊣

Now define $(E|A)$-complete sections $Y, Z \subseteq A$ by

$$Y = \{y \in A : I_{[y]_{F'}}(f) < I_{[y]_E}(f) - \epsilon/4\}$$

and

$$Z = \{z \in A : I_{[z]_{F'}}(f) > I_{[z]_E}(f) + \epsilon/4\},$$

noting that Y and Z are disjoint from $\mathrm{dom}(F'')$, thus $F|Y = F'|Y$ and $F|Z = F'|Z$. By Proposition 2.14 and Sublemma 4.4, we may assume that there exists $x \in A$ such that the restrictions of D_F to $(E/F)|(Y/F)$ and $(E/F)/(Z/F)$ are dense around every point of $[x]_E \cap Y$ and $[x]_E \cap Z$.

Fix $y \in [x]_E \cap Y'$ and $z \in [x]_E \cap Z'$, choose $m, n \in \mathbb{N}$ such that

$$2/3 \leq (m/n)\, D([y]_F, [z]_F) \leq 3/2,$$

and choose $\delta > 0$ sufficiently small that

$$\delta/(m|[y]_F|_x), \delta/(n|[z]_F|_x) < 1/2.$$

Now fix pairwise F-inequivalent elements $y_i \in [x]_E \cap Y$ and $z_j \in [x]_E \cap Z$ such that

$$\forall i, j \in \mathbb{N}\ \ (1 \leq D([y_i]_F, [y]_F), D([z_j]_F, [z]_F) \leq 1 + \delta)\,.$$

Set $Y' = \bigcup_{i<m} [y_i]_F$ and $Z' = \bigcup_{j<n} [z_i]_F$, and note that

$$m|[y]_F|_x \leq |Y'|_x \leq m|[y]_F|_x + m\delta$$

and

$$n|[z]_F|_x \leq |Z'|_x \leq n|[z]_F|_x + n\delta,$$

thus

$$\frac{m|[y]_F|_x}{n(|[z]_F|_x + \delta)} \leq \frac{\sum_{i<m} |[y_i]_F|_x}{\sum_{j<n} |[z_j]_F|_x} \leq \frac{m(|[y]_F|_x + \delta)}{n|[z]_F|_x}.$$

As the middle quantity is by definition $D(Y', Z')$, we have that

$$D(Y', Z') \leq (m/n)D([y]_F, [z]_F) + \delta/(n|[z]_F|_x) \leq 2$$

and

$$D(Z', Y') \leq (n/m)D([z]_F, [y]_F) + \delta/(m|[y]_F|_x) \leq 2,$$

so $D(Y' \cup Z', Y'), D(Y' \cup Z', Z') \leq 3$. It follows that

$$\begin{aligned}
I_{Y' \cup Z'}(f) &= D(Y', Y' \cup Z')I_{Y'}(f) + D(Z', Y' \cup Z')I_{Z'}(f) \\
&\leq (1/3)I_{Y'}(f) + (2/3)I_{Z'}(f) \\
&\leq (1/3)(I_{[x]_E}(f) - \epsilon/4) + (2/3)(I_{[x]_E}(f) + \epsilon/2) \\
&= I_{[x]_E}(f) + \epsilon/4,
\end{aligned}$$

and similarly,

$$\begin{aligned} I_{Y'\cup Z'}(f) &= D(Y', Y'\cup Z')I_{Y'}(f) + D(Z', Y'\cup Z')I_{Z'}(f) \\ &\geq (2/3)I_{Y'}(f) + (1/3)I_{Z'}(f) \\ &\geq (2/3)(I_{[x]_E}(f) - \epsilon/2) + (1/3)(I_{[x]_E}(f) + \epsilon/4) \\ &= I_{[x]_E}(f) - \epsilon/4, \end{aligned}$$

thus $|I_{Y'\cup Z'}(f) - I_{[x]_E}(f)| \leq \epsilon/4$, contradicting the Φ-maximality of F''. ⊣

Now fix a sequence of functions $f_n \in \mathscr{F}$ such that

$$\forall n \in \mathbb{N}\ \left(\mathscr{F} = \{f_m\}_{m\geq n}\right),$$

put $F_0 = \Delta(X) = \{(x,x)\}_{x\in X}$, and given a finite Borel equivalence relation $F_n \subseteq E$, apply Lemma 4.3 finitely many times to produce an E-invariant Borel set $B_n \subseteq X$ on which E admits a $\leq_D$-discretely linearly ordered complete section and a finite Borel equivalence relation $F_n \subseteq F_{n+1} \subseteq E$ which is $(1/n)$-approximating for f_n.

It is clear that the restriction of E to $B = \bigcup_{n\in\mathbb{N}} B_n$ admits a Borel complete section $C \subseteq B$ such that the restriction of $\leq_D$ to each class of $E|C$ is a discrete linear order. Moreover, the sequence $\langle I_{[x]_{F_n}}(f)\rangle_{n\in\mathbb{N}}$ converges uniformly for $x \in X \setminus B$. Letting $I_x(f)$ be this limit, it follows from Proposition 4.1 that

$$\int f\ d\mu(x) = \lim_{n\to\infty} \int I_{[x]_{F_n}}(f)\ d\mu(x) = \int I_x(f)\ d\mu(x).$$

Note that for all $\epsilon > 0$ and $n \in \mathbb{N}$ sufficiently large, F_n is ϵ-approximating for f. It easily follows that $I_x(f)$ is E-invariant. ⊣

Now we are ready to prove Ditzen's theorem:

Theorem 4.5 (Ditzen). *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E admits a D-invariant probability measure. Then the set $\mathscr{E}\mathscr{I}(D)$ of D-invariant, E-ergodic probability measures on X is non-empty and Borel, and there is a [surjective] Borel function $\pi : X \to \mathscr{E}\mathscr{I}(D)$ such that*

1. *If μ is D-invariant, then $\mu = \int \pi(x)\ d\mu(x)$.*

2. *If μ is also E-ergodic, then $\forall^*_\mu x \in X\ (\pi(x) = \mu)$.*

Proof (Miller). Without loss of generality, we may assume that $X = \mathscr{C}$. Let $\mathscr{F}$ be the set of characteristic functions of basic clopen sets. By Theorem 4.2, there is a co-D-negligible E-invariant Borel set $A \subseteq X$ and bounded E-invariant Borel functions $I(f) : A \to \mathbb{R}$, for $f \in \mathscr{F}$, such that

$$\int f(x)\ d\mu(x) = \int I_x(f)\ d\mu(x),$$

for every D-invariant probability measure μ on X. It follows from the proof of Theorem 4.2 that

$$I_x(\mathbb{1}_{\mathscr{C}}) = 1 \text{ and } \forall s \in 2^{<\mathbb{N}}\ \big(I_x(\mathbb{1}_{\mathscr{N}_s}) = I_x(\mathbb{1}_{\mathscr{N}_{s0}}) + I_x(\mathbb{1}_{\mathscr{N}_{s0}})\big),$$

for all $x \in A$. It now follows from Exercise 17.7 of Kechris [51] that each assignment

$$\mathscr{N}_s \mapsto I_x(\mathbb{1}_{\mathscr{N}_s})$$

uniquely determines a probability measure μ_x on X. It follows that

$$\forall s \in 2^{<\mathbb{N}}\ \left(\mu(\mathscr{N}_s) = \int \mu_x(\mathscr{N}_s)\ d\mu(x)\right),$$

for every D-invariant probability measure μ on X. In particular, if μ is also E-ergodic, then

$$\forall s \in 2^{<\mathbb{N}}\, \forall^*_\mu x \in A\ (\mu(\mathscr{N}) = \mu_x(\mathscr{N}_s)),$$

thus $\forall^*_\mu x \in A\ (\mu = \mu_x)$. It follows that

$$B = \{x \in A : \mu_x \text{ is not a } D\text{-invariant, } E\text{-ergodic probability measure}\}$$

is null with respect to every D-invariant, E-ergodic probability measure on X.

Lemma 4.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and $B \subseteq X$ is a Borel set which is null with respect to every D-invariant, E-ergodic probability measure on X. Then B is null with respect to every D-invariant probability measure on X.*

Proof. As usual, we may extend the topology of X, while maintaining its underlying Borel structure, so that B is open. By Feldman-Moore [36], there is a countable group $G \leq [E]$ which generates E. Fix an enumeration $\langle B_n \rangle_{n \in \mathbb{N}}$ of a G-invariant algebra of sets $\mathscr{B}$ which forms a basis for the new topology such that $B \in \mathscr{B}$, and define $d : \mathscr{I}(D)^2 \rightarrow \mathbb{R}$ by

$$d(\mu, \nu) = \sum_{n \in \mathbb{N}} \frac{|\mu(B_n) - \nu(B_n)|}{2^n}.$$

We claim that $(\mathscr{I}(D), d)$ is a compact metric space. It is clear that d is symmetric and satisfies the triangle inequality. As any probability measure on X is determined by its values on the elements of $\mathscr{B}$, it follows that

$$\mu = \nu \Leftrightarrow d(\mu, \nu) = 0,$$

thus d is a metric.

As d is clearly complete, it only remains to check that it is *totally bounded*, i.e., that X can be covered with finitely many ϵ-balls, for all $\epsilon > 0$ (see Proposition 4.2 of Kechris [51]). Fix $n > 4/\epsilon$, let S be the set of functions from $\{0, \ldots, n\}$ into itself, and for each $s \in S$, fix $\mu_s \in \mathscr{I}(D)$ such that

$$\forall i < n \ (s_i/n \leq \mu_s(B_i) \leq (s_i + 1)/n),$$

if such a probability measure exists. It is clear that for each $\mu \in \mathscr{I}(D)$, there exists $s \in S$ such that

$$\forall i < n \ (|\mu_s(B_i) - \mu(B_i)| < \epsilon/4),$$

and it follows that

$$\begin{aligned} d(\mu, \mu_s) &= \sum_{i \leq n} |\mu(B_i) - \mu_s(B_i)|/2^i + \sum_{i > n} |\mu(B_i) - \mu_s(B_i)|/2^i \\ &< \sum_{i \leq n} \epsilon/2^{i+2} + \sum_{i > n} 1/2^i \\ &< \epsilon/2 + 1/2^n \\ &< \epsilon, \end{aligned}$$

thus the sets of the form $\mathscr{B}(\mu_s, \epsilon)$ form the desired cover.

Now that we have established that $(\mathscr{I}(D), d)$ is compact, set

$$\alpha = \sup_{\mu \in \mathscr{I}(D)} \mu(B),$$

and define

$$A = \{\mu \in \mathscr{I}(D) : \mu(B) = \alpha\}.$$

As the map $\mu \mapsto \mu(B)$ is continuous, it follows that A is a non-empty compact convex subspace of $(\mathscr{I}(D), d)$. It now follows from the Krein-Millman Theorem that A has an extreme point μ. Since every extreme point of A is clearly E-ergodic, it follows that $\alpha = 0$. $\dashv$

It follows that the set of D-invariant, E-ergodic probability measures on X is non-empty and

$$\mathscr{E}\mathscr{I}(D) = \{\mu \in P(X) : \mu \text{ is } D\text{-invariant and } \forall^*_\mu x \in A \ (\mu = \mu_x)\},$$

thus $\mathscr{E}\mathscr{I}(D)$ is Borel. Fix a D-invariant, E-ergodic probability measure μ, and set

$$\pi(x) = \begin{cases} \mu_x & \text{if } x \in A \setminus B, \\ \mu & \text{otherwise.} \end{cases}$$

Clearly π is as desired. $\dashv$

Theorem 4.2 can also be used to reduce Ditzen's theorem to that of Kifer-Pirogov. This can be seen via the following strengthening of Lemma 9.3.2 of Zimmer [82]:

Theorem 4.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. Then there is a hyperfinite equivalence relation $F \subseteq E$ such that $\mathscr{E}\mathscr{I}(D) = \mathscr{E}\mathscr{I}(D|F)$, thus $\mathscr{I}(D) = \mathscr{I}(D|F)$.*

Proof. Without loss of generality, we may assume that X is a Polish ultrametric space. Let $\mathscr{B}$ be the (countable) set of open balls of rational radius, and apply Theorem 4.2 to obtain an E-invariant Borel set $A \subseteq X$ off of which E admits a $\leq_D$-discretely linearly ordered complete section, and an increasing sequence of finite Borel subequivalence relations $F_n \subseteq E$ such that for all $x \in A$, $B \in \mathscr{B}$, and D-invariant probability measure $\mu \in P(X)$:

1. $\langle I_{[x]_{F_n}}(\mathbb{1}_B)\rangle_{n\in\mathbb{N}}$ converges uniformly in x to some $I_x(\mathbb{1}_B) \in \mathbb{R}$.

2. $I_x(\mathbb{1}_B)$ is E-invariant.

3. $\int \mathbb{1}_B(x)\ d\mu(x) = \int I_x(\mathbb{1}_B)\ d\mu(x)$.

Setting $\mu_x(B) = I_x(\mathbb{1}_B)$, it follows that if $\mu \in \mathscr{E}\mathscr{I}(D)$, then

$$\forall B \in \mathscr{B}\, \forall^*_\mu x \in B\ (\mu(B) = \mu_x(B)).$$

We will show that

$$F = \bigcup_{n\in\mathbb{N}} F_n \cup (E|(X \setminus A))$$

is the desired hyperfinite subequivalence relation.

Suppose that $\mu \in \mathscr{E}\mathscr{I}(D)$ and $A \subseteq X$ is a non-null Borel set. We will show that $\mu([A]_F) = 1$. Fix $0 < \epsilon < 1/2$. By Lemma 2.6, we can find $B \in \mathscr{B}$ such that

$$\mu(A \cap B) > (1 - \epsilon^2)\mu(B),$$

and it follows from (1) that we can find $n \in \mathbb{N}$ such that

$$\forall^*_\mu x \in X\ \left(\left|\mu_{[x]_{F_n}}(B) - \mu(B)\right| < \epsilon\mu(B)\right),$$

where $\mu_S(B) = I_S(\mathbb{1}_B)$. Now define $C \subseteq X$ by

$$C = \{x \in X : \mu_{[x]_{F_n}}(A \cap B) > (1 - \epsilon/(1 - \epsilon))\mu_{[x]_{F_n}}(B)\},$$

and observe that

$$\begin{aligned}
(1-\epsilon^2)\mu(B) &< \mu(A \cap B) \\
&= \int \mu_{[x]_{F_n}}(A \cap B)\ d\mu(x) \\
&= \int_C \mu_{[x]_{F_n}}(A \cap B)\ d\mu(x) + \int_{X\setminus C} \mu_{[x]_{F_n}}(A \cap B)\ d\mu(x) \\
&\leq \int_C \mu_{[x]_{F_n}}(B)\ d\mu(x) + \left(1 - \left(\frac{\epsilon}{1-\epsilon}\right)\right) \int_{X\setminus C} \mu_{[x]_{F_n}}(B)\ d\mu(x) \\
&= \mu(B) - \left(\frac{\epsilon}{1-\epsilon}\right) \int_{X\setminus C} \mu_{[x]_{F_n}}(B)\ d\mu(x) \\
&\leq \mu(B) - \left(\frac{\epsilon}{1-\epsilon}\right) \int_{X\setminus C} (1-\epsilon)\mu(B)\ d\mu(x) \\
&= \mu(B) - \left(\frac{\epsilon}{1-\epsilon}\right) \mu(X \setminus C)(1-\epsilon)\mu(B) \\
&= \mu(B)\left(1 - \epsilon\mu(X \setminus C)\right),
\end{aligned}$$

thus $1 - \epsilon\mu(X \setminus C) > 1 - \epsilon^2$, and it follows that $\mu(C) > 1 - \epsilon$.

Noting that for almost all $x \in B$ we have that

$$\begin{aligned}\mu_{[x]_{F_n}}(A \cap B) &> (1-\epsilon)\mu_{[x]_{F_n}}(B)\\ &> (1-\epsilon)^2\mu(B)\\ &> 0,\end{aligned}$$

it follows that almost all of C is contained in $[A \cap B]_{F_n}$. In particular, we have that

$$\begin{aligned}\mu([A]_F) &\geq \mu([A \cap B]_{F_n})\\ &\geq \mu(C)\\ &> 1-\epsilon,\end{aligned}$$

and as $0 < \epsilon < 1/2$ was arbitrary, it follows that $\mu([A]_F) = 1$. $\dashv$

5 Existence of D-invariant probability measures

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. A map $f \in [\![E]\!]$ is D-*invariant* if

$$\forall x \in \mathrm{dom}(f)\ (D(f(x), x) = 1).$$

We will use $[D]$ and $[\![D]\!]$ to denote the D-invariant elements of $[E]$ and $[\![E]\!]$, respectively. The following simple example provides a significant obstruction to a generalization of Theorem 3.10:

Example 5.1. Suppose that $X = \mathbb{N}$ and $E = (\mathbb{N}^+)^2$. Define $D : E \to \mathbb{R}^+$ by

$$D(m, n) = m/n.$$

Clearly D is a Borel cocycle, E is D-aperiodic, E admits no D-invariant Borel probability measures, and

$$[\![D]\!] = \{\mathrm{id}|S : S \subseteq \mathbb{N}\},$$

thus $[\![D]\!]$ contains no compressions of E.

A natural first reaction to this example is to hope that if $[\![D]\!]$ is sufficiently trivial then there is no D-invariant probability measure, and then to try to push through an analog of Theorem 3.10 outside of this special case. Such an approach is implausible, however, for Example 2.15 provides a setting in which $[\![D]\!]$ is trivial but there is a D-invariant probability measure.

So it seems that if we are to have any hope of coming up with an analog of Theorem 3.10, $[\![D]\!]$ must be enriched in such a manner that points which cannot be mapped to one another via a D-invariant function can be split into pieces so that fractions of these points can be mapped to one another. This is not so different from the invariant case: If G is a countable group of Borel automorphisms for which there is no G-invariant probability measure, then it is necessary to pass to the full group of E_G in order to find a compression. The purpose of this is to allow us to begin with two sets $A, B \subseteq X$ which do not map to one another via an element of G, break them up into pieces, and then map these pieces to one another via different elements of G. The additional constraint of D-invariance simply forces us to go a step further, and break up the points of X themselves.

We will use $\mathfrak{B}$ to denote the set of Borel functions $b : X \to [0, 1]$. We will think of the elements of $\mathfrak{B}$ as *fractional Borel subsets of* X, with $b(x)$ specifying the fraction of x included in b. We will use $\leq$ to denote the partial ordering of $\mathfrak{B}$ given by

$$a \leq b \Leftrightarrow \forall x \in X \ (a(x) \leq b(x)).$$

The minimal element of $(\mathfrak{B}, \leq)$ is the constantly 0 function $\mathbb{0}$, and the maximal element of $(\mathfrak{B}, \leq)$ is the constantly 1 function $\mathbb{1}$. We will use $+$ and $-$ to denote the binary operations on $\mathfrak{B}$ given by

$$[a + b](x) = \min(a(x) + b(x), 1)$$

and

$$[a - b](x) = \max(a(x) - b(x), 0).$$

It is straightforward to check that the map $B \mapsto \mathbb{1}_B$ provides an embedding of the Borel subsets of X under union into $\mathfrak{B}$ under addition.

Given a Borel cocycle $D : E \to \mathbb{R}^+$, we will think of each non-negative function $\varphi : E \to \mathbb{R}$ as specifying D-invariant mappings between (possibly fractional) multiples of (the masses associated with) points of X. Specifically, if $r = \varphi(x, y)$, then we think of φ as sending r copies of x to $rD(x, y)$ copies of y. The *domain of* φ is a real-valued function on X, given by

$$[\mathrm{dom}_D(\varphi)](x) = \sum_{y \in [x]_E} \varphi(x, y),$$

and similarly, the *range of* φ is given by

$$[\mathrm{rng}_D(\varphi)](y) = \sum_{x \in [y]_E} \varphi(x, y)D(x, y).$$

The *fractional full semigroup of* D is

$$[\![D]\!] = \{\varphi : E \to [0, 1] \text{ Borel} : \mathrm{dom}_D(\varphi), \mathrm{rng}_D(\varphi) \in \mathfrak{B}\}.$$

Given $\varphi \in [\![D]\!]$ and $b \leq \mathrm{dom}_D(\varphi)$, the *restriction of* φ *to* b is

$$[\varphi \restriction b](x, y) = \varphi(x, y)D(x, y)b(x)/[\mathrm{dom}_D(\varphi)](x),$$

as well as the *image of* b *under* φ,

$$\varphi[b] = \mathrm{rng}_D(\varphi \restriction b).$$

Given $\varphi, \psi : E \to \mathbb{R}^+$ such that $\mathrm{dom}_D(\varphi) \leq \mathrm{rng}_D(\psi)$, the *composition* of φ, ψ is

$$\varphi * \psi(x, y) = \sum_{z \in [x]_E} \psi(x, z)\varphi(z, y),$$

Also, the *inverse of* $\varphi \in [\![D]\!]$ is given by

$$\varphi^*(x, y) = \varphi(y, x)D(y, x).$$

Note that $[\![D]\!]$ embeds into $[\![D]\!]$ via the map $f \mapsto \varphi_f$, where

$$\varphi_f(x, y) = \begin{cases} 1 & \text{if } x \in \mathrm{dom}(f) \text{ and } f(x) = y, \\ 0 & \text{otherwise.} \end{cases}$$

The image of $[D]$ under this embedding is the subsemigroup of the image of $[\![D]\!]$ of maps for which

$$\mathrm{dom}_D(\varphi) = \mathbb{1} \text{ and } \mathrm{rng}_D(\varphi) = \mathbb{1}.$$

A map has *full domain* if it satisfies the former condition, and *full range* if it satisfies the latter.

A fractional Borel set $b \in \mathfrak{B}$ is a *complete section for E* if

$$B_b = \{x \in B : b(x) > 0\}$$

is a complete section for E, and B is a *co-complete section for E* if its *complement* $\mathbb{1} - b$ is a complete section. A *D-compression of E* is a map $\varphi \in [\![D]\!]$ with full domain and co-complete range. The equivalence relation E is *D-compressible* if it admits a D-compression.

The following fact is the analog of Nadkarni's theorem [62]:

Theorem 5.2. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E is D-aperiodic. Then exactly one of the following holds:*

1. *E admits a D-invariant probability measure.*
2. *E is D-compressible.*

Proof. To see $(2) \Rightarrow \neg(1)$ suppose, towards a contradiction, that $\varphi \in [\![D]\!]$ is a D-compression of E and μ is a D-invariant probability measure on X. Fix a sequence of automorphisms $f_n \in [E]$ and a sequence of Borel sets $B_n \subseteq X$ such that the sets of the form graph$(f_n|B_n)$ partition E, and observe that

$$\begin{aligned}
\int [\mathrm{rng}_D(\varphi)](y)\ d\mu(y) &= \int \sum_{x\in[y]_E} \varphi(x,y)D(x,y)\ d\mu(y) \\
&= \int \sum_{n\in\mathbb{N}} \varphi(f_n^{-1}(y),y)D(f_n^{-1}(y),y)\mathbb{1}_{f_n(B_n)}(y)\ d\mu(y) \\
&= \sum_{n\in\mathbb{N}} \int_{f_n(B_n)} \varphi(f_n^{-1}(y),y)D(f_n^{-1}(y),y)\ d\mu(y) \\
&= \sum_{n\in\mathbb{N}} \int_{f_n(B_n)} \varphi(f_n^{-1}(y),y)\ d(f_n)_*\mu(y) \\
&= \sum_{n\in\mathbb{N}} \int_{B_n} \varphi(x,f_n(x))\ d\mu(x) \\
&= \int \sum_{n\in\mathbb{N}} \varphi(x,f_n(x))\mathbb{1}_{B_n}(x)\ d\mu(x) \\
&= \int \sum_{y\in[x]_E} \varphi(x,y)\ d\mu(x) \\
&= \int [\mathrm{dom}_D(\varphi)](x)\ d\mu(x).
\end{aligned}$$

As $\mathrm{dom}_D(\varphi) = \mathbb{1}$, it follows that $\mathrm{rng}_D(\varphi) = \mathbb{1}$ μ-almost everywhere. As $\mathrm{rng}_D(\varphi)$ is a co-complete section for E, this contradicts the fact that μ is E-quasi-invariant.

To see $\neg(1) \Rightarrow (2)$, we will first use an enhanced version of the proof of Theorem 4.5 to show that if there are no D-invariant probability measures on X, then X can be partition into countably many E-invariant Borel sets on which we obtain very specific sorts of witnesses to D-negligibility. We will then describe how to build D-compressions from such witnesses.

Suppose that $B \subseteq X$ is an E-invariant Borel set.

1. A *witness to D-negligibility of B of type 1* is a Borel complete section $A \subseteq B$ such that the restriction of $<_D$ to each class of $E|A$ is a discrete linear order.

2. A *witness to D-negligibility of B of type 2* is a partition of B into Borel sets $B_n \subseteq B$ and an increasing sequence of finite Borel equivalence relations $F_n \subseteq E$ such that

   (a) For all $x \in X$ and all B in the algebra $\mathscr{B}$ generated by the B_n's, $\mu_{[x]_{F_n}}(B)$ converges uniformly in x to some real number $\mu_x(B)$.

(b) For each $B \in \mathscr{B}$, the map $x \mapsto \mu_x(B)$ is E-invariant.

(c) $\sum_{n \in \mathbb{N}} \mu_x(B_n) < 1$.

3. A *witness to D-negligibility of B of type 3* is a finite Borel equivalence relation $F \subseteq E$, a Borel automorphism $f \in [E]$, and a Borel complete section $A \subseteq B$ such that for all $x \in B$,

$$D(f([x]_F) \cap A, [x]_F \cap A) > 1.$$

Lemma 5.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle which admits no invariant probability measures. Then X can be partitioned into countably many E-invariant Borel sets which admit witnesses to D-negligibility.*

Proof. We will gradually strip away verifiably D-negligible E-invariant Borel subsets of X until nothing remains. By Feldman-Moore [36], there is a countable group $G \leq [E]$ which generates E. For each $g \in G$, define $D_g : X \to \mathbb{R}^+$ by

$$D_g(y) = D(g(y), y),$$

and for each $n > 0$, set

$$X_{gn} = \{x \in X : 1/n \leq D_g(x) \leq n\}.$$

By Exercise 13.12 of Kechris [51], we can find a Polish ultrametric d on X, compatible with its underlying Borel structure, in which each element of G is a homeomorphism, each D_g is continuous, and each X_{gn} is clopen. Let $\mathscr{U}$ be a countable G-invariant algebra of clopen subsets of X which contains all clopen balls of rational diameter and every X_{gn}, for $g \in G$ and $n \in \mathbb{N}$. Let $\mathscr{F}$ be the closure of the family of functions of the form $\mathbb{1}_U$ and D_g, for $U \in \mathscr{U}$ and $g \in G$, under pairwise multiplication. Note that each element of $\mathscr{F}$ is continuous. By Theorem 4.2, there is an E-invariant Borel set $X_1 \subseteq X$ whose complement is verifiably D-negligible of type 1, and an increasing sequence of finite Borel equivalence relations $F_n \subseteq E|X_1$ such that for all $x \in X_1$, $f \in \mathscr{F}$, and $\mu \in \mathscr{I}(D)$:

1. $\langle I_{[x]_{F_n}}(f)\rangle_{n\in\mathbb{N}}$ converges uniformly in $x \in X_1$ to some $I_x(f) \in \mathbb{R}$.

2. $I_x(f)$ is E-invariant.

3. $\int f(x)\ d\mu(x) = \int I_x(f)\ d\mu(x)$.

We will use $\mu_x(U)$ to denote $I_x(1_U)$. For $x \in X_1$ and $B \subseteq X$ Borel, put

$$\mu_x^*(B) = \inf \left\{ \sum_{V \in \mathscr{V}} \mu_x(V) : \mathscr{V} \subseteq \mathscr{U} \text{ covers } B \right\}.$$

Sublemma 5.4. *For all $x \in X_1$, the map μ_x^* is a measure on X.*

Proof. First, we will show that μ_x^* is an *outer measure*. That is,

1. $\mu_x^*(\emptyset) = 0$.

2. $\forall A \subseteq B \subseteq X$ Borel $(\mu_x^*(A) \leq \mu_x^*(B))$.

3. $\forall B_0, B_1, \ldots \subseteq X$ Borel $\left(\mu_x^*\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n\in\mathbb{N}} \mu_x^*(B_n)\right)$.

Conditions (1) and (2) follow trivially from the definition of μ_x^*. To see (3), fix $\epsilon > 0$ and covers $\mathscr{V}_n \subseteq \mathscr{U}$ of B_n such that

$$\mu_x^*(B_n) \geq \sum_{V \in \mathscr{V}_n} \mu_x(V) - \epsilon/2^{n+1},$$

and observe that $\mathscr{V} = \bigcup_{n\in\mathbb{N}} \mathscr{V}_n$ is a cover of $\bigcup_{n\in\mathbb{N}} B_n$ and

$$\mu_x^*\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{V\in\mathscr{V}} \mu_x(V) \leq \epsilon + \sum_{n\in\mathbb{N}} \mu_x^*(B_n),$$

thus $\mu_x^*\left(\bigcup_{n\in\mathbb{N}} B_n\right) \leq \sum_{n\in\mathbb{N}} \mu_x^*(B_n)$.

In fact, μ_x^* is a *metric outer measure*. That is,

$$\forall A, B \subseteq X \text{ Borel } \big(d(A,B) > 0 \Rightarrow \mu_x^*(A \cup B) = \mu_x^*(A) + \mu_x^*(B)\big),$$

where

$$d(A,B) = \inf_{x\in A, y\in B} d(x,y).$$

To see this, fix $\epsilon > 0$ and find a cover $\mathscr{V} \subseteq \mathscr{U}$ of $A \cup B$ with

$$\sum_{V \in \mathscr{V}} \mu_x(V) < \mu_x^*(A \cup B) + \epsilon.$$

Let $\langle V_n \rangle_{n \in \mathbb{N}}$ be an enumeration of $\mathscr{V}$, and for each $n \in \mathbb{N}$, find a family $\mathscr{W}_n \subseteq \mathscr{U}$ of sets of diameter less than $d(A, B)$ which partitions V_n. Set

$$\mathscr{W} = \bigcup_{n \in \mathbb{N}} \mathscr{W}_n,$$

and define

$$\mathscr{W}_A = \{W \in \mathscr{W} : A \cap W \neq \emptyset\}$$

and

$$\mathscr{W}_B = \{W \in \mathscr{W} : B \cap W \neq \emptyset\},$$

noting that $\mathscr{W}_A \cap \mathscr{W}_B = \emptyset$. Now observe that

$$\begin{aligned} \mu_x^*(A) + \mu_x^*(B) &\leq \sum_{W \in \mathscr{W}_A} \mu_x(W) + \sum_{W \in \mathscr{W}_B} \mu_x(W) \\ &\leq \sum_{W \in \mathscr{W}} \mu_x(W) \\ &= \sum_{n \in \mathbb{N}} \sum_{W \in \mathscr{W}_n} \mu_x(W). \end{aligned}$$

Fix $n \in \mathbb{N}$ and an enumeration W_{mn} of $\mathscr{W}_n$, and note that since μ_x is finitely additive,

$$\begin{aligned} \sum_{W \in \mathscr{W}_n} \mu_x(W) &= \lim_{\ell \to \infty} \sum_{m < \ell} \mu_x(W_{mn}) \\ &= \lim_{\ell \to \infty} \mu_x \left(\bigcup_{m < \ell} W_{mn} \right) \\ &\leq \lim_{\ell \to \infty} \mu_x \left(\bigcup_{m \in \mathbb{N}} W_{mn} \right) \\ &= \mu_x(V_n), \end{aligned}$$

thus

$$\mu_x^*(A) + \mu_x^*(B) \leq \sum_{n \in \mathbb{N}} \mu_x(V_n) < \mu_x^*(A \cup B) + \epsilon,$$

and it follows that $\mu_x^*(A) + \mu_x^*(B) \leq \mu_x^*(A \cup B)$.

It now follows from Example 4 of §17.B of Kechris [51] that μ_x^* is a measure. $\dashv$

Now let $\mathscr{U}_n \subseteq \mathscr{U}$ denote the set of clopen balls of diameter $1/n$, and note that there is an E-invariant Borel set $X_2 \subseteq X_1$ such that $X_1 \setminus X_2$ is verifiably D-negligible of type 2, and

$$\sum_{U \in \mathscr{U}_n} \mu_x(U) = 1,$$

for all $n \in \mathbb{N}$ and $x \in X_2$. More generally, as $\mu_x(U) + \mu_x(X \setminus U) = 1$ for all $U \in \mathscr{U}$, it follows that

$$\mu_x(U) = \sum_{V \in U_n} \mu_x(U \cap V).$$

Sublemma 5.5. *For all $x \in X_2$ and $f \in \mathscr{F}$, $I_x(f) = \int f(y)\ d\mu_x^*(y)$.*

Proof. We will begin with the special case that $f = \mathbb{1}_U$, for some $U \in \mathscr{U}$. It is enough to show that $\mu_x^*(U) \geq \mu_x(U) - \epsilon$, for all $\epsilon > 0$ and $x \in X_2$. For each $n \in \mathbb{N}$, fix a finite pairwise disjoint family $\mathscr{V}_n \subseteq \mathscr{U}_n$ of subsets of U such that

$$\mu_x\Big(U \cap \bigcup \mathscr{V}_n\Big) > \mu_x(U) - \epsilon/2^{n+1}.$$

Set $K_n = U \cap \bigcap_{m<n} \bigcup \mathscr{V}_n$, and observe that $K = \bigcap_{n \in \mathbb{N}} K_n$ is totally bounded and closed, thus compact. Now suppose, towards a contradiction, that

$$\mu_x^*(K) < \mu_x(U) - \epsilon.$$

Then there is a cover of K by pairwise disjoint sets $V_n \in \mathscr{U}$ such that

$$\sum_{n \in \mathbb{N}} \mu_x^*(V_n) < \mu_x(U) - \epsilon.$$

By compactness, there is a finite subcover $V_1, \ldots, V_n$. Letting V be the union of the sets in this subcover, it follows that $V \in \mathscr{U}$ and $\mu_x^*(V) < \mu_x(U) - \epsilon$.

We claim that $K_n \subseteq V$ for some $n \in \mathbb{N}$. As

$$\mu_x^*(V) < \mu_x(U) - \epsilon < \mu_x^*(K_n)$$

and μ_x^* is monotonic, this will give the desired contradiction. So suppose, towards a contradiction, that for each $n \in \mathbb{N}$ there exists $x_n \in K_n \setminus V$. It follows that for each

$n \in \mathbb{N}$, the closure of $X_n = \{x_m\}_{m>n}$ is compact, thus

$$\bigcap_{n\in\mathbb{N}} \overline{X_n} \subseteq K \setminus V \neq \emptyset,$$

which contradicts the fact that $K \subseteq V$.

Now we are ready for the general case. We will proceed via a straightforward approximation argument. Unfortunately, we need to make several approximations before getting to the main calculation, and the order in which these approximations are made is not the same as the order in which they are used. In order to help the reader keep track of things, we will therefore associate with each approximation a number which indicates the order in which it is used in the final calculation.

Suppose that $f \in \mathscr{F}$, fix $\epsilon > 0$, and note that by the continuity of f, there is a partition $\mathscr{V} \subseteq \mathscr{U}$ of X and real numbers f_V, for $V \in \mathscr{V}$, such that

$$\forall x \in X_2 \, \forall V \in \mathscr{V} \, \forall y \in V \ \left(|f(y) - f_V| \leq \epsilon\right).$$

Noting that for each $V \in \mathscr{V}$,

$$\begin{aligned} \left| \int_V f(y) d\mu_x^*(y) - f_V \mu_x^*(V) \right| &= \left| \int_V f(y) \ d\mu_x^*(y) - \int_V f_V \ d\mu_x^*(y) \right| \\ &\leq \int_V |f(y) - f_V| \ d\mu_x^*(y) \\ &\leq \epsilon \mu_x^*(V), \end{aligned}$$

it follows that

$$\left| \int f(y) \ d\mu_x^*(y) - \sum_{V\in\mathscr{V}} f_V \mu_x^*(V) \right| \leq \epsilon. \tag{1}$$

Fix finite sets $\mathscr{V}_n \subseteq \mathscr{V}$ and a partition of X_2 into E-invariant Borel sets $X_2^{(n)}$ with

$$\sum_{V\in\mathscr{V}\setminus\mathscr{V}_n} f_V \mu_x^*(V) \leq \epsilon, \tag{2}$$

$$\sum_{V\in\mathscr{V}\setminus\mathscr{V}_n} I_x(f\mathbb{1}_V) \leq \epsilon, \tag{7}$$

and

$$\mu_x\big(X \setminus \bigcup \mathscr{V}_n\big) \leq \epsilon / \sup_{x\in X} f(x), \tag{9}$$

for all $n \in \mathbb{N}$ and $x \in X_2^{(n)}$ (actually, (7) follows from (9)). It follows from the $f = \mathbb{1}_U$ case that

$$f_V \mu_x^*(U) = f_V \mu_x(U) = I_x(f_V \mathbb{1}_V). \tag{3}$$

Now, for all $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that for all $x \in X_2^{(n)}$ and $V \in \mathscr{V}_n$,

$$\Big|\mu_x\big(X \setminus \bigcup \mathscr{V}_n\big) - \mu_{[x]_{F_{k_n}}}\big(X \setminus \bigcup \mathscr{V}_n\big)\Big| \leq \epsilon / \sup_{x\in X} f(x) \tag{8}$$

and

$$\Big|I_x(f_V \mathbb{1}_V) - I_{[x]_{F_{k_n}}}(f_V \mathbb{1}_V)\Big| + \Big|I_x(f) - I_{[x]_{F_{k_n}}}(f)\Big| \leq \epsilon / |\mathscr{V}_n|,$$

from which it follows that

$$\left|\sum_{V\in\mathscr{V}_n} I_x(f_V \mathbb{1}_V) - \sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f_V \mathbb{1}_V)\right| \leq \epsilon \tag{4}$$

and

$$|I_x(f) - I_{[x]_{F_{k_n}}}(f)| \leq \epsilon. \tag{6}$$

Noting that for all $k \in \mathbb{N}$,

$$\begin{aligned}
\Big|I_{[x]_{F_k}}(f_V \mathbb{1}_V) - I_{[x]_{F_k}}(f \mathbb{1}_V)\Big| &= \left|\frac{\sum_{y\in[x]_{F_k}} (f_V - f(y))\mathbb{1}_V(y) D(y,x)}{\sum_{y\in[x]_{F_k}} D(y,x)}\right| \\
&\leq \frac{\sum_{y\in[x]_{F_k}} |f_V - f(y)|\mathbb{1}_V(y) D(y,x)}{\sum_{y\in[x]_{F_k}} D(y,x)} \\
&\leq \epsilon \mu_{[x]_{F_k}}(V),
\end{aligned}$$

it follows that

$$\left|\sum_{V\in\mathscr{V}_n} I_{[x]_{F_k}}(f_V \mathbb{1}_V) - \sum_{V\in\mathscr{V}_n} I_{[x]_{F_k}}(f \mathbb{1}_V)\right| \leq \epsilon. \tag{5}$$

Finally, we are ready for the main calculation:

$$
\begin{aligned}
\left|\int f(y)\ d\mu_x^*(y) - I_x(f)\right| &\leq \left|\int f(y)\ d\mu_x^*(y) - \sum_{V\in\mathscr{V}} f_V\mu_x^*(V)\right| + \\
&\qquad \left|\sum_{V\in\mathscr{V}} f_V\mu_x^*(V) - I_x(f)\right| \\
&\leq \epsilon + \sum_{V\in\mathscr{V}\setminus\mathscr{V}_n} f_V\mu_x^*(V) + \left|\sum_{V\in\mathscr{V}_n} f_V\mu_x^*(V) - I_x(f)\right| \\
&\leq 2\epsilon + \left|\sum_{V\in\mathscr{V}_n} I_x(f_V\mathbb{1}_V) - I_x(f)\right| \\
&\leq 2\epsilon + \left|\sum_{V\in\mathscr{V}_n} I_x(f_V\mathbb{1}_V) - \sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f_V\mathbb{1}_V)\right| + \\
&\qquad \left|\sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f_V\mathbb{1}_V) - I_x(f)\right| \\
&\leq 3\epsilon + \left|\sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f_V\mathbb{1}_V) - \sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f\mathbb{1}_V)\right| + \\
&\qquad \left|\sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f\mathbb{1}_V) - I_x(f)\right| \\
&\leq 4\epsilon + |I_x(f) - I_{[x]_{F_{k_n}}}(f)| + \\
&\qquad \left|\sum_{V\in\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f\mathbb{1}_V) - I_{[x]_{F_{k_n}}}(f)\right| \\
&\leq 5\epsilon + \sum_{V\in\mathscr{V}\setminus\mathscr{V}_n} I_{[x]_{F_{k_n}}}(f\mathbb{1}_V) \\
&\leq 5\epsilon + \mu_{[x]_{F_{k_n}}}\left(X\setminus\bigcup\mathscr{V}_n\right)\sup_{x\in X} f(x) \\
&\leq 5\epsilon + \left|\mu_{[x]_{F_{k_n}}}\left(X\setminus\bigcup\mathscr{V}_n\right) - \mu_x\left(X\setminus\bigcup\mathscr{V}_n\right)\right|\sup_{x\in X} f(x) + \\
&\qquad\quad \mu_x\left(X\setminus\bigcup\mathscr{V}_n\right)\sup_{x\in X} f(x) \\
&\leq 7\epsilon.
\end{aligned}
$$

As $\epsilon > 0$ was arbitrary, it follows that $I_x(f) = \int f(y)\ d\mu_x^*(y)$. ⊣

In particular, it follows that μ_x^* is a probability measure on X, for every $x \in X_2$.

As E admits no D-invariant probability measures, it must be the case that

$$\forall x \in X_2\ (\mu_x^* \text{ is not } D\text{-invariant}).$$

By Proposition 2.10, for each $x \in X_2$ there exists $g \in G$ and $U \in \mathscr{U}$ such that

$$\mu_x^*(g(U)) \neq \int_U D_g(y)\ d\mu_x^*(y).$$

By choosing $n \in \mathbb{N}$ sufficiently large and setting $V = U \cap X_{gn}$, it follows that

$$\mu_x^*(g(V)) \neq \int_V D_g(y)\ d\mu_x^*(y).$$

By Sublemma 5.5, we can find a countable partition $\mathscr{X}$ of X into E-invariant Borel sets and natural numbers k_B, for $B \in \mathscr{X}$, such that for all $B \in \mathscr{X}$, one of the following holds:

1. $\forall x \in B\ \big(\mu_{[x]_{F_{k_B}}}(g(V)) < I_{[x]_{F_{kn}}}(D_g \mathbb{1}_V)\big)$.
2. $\forall x \in B\ \big(\mu_{[x]_{F_{k_B}}}(g(V)) > I_{[x]_{F_{kn}}}(D_g \mathbb{1}_V)\big)$.

Noting that

$$\mu_{[x]_{F_{k_B}}}(g(V)) = \frac{\sum_{y \in [x]_{F_{k_B}} \cap g(V)} D(y,x)}{\sum_{y \in [x]_{F_{k_B}}} D(y,x)}$$

and

$$I_{[x]_{F_{kn}}}(D_g \mathbb{1}_V) = \frac{\sum_{y \in [x]_{F_{k_B}} \cap V} D(g(y),y)D(y,x)}{\sum_{y \in [x]_{F_{k_B}}} D(y,x)} = \frac{\sum_{y \in g([x]_{F_{k_B}}) \cap g(V)} D(y,x)}{\sum_{y \in [x]_{F_{k_B}}} D(y,x)},$$

it follows that the above two conditions are equivalent to:

1. $\forall x \in B\ \big(D(g([x]_{F_{kn}}) \cap g(V), [x]_{F_{kn}} \cap g(V)) > 1\big)$.
2. $\forall x \in B\ \big(D(g([x]_{F_{kn}}) \cap g(V), [x]_{F_{kn}} \cap g(V)) < 1\big)$.

In case (1), $\langle F_{kn}, g, g(V)\rangle$ is a type 3 witness to D-negligibility. In case (2),

$$\big\langle g(F_{kn}), g^{-1}, g(V)\big\rangle$$

is a type 3 witness to D-negligibility. $\dashv$

To complete the proof of the theorem, it is now sufficient to show the following:

Lemma 5.6. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E can be partitioned into countably many E-invariant Borel sets which admit witnesses to D-negligibility. Then E is D-compressible.*

Proof. It is trivial to check that E must be D-aperiodic. There are now essentially four cases:

- E is smooth: Let B be a Borel transversal of E, and note that we can easily build a pairwise disjoint sequence of Borel complete sections $B_n \subseteq X$ for E, with $B_0 = B$, such that

  $$\forall n \in \mathbb{N}\, \forall x \in X \ \ (B_n \cap [x]_E \text{ is } D\text{-finite and } D(B_{n+1} \cap [x]_E, B_n \cap [x]_E) > 1) .$$

  For each $n \in \mathbb{N}$, fix a Borel function $\varphi_n : E \cap (B_n \times B_{n+1}) \to [0, 1]$ such that:

  1. $\forall x \in B_n \ \left(\sum_{y \in B_{n+1} \cap [x]_E} \varphi_n(x, y) = 1\right)$.
  2. $\forall y \in B_{n+1} \ \left(\sum_{x \in B_n \cap [y]_E} \varphi_n(x, y) \leq 1\right)$.

  Now define $\varphi \in [\![D]\!]$ by

  $$\varphi(x, y) = \begin{cases} \varphi_n(x, y) & \text{if } n \in \mathbb{N},\ x \in B_n, \text{ and } y \in B_{n+1}, \\ x & \text{if } x \notin \bigcup_{n \in \mathbb{N}} B_n \text{ and } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

  It is clear that φ is a D-compression of E.

- X admits a witness to D-negligibility of type 1: Fix a Borel complete section $B \subseteq X$ for E such that the restriction of $<_D$ to each class of $E|B$ is a discrete linear order. As we have already handled the smooth case, we may assume that each of these restrictions is of type $\mathbb{Z}$. Let $+ : B \to B$ be the corresponding successor function, define $f \in [E]$ by

  $$f(x) = \begin{cases} x^+ & \text{if } x \in B, \\ x & \text{otherwise,} \end{cases}$$

and set

$$\varphi(x,y) = \begin{cases} \varphi_f(x,y) & \text{if } x \in B, \\ \varphi_{\mathrm{id}}(x,y) & \text{otherwise.} \end{cases}$$

Clearly φ is a D-compression of E.

- X admits a witness to D-negligibility of type 2: Fix a partition of X into Borel sets $B_n \subseteq X$ and an increasing sequence of finite Borel equivalence relations of $F_n \subseteq E$ such that:

  1. For all $x \in X$ and all B in the algebra $\mathscr{B}$ generated by the B_n's, $\mu_{[x]_{F_n}}(B)$ converges uniformly in x to some real number $\mu_x(B)$.
  2. For each $B \in \mathscr{B}$, the map $x \mapsto \mu_x(B)$ is E-invariant.
  3. $\sum_{n\in\mathbb{N}} \mu_x(B_n) < 1$.

  Clearly we can find a countable partition $\mathscr{X}$ of X into E-invariant Borel sets such that for each $B \in \mathscr{X}$, there exists $\epsilon_B > 0$ and $k_B \in \mathbb{N}$ such that

  $$\mu_x(B_k) \leq \inf_{n\in\mathbb{N}} \mu_x \left(\bigcup_{k\geq n} B_k \right) - \epsilon_B,$$

  for all $x \in B$ and $k \geq k_B$. Moreover, we can ensure that

  $$B_{k_B} \text{ is a complete section for } E|B.$$

For $a, b \in \mathfrak{B}$, we will write $a \preceq b$ if there exists $\varphi \in [\![D]\!]$ such that

$$\mathrm{dom}_D(\varphi) = a \text{ and } \mathrm{rng}_D(\varphi) \leq b.$$

We will refer to such maps as *injections from a into b.*

Sublemma 5.7. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, $a, b \in \mathfrak{B}$, and $F \subseteq E$ is a finite Borel subequivalence relation of E such that $I_{[x]_F}(a) \leq I_{[x]_F}(b)$, for all $x \in X$. Then there is an injection $\varphi \in [\![D]\!]$ of a into b such that*

$$\forall (x,y) \in E \ (\varphi(x,y) > 0 \Rightarrow xFy).$$

Proof. Noting that $I_{[x]_F}(a) \leq I_{[x]_F}(b)$ implies that for all $x \in X$,

$$\sum_{y \in [x]_F} a(y)|y|_x \leq \sum_{y \in [x]_F} b(y)|y|_x,$$

the sublemma then follows easily from the smoothness of F. $\dashv$

Now fix $B \in \mathscr{X}_0$. We will recursively construct injections $\varphi_k \in [\![D]\!]$ from B_k into $\bigcup_{\ell > k_B} B_\ell$, for $k \geq k_B$, such that

$$\sum_{k \geq k_B} \mathrm{rng}_D(\varphi_k) \leq \sum_{k > k_B} \mathbb{1}_{B_k}.$$

Granting that we have accomplished this, set $\varphi_k = \varphi_{\mathrm{id}}$ for $k < k_B$, let $k(x)$ be the unique natural number such that $x \in B_{k(x)}$, and observe that

$$\varphi(x, y) = \varphi_{k(x)}(x, y)$$

is the desired D-compression. In building the φ_k's, we will also ensure that

$$\forall^\infty \ell \in \mathbb{N}\, \forall x, y \in X \ (\varphi_k(x, y) > 0 \Rightarrow x F_\ell y),$$

which implies that

$$I_x(\mathrm{dom}_D(\varphi_k)) = I_x(\mathrm{rng}_D(\varphi_k)).$$

Suppose that we have accomplished this for $k_B \leq \ell < k$, and note that for all $B \in \mathscr{X}$ and $x \in B$,

$$\begin{aligned} I_x(\mathbb{1}_{B_k}) &\leq \sum_{j \geq k} I_x(\mathbb{1}_{B_j}) - \epsilon_B \\ &= \sum_{j \geq k} I_x(\mathbb{1}_{B_j}) + \sum_{k_B \leq j < k} I_x(\mathrm{dom}(\varphi_j)) - \sum_{k_B \leq j < k} I_x(\mathrm{rng}(\varphi_j)) - \epsilon_B \\ &= \sum_{j \geq k_B} I_x(\mathbb{1}_{B_j}) - \sum_{k_B \leq j < k} \mathrm{rng}(\varphi_j) - \epsilon_B. \end{aligned}$$

It follows that for $\ell \in \mathbb{N}$ sufficiently large,

$$I_{[x]_{F_\ell}}(\mathbb{1}_{B_k}) \leq \sum_{j \geq k_B} I_{[x]_{F_\ell}}(B_j) - \sum_{k_B \leq j < k} I_{[x]_{F_\ell}}(\mathrm{rng}(\varphi_j)) - \epsilon_B.$$

By Sublemma 5.7, there is an injection $\varphi_k \in [\![D]\!]$ of $\mathbb{1}_{B_k}$ into

$$\sum_{j \geq k_B} \mathbb{1}_{B_j} - \sum_{k_B \leq j < k} \mathrm{rng}_D(\varphi_j)$$

such that $\forall (x, y) \in E\ (\varphi_k(x, y) > 0 \Rightarrow x F_\ell y)$.

- X admits a witness to D-negligibility of type 3: Fix a finite Borel equivalence relation $F \subseteq E$, a Borel automorphism $f \in [E]$, and a Borel complete section $B \subseteq X$ such that

$$\forall x \in X \ \big(D(f([x]_F) \cap B, [x]_F \cap B) > 1\big).$$

  Define $R \subseteq E|B$ by

$$R = \{(x, y) \in E : x, y \in B \text{ and } y \in f([x]_F)\},$$

  and find a Borel function $\psi : R \to [0, 1]$ such that:

  1. $\forall x \in B \ \big(\sum_{y \in R_x} \psi(x, y) = 1\big)$.
  2. $\forall y \in B \ (\sum_{x \in R^y} \psi(x, y) \leq 1)$.

  Now define $\varphi \in [\![D]\!]$ by

$$\varphi(x, y) = \begin{cases} \psi(x, y) & \text{if } x \in B \text{ and } y \in R_x, \\ 1 & \text{if } x \notin B, \\ 0 & \text{otherwise.} \end{cases}$$

  It is clear that φ is a D-compression of E.

For the general case, partition X into countably many E-invariant Borel sets which fall into one of these categories, and paste the D-compressions together. $\dashv$

Remark 5.8. In the invariant case, it is not difficult to modify the proof of Lemma 5.6 to show that whenever its hypotheses are satisfied, there is a compression of E. This gives a new proof of Theorem 3.10, although it is essentially the same proof as that of Nadkarni [61] in the hyperfinite case.

Next, we note that the results of Becker-Kechris (see [52]) on paradoxicality also have analogs in the D-invariant setting. We say that E is *D-paradoxical* if there exist $a, b \in \mathfrak{B}$ such that $a + b = \mathbb{1}$ and $a \approx b \approx \mathbb{1}$. We refer to such a pair a, b as a *D-paradoxical decomposition*.

Theorem 5.9. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and $D : E \to \mathbb{R}^+$ is a Borel cocycle. Then the following are equivalent:*

1. *There is no D-invariant probability measure on X.*

2. *E is D-paradoxical.*

Proof. To see (2) $\Rightarrow$ (1), simply note that $a \approx b \approx \mathbb{1}$ implies that both a and b are complete sections for E, thus any injection of $\mathbb{1}$ into a or b is necessarily a D-compression.

To see (1) $\Rightarrow$ (2), suppose that $\varphi \in [\![D]\!]$ is a D-compression of E, set

$$a_n = \varphi^n[\mathbb{1} - \varphi[\mathbb{1}]],$$

put

$$a_\infty = \lim_{n\to\infty} \varphi^n[\mathbb{1}],$$

and define

$$a = \sum_{n\in\mathbb{N}} a_{2n} \text{ and } b = a_\infty + \sum_{n\in\mathbb{N}} a_{2n+1}.$$

It is clear that $a \preceq b \preceq \mathbb{1}$. We claim that $\mathbb{1} \preceq a$. To see this, fix a countable sequence of Borel automorphisms $f_n : X \to X$ such that

$$\forall n \in \mathbb{N} \left(E = \bigcup_{m\geq n} \text{graph}(f_m) \right),$$

put $\varphi_n = \varphi_{f_n} \restriction a_0$, and observe that

$$\forall x \in X \left(\sum_{n\in\mathbb{N}} [\text{rng}_D(\varphi_n)](x) = \infty \right).$$

Setting $b_n = \mathbb{1} - \sum_{m<n} \text{rng}_D(\varphi_n)$ and $\psi_n = \varphi_n^* \restriction b_n$, it follows that $\sum_{n\in\mathbb{N}} \text{dom}(\psi_n) = \mathbb{1}$ and $\text{rng}(\psi_n) \leq a_0$ for each $n \in \mathbb{N}$, thus

$$\pi(x, y) = \sum_{n\in\mathbb{N}} [\varphi^n * \psi_n](x, y)$$

is an injection of $\mathbb{1}$ into a.

It only remains to note the following variant of the Schröder-Bernstein Theorem:

Lemma 5.10. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, $a, b \in \mathfrak{B}$ are Borel, and $a \preceq b \preceq a$. Then $a \approx b$.*

Proof. Fix injections φ and ψ in $[\![D]\!]$ of a into b and b into a. Set

$$a_n = (\psi * \varphi)^n[a] - (\psi * \varphi)^n * \psi[b]$$

and

$$b_n = (\psi * \varphi)^n * \psi[b] - (\psi * \varphi)^{n+1}[a],$$

put $a_\infty = \sum_{n \in \mathbb{N}} a_n$, $b_\infty = \sum_{n \in \mathbb{N}} b_n$, and $c_\infty = \lim_{n \to \infty} (\psi * \varphi)^n$, and define

$$\pi = \varphi \restriction a_\infty + \psi^* \restriction (b_\infty + c_\infty).$$

Now note that

$$\begin{aligned} \mathrm{dom}_D(\pi) &= \mathrm{dom}_D(\varphi \restriction a_\infty + \psi^* \restriction (b_\infty + c_\infty)) \\ &= a_\infty + b_\infty + c_\infty \\ &= a, \end{aligned}$$

and

$$\begin{aligned} \mathrm{rng}_D(\pi) &= \mathrm{rng}_D(\varphi \restriction a_\infty + \psi^* \restriction (b_\infty + c_\infty)) \\ &= \sum_{n \in \mathbb{N}} \mathrm{rng}_D(\varphi \restriction a_n) + \sum_{n \in \mathbb{N}} \mathrm{rng}_D(\psi^* \restriction b_n) + \mathrm{rng}_D(c_\infty) \\ &= \sum_{n \in \mathbb{N}} \varphi * (\psi * \varphi)^n[a] - \varphi * (\psi * \varphi)^n * \psi[b] + \\ &\quad \sum_{n \in \mathbb{N}} \psi^* * (\psi * \varphi)^n * \psi[b] - \psi^* * (\psi * \varphi)^{n+1}[a] + \psi^*\Big[\lim_{n \to \infty} (\psi * \varphi)^n[a]\Big] \\ &= \sum_{n \in \mathbb{N}} (\varphi * \psi)^n * \varphi[a] - (\varphi * \psi)^{n+1}[b] + \sum_{n \in \mathbb{N}} (\varphi * \psi)^n[b] - (\varphi * \psi)^n * \varphi[a] + \\ &\quad \lim_{n \to \infty} (\varphi * \psi)^n * \varphi[a] \\ &= b. \end{aligned}$$

It follows that $a \approx b$. $\dashv$

Remark 5.11. As in the invariant case, it is not hard to modify the above proof to show that the D-compressibility of E is equivalent to the existence of a partition of unity into fractional Borel sets $b_n \in \mathfrak{B}$ such that $\forall m, n \in \mathbb{N}\ (b_m \approx b_n)$.

Suppose $f : X \to X$ is Borel. A probability measure μ on X is *f-invariant* if

$$\forall B \subseteq X \text{ Borel } (\mu(f^{-1}(B)) = \mu(B)).$$

When f is injective, the invariance of μ with respect to f is equivalent to the invariance of μ with respect to the corresponding orbit equivalence relation. This is false for many-to-one functions, however. As an application of Theorem 5.2, we will now answer the following:

Question 5.12 (Nadkarni). *Suppose that X is a Polish space and $f : X \to X$ is a countable-to-one Borel function. Is there a version of compressibility which is equivalent to the existence of an f-invariant probability measure?*

The first step towards answering this question is to give an equivalent version of invariance in terms of cocycles. The *tail equivalence relation associated with f* is

$$x E_t(f) y \Leftrightarrow \exists m, n \in \mathbb{N}\ (f^m(x) = f^n(y)).$$

Proposition 5.13. *Suppose X is a Polish space, $f : X \to X$ is a countable-to-1 Borel automorphism, $D : E_t(f) \to \mathbb{R}^+$ is a Borel cocycle, and μ is a D-invariant probability measure on X. Then*

$$\mu \textit{ is } f\textit{-invariant} \Leftrightarrow \forall^*_\mu x \in X \left(\sum_{f(y)=x} D(y, x) = 1 \right).$$

Proof. Fix a partition of X into Borel sets X_n, and note that for $B \subseteq X$ Borel,

$$\begin{aligned} \mu(f^{-1}(B)) &= \sum_{n \in \mathbb{N}} \mu(f^{-1}(B) \cap X_n) \\ &= \sum_{n \in \mathbb{N}} \int_{B \cap f(X_n)} D(f^{-1}(x), x)\ d\mu(x) \\ &= \int_B \sum_{f(y)=x} D(y, x)\ d\mu(x). \end{aligned}$$

It follows that if $\sum_{f(y)=x} D(y,x) = 1$ for μ-almost every $x \in X$, then μ is f-invariant. Conversely, if μ is f-invariant then for all Borel sets $B \subseteq X$,

$$\int_B 1 \; d\mu(x) = \mu(B) = \mu(f^{-1}(B)) = \int_B \sum_{f(y)=x} D(y,x) \; d\mu(x),$$

thus $\sum_{f(y)=x} D(y,x) = 1$ for μ-almost all $x \in X$. $\dashv$

Accordingly, we say that a cocycle $D : E_t(f) \to \mathbb{R}^+$ is *f-invariant* if

$$\sum_{y \in [x]_E} D(y,x) = 1,$$

for all $x \in X$. Of course, when f is injective this information alone completely determines D. In the many-to-1 case, this puts a serious limitation on the sorts of cocycles which can appear, but certainly does not fully determine them. In particular, f-invariance has nothing to say about the restriction of the cocycle to the *smooth equivalence relation associated with* f, which is given by

$$x E_s(f) y \Leftrightarrow f(x) = f(y).$$

In some sense, however, this is the only missing information:

Proposition 5.14. *Suppose that X is a Polish space, $f : X \to X$ is a Borel automorphism, and $D : E_s(f) \to \mathbb{R}^+$ is a Borel cocycle. Then there is at most one way of extending D to an f-invariant cocycle.*

Proof. Suppose that $D' : E_t(f) \to \mathbb{R}^+$ is such an extension. Then

$$D'([x]_{E_s(f)}, f(x)) = 1,$$

thus

$$D'(x, f(x)) = D(x, [x]_{E_s(f)}) D'([x]_{E_s(f)}, f(x)) = D(x, [x]_{E_s(f)}).$$

It follows that

$$D'(x, f^m(x)) = \prod_{i<m} D'(f^i(x), f^{i+1}(x)) = \prod_{i<m} D(f^i(x), [f^i(x)]_{E_s(f)}).$$

Now suppose $xE_t(f)y$, find $m, n \in \mathbb{N}$ such that $f^m(x) = f^n(y)$, and note that

$$\begin{aligned} D'(x,y) &= D'(x, f^m(x))D'(f^m(x), f^n(y))D'(f^n(y), y) \\ &= \prod_{i<m} D(f^i(x), [f^i(x)]_{E_s(f)}) / \prod_{j<n} D(f^j(x), [f^j(x)]_{E_s(f)}), \end{aligned}$$

thus D' is completely determined by D. ⊣

Finally, we are ready to answer Nadkarni's question:

Theorem 5.15. *Suppose X is a Polish space and $f : X \to X$ is an aperiodic $\aleph_0$-to-1 Borel function. Then exactly one of the following holds:*

1. *There is an f-invariant probability measure on X.*
2. *For every Borel set $B \subseteq X$ with $f(B) = B$ and every Borel cocycle $D : E_s(f|B) \to \mathbb{R}^+$ which has an $f|B$-invariant extension $D' : E_t(f|B) \to \mathbb{R}^+$, there is a D'-compression of $E_t(f|B)$.*

Proof. To see (1) ⇒ ¬(2), suppose that μ is an f-invariant probability measure on X. By Proposition 2.1, there is a conull Borel $E_t(f)$-complete section $A \subseteq X$ such that $\mu|A$ is $E_t(f)|A$-quasi-invariant. As $f^{-1}(A \setminus f(A)) \cap A = \emptyset$, it follows that

$$\mu(A \setminus f(A)) = \mu(f^{-1}(A \setminus f(A))) = 0,$$

thus

$$\begin{aligned} \mu([A \setminus f(A)]_{E_t(f)}) &= \mu([A \setminus f(A)]_{E_t(f)} \cap A) \\ &= \mu([A \setminus f(A)]_{E_t(f)|A}) \\ &= 0. \end{aligned}$$

Similarly, as $\mu(A \setminus f^{-1}(A)) \leq \mu(f^{-1}(f(A) \setminus A)) = 0$, it follows that

$$\begin{aligned} \mu([f(A) \setminus A]_{E_t(f)}) &= \mu([f^{-1}(f(A) \setminus A)]_{E_t(f)}) \\ &= \mu([[A]_{E_s(f)} \setminus f^{-1}(A)]_{E_t(f)}) \\ &= \mu([A \setminus f^{-1}(A)]_{E_t(f)}) \\ &= \mu([A \setminus f^{-1}(A)]_{E_t(f)} \cap A) \\ &= \mu([A \setminus f^{-1}(A)]_{E_t(f)|A}) \\ &= 0. \end{aligned}$$

Hence, there is a conull $E_t(f)$-invariant Borel set $A' \subseteq X$ such that $A \cap A' = f(A) \cap A'$. Setting $B = A \cap A'$, it follows that $B = f(B)$. Letting $D' : E_t(f|B) \to \mathbb{R}^+$ be the cocycle associated with $\mu|B$, it follows that $E_t(f|B)$ is not D'-compressible.

To see $\neg(2) \Rightarrow (1)$, suppose that $B \subseteq X$ is a conull Borel $E_t(f)$-complete section, $f(B) = B$, $D : E_t(f|B) \to \mathbb{R}^+$ is an f-invariant Borel cocycle, and $E_t(f|B)$ is not D-compressible. By Theorem 5.2, there is a D-invariant probability measure μ. It follows that $A \mapsto \mu(A \cap B)$ is an f-invariant probability measure. $\dashv$

We will close this section by showing that whenever E is D-aperiodic, there is a D-negligible E-invariant comeager Borel set $C \subseteq X$. This generalizes a theorem of Wright [80], who proved the special case when X is a perfect Polish space, $D = 1$, and E is generated by a countable group of homeomorphisms with a dense orbit.

Theorem 5.16 (Kechris-Miller). *Suppose X is a Polish space, E is a countable Borel equivalence relation on X, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and E is D-aperiodic. Then there is an invariant comeager Borel set $C \subseteq X$ and a smooth D-aperiodic Borel subequivalence relation of $E|C$. In particular, $E|C$ admits no D-invariant probability measures.*

Proof. Fix a decreasing, vanishing sequence of Borel sets $A_n \subseteq X$ which are complete sections for E. Recursively define Borel functions $k_n : X \to \mathbb{N}$ by setting $k_0(x) = 0$ and

$$k_{n+1}(x) = \min\{k \in \mathbb{N} : A_k \cap [x]_E \subsetneq A_{k_n(x)} \cap [x]_E\}.$$

It is clear that the sets

$$B_n = \{x \in X : x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}\}.$$

are complete sections for E which partition X. By neglecting an E-invariant Borel set on which E is D-periodic and thus smooth, we may assume each $E|B_n$ is D-aperiodic.

Let $c : [E]^{<\infty} \to \mathbb{N}$ be the Borel $\aleph_0$-coloring of the graph G from the proof of Proposition 3.1. For $\alpha \in \mathbb{N}^\mathbb{N}$, recursively define an increasing sequence of fsr's $F_n^\alpha \subseteq E$ by setting $F_0^\alpha = \Delta|B_0$, and putting $xF_{n+1}^\alpha y$ if either $xF_n^\alpha y$ or

$$\exists S \in [E]^{<\infty} \ \big(c(S) = n \text{ and } x, y \in S \text{ and } \exists z \in B_0 \exists T \subseteq C_n \ (S = [z]_{F_\alpha^n} \cup T)\big),$$

where $C_n = (B_1 \cup \cdots \cup B_n) \setminus \mathrm{dom}(F^n_\alpha)$. Setting $F^\alpha_\infty = \bigcup_{n\in\mathbb{N}} F^\alpha_n$, we claim that

$$\forall^* \alpha \in \mathbb{N}^\mathbb{N} \forall^* x \in X \ (|[x]_{F^\alpha_\infty}|_x = \infty). \tag{†}$$

Granting this, it follows that there exists $\alpha \in \mathbb{N}^\mathbb{N}$ such that

$$B = \{x \in X : |[x]_{F^\alpha_\infty}| = \infty\}$$

is comeager. Set $C = [B]_E$, fix Borel automorphisms $f_n : X \to X$ such that

$$f_0 = \mathrm{id} \text{ and } E = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n),$$

let $n(x)$ be the least natural number such that $f_{n(x)}(x) \in B$, and define $F \subseteq E|C$ by

$$xFy \Leftrightarrow f_{n(x)} F^\alpha_\infty f_{n(y)}.$$

As B_0 is a transversal of F, it follows that F is smooth and D-aperiodic.

It remains to prove (†). It is enough to show that

$$\forall x \in X\, \forall^* \alpha \in \mathbb{N}^\mathbb{N} \ (|[x]_{F^\alpha_\infty}|_x = \infty),$$

by the Kuratowski-Ulam Theorem (see Theorem 8.41 of Kechris [51]). Noting that

$$\{\alpha \in \mathbb{N}^\mathbb{N} : |[x]_{F^\alpha_\infty}|_x = \infty\} = \bigcap_{n\in\mathbb{N}} \{\alpha \in \mathbb{N}^\mathbb{N} : |[x]_{F^\alpha_\infty}|_x > n\},$$

and that the latter sets are clearly open, it suffices to show that each

$$\{\alpha \in \mathbb{N} : |[x]_{F^\alpha_\infty}|_x > n\}$$

is dense. So suppose that $\mathscr{N}_s$ is a basic neighborhood of $\mathbb{N}^\mathbb{N}$ with $x \in \bigcup_{n<|s|} B_n$. Defining $F^s_{|s|}$ as before, it follows that we can find $S \in [E]^{<\infty}$, containing x, which is the union of a single $F^s_{|s|}$-class with a set $T \subseteq B_{|s|+1} \setminus \mathrm{dom}(F^s_{|s|})$ such that $|T|_x > n$. Letting $s' = s^\frown\langle c(S)\rangle$, it follows that

$$\forall \alpha \in \mathscr{N}_{s'} \ (|[x]_{F^\alpha_\infty}|_x > n),$$

which completes the proof. ⊣

6 Ends of Graphs

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. A *graphing* of E is a Borel graph $\mathscr{G} \subseteq X^2$ whose connected components are exactly the equivalence classes of E. Strengthening results of Adams [1] and Paulin [65] in the measure-theoretic context, here we will examine what sort of information about E can be extracted from certain features of $\mathscr{G}$.

A *path* through $\mathscr{G}$ is a sequence $\langle x_0, x_1, \ldots, x_n \rangle$ of distinct vertices of $\mathscr{G}$, with $(x_i, x_{i+1}) \in \mathscr{G}$ for all $i < n$. A *ray* through $\mathscr{G}$ is an infinite sequence α of distinct vertices of $\mathscr{G}$, with $(\alpha_n, \alpha_{n+1}) \in \mathscr{G}$ for all n. Two rays α, β are *end-equivalent* if for any finite set S of vertices of $\mathscr{G}$, there is a $\mathscr{G}$-path from α to β which avoids S. Equivalently, α and β are end-equivalent if there are infinitely many paths from α to β, no two of which have any intermediate vertices in common. The ends of $\mathscr{G}$ are simply the end equivalence-classes of $\mathscr{G}$.

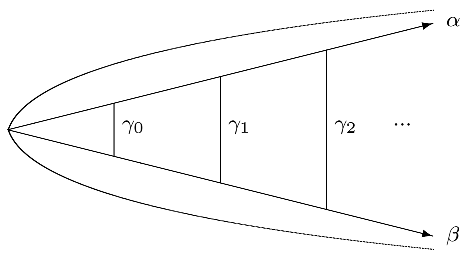

Figure 3.1: α, β are end-equivalent if there is an infinite ladder of paths between them.

If $(\mathscr{T}, x)$ is a rooted tree, then the ends of $(\mathscr{T}, x)$ can be identified with the branches of $(\mathscr{T}, x)$. To see this, simply note that every end-class contains some ray beginning at x, this ray is necessarily a branch of $(\mathscr{T}, x)$, and any two such branches which are end-equivalent are identical.

When $\mathscr{G}$ is connected, there is a more substantial sense in which the ends of a graph generalize the branches of a tree. For each ray x through $\mathscr{G}$ and finite vertex set S, let $\mathscr{N}_S^x$ be the set of ends of $\mathscr{G}$ which contain a ray who is connected to x via a $\mathscr{G}$-path which avoids S. It is straightforward to check that the topology τ generated by the $\mathscr{N}_S^x$'s is a zero-dimensional, Hausdorff topology on the ends of $\mathscr{G}$. Moreover, if $\mathscr{G}$ is countable then τ is second countable, and if $\mathscr{G}$ is locally finite then τ is compact.

One might refer to the ends that we have defined above as combinatorial ends. There is also a topological notion of an end which is in some sense more natural. For locally finite graphs, the two notions do not differ. On the other hand, most of the theorems we shall consider fail for relatively uninteresting reasons if we substitute topological ends for combinatorial ends. The reader is encouraged to look to Diestel-Kühn [22] for more on this distinction.

Now suppose that $\mathscr{G}$ is a Borel graph on a standard Borel space X. We use $[\mathscr{G}]^\infty$ to denote the standard Borel space of rays through $\mathscr{G}$, and we use $\mathscr{E}_\mathscr{G}$ to denote the end-equivalence relation on $[\mathscr{G}]^\infty$. Although we will prove a variety of stronger results, the primary information we wish to get across is summarized in Figure 3.2.

Feature of each component of $\mathscr{G}$	Information about E
Locally finite and exactly 1 end	Aperiodic
Exactly 2 ends	Hyperfinite
At least 3 ends, but only finitely many	Smooth
At least 3 ends, but fewer than perfectly many	Compressible

Figure 3.2: Features of $\mathscr{G}$ which determine information about $E_\mathscr{G}$.

Theorem 6.1. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a locally finite Borel graphing of E whose components each have exactly 1 end.*

Proof. We begin with the case that E is compressible. By Jackson-Kechris-Louveau [48], we can find a locally finite graphing $\mathscr{H}$ of E. Define a graphing $\mathscr{G}$ of $E \times I(\mathbb{N})$ by putting $((x, m), (y, n)) \in \mathscr{G}$ in case

$$((x, y) \in \mathscr{H} \text{ and } m = n) \text{ or } (x = y \text{ and } m = n \pm 1).$$

Clearly $\mathscr{G}$ is locally finite. We will show that every component of $\mathscr{G}$ has exactly one end. Suppose that C is an equivalence-class of $E \times I(\mathbb{N})$, $\langle (x_i, m_i) \rangle, \langle (y_i, n_i) \rangle$

are rays of $\mathscr{G}|C$, and S is a finite subset of C. Fix $i \in \mathbb{N}$ sufficiently large that $(x_i, m), (y_i, n) \notin S$ for all $m \geq m_i$ and $n \geq n_i$, and fix $\ell \geq m_i, n_i$ such that

$$\forall (z, k) \in S \ (k < \ell).$$

It is clear that the "vertical" $\mathscr{G}$-path from (x_i, m_i) to (x, ℓ) and the "vertical" $\mathscr{G}$-path from (y_i, n_i) to (y_i, ℓ) avoids S, as does any "horizontal" path from (x, ℓ) to (y, ℓ), thus there is a path from (x_i, m_i) to (y_i, n_i) which avoids S.

For the general case, we employ a similar idea. By Jackson-Kechris-Louveau [48], we can partition X into complete sections $B_n \subseteq X$ for E. Let $n(x)$ be the unique natural number such that $x \in B_{n(x)}$, and note that by Jackson-Kechris-Louveau [48], we can find locally finite graphings $\mathscr{G}_n$ of $E|B_n$. By Theorem 18.10 of Kechris [51], we can find a Borel function $\varphi : X \to X$ such that $n(\varphi(x)) = n(x) + 1$. By removing an E-invariant Borel set on which E is compressible, we may assume that φ is finite-to-1. Now define a graphing $\mathscr{G}$ of E by

$$\mathscr{G} = \big\{(x, y) \in E : \varphi(x) = y \text{ or } \varphi(y) = x \text{ or } \exists n \in \mathbb{N} \ ((x, y) \in \mathscr{G}_n)\big\}.$$

Clearly $\mathscr{G}$ is locally finite. Again, we will show that every component of $\mathscr{G}$ has exactly one end. Suppose that C is an equivalence-class of E, $\langle (x_i, m_i) \rangle_{i \in \mathbb{N}}, \langle (y_i, n_i) \rangle_{i \in \mathbb{N}}$ are rays of $\mathscr{G}|C$, and $S \subseteq C$ is finite. Fix $i \in \mathbb{N}$ sufficiently large that

$$\forall n \in \mathbb{N} \ (\varphi^n(x_i), \varphi^n(y_i) \notin S),$$

and fix $\ell \geq n(x_i), n(y_i)$ sufficiently large that

$$\forall z \in S \ (n(z) < \ell).$$

Clearly the path along φ from x_i to B_ℓ avoids S, as does the path along φ from y_i to B_ℓ. As the corresponding elements of B_ℓ are $\mathscr{G}_\ell$-connected via a path which avoids S, it follows that there is a path from x_i to y_i which avoids S. $\dashv$

Remark 6.2. As noted by Adams [1], the situation is much different for treeings $\mathscr{T}$ with 1 end. This is because the function f which sends x to its unique $\mathscr{T}$-neighbor in the direction of the end is a Borel function which induces $\mathscr{T}$, thus $E = E_t(f)$ must be hyperfinite by Jackson-Kechris-Louveau [48].

Next we turn to the case of 2 ends:

Theorem 6.3. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a Borel graphing of E, and $\mathscr{B} \subseteq [\mathscr{G}]^\infty$ is an $\mathscr{E}_\mathscr{G}$-invariant Borel set consisting of exactly 2 end-classes of every component of $\mathscr{G}$. Then there is a Borel E-complete section $B \subseteq X$ and a Borel forest $\mathscr{L} \subseteq \mathscr{G}|B$ whose restriction to each class of E is a segment, a ray, or a line. In particular, E is hyperfinite.*

Proof. A set $S \in [E]^{<\infty}$ *disconnects $\mathscr{B}$-ends* if there are rays $\alpha, \beta \in \mathscr{B}$ through $\mathscr{G}|[S]_E$ which are not connected by a $\mathscr{G}$-path that avoids S. Let $F \subseteq E$ be a maximal fsr whose classes are $\mathscr{G}$-connected and disconnect $\mathscr{B}$-ends. The main observation is that, for each E-class C, there is a canonical way of picking out a tree $\mathscr{T} \subseteq \mathscr{G}|C$ of vertex degree ≤ 2, which passes through each $(F|C)$-class exactly once. To see this, we must first establish several facts regarding the manner in which the classes of $F|C$ sit within $\mathscr{G}|C$.

For $S \in [E]^{<\infty}$, define

$$\mathscr{G}_S = \{(x, y) \in \mathscr{G}|C : x, y \notin S\},$$

and put $\mathscr{B}_S = \mathscr{B} \cap [\mathscr{G}_S|C]^\infty$.

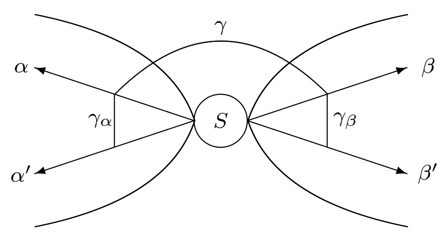

Figure 3.3: If α, β are $\mathscr{G}_S$-connected, then so too are α', β'.

Lemma 6.4. *Each $S \in C/F$ disconnects every pair of end-inequivalent rays of $\mathscr{B}_S$.*

Proof. Fix end-inequivalent rays $\alpha, \beta \in \mathscr{B}_S$ and suppose, towards a contradiction, that there is a $\mathscr{G}_S$-path γ from α to β. Fix $(\mathscr{G}|C)$-rays $\alpha' \in [\alpha]_{\mathscr{E}_\mathscr{G}}, \beta' \in [\beta]_{\mathscr{E}_\mathscr{G}}$ which are disconnected by S, and note that by removing initial segments, we may assume that

α', β' are rays through $\mathscr{G}_S$. Let γ_α be a $\mathscr{G}_S$-path from α' to α whose terminal point is the initial point of γ, and let γ_β be a $\mathscr{G}_S$-path from β to β' whose initial point is the terminal point of γ. Then $\gamma_\alpha\gamma\gamma_\beta$ is a $\mathscr{G}_S$-path from α' to β', a contradiction. $\dashv$

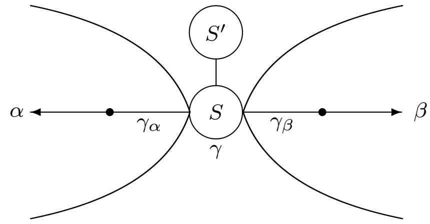

Figure 3.4: If S' is not $\mathscr{G}_S$-connected to α or β, it cannot disconnect them.

Lemma 6.5. *If S, S' are distinct classes of $F|C$, then S' is $\mathscr{G}_S$-connected to a ray though $\mathscr{G}_S$.*

Proof. Fix end-inequivalent rays $\alpha, \beta \in \mathscr{B}_S$, let γ_α be a $\mathscr{G}$-path of minimal length from α to S, let γ_β be a $\mathscr{G}$-path of minimal length from S to β, and let γ be a $(\mathscr{G} \setminus \mathscr{G}_S)$-path from the terminal point of γ_α to the initial point of γ_β. Then $\gamma_\alpha\gamma\gamma_\beta$ is a $\mathscr{G}_{S'}$-path from α to β. Together with Lemma 6.4, this contradicts the fact that S' disconnects ends. $\dashv$

Lemma 6.6. *Suppose that S, S', S'' are distinct classes of $F|C$, and S' is $\mathscr{G}_S$-connected to S''. After reversing the roles of S', S'' if necessary, every $\mathscr{G}$-path from S to S' goes through S''.*

Proof. Let α, β be end-inequivalent rays of $\mathscr{B}_{S \cup S' \cup S''}$. Combining our last observation with the fact that S', S'' are $\mathscr{G}_S$-connected, it follows that, after reversing the roles of α, β if necessary, we may assume that S', S'' are both $\mathscr{G}_S$-connected to β.

Let $\gamma_{\beta,S'}$ be a path of minimal length from β to $S' \cup S''$, noting that by reversing the roles of S', S'' if necessary, we may assume that $\gamma_{\beta,S'}$ avoids S''. Now suppose, towards a contradiction, that there is a path $\gamma_{S',S}$ from S' to S which avoids S''. Let $\gamma_{S,\alpha}$ be a $\mathscr{G}$-path of minimal length from S to α, let γ_S be a $(\mathscr{G} \setminus \mathscr{G}_S)$-path from the

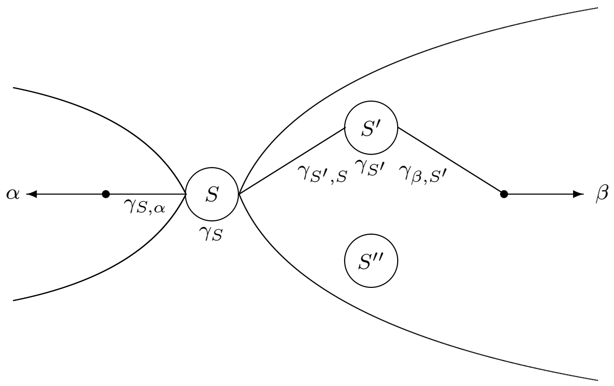

Figure 3.5: If $\gamma_{S',S}$ avoids S'', then α, β are $\mathscr{G}_{S''}$-connected.

terminal point of $\gamma_{S',S}$ to the initial point of $\gamma_{S,\alpha}$, and let $\gamma_{S'}$ be a $(\mathscr{G} \setminus \mathscr{G}_{S'})$-path from the terminal point of $\gamma_{\beta,S'}$ to the initial point of $\gamma_{S',S}$. Then $\gamma_{\beta,S'}\gamma_{S'}\gamma_{S',S}\gamma_S\gamma_{S,\alpha}$ is a $\mathscr{G}_{S''}$-path from β to α, contradicting Lemma 6.4. $\dashv$

Now let d be the graph metric on $\mathscr{G}|C$, let $\mathscr{F}$ be the set of classes of $F|C$, and let $\mathscr{T}_C$ be the set of all distinct pairs $(S, S') \in \mathscr{F}^2$ such that

$$\forall S'' \in \mathscr{F}\ (S', S'' \text{ are } \mathscr{G}_S\text{-connected} \Rightarrow d(S, S') \leq d(S, S'')).$$

Lemma 6.7. *$\mathscr{T}_C$ is a tree whose vertices are all of degree ≤ 2.*

Proof. By our previous observations, it suffices to show that $\mathscr{T}_C$ is symmetric and contains no cycles. To see that $\mathscr{T}_C$ is symmetric, suppose that $(S, S') \notin \mathscr{T}_C$, and find $S'' \in \mathscr{F}$ such that S', S'' are $\mathscr{G}_S$-connected and $d(S, S'') < d(S, S')$. Then S, S'' are $\mathscr{G}_{S'}$-connected and every $\mathscr{G}$-path from S' to S goes through S'', thus $d(S', S'') < d(S', S)$, so $(S', S) \notin \mathscr{T}_C$.

Now suppose, towards a contradiction, that there is a $\mathscr{T}_C$-cycle $\langle S_i \rangle_{i<n}$ of length $n \geq 3$. Put $S = \bigcup_{i<n} S_i$, let α, β be end-inequivalent rays in $\mathscr{B}_S$, let γ_α be a $\mathscr{G}$-path of minimal length from α to S, let γ_β be a $\mathscr{G}$-path of minimal length from S to β, and find S_n in our $\mathscr{T}_C$-cycle which avoids $\gamma_\alpha, \gamma_\beta$. Then there is a $\mathscr{G}$-path γ from the terminal point of γ_α to the initial point of γ_β, and it follows that $\gamma_\alpha\gamma\gamma_\beta$ is a $\mathscr{G}_{S_n}$-path from α to β, contradicting the fact that S_n disconnects α, β. $\dashv$

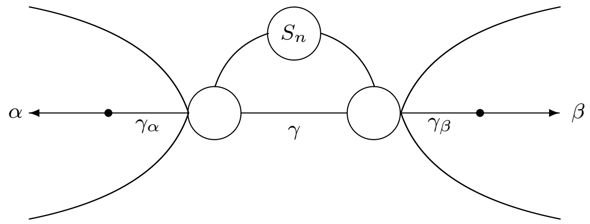

Figure 3.6: Not all elements of a $\mathscr{T}_C$-cycle can disconnect α, β.

Finally we are ready to build the desired forest. Set $\mathscr{T} = \bigcup \mathscr{T}_C \subseteq ([E]^{<\infty})^2$, noting that $\mathscr{T}$ is a Borel forest. Now associate, in a Borel manner, with each $(S, S') \in \mathscr{T}$ a $\mathscr{G}$-path $\gamma_{S,S'}$ of minimal length connecting S, S' such that $\gamma_{S,S'} = \gamma_{S',S}$. Also associate, in a Borel manner, with each F-class S a $(\mathscr{G} \setminus \mathscr{G}_S)$-path γ_S which connects $\gamma_{S,S'}, \gamma_{S,S''}$, where S', S'' are the $\mathscr{T}$-neighbors of S (if S does not have two $\mathscr{T}$-neighbors, let $\gamma_S = \emptyset$). Setting

$$\mathscr{L} = \{(x, y) \in E : \exists (S, S') \in \mathscr{T} \ ((x, y) \text{ occurs in } \gamma_S \text{ or } \gamma_{S,S'})\},$$

it easily follows that $\mathscr{L}$ is a Borel subgraph of $\mathscr{G} | \operatorname{dom}(\mathscr{L})$ whose restriction to any class of E is a tree of vertex degree ≤ 2. $\dashv$

Remark 6.8. Lemma 3.19 of Jackson-Kechris-Louveau [48] provides a sort of converse for Theorem 6.3. It implies that under (CH), if E is hyperfinite, then there is a universally measurable $\mathscr{E}_{\mathscr{G}}$-invariant set $\mathscr{B} \subseteq [\mathscr{G}]^\infty$ which consists of 1 or 2 end-classes of almost every component of $\mathscr{G}$. In fact, this is true for graphings as well. This follows from the simple fact that every Borel graphing of a hyperfinite equivalence relation has a spanning Borel subforest.

Remark 6.8 leads to the following question:

Question 6.9. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, and every Borel graphing of E has a spanning Borel subforest. Must E be hyperfinite?*

Here we simply note the following related fact:

Proposition 6.10 (Kechris-Miller). *Suppose X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a Borel graphing of E, and $n \in \mathbb{N}$. Then there is a spanning Borel subgraphing $\mathscr{H} \subseteq \mathscr{G}$ which has no cycles of length $\leq n$.*

Proof. Let $\deg_{\mathscr{G}}(x)$ denote the vertex degree of x. We say that $\mathscr{G}$ is *bounded* if

$$\sup_{x \in X} \deg_{\mathscr{G}}(x) < \infty.$$

Lemma 6.11. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a bounded Borel graphing of E, $n \in \mathbb{N}$, and $\mathscr{H} \subseteq \mathscr{G}$ is a Borel graph with no cycles of length $\leq n$. Then there is a spanning Borel graph $\mathscr{H} \subseteq \mathscr{H}' \subseteq \mathscr{G}$ which has no cycles of length $\leq n$.*

Proof. Define $\mathscr{X} \subseteq [E]^{<\infty}$ by

$$\mathscr{X} = \{S \in [E]^{<\infty} : S \text{ is a } \mathscr{G}\text{-cycle of length } \leq n\},$$

and define a graph $\mathfrak{G}$ on $\mathscr{X}$ by

$$(S, T) \in \mathfrak{G} \Leftrightarrow (S \neq T \text{ and } S \cap T \neq \emptyset).$$

As $\mathscr{G}$ is bounded, so too is $\mathfrak{G}$. It follows from Proposition 4.6 of Kechris-Solecki-Todorcevic [56] that for $m \in \mathbb{N}$ sufficiently large, there is a Borel coloring $c : \mathscr{X} \to \{0, \ldots, m\}$ of $\mathfrak{G}$.

Put $\mathscr{G}_0 = \mathscr{G}$, and given $\mathscr{G}_k \subseteq \cdots \subseteq \mathscr{G}_0$, set

$$\mathscr{X}_k = \{S \in \mathscr{X} : c(S) = k \text{ and } S \text{ is an } \mathscr{G}_k\text{-cycle}\}.$$

Fix a Borel assignment $S \mapsto (x_S, y_S) \in (\mathscr{G}_k \setminus \mathscr{H})|S$ of edges to be cut from $S \in \mathscr{X}_k$, and set

$$\mathscr{G}_{k+1} = \mathscr{G}_k \setminus \bigcup_{S \in \mathscr{X}_k} \{(x_S, y_S), (y_S, x_S)\}.$$

Clearly $\mathscr{H}' = \mathscr{G}_{m+1}$ is the desired subgraph of $\mathscr{G}$. ⊣

By Feldman-Moore [36], there is an increasing, exhaustive sequence of bounded Borel graphings $\mathscr{G}_k \subseteq \mathscr{G}$. Put $\mathscr{H}_0 = \emptyset$, and given a graph $\mathscr{H}_k \subseteq \mathscr{G}_k$ with no cycles of length $\leq n$, apply Lemma 6.11 to find a spanning Borel subgraph $\mathscr{H}_k \subseteq \mathscr{H}_{k+1} \subseteq \mathscr{G}_k$ with no cycles of length $\leq n$. It is clear that $\mathscr{H} = \bigcup_{k \in \mathbb{N}} \mathscr{H}_k$ is as desired. ⊣

Next, we turn to graphs with more ends:

Theorem 6.12. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a Borel graphing of E, and $\mathscr{B} \subseteq [\mathscr{G}]^\infty$ is an $\mathscr{E}_\mathscr{G}$-invariant Borel set consisting of at least three but only finitely many ends from every component of $\mathscr{G}$. Then E is smooth.*

Proof. A $\mathscr{G}$-connected set $S \in [E]^{<\infty}$ is a *$\mathscr{B}$-isolator* if no ray $\alpha \in \mathscr{B}$ through $\mathscr{G} | [S]_E$ is $\mathscr{G}_S$-connected to another ray in $\mathscr{B}$ with which it is not end-equivalent. Let $F \subseteq E$ be a maximal fsr whose classes are $\mathscr{G}$-connected $\mathscr{B}$-isolators. It follows from the maximality of F and the fact that $\mathscr{B}$ contains a non-zero, finite number of ends of each component of $\mathscr{G}$ that every E-class contains an F-class. As no two disjoint finite subsets of an E-class can both be $\mathscr{B}$-isolators, it follows that every E-class contains exactly one F-class, thus E is smooth. ⊣

Remark 6.13. Theorem 6.12 was noted independently by Blanc [15].

Given a Borel cocycle $D : E \to \mathbb{R}^+$, a *D-ray* of $\mathscr{G}$ is simply a D-infinite ray of $\mathscr{G}$. We use $[\mathscr{G}]^\infty_D$ to denote the standard Borel space of D-rays through $\mathscr{G}$. The D-ends of $\mathscr{G}$ are simply the equivalence classes of $\mathscr{E}_\mathscr{G} | [\mathscr{G}]^\infty_D$.

Theorem 6.14. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a Borel graphing of E, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and $\mathscr{B}$ is an $\mathscr{E}_\mathscr{G}$-invariant Borel subset of $[\mathscr{G}]^\infty$. Then off of a D-negligible E-invariant Borel set, $\mathscr{B}$ contains 0,1,2, or perfectly many D-ends of every connected component of $\mathscr{G}$.*

Proof. It is enough to show that if $\mathscr{B}$ contains at least three D-ends and at least one isolated D-end of every component of $\mathscr{G}$, then X is D-negligible. In fact, we will show that there is a Borel E-complete section $B \subseteq X$ and a smooth D-aperiodic subequivalence relation of $E|B$. This easily implies that X is D-negligible.

Let $F \subseteq E$ be a maximal fsr whose classes are $\mathscr{B}$-isolators. As before, if C is an E-class, then no two F-classes $S_0, S_1 \subseteq C$ can isolate the same end of $\mathscr{B}$. Moreover, if we associate with each F-class S a D-ray $\alpha_S \in \mathscr{B}$ which it isolates from $\mathscr{B}$, as well as the $\mathscr{G}_S$-connected component X_S of α_S, then it follows that $S_0 \neq S_1 \Rightarrow \alpha_{S_0} \cap \alpha_{S_1} = \emptyset$. Setting $B = \bigcup_{x \in \mathrm{dom}(F)} X_{[x]_F}$ and defining $F' \subseteq E$ by

$$xF'y \Leftrightarrow \exists x \in \mathrm{dom}(F)\ (x, y \in X_{[x]_F}),$$

it follows that F' is a smooth D-aperiodic subequivalence relation of $E|B$. ⊣

A set $B \subseteq X$ is *dense in the D-ends of $\mathscr{G}$* if for every finite set $S \subseteq X$, B intersects every D-infinite connected component of $\mathscr{G}_S$. The following fact can be viewed as a generalization of the Poincaré recurrence lemma:

Theorem 6.15. *Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathscr{G}$ is a Borel graphing of E, $D : E \to \mathbb{R}^+$ is a Borel cocycle, and B is a Borel complete section for E. Then off of a D-negligible E-invariant Borel set, B is dense in the D-ends of $\mathscr{G}$.*

Proof. A $\mathscr{G}$-connected set $S \in [E]^{<\infty}$ is a *B-isolator* if there is a D-infinite connected component of $\mathscr{G}_S$ which is disjoint from B. It is enough to show that if every E-class contains a B-isolator, then X is D-negligible. In fact, we will show that there is a Borel E-complete section $A \subseteq X$ and a smooth D-aperiodic subequivalence relation of $E|B$. This easily implies that X is D-negligible.

Let $F \subseteq E$ be a maximal fsr whose classes are B-isolators, and associate with each $x \in \mathrm{dom}(F)$ the set

$$I_x = \{y \in [x]_E : [y]_{\mathscr{G}_{[x]_F}} \text{ is } D\text{-infinite and disjoint from } B\}.$$

Although it need not be the case that $I_x \cap I_y = \emptyset$ when x, y are F-inequivalent, the following two lemmas essentially allow us to proceed as if this were the case:

Lemma 6.16. *If $I_x \cap I_y \neq \emptyset$, then either $I_x \subseteq I_y$ or $I_y \subseteq I_x$.*

Proof. Of course, we may assume that x, y are F-inequivalent. Fix $z \in I_x \cap I_y$ and let γ be a $\mathscr{G}$-path of minimal length from z to $[x]_F \cup [y]_F$. By reversing the roles of x, y is necessary, we may assume that γ avoids $[x]_F$, and therefore that y is $\mathscr{G}_{[x]_F}$-connected to z. It then follows that $I_y \subseteq [z]_{\mathscr{G}_{[x]_F}} \subseteq I_x$. ⊣

Lemma 6.17. *For each* $x \in \operatorname{dom}(F)$*, there exists* $y \in \operatorname{dom}(F)$ *such that*

$$I_x \subseteq I_y \text{ and } \forall z \in \operatorname{dom}(F)\ (I_y \cap I_z \neq \emptyset \Rightarrow I_z \subseteq I_y).$$

Proof. Suppose, towards a contradiction, that the lemma fails. It then follows from Lemma 6.16 that there exists $x_0, x_1, \ldots \in [x]_E$ such that

$$I_x \subsetneq I_{x_0} \subsetneq I_{x_1} \subsetneq \cdots .$$

Fix $w \in [x]_E \cap B$ and let γ be a $\mathscr{G}$-path of minimal length from w to the set

$$I = \bigcup_{n \in \mathbb{N}} I_{x_n}.$$

Fix $n \in \mathbb{N}$ sufficiently large that γ avoids $[x_n]_F$. It then follows that w is $\mathscr{G}_{[x_n]}$-connected to an element of I_{x_n}, a contradiction. ⊣

It now follows that the set

$$Y = \{y \in \operatorname{dom}(F) : \forall x \in \operatorname{dom}(F)\ (I_x \cap I_y \neq \emptyset \Rightarrow I_x \subseteq I_y)\}$$

is an E-complete section. Set $A = \bigcup_{y \in Y} I_y$, and define F' on A by

$$xF'z \Leftrightarrow \exists y \in Y\ (x, z \in I_y \text{ and } x, z \text{ are } \mathscr{G}_{[y]_F}\text{-connected}).$$

It is clear that F' is a smooth D-aperiodic subequivalence relation of $E|A$. ⊣

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