ON THE EXISTENCE OF SMALL ANTICHAINS FOR DEFINABLE QUASI-ORDERS

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Abstract. We generalize Kada's definable strengthening of Dilworth's characterization of the class of quasi-orders admitting an antichain of a given finite cardinality.

Introduction

A binary relation R on a set X is a *quasi-order* if it is reflexive and transitive. Two points $x, y \in X$ are *R-comparable* if $x \mathrel{R} y$ or $y \mathrel{R} x$, and *R-incomparable* otherwise. A set $Y \subseteq X$ is an *R-chain* if any two points of Y are R-comparable, and an *R-antichain* if any two distinct points of Y are R-incomparable.

Dilworth showed that if $k \in \mathbb{Z}^+$, X is finite, and there is no R-antichain of cardinality $k+1$, then there is a cover $(C_i)_{i<k}$ of X by R-chains (see [Dil50, Theorem 1.1]).

A subset of a topological space X is *Borel* if it is in the σ-algebra generated by the topology τ_X of X, *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$, and *$\aleph_0$-universally Baire* if its preimage under any continuous function $\phi \colon 2^\mathbb{N} \to X$ has the Baire property.

Here we establish the following strengthening of Dilworth's theorem:

Theorem 1. *Suppose that $k \in \mathbb{Z}^+$, X is a Hausdorff space, and R is an $\aleph_0$-universally-Baire quasi-order on X whose incomparability relation is analytic. Then exactly one of the following holds:*

1. *There is a cover $(C_i)_{i<k}$ of X by Borel R-chains.*
2. *There is an R-antichain of cardinality $k+1$.*

The *equivalence relation* on X associated with R is that with respect to which two points $x, y \in X$ are equivalent if $x \mathrel{R} y$ and $y \mathrel{R} x$, and the *strict relation* associated with R is that with respect to which two points $x, y \in X$ are related if $x \mathrel{R} y$ but $\neg y \mathrel{R} x$. Kada established the special case of Theorem 1 in which the strict quasi-order

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is co-analytic and both the equivalence and incomparability relations are analytic (see [Kad89, Theorem 1′]). Whereas his intricate argument relied heavily upon recursion-theoretic methods, we utilize only elementary ideas and the $\mathbb{G}_0$ dichotomy (see [KST99, Theorem 6.3]), which itself has a classical proof (see [Mil11, Theorem 8]).

A subset of an analytic Hausdorff space is $\boldsymbol{\Sigma}^1_1$ if it is analytic. More generally, for each $n \in \mathbb{Z}^+$, a subset of an analytic Hausdorff space is $\boldsymbol{\Pi}^1_n$ if its complement is $\boldsymbol{\Sigma}^1_n$, and $\boldsymbol{\Sigma}^1_{n+1}$ if it is a continuous image of a $\boldsymbol{\Pi}^1_n$ subset of an analytic Hausdorff space. A subset of an analytic Hausdorff space is $\boldsymbol{\Delta}^1_n$ if it is both $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$. Souslin's theorem ensures that the families of Borel and $\boldsymbol{\Delta}^1_1$ sets coincide (see, for example, [Kec95, Theorem 28.1]). The axiom of determinacy ($\mathtt{AD}$) implies that the family of $\boldsymbol{\Delta}^1_{2n+1}$ sets has a rich structural theory analogous to that of the Borel sets (see, for example, [Jac08]).

We also obtain the following analog of Theorem 1 under determinacy:

Theorem 2 ($\mathtt{AD}$). *Suppose that $k \in \mathbb{Z}^+$, $n \in \mathbb{N}$, X is an analytic Hausdorff space, and R is a quasi-order on X whose incomparability relation is $\boldsymbol{\Sigma}^1_{2n+1}$. Then exactly one of the following holds:*

1. *There is a cover $(C_i)_{i<k}$ of X by $\boldsymbol{\Delta}^1_{2n+1}$ R-chains.*
2. *There is an R-antichain of cardinality $k+1$.*

In addition, we generalize Dilworth's theorem to arbitrary quasi-orders on analytic Hausdorff spaces under the strengthening of determinacy where the players specify elements of $\mathbb{R}$ instead of $\mathbb{N}$ ($\mathtt{AD}_\mathbb{R}$):

Theorem 3 ($\mathtt{AD}_\mathbb{R}$). *Suppose that $k \in \mathbb{Z}^+$, X is an analytic Hausdorff space, and R is a quasi-order on X. Then exactly one of the following holds:*

1. *There is a cover $(C_i)_{i<k}$ of X by R-chains.*
2. *There is an R-antichain of cardinality $k+1$.*

In §1, we establish Theorem 1. In §2, we describe the minor alterations to the proof necessary to obtain Theorems 2 and 3. We work in the base theory $\mathtt{ZF} + \mathtt{DC}$ throughout.

1. The classical case

A binary relation G on a set X is a *graph* if it is irreflexive and symmetric. A (Y-)*coloring* of G is a function $c \colon X \to Y$ such that $w \mathrel{G} x \implies c(w) \neq c(x)$ for all $w, x \in X$. The *chromatic number* of G, written $\chi(G)$, is the least cardinal κ for which there is a κ-coloring of G (if such a cardinal exists). We use $\chi_{\mathrm{fin}}(G)$ to denote the supremum of the chromatic numbers of the graphs of the form $G \restriction F$, where

$F \subseteq X$ is a finite set. We use G^* to denote the supergraph of G with respect to which two points $x, y \in X$ are related if and only if there is a finite superset $F \subseteq X$ of $\{x, y\}$ such that $c(x) \neq c(y)$ for every $\chi_{\mathrm{fin}}(G)$-coloring c of $G \upharpoonright F$. Note that if $\chi_{\mathrm{fin}}(G) = \aleph_0$, then $G = G^*$.

Given a set $R \subseteq X \times Y$, define $R^{-1} = \{(y, x) \in Y \times X \mid x \mathrel{R} y\}$ and $R^{\pm 1} = R \cup R^{-1}$.

Proposition 4. *Suppose that X is a set, G is a graph on X, and $G' \subseteq G^*$ is finite. Then there is a finite set $F \subseteq X$ containing $\bigcup_{i<2} \mathrm{proj}_i(G')$ such that every $\chi_{\mathit{fin}}(G)$-coloring c of $G \upharpoonright F$ is a coloring of $(G')^{\pm 1}$.*

Proof. For all $(x, y) \in G'$, fix a finite superset $F_{(x,y)} \subseteq X$ of $\{x, y\}$ such that $c(x) \neq c(y)$ for every $\chi_{\mathrm{fin}}(G)$-coloring c of $G \upharpoonright F_{(x,y)}$, and observe that the set $F = \bigcup_{(x,y) \in G'} F_{(x,y)}$ is as desired. ⊠

A set $Y \subseteq X$ is a *G-clique* if any two distinct points of Y are G-related, and *G-independent* if no two points of Y are G-related.

Proposition 5. *Suppose that X is a set, G is a graph on X, and $C \subseteq X$ is a finite G^*-clique. Then $|C| \leq \chi_{\mathit{fin}}(G)$.*

Proof. By Proposition 4, there is a finite set $F \subseteq X$ containing C such that $c \upharpoonright C$ is injective for every $\chi_{\mathrm{fin}}(G)$-coloring c of $G \upharpoonright F$, in which case the pigeon-hole principle ensures that $|C| \leq \chi_{\mathrm{fin}}(G)$. ⊠

The *horizontal sections* of R are the sets $R^y = \{x \in X \mid x \mathrel{R} y\}$, where $y \in Y$. The *vertical sections* are the sets $R_x = \{y \in Y \mid x \mathrel{R} y\}$, where $x \in X$.

Proposition 6. *Suppose that X is a set, G is a graph on X for which $\chi_{\mathit{fin}}(G) < \aleph_0$, $x, y \in X$, and there is a G^*-clique $C \subseteq G^*_x \cup G^*_y$ of cardinality $\chi_{\mathit{fin}}(G)$. Then $x \mathrel{G^*} y$.*

Proof. Proposition 4 yields a finite set $F \subseteq X$ containing $C \cup \{x, y\}$ such that $c \upharpoonright C$ is injective and $\forall w \in \{x, y\} \forall z \in C \cap G^*_w \; c(w) \neq c(z)$ for every $\chi_{\mathrm{fin}}(G)$-coloring c of $G \upharpoonright F$. But if c is such a coloring, then $c(C) = \chi_{\mathrm{fin}}(G)$, so $c(x) \in c(C \cap G^*_y)$, thus $c(x) \neq c(y)$, hence $x \mathrel{G^*} y$. ⊠

We use $\|_R$, $\equiv_R$, $\perp_R$, and $<_R$ to denote the comparability, equivalence, incomparability, and strict relations associated with R.

Proposition 7. *Suppose that X is a set and R is a quasi-order on X. Then $R \setminus \perp^*_R$ is transitive.*

Proof. Suppose, towards a contradiction, that there exist $x, y, z \in X$ for which $x \mathrel{(R \setminus \perp^*_R)} y \mathrel{(R \setminus \perp^*_R)} z$, as well as a finite superset $F \subseteq X$ of $\{x, z\}$ such that $c(x) \neq c(z)$ for every $\chi_{\mathrm{fin}}(\perp_R)$-coloring c of $\perp_R \upharpoonright F$.

Then $x \mathrel{R} z$, so x and z are not $\perp_R$-related, thus $\chi_{\mathrm{fin}}(\perp_R) < \aleph_0$. For all $w \in \{x, z\}$, the fact that w and y are not $\perp_R^*$-related yields a $\chi_{\mathrm{fin}}(\perp_R)$-coloring c_w of $\perp_R \restriction (F \cup \{y\})$ for which $c_w(w) = c_w(y) = 0$, in which case the set $C_w = c_w^{-1}(\{0\})$ is an R-chain containing $\{w, y\}$ for which $(F \cup \{y\}) \setminus C_w$ is a union of the R-chains $c_w^{-1}(\{i\})$, for $0 < i < \chi_{\mathrm{fin}}(\perp_R)$, and therefore does not contain an R-antichain of cardinality $\chi_{\mathrm{fin}}(\perp_R)$. Then the set $C_0 = (C_x \cap R^y) \cup (C_z \cap R_y)$ is an R-chain containing $\{x, z\}$, so $(F \cup \{y\}) \setminus C_0$ is not a union of a sequence $(C_i)_{0<i<\chi_{\mathrm{fin}}(\perp_R)}$ of R-chains, since otherwise the function $c \colon F \to \chi_{\mathrm{fin}}(\perp_R)$, given by $c \restriction (C_i \cap F) = i$ for all $i < \chi_{\mathrm{fin}}(\perp_R)$, is a $\chi_{\mathrm{fin}}(\perp_R)$-coloring of $\perp_R \restriction F$ for which $c(x) = c(z)$. Dilworth's theorem therefore yields an R-antichain $A \subseteq (F \cup \{y\}) \setminus C_0$ of cardinality $\chi_{\mathrm{fin}}(\perp_R)$. Fix $u \in A \cap C_x$ and $w \in A \cap C_z$. As $u, w \notin C_0$, it follows that neither $u \mathrel{R} y$ nor $y \mathrel{R} w$, so the fact that C_x and C_z are R-chains ensures that $w <_R y <_R u$, contradicting the fact that A is an R-antichain. ⊠

Define $[x, y]_R = \{z \in X \mid x \mathrel{R} z \mathrel{R} y\}$ and $(x, y]_R = [x, y]_R \setminus [x]_{\equiv_R}$. We use $\frown$, $\sqsubseteq$, and (i) to denote concatenation, extension, and the sequence of length one whose sole entry is i. Fix sequences $s_n \in 2^n$ that are *dense* in $2^{<\mathbb{N}}$, in the sense that $\forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ s \sqsubseteq s_n$, and define $\mathbb{G}_0 = \{(s_n \frown (i) \frown c, s_n \frown (1-i) \frown c) \mid c \in 2^{\mathbb{N}},\ i < 2, \text{ and } n \in \mathbb{N}\}$.

Proposition 8. *Suppose that X is a topological space, R is an $\aleph_0$-universally-Baire quasi-order on X that does not have antichains of arbitrarily large finite cardinality, and $\perp_R^*$ is $\aleph_0$-universally Baire. Then there is no continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $\mathbb{G}_0$ to $\perp_R^*$.*

Proof. As Dilworth's theorem ensures that $\chi_{\mathrm{fin}}(\perp_R) < \aleph_0$, it is sufficient to show that if $\phi \colon 2^{\mathbb{N}} \to X$ is a continuous homomorphism from $\mathbb{G}_0$ to $\perp_R^*$, then there exists $x \in \phi(2^{\mathbb{N}})$ for which there is a continuous homomorphism from $\mathbb{G}_0$ to $\perp_R^* \restriction (\phi(2^{\mathbb{N}}) \cap (\perp_R^*)_x)$, since $\chi_{\mathrm{fin}}(\perp_R)$ applications of this fact yield a $\perp_R^*$-clique of cardinality $\chi_{\mathrm{fin}}(\perp_R) + 1$, contradicting Proposition 5.

Letting G' be the pullback of $\perp_R^*$ through $\phi \times \phi$, it is sufficient to find $c \in 2^{\mathbb{N}}$ for which G'_c has the Baire property and is not meager, as the proof of [KST99, Proposition 6.2] ensures that every $\mathbb{G}_0$-independent set with the Baire property is meager, so [KST99, Theorem 6.3] would then yield a continuous homomorphism $\psi \colon 2^{\mathbb{N}} \to G'_c$ from $\mathbb{G}_0$ to $\mathbb{G}_0 \restriction G'_c$ (although the existence of such a function also follows from a straightforward recursive construction), in which case the point $x = \phi(c)$ and the homomorphism $\phi \circ \psi$ are as desired.

Suppose, towards a contradiction, that every vertical section of G' with the Baire property is meager, and let R' be the pullback of R

through $\phi \times \phi$. As $\perp_R^*$ and R are $\aleph_0$-universally Baire, the horizontal and vertical sections of G' and R' all have the Baire property. As $\perp_{R'} \subseteq G'$, every vertical section of $\perp_{R'}$ is meager, and the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) ensures that $\|_{R'}$ is comeager, so R' is not meager.

Lemma 9. *There exists $(b, d) \in \mathbb{G}_0$ for which $[b, d]_{R'}$ is not meager.*

Proof. It is trivial to check that the binary relation S' on $2^{\mathbb{N}}$ given by $c \mathrel{S'} d \iff \forall^* b \in 2^{\mathbb{N}} \ (b \mathrel{R'} c \implies b \mathrel{R'} d)$ is a quasi-order. Moreover, if $(d, c) \in {\sim}S'$, then $\exists^* b \in 2^{\mathbb{N}} \ (b \mathrel{R'} d$ and $\neg b \mathrel{R'} c)$, so the fact that $(\perp_{R'})_c$ is meager ensures that $(c, d]_{R'}$ is not meager. We can therefore assume that $\mathbb{G}_0 \subseteq S'$, so $\mathbb{G}_0 \subseteq \equiv_{S'}$. As the smallest equivalence relation on $2^{\mathbb{N}}$ containing $\mathbb{G}_0$ is $\mathbb{E}_0$ (by a straightforward inductive argument), it follows that $\mathbb{E}_0 \subseteq \equiv_{S'}$. The Kuratowski-Ulam and Montgomery-Novikov theorems (see, for example, [Kec95, Theorem 16.1]) ensure that for all $t \in 2^{<\mathbb{N}}$, the set $B_t = \{c \in 2^{\mathbb{N}} \mid \forall^* b \in \mathcal{N}_t \ b \mathrel{R'} c\}$ has the Baire property. As these sets are $\equiv_{S'}$-invariant, and therefore $\mathbb{E}_0$-invariant, they are either meager or comeager (see, for example, [Kec95, Theorem 8.47]). Define $T = \{t \in 2^{<\mathbb{N}} \mid B_t \text{ is comeager}\}$. Then the comeager set $C = \bigcap_{t \in T} B_t \cap \bigcap_{t \in \sim T} {\sim}B_t$ is an $\equiv_{S'}$-class, since $c \equiv_{S'} d \iff \forall t \in 2^{<\mathbb{N}} \ (c \in B_t \iff d \in B_t)$ for all $c, d \in 2^{\mathbb{N}}$ (see, for example, [Kec95, Proposition 8.26]). Fixing $t, u \in 2^{<\mathbb{N}}$ with the property that $R' \cap (\mathcal{N}_t \times \mathcal{N}_u)$ is comeager in $\mathcal{N}_t \times \mathcal{N}_u$, the Kuratowski-Ulam theorem implies that $\forall^* c \in \mathcal{N}_u \forall^* b \in \mathcal{N}_t \ b \mathrel{R'} c$, so $t \in T$, thus $\forall^* b, c \in \mathcal{N}_t \ b \mathrel{R'} c$, hence there is an $\equiv_{R'}$-class $C' \subseteq 2^{\mathbb{N}}$ that is comeager in $\mathcal{N}_s$. But non-meager subsets of $2^{\mathbb{N}}$ with the Baire property are not $\mathbb{G}_0$-independent, and any $(b, d) \in \mathbb{G}_0 \restriction C'$ is as desired. ⊠

As $b \mathrel{G'} d$, Proposition 7 ensures that $\forall c \in [b, d]_{R'} \ (b \mathrel{G'} c$ or $c \mathrel{G'} d)$, so G'_b or G'_d is not meager, the desired contradiction. ⊠

Remark 10. A similar approach can be used to eliminate the need for multiple applications of the $\mathbb{G}_0$ dichotomy, and therefore the need to assume that $\mathtt{add}(\mathcal{M}) < \kappa$, in [MV19] (see [Mil20, Propositions 1.6.17 and 1.6.19]).

Proposition 11. *Suppose that X is a set and R is a quasi-order on X. Then $\perp_R^*$ is $\equiv_R$-invariant.*

Proof. It is sufficient to show that if $x \equiv_R x'$ and $\neg x \perp_R^* y$, then $\neg x' \perp_R^* y$. Towards this end, given a finite superset $F \subseteq X$ of $\{x', y\}$, fix a $\chi_{\text{fin}}(\perp_R)$-coloring c of $\perp_R \restriction (F \cup \{x\})$ for which $c(x) = c(y)$, and observe that the extension c' of $c \restriction (F \setminus \{x'\})$, given by $c'(x') = c(x)$, is a $\chi_{\text{fin}}(\perp_R)$-coloring of $\perp_R \restriction F$ for which $c'(x') = c'(y)$. ⊠

Proposition 12. *Suppose that X is a set, R is a quasi-order on X that does not have antichains of arbitrarily large finite cardinality, $A \subseteq X$ is an R-antichain of cardinality $\chi_{fin}(\perp_R)$, and $Y \subseteq X$ is $\perp_R^*$-independent. Then there exists $x \in A$ for which $\{x\} \cup Y$ is $\perp_R^*$-independent.*

Proof. Suppose, towards a contradiction, that there exists a function $\phi \colon A \to Y$ whose graph is contained in $\perp_R^*$. As Dilworth's theorem ensures that $\chi_{\text{fin}}(\perp_R) < \aleph_0$, it follows that A is a maximal R-antichain, and is therefore the union of the sets $A' = \{x \in A \mid A \cap R^{\phi(x)} \neq \emptyset\}$ and $A'' = \{x \in A \mid A \cap R_{\phi(x)} \neq \emptyset\}$.

Lemma 13. *The sets A' and A'' are disjoint, so $A \cap \phi(A) = \emptyset$.*

Proof. Suppose, towards a contradiction, that there exists $x \in A' \cap A''$, and fix $y, z \in A$ for which $y \mathrel{R} \phi(x) \mathrel{R} z$. As A is an R-antichain, it follows that $y = z$, so $\phi(x) \equiv_R y$, thus Proposition 11 yields that $\phi(x) \perp_R^* \phi(y)$, contradicting the $\perp_R^*$-independence of Y.

It only remains to note that if there exists $x \in A$ for which $\phi(x) \in A$, then $x \in A' \cap A''$, a contradiction. ⊠

Lemma 14. *If $w', x' \in A'$ and $\phi(x') \mathrel{R} \phi(w')$, then $w' \perp_R^* \phi(x')$.*

Proof. If w' and $\phi(x')$ are not $\perp_R^*$-related, then $w' \parallel_R \phi(x')$, so Lemma 13 ensures that $w' \ (R \setminus \perp_R^*) \ \phi(x')$. But the $\perp_R^*$-independence of Y implies that $\phi(x') \ (R \setminus \perp_R^*) \ \phi(w')$, thus Proposition 7 yields that w' and $\phi(w')$ are not $\perp_R^*$-related, a contradiction. ⊠

Lemma 15. *If $w'', x'' \in A''$ and $\phi(w'') \mathrel{R} \phi(x'')$, then $w'' \perp_R^* \phi(x'')$.*

Proof. If w'' and $\phi(x'')$ are not $\perp_R^*$-related, then $w'' \parallel_R \phi(x'')$, so Lemma 13 ensures that $\phi(x'') \ (R \setminus \perp_R^*) \ w''$. But the $\perp_R^*$-independence of Y implies that $\phi(w'') \ (R \setminus \perp_R^*) \ \phi(x'')$, thus Proposition 7 yields that $\phi(w'')$ and w'' are not $\perp_R^*$-related, a contradiction. ⊠

If $A' \neq \emptyset$, then the fact that Y is an R-chain yields $x' \in A'$ for which $\phi(x')$ is $(R \restriction \phi(A'))$-minimal, so Lemma 14 ensures that $A' \cup \{\phi(x')\}$ is an $\perp_R^*$-clique, and since Lemma 13 implies that $\phi(x') \notin A'$, Proposition 5 yields that $|A'| < \chi_{\text{fin}}(\perp_R)$. Similarly, if $A'' \neq \emptyset$, then the fact that Y is an R-chain yields $x'' \in A''$ for which $\phi(x'')$ is $(R \restriction \phi(A''))$-maximal, so Lemma 15 ensures that $A'' \cup \{\phi(x'')\}$ is an $\perp_R^*$-clique, and since Lemma 13 implies that $\phi(x'') \notin A''$, Proposition 5 implies that $|A''| < \chi_{\text{fin}}(\perp_R)$. It follows that A' and A'' are non-empty, so there are indeed $x' \in A'$ and $x'' \in A''$ for which $\phi(x')$ is $(R \restriction \phi(A'))$-minimal and $\phi(x'')$ is $(R \restriction \phi(A''))$-maximal. As $A \subseteq (\perp_R^*)_{\phi(x')} \cup (\perp_R^*)_{\phi(x'')}$ by Lemmas 14 and 15, Proposition 6 implies that $\phi(x') \perp_R^* \phi(x'')$, contradicting the $\perp_R^*$-independence of Y. ⊠

For each $k \in \mathbb{N}$, let $[X]^k$ denote the family of all subsets of X of cardinality k, equipped with the topology generated by the sets of the form $\{F \in [X]^k \mid \exists \pi \colon F \hookrightarrow \mathcal{F} \ \forall x \in F \ x \in \pi(x)\}$, where $\mathcal{F} \in [\tau_X]^k$. Let $[X]^{\leq k}$ denote the disjoint union of the spaces of the form $[X]^j$, for $j \leq k$. Similarly, let $[X]^{<\aleph_0}$ denote the disjoint union of the spaces of the form $[X]^k$, for $k \in \mathbb{N}$. A set $Y \subseteq X$ *punctures* a family $\mathcal{F} \subseteq [X]^{<\aleph_0}$ if $F \cap Y \neq \emptyset$ for all $F \in \mathcal{F}$.

Proposition 16. *Suppose that X is a Hausdorff space, G is an analytic graph on X that admits a Borel coloring $c \colon X \to \mathbb{N}$, and $\mathcal{F} \subseteq [X]^{<\aleph_0}$ is an analytic set with the property that for every G-independent set $Y \subseteq X$, the corresponding set $\{x \in X \mid \{x\} \cup Y$ is G-independent$\}$ punctures $\mathcal{F}$. Then every G-independent Borel subset of X is contained in a G-independent Borel subset of X that punctures $\mathcal{F}$.*

Proof. For each natural number k and G-independent set $Y \subseteq X$, we use $\mathcal{F}^k_Y$ to denote the family of sets $F \in \mathcal{F}$ with the property that $|\{x \in F \mid \{x\} \cup Y \text{ is not } G\text{-independent}\}| \geq |F| - k$. Note that $\mathcal{F}^0_Y = \emptyset$ and $\mathcal{F} \cap [X]^{\leq k} \subseteq \mathcal{F}^k_Y$, since $|F| - k \leq 0$ for all $F \in [X]^{\leq k}$. It is sufficient to show that for all $k \in \mathbb{N}$, every G-independent Borel set $B \subseteq X$ that punctures $\mathcal{F}^k_B$ is contained in a G-independent Borel set $C \subseteq X$ that punctures $\mathcal{F}^{k+1}_C$, as repeated application of this fact yields an increasing sequence of G-independent Borel supersets $B_k \subseteq X$ of any given G-independent Borel subset of X that puncture $\mathcal{F}^k_{B_k}$, in which case the G-independent set $\bigcup_{k \in \mathbb{N}} B_k$ punctures $\bigcup_{k \in \mathbb{N}} \mathcal{F}^k_{B_k} \supseteq \bigcup_{k \in \mathbb{N}} \mathcal{F} \cap [X]^{\leq k} = \mathcal{F}$.

Suppose that $k \in \mathbb{N}$, we have already established the aforementioned fact strictly below k, and $B \subseteq X$ is a G-independent Borel set that punctures $\mathcal{F}^k_B$. Fix natural numbers i_j such that $\forall i \in \mathbb{N} \exists^\infty j \in \mathbb{N} \ i = i_j$, and define $B'_0 = B$. Given $j \in \mathbb{N}$ and a G-independent Borel set $B'_j \subseteq X$ that punctures $\mathcal{F}^k_{B'_j}$, let A'_j be the set of $x \in X$ for which there exists $F \in \mathcal{F}$, disjoint from B'_j, with the property that $x \in F$ and $|\{y \in F \setminus \{x\} \mid B'_j \cup \{y\} \text{ is not } G\text{-independent}\}| \geq |F| - (k+1)$. As B'_j punctures $\mathcal{F}^k_{B'_j}$, no such F is in $\mathcal{F}^k_{B'_j}$, so $B'_j \cup \{x\}$ is G-independent for any such x, thus $(A'_j \cap c^{-1}(\{i_j\})) \cup B'_j$ is also G-independent. As the latter set is analytic, it is contained in a G-independent Borel set (see, for example, the proof of [Mil11, Proposition 2]), in which case k applications of the induction hypothesis yield a G-independent Borel set $B'_{j+1} \subseteq X$ containing $(A'_j \cap c^{-1}(\{i_j\})) \cup B'_j$ that punctures $\mathcal{F}^k_{B'_{j+1}}$.

To see that the G-independent Borel set $C = \bigcup_{j \in \mathbb{N}} B'_j$ punctures $\mathcal{F}^{k+1}_C$, observe that if $F \in \mathcal{F}^{k+1}_C$, then the hypothesis on $\mathcal{F}$ yields a point $x \in F$ for which $C \cup \{x\}$ is G-independent, as well as $j \in \mathbb{N}$

for which $F \in \mathcal{F}^{k+1}_{B'_j}$, and $j' \geq j$ for which $i_{j'} = c(x)$, in which case $B'_{j'} \cap F \neq \emptyset$ or $x \in B'_{j'+1}$. ⊠

The *Borel chromatic number* of a graph G on X is the least cardinal $\chi_{\mathrm{B}}(G)$ of the form $|Y|$, where Y is an analytic Hausdorff space for which there exists a Borel Y-coloring of G (if such a space exists).

Proposition 17. *Suppose that X is a Hausdorff space and R is a quasi-order on X with the property that $\perp_R$ is analytic and $\chi_B(\perp^*_R) \leq \aleph_0$. Then $\chi_B(\perp^*_R) = \chi_{fin}(\perp_R)$.*

Proof. As the case $\chi_{\mathrm{fin}}(\perp_R) \in \{1, \aleph_0\}$ is trivial, suppose that $k \in \mathbb{Z}^+$, we have already established the proposition for $\chi_{\mathrm{fin}}(\perp_R) \leq k$, and $\chi_{\mathrm{fin}}(\perp_R) = k + 1$. As $\perp^*_R$ is analytic, Propositions 12 and 16 yield an $\perp^*_R$-independent Borel set $B \subseteq X$ that intersects every R-antichain of cardinality $k+1$. As Dilworth's theorem ensures that $\chi_{\mathrm{fin}}(\perp_R \restriction {\sim}B) = k$, the induction hypothesis yields a Borel k-coloring c of $(\perp_R \restriction {\sim}B)^*$. Observe that $\perp^*_R \restriction {\sim}B \subseteq (\perp_R \restriction {\sim}B)^*$, for if $x, y \in {\sim}B$ and $F \subseteq X$ is a finite set containing $\{x, y\}$ such that $d(x) \neq d(y)$ for every $(k+1)$-coloring d of $\perp_R \restriction F$, then $F \setminus B$ is a finite set containing $\{x, y\}$ such that $d(x) \neq d(y)$ for every k-coloring d of $\perp_R \restriction (F \setminus B)$. In particular, it follows that the extension of c to X with constant value k on B is a Borel $(k+1)$-coloring of $\perp^*_R$. ⊠

As every analytic subset of a topological space is $\aleph_0$-universally Baire (see, for example, [Kec95, Theorem 21.6]), Theorem 1 follows from Proposition 8, the $\mathbb{G}_0$ dichotomy, and Proposition 17.

2. Generalizations under determinacy

Given an ordinal α, a subset of a topological space X is *α-Borel* if it is in the closure of τ_X under complements and unions of length strictly less than α. Given an aleph κ, a topological space is *κ-Souslin* if it is a continuous image of a closed subset of $\kappa^{\mathbb{N}}$.

For all $n > 0$, let $\boldsymbol{\delta}^1_n$ denote the supremum of the lengths of well-orders of the form $R/{\equiv_R}$, where R is a $\boldsymbol{\Delta}^1_n$ quasi-order on an analytic Hausdorff space. The axiom of determinacy ensures that the $\boldsymbol{\Delta}^1_{2n+1}$ and $\boldsymbol{\delta}^1_{2n+1}$-Borel subsets of analytic Hausdorff spaces coincide. It also yields an aleph $\boldsymbol{\lambda}^1_{2n+1}$ for which $\boldsymbol{\delta}^1_{2n+1} = (\boldsymbol{\lambda}^1_{2n+1})^+$, and implies that the $\boldsymbol{\Sigma}^1_{2n+1}$ and $\boldsymbol{\lambda}^1_{2n+1}$-Souslin subsets of analytic Hausdorff spaces coincide (see, for example, [Jac08]).

A *tree* on a set I is a set $T \subseteq I^{<\mathbb{N}}$ that is *closed under initial segments*, in the sense that $\forall t \in T \forall n < |t| \; t \restriction n \in T$. A *subtree* of T is a tree

$S \subseteq T$ on I. A *branch* through T is a sequence $x \in I^{\mathbb{N}}$ such that $\forall n \in \mathbb{N}\ x \restriction n \in T$. A tree is *well-founded* if it has no branches.

The *pruning derivative* associates with each tree T on a set I the subtree $T' = \{t \in T \mid \exists i \in I\ t \frown (i) \in T\}$. The *iterates* of the pruning derivative are given by $T^{(0)} = T$, $T^{(\alpha+1)} = (T^{(\alpha)})'$ for all ordinals α, and $T^{(\lambda)} = \bigcap_{\alpha<\lambda} T^{(\alpha)}$ for all limit ordinals λ. The *pruning rank* of T is the least ordinal $\rho(T)$ for which $T^{(\rho(T))} = T^{(\rho(T)+1)}$. A straightforward induction shows that T is well-founded if and only if $T^{(\rho(T))} = \emptyset$. For each $t \in T$, let $\rho_T(t)$ denote the largest ordinal for which $t \in T^{(\rho_T(t))}$ (if such an ordinal exists).

An $(\alpha+1)$-*Borel code* for a subset of X is a pair (f, T), where T is a well-founded tree on $\alpha \times \alpha$ and f is a function associating to each sequence $t \in {\sim}T$ a subset of X that is closed or open. Given such a code, we recursively define $f^{(\beta)}$ on ${\sim}T^{(\beta)}$ by setting $f^{(0)} = f$, letting $f^{(\beta+1)}$ be the extension of $f^{(\beta)}$ given by $f^{(\beta+1)}(t) = \bigcup_{\gamma<\alpha} \bigcap_{\delta<\alpha} f^{(\beta)}(t \frown ((\gamma, \delta)))$ whenever $\rho_T(t) = \beta$ for all ordinals β, and defining $f^{(\lambda)} = \bigcup_{\beta<\lambda} f^{(\beta)}$ for all limit ordinals λ. The $(\alpha+1)$-Borel set *coded* by (f, T) is $f^{(\rho(T))}(\emptyset)$.

The proof of Souslin's theorem shows that there is a function sending each pair of functions witnessing that a set and its complement are κ-Souslin to a $(\kappa+1)$-Borel code for the set. Under AD, the coding lemma (see [Mos09, Lemma 7D.5]) and projective uniformization (see, for example, [Kec95, Theorem 39.9]) can be used to obtain a function sending each $(\boldsymbol{\lambda}^1_{2n+1}+1)$-Borel code for a subset of an analytic Hausdorff space to a function witnessing that the encoded set is $\boldsymbol{\lambda}^1_{2n+1}$-Souslin.

Proposition 18 (AD)**.** *Suppose that $n \in \mathbb{N}$, X is an analytic Hausdorff space, G is a $\boldsymbol{\Sigma}^1_{2n+1}$ graph on X that admits a $\boldsymbol{\Delta}^1_{2n+1}$ coloring $c \colon X \to \boldsymbol{\lambda}^1_{2n+1}$, and $\mathcal{F} \subseteq [X]^{<\aleph_0}$ is a $\boldsymbol{\Sigma}^1_{2n+1}$ set with the property that for every G-independent set $Y \subseteq X$, the corresponding set $\{x \in X \mid \{x\} \cup Y$ is G-independent$\}$ punctures $\mathcal{F}$. Then every G-independent $\boldsymbol{\Delta}^1_{2n+1}$ subset of X is contained in a G-independent $\boldsymbol{\Delta}^1_{2n+1}$ subset of X that punctures $\mathcal{F}$.*

Proof. We proceed essentially as in the proof of Proposition 16. The first paragraph remains unchanged. The induction beginning in the second paragraph, however, has length $\boldsymbol{\lambda}^1_{2n+1}$ instead of ω, which is problematic because naively applying [Mil11, Proposition 2] at each stage of the induction requires too large a fragment of the axiom of choice. This problem can be alleviated by using the above remarks to keep track of codes for the sets B'_j that are built along the way, which can be achieved because the proof of [Mil11, Proposition 2] utilizes little more than Souslin's theorem. ⊠

Proposition 18 gives rise to an analogous version of Proposition 17. As every subset of a topological space is $\aleph_0$-universally Baire under $\mathtt{AD}$ (see, for example, [Mos09, Theorem 7D.2]), this can be combined with Proposition 8 and Kanovei's generalization of the $\mathbb{G}_0$ dichotomy (see [Kan97], although the elementary proof of [Mil11, Theorem 8] can be adapted to obtain the special cases we need by keeping track of codes as above) to establish Theorem 2.

By eliminating the outer induction and the use of [Mil11, Proposition 2] in the proof of Proposition 16, one obtains a proof of the weaker result without definability conditions on the sets involved. Moreover, this proof trivially generalizes to colorings $c \colon X \to \kappa$, for any aleph κ, and gives rise to an analogous version of Proposition 17. As a result of Woodin's ensures that every subset of an analytic Hausdorff space is κ-Souslin, for some aleph κ, under $\mathtt{AD}_{\mathbb{R}}$ (see, for example, [Kan03, Theorem 32.23]), this can be combined with Proposition 8 and the weakening of Kanovei's generalization of the $\mathbb{G}_0$ dichotomy in which there are no definability constraints on the coloring (which follows from the simplification of the proof of [Mil11, Theorem 8] in which the use of Souslin's theorem is eliminated) to establish Theorem 3.

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