COORDINATEWISE DECOMPOSITION OF GROUP-VALUED BOREL FUNCTIONS

BENJAMIN D. MILLER

Abstract. Answering a question of Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [6], we characterize the Borel sets $S \subseteq X \times Y$ on which every Borel function $f : S \to \mathbb{C}$ is of the form $uv|S$, where $u : X \to \mathbb{C}$ and $v : Y \to \mathbb{C}$ are Borel.

Suppose that $S \subseteq X \times Y$ and Γ is a group. A *coordinatewise decomposition* of a function $f : S \to \Gamma$ is a pair (u, v), where $u : X \to \Gamma$, $v : Y \to \Gamma$, and

$$\forall (x, y) \in S \ (f(x, y) = u(x)v(y)).$$

While our main goal here is to study coordinatewise decompositions in the descriptive set-theoretic context, we will first study the existence of coordinatewise decompositions without imposing any definability restrictions.

For the sake of notational convenience, we will assume that $X \cap Y = \emptyset$. The *graph associated with* S is the graph on the set $Z_S = X \cup Y$ given by $\mathcal{G}_S = S \cup S^{-1}$. The following fact was proven essentially by Cowsik-Kłopotowski-Nadkarni [1]:

Proposition 1. *Suppose that X, Y are disjoint, $S \subseteq X \times Y$, and Γ is a non-trivial group. Then the following are equivalent:*

1. *Every function $f : S \to \Gamma$ admits a coordinatewise decomposition;*
2. *$\mathcal{G}_S$ is acyclic.*

Proof. To see $\neg(2) \Rightarrow \neg(1)$ note that, by reversing the roles of X and Y if necessary, we can assume that there is a proper cycle of the form $x_0, y_0, x_1, y_1, \ldots, x_{n+1} = x_0$ through $\mathcal{G}_S$. Fix $\gamma_0 \in \Gamma \setminus \{1_\Gamma\}$, define $f : S \to \Gamma$ by

$$f(x, y) = \begin{cases} \gamma_0 & \text{if } (x, y) = (x_0, y_0), \\ 1_\Gamma & \text{otherwise,} \end{cases}$$

and suppose that (u, v) is a coordinatewise decomposition of f. Then

$$\begin{aligned} \gamma_0 &= f(x_0, y_0)f(x_1, y_0)^{-1} \cdots f(x_n, y_n)f(x_{n+1}, y_n)^{-1} \\ &= (u(x_0)v(y_0))(u(x_1)v(y_0))^{-1} \cdots (u(x_n)v(y_n))(u(x_{n+1})v(y_n))^{-1} \\ &= u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1} \\ &= u(x_0)u(x_{n+1})^{-1} \\ &= 1_\Gamma, \end{aligned}$$

which contradicts our choice of γ_0.

To see (2) $\Rightarrow$ (1), let E_S be the equivalence relation whose classes are the connected components of $\mathcal{G}_S$, fix a *transversal* $B \subseteq Z_S$ of E_S (i.e., a set which intersects every E_S-class in exactly one point), and define $B_n \subseteq Z$ by

$$B_n = \{z \in Z : d_S(z, B) = n\},$$

where d_S denotes the graph metric associated with $\mathcal{G}_S$. For $z \in B_{n+1}$, let $g(z)$ denote the unique $\mathcal{G}$-neighbor of z in B_n, and define recursively $u : X \to \Gamma, v : Y \to \Gamma$ by

$$u(x) = \begin{cases} 1_\Gamma & \text{if } x \in B, \\ f(x, g(x))v(g(x))^{-1} & \text{otherwise,} \end{cases}$$

and

$$v(y) = \begin{cases} 1_\Gamma & \text{if } y \in B, \\ u(g(y))^{-1}f(g(y), y) & \text{otherwise.} \end{cases}$$

To see that (u, v) is a coordinatewise decomposition of f, suppose that $(x, y) \in S$ and note that either $g(x) = y$ or $g(y) = x$. In the former case, it follows that $u(x) = f(x, y)v(y)^{-1}$, thus $f(x, y) = u(x)v(y)$. In the latter case, it follows that $v(y) = u(x)^{-1}f(x, y)$, thus $f(x, y) = u(x)v(y)$. $\square$

As a corollary of the proof of Proposition 1, we obtain a sufficient condition for the existence of Borel coordinatewise decompositions:

Corollary 2. *Suppose that X and Y are Polish spaces, $S \subseteq X \times Y$ is Borel, $\mathcal{G}_S$ is acyclic, and E_S admits a Borel transversal. Then every standard Borel group-valued Borel function on S admits a Borel coordinatewise decomposition.*

Proof. It is sufficient to check that if $f : S \to \Gamma$ is a standard Borel group-valued Borel function, then the functions u and v constructed in the proof of Proposition 1 are Borel. Letting $B_n \subseteq Z_S$ and $g : Z_S \to Z_S$ be as constructed above, it follows from the fact that $\mathcal{G}_S$ is acyclic that

$$\begin{aligned} z \in B_{n+1} \quad &\Leftrightarrow \quad z \notin \bigcup_{i \leq n} B_i \text{ and } \exists w \in B_n \ ((z, w) \in \mathcal{G}) \\ &\Leftrightarrow \quad z \notin \bigcup_{i \leq n} B_i \text{ and } \exists! w \in B_n \ ((z, w) \in \mathcal{G}), \end{aligned}$$

and it follows from results of Souslin and Lusin (see, for example, Theorems 14.11 and 18.11 of Kechris [5]) that each of these sets is Borel. As

$$\text{graph}(g) = \bigcup_{n \in \mathbb{N}} \mathcal{G}_S \cap (B_{n+1} \times B_n),$$

it follows that g is Borel as well (see, for example, Theorem 14.12 of Kechris [5]), and this easily implies that u and v are Borel. $\square$

Our main theorem is that the sufficient condition given in Corollary 2 is also necessary to guarantee the existence of Borel coordinatewise decompositions:

Theorem 3. *Suppose that X, Y are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, and Γ is a non-trivial standard Borel group. Then the following are equivalent:*

1. *Every Borel function $f : S \to \Gamma$ admits a Borel coordinatewise decomposition;*
2. *$\mathcal{G}_S$ is acyclic and E_S admits a Borel transversal.*

Proof. As (2) $\Rightarrow$ (1) follows from Corollary 2, we need only show that (1) $\Rightarrow$ (2). As the map f described in the proof of $\neg$(2) $\Rightarrow$ $\neg$(1) of Proposition 1 is clearly Borel, it follows that $\mathcal{G}_S$ is acyclic, thus E_S is Borel (by Theorems 14.11 and 18.11 of Kechris [5]).

Fix a non-trivial countable subgroup $\Delta \leq \Gamma$, endow Δ with the discrete topology, and endow $\Delta^{\mathbb{N}}$ with the corresponding product topology. Define E_0^Δ on $\Delta^{\mathbb{N}}$ by

$$\alpha E_0^\Delta \beta \Leftrightarrow \exists n \in \mathbb{N}\, \forall m > n\ (\alpha(m) = \beta(m)),$$

and define $F_0^\Delta \subseteq E_0^\Delta$ on $\Delta^{\mathbb{N}}$ by

$$\alpha F_0^\Delta \beta \Leftrightarrow \exists n \in \mathbb{N}\ \left(\alpha(0) \cdots \alpha(n) = \beta(0) \cdots \beta(n) \text{ and } \forall m > n\ (\alpha(m) = \beta(m))\right).$$

Let Δ act freely on $\Delta^{\mathbb{N}}$ by left multiplication on the 0^{th}-coordinate, i.e.,

$$\delta \cdot \alpha = (\delta\alpha(0), \alpha(1), \alpha(2), \ldots).$$

Lemma 4. *The action of Δ on $\Delta^{\mathbb{N}}$ induces a free action of Δ on $\Delta^{\mathbb{N}}/F_0^\Delta$.*

Proof. It is enough to show that

$$\forall \delta \in \Delta\, \forall \alpha, \beta \in \Delta^{\mathbb{N}}\ (\alpha F_0^\Delta \beta \Rightarrow \delta \cdot \alpha F_0^\Delta \delta \cdot \beta).$$

Towards this end, suppose that $\delta \in \Delta$ and $(\alpha, \beta) \in F_0^\Delta$, fix $n \in \mathbb{N}$ such that

$$\alpha(0) \cdots \alpha(n) = \beta(0) \cdots \beta(n) \text{ and } \forall m > n\ (\alpha(m) = \beta(m)),$$

and note that

$$\delta\alpha(0) \cdots \alpha(n) = \delta\beta(0) \cdots \beta(n) \text{ and } \forall m > n\ (\alpha(m) = \beta(m)),$$

thus $\delta \cdot \alpha F_0^\Delta \delta \cdot \beta$. □

Suppose now that $F \subseteq E$ are Borel equivalence relations on a Polish space Z. We say that a set $B \subseteq Z$ is *F-invariant* if $\forall z_1 \in B\, \forall z_2 \in Z\ (z_1 F z_2 \Rightarrow z_2 \in B)$, and $B \subseteq Z$ is an *E-complete section* if $\forall z_1 \in Z\, \exists z_2 \in B\ (z_1 E z_2)$. We say that E is *relatively ergodic* over F if there is no Borel way of choosing a non-empty proper subset of the F-classes within each E-class, i.e., if there is no F-invariant Borel set $B \subseteq Z$ such that both B and $Z \setminus B$ are E-complete sections.

Lemma 5. *E_0^Δ is relatively ergodic over F_0^Δ.*

Proof. Suppose, towards a contradiction, that $B \subseteq \Delta^{\mathbb{N}}$ is an F_0^Δ-invariant Borel set such that both B and $\Delta^{\mathbb{N}} \setminus B$ are E_0^Δ-complete sections. As B is an E_0^Δ-complete section, it follows that B is non-meager, thus there exists $s \in \Delta^{<\mathbb{N}}$ such that B is comeager in $\mathcal{N}_s$. Define $C \subseteq \Delta^{\mathbb{N}}$ by

$$C = \Delta^{\mathbb{N}} \setminus [\mathcal{N}_s \setminus B]_{E_0^{\Delta}},$$

and observe that C is an E_0^{Δ}-invariant comeager Borel set and $\mathcal{N}_s \cap C \subseteq B \cap C$. It only remains to show that $C \subseteq B$, which implies that $\Delta^{\mathbb{N}} \setminus B$ is meager and therefore contradicts the fact that $\Delta^{\mathbb{N}} \setminus B$ is an E_0^{Δ}-complete section. Towards this end, put $n = |s|$, and given any $\alpha \in C$, define $\delta \in \Delta$ by

$$\delta = (s(0) \cdots s(n-1))^{-1}(\alpha(0) \cdots \alpha(n)).$$

As $\alpha F_0^{\Delta} s(0) \dots s(n-1)\delta\alpha(n+1)\alpha(n+2)\dots$, it follows that $\alpha \in B$. □

Suppose now that E_1 and E_2 are Borel equivalence relations on Polish spaces Z_1 and Z_2, respectively. A *reduction* of E_1 into E_2 is a function $\pi : Z_1 \to Z_2$ such that $\forall z, z' \in Z_1\ (z E_1 z' \Leftrightarrow \pi(z) E_2 \pi(z'))$. An *embedding* is an injective reduction. Let E_0 denote the equivalence relation on $2^{\mathbb{N}}$ which is given by

$$\alpha E_0 \beta \Leftrightarrow \exists n \in \mathbb{N}\, \forall m > n\ (\alpha(m) = \beta(m)).$$

While our next lemma follows from the much more general results of Dougherty-Jackson-Kechris [2], it is easy enough to prove directly:

Lemma 6. *There is a Borel embedding $\pi_1 : \Delta^{\mathbb{N}} \to 2^{\mathbb{N}}$ of E_0^{Δ} into E_0.*

Proof. Fix an enumeration (k_n, δ_n) of $\mathbb{N} \times \Delta$, and define $\pi_1 : \Delta^{\mathbb{N}} \to 2^{\mathbb{N}}$ by

$$[\pi_1(\alpha)](n) = \begin{cases} 1 & \text{if } \alpha(k_n) = \delta_n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that π_1 is the desired embedding. □

Now suppose, towards a contradiction, that E_S has no Borel transversal.

Lemma 7. *There is a Borel embedding $\pi_2 : 2^{\mathbb{N}} \to Z_S$ of E_0 into $E_S|X$.*

Proof. An equivalence relation E on a Polish space Z is said to be *smooth* if there is a Borel reduction of E into the trivial equivalence relation $\Delta(\mathbb{R}) = \{(x, x) : x \in \mathbb{R}\}$, or equivalently, if E admits a Borel *separating family*, i.e., a family $B_0, B_1, \dots$ of Borel subsets of Z such that

$$\forall z_1, z_2 \in Z\ (z_1 E z_2 \Leftrightarrow \forall n \in \mathbb{N}\ (z_1 \in B_n \Leftrightarrow z_2 \in B_n)).$$

Suppose, towards a contradiction, that there is no Borel embedding of E_0 into $E_S|X$. As E_S is Borel, so too is $E_S|X$. It follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that $E_S|X$ is smooth. Fix a Borel separating family $B_0, B_1, \dots$ for $E_S|X$, and observe that the sets

$$A_n = B_n \cup \{y \in Y : \exists x \in B_n\ ((x, y) \in S)\}$$

form a $\mathbf{\Sigma^1_1}$ separating family for $E_S|(X \cup \mathrm{proj}_Y[S])$, where $\mathrm{proj}_Y : X \times Y \to Y$ denotes the projection function. It then follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that E_S is smooth. As $\mathcal{G}_S$ is acyclic, it follows from Hjorth [4] (see also Miller [7]) that E_S admits a Borel transversal, which contradicts our assumption that it does not. □

For $x_1 E_S x_2$, we say that z is *$\mathcal{G}_S$-between* x_1 and x_2 if z lies along the unique injective $\mathcal{G}_S$-path from x_1 to x_2. Define $B \subseteq Z_S$ by

$$B = \{z \in Z_S : \exists x_1, x_2 \in \operatorname{rng}(\pi_2 \circ \pi_1) \ (z \text{ is } \mathcal{G}_S\text{-between } x_1 \text{ and } x_2)\}.$$

As $\mathcal{G}_S$ is acyclic and $\operatorname{rng}(\pi_2 \circ \pi_1)$ intersects every E_S-class in a countable set, it follows that B is Borel. As $E_S \cap (B \times \operatorname{rng}(\pi_2 \circ \pi_1))$ has countable sections, the Lusin-Novikov uniformization theorem (see, for example, §18 of Kechris [5]) ensures that it has a Borel uniformization $\pi_3 : B \to \operatorname{rng}(\pi_2 \circ \pi_1)$. We can clearly assume that $\pi_3 | \operatorname{rng}(\pi_2 \circ \pi_1) = \text{id}$. Define $\pi : B \to \Delta^{\mathbb{N}}$ by

$$\pi = (\pi_2 \circ \pi_1)^{-1} \circ \pi_3,$$

and finally, define $f : S \to \Delta$ by

$$f(x, y) = \begin{cases} 1_\Gamma & \text{if } x \notin B \text{ or } y \notin B, \text{ and} \\ \delta & \text{if } x, y \in B \text{ and } \delta \cdot \pi(y) F_0^\Delta \pi(x). \end{cases}$$

Now suppose, towards a contradiction, that there is a Borel coordinatewise decomposition (u, v) of f.

Lemma 8. *Suppose that $x, x' \in B \cap X$ and $x E_S x'$. Then:*

1. $u(x)u(x')^{-1} \in \Delta$.
2. $u(x)u(x')^{-1} \cdot \pi(x') F_0^\Delta \pi(x)$.

Proof. Let $x_0, y_0, \ldots, x_n, y_n, x_{n+1}$ be the unique $\mathcal{G}_S$-path from x to x'. To see (1), observe that for all $i \leq n$,

$$\begin{aligned} u(x_i)u(x_{i+1})^{-1} &= (u(x_i)v(y_i))(u(x_{i+1})v(y_i))^{-1} \\ &= f(x_i, y_i) f(x_{i+1}, y_i)^{-1}, \end{aligned}$$

thus $u(x_i)u(x_{i+1})^{-1} \in \Delta$. Noting that

$$u(x_0)u(x_{n+1})^{-1} = u(x_0)u(x_1)^{-1} u(x_1)u(x_2)^{-1} \cdots u(x_n)u(x_{n+1})^{-1},$$

it follows that $u(x)u(x')^{-1} \in \Delta$.

To see (2), recall that Δ acts freely on $\Delta^{\mathbb{N}}/F_0^\Delta$, thus for all $i \leq n$,

$$\begin{aligned} u(x_i)u(x_{i+1})^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} &= f(x_i, y_i) f(x_{i+1}, y_i)^{-1} \cdot [\pi(x_{i+1})]_{F_0^\Delta} \\ &= f(x_i, y_i) \cdot [\pi(y_i)]_{F_0^\Delta} \\ &= [\pi(x_i)]_{F_0^\Delta}. \end{aligned}$$

It then follows that

$$\begin{aligned} u(x_0)u(x_{n+1})^{-1} \cdot [\pi(x_{n+1})]_{F_0^\Delta} &= u(x_0)u(x_1)^{-1} \cdots u(x_n)u(x_{n+1})^{-1} \cdot [\pi(x_{n+1})]_{F_0^\Delta} \\ &= u(x_0)u(x_1)^{-1} \cdots u(x_{n-1})u(x_n)^{-1} \cdot [\pi(x_n)]_{F_0^\Delta} \\ &\ \ \vdots \\ &= [\pi(x_0)]_{F_0^\Delta}, \end{aligned}$$

which completes the proof of the lemma. □

Define now $w : \Delta^{\mathbb{N}} \to \Gamma$ by $w = u \circ \pi_2 \circ \pi_1$. Fix a countable Borel separating family $\Gamma_0, \Gamma_1, \ldots$ for Γ, and define $n : \Delta^{\mathbb{N}} \to \Gamma$ by

$$n(\alpha) = \min\{n \in \mathbb{N} : \exists \delta_1, \delta_2 \in \Delta \ (\delta_1 w(\alpha) \in \Gamma_n \text{ and } \delta_2 w(\alpha) \notin \Gamma_n)\}.$$

Lemma 8 ensures that if $\alpha E_0^{\Delta} \beta$, then $w(\alpha)w(\beta)^{-1} \in \Delta$, thus

$$\begin{aligned} \Delta w(\alpha) &= \Delta w(\alpha) w(\beta)^{-1} w(\beta) \\ &= \Delta w(\beta), \end{aligned}$$

and it follows that $n(\alpha) = n(\beta)$. As $\pi_3 | \operatorname{rng}(\pi_2 \circ \pi_1) = \mathrm{id}$, Lemma 8 ensures also that $w(\alpha)w(\beta)^{-1} \cdot \beta F_0^{\Delta} \alpha$. It follows that if $\alpha = \delta \cdot \beta$, then $w(\alpha)w(\beta)^{-1} = \delta$, thus $w(\alpha) = \delta w(\beta)$. Defining $A \subseteq \Delta^{\mathbb{N}}$ by

$$A = \{\alpha \in \Delta^{\mathbb{N}} : w(\alpha) \in \Gamma_{n(x)}\},$$

it follows that both A and $\Delta^{\mathbb{N}} \setminus A$ are E_0^{Δ}-complete sections. As A is clearly F_0^{Δ}-invariant, it follows that E_0^{Δ} is not relatively ergodic over F_0^{Δ}, which contradicts Lemma 5, and therefore completes the proof of the theorem. □

Kłopotowski-Nadkarni-Sarbadhikari-Srivastava [6] have studied coordinatewise decomposition using another equivalence relation L which, modulo straightforward identifications, is the equivalence relation whose classes are the connected components of the dual graph $\breve{\mathcal{G}}_S$ on S, which is given by

$$\breve{\mathcal{G}}_S = \{((x_1, y_1), (x_2, y_2)) \in S \times S : (x_1, y_1) \neq (x_2, y_2) \text{ and } (x_1 = x_2 \text{ or } y_1 = y_2)\}.$$

The equivalence classes of L are called the *linked components* of S, and the linked components of S are said to be *uniquely linked* if $\mathcal{G}_S$ is acyclic.

Conjecture 9 (Kłopotowski-Nadkarni-Sarbadhikari-Srivastava). *Suppose that X, Y are disjoint Polish spaces and $S \subseteq X \times Y$ is Borel. Then the following are equivalent:*

1. *Every Borel function $f : S \to \mathbb{C}$ admits a Borel coordinatewise decomposition;*
2. *The linked components of S are uniquely linked and L is smooth.*

In light of Theorem 3 and the above remarks, the following observation implies that Conjecture 9 is indeed correct:

Proposition 10. *Suppose that X and Y are disjoint Polish spaces, $S \subseteq X \times Y$ is Borel, and $\mathcal{G}_S$ is acyclic. Then the following are equivalent:*

1. *E_S admits a Borel transversal;*
2. *L is smooth.*

Proof. To see (1) ⇒ (2), suppose that E_S admits a Borel transversal $B \subseteq Z_S$. Let $\pi_1 : Z_S \to Z_S$ be the function which sends z to the unique element of $B \cap [z]_{E_S}$, and let $\pi_2 = \operatorname{proj}_X |S$. Then π_1 is a Borel reduction of E_S into $\Delta(Z_S)$ and π_2 is a Borel reduction of L into E_S, thus $\pi_1 \circ \pi_2$ is a Borel reduction of L into $\Delta(Z_S)$, so L is smooth.

To see (2) $\Rightarrow$ (1), suppose that L is smooth, and fix a Borel reduction $\pi_1 : S \to \mathbb{R}$ of L into $\Delta(\mathbb{R})$. Put $Z = \mathrm{proj}_X[S] \cup \mathrm{proj}_Y[S]$. By the Jankov-von Neumann uniformization theorem (see, for example, §18 of Kechris [5]), there is a $\sigma(\mathbf{\Sigma^1_1})$-measurable reduction $\pi_2 : Z \to S$ of $E_S|Z$ into L, thus $\pi_1 \circ \pi_2$ is a $\sigma(\mathbf{\Sigma^1_1})$-measurable reduction of $E_S|Z$ into $\Delta(\mathbb{R})$. It then follows from Theorem 1.1 of Harrington-Kechris-Louveau [3] that E_S is smooth. As $\mathcal{G}_S$ is acyclic, it then follows from Hjorth [4] (see also Miller [7]) that E_S admits a Borel transversal. □

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