ON THE EXISTENCE OF COCYCLE-INVARIANT BOREL PROBABILITY MEASURES

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Abstract. We show that a natural generalization of compressibility is the sole obstruction to the existence of a cocycle-invariant Borel probability measure.

Introduction

Suppose that X is a standard Borel space and $T \colon X \to X$ is a Borel automorphism of X. A Borel measure μ on X is *T-invariant* if $\mu(T(B)) = \mu(B)$ for all Borel sets $B \subseteq X$. The characterization of the class of Borel automorphisms of standard Borel spaces admitting an invariant Borel probability measure is a fundamental problem going back to Hopf (see [Hop32]).

A *compression* of an equivalence relation E on X is an injection $\phi \colon X \to X$ sending each E-class into a proper subset of itself. Building on work of Murray-von Neumann (see [MVN36]), Nadkarni has shown that the existence of a Borel compression of the orbit equivalence relation E_T^X induced by T is the sole obstruction to the existence of a T-invariant Borel probability measure (see [Nad90]).

Suppose that E is a Borel equivalence relation on X that is *countable*, in the sense that all of its equivalence classes are countable. A Borel measure μ on X is *E-invariant* if it is T-invariant for all Borel automorphisms $T \colon X \to X$ whose graphs are contained in E. It is easy to see that a Borel measure is T-invariant if and only if it is E_T^X-invariant. Becker-Kechris have pointed out that Nadkarni's argument yields the more general fact that the existence of a Borel compression of E is the sole obstruction to the existence of an E-invariant Borel probability measure (see [BK96, Theorem 4.3.1]).

An equivalence relation is *aperiodic* if all of its classes are infinite. A set $Y \subseteq X$ is *E-complete* if it intersects every E-class in at least one point, and a set $Y \subseteq X$ is a *partial transversal* of E if it intersects

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every E-class in at most one point. A *transversal* of E is an E-complete partial transversal of E. The Lusin-Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]) ensures that there is a Borel transversal of E if and only if X is the union of countably-many Borel partial transversals of E. We say that E is *smooth* if it satisfies these equivalent conditions. Dougherty-Jackson-Kechris have pointed out that the existence of a Borel compression of E is equivalent to the existence of an aperiodic smooth Borel subequivalence relation of E (see [DJK94, Proposition 2.5]), thereby obtaining another characterization of the class of countable Borel equivalence relations on standard Borel spaces admitting an invariant Borel probability measure.

A substantially weaker notion than E-invariance is that of *E-quasi-invariance*, where one asks that $\mu(T(B)) = 0 \iff \mu(B) = 0$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T \colon X \to X$ whose graphs are contained in E. Given a group Γ, we say that a function $\rho \colon E \to \Gamma$ is a *cocycle* if $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x \mathrel{E} y \mathrel{E} z$. Given a Borel cocycle $\rho \colon E \to (0, \infty)$, we say that a Borel measure μ on X is *ρ-invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \; d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel automorphisms $T \colon X \to X$ whose graphs are contained in E. Clearly E-invariance is equivalent to invariance with respect to the constant cocycle, whereas the Radon-Nikodym Theorem (see, for example, [Kec95, §17.A]) and the Feldman-Moore observation that countable Borel equivalence relations on standard Borel spaces are orbit equivalence relations induced by Borel actions of countable groups (see [FM77, Theorem 1]) ensure that E-quasi-invariance is equivalent to invariance with respect to some Borel cocycle $\rho \colon E \to (0, \infty)$ (see, for example, [KM04, §8]). A characterization of the class of Borel cocycles $\rho \colon E \to (0, \infty)$ admitting an invariant Borel probability measure was provided in [Mil08a]. Here we investigate more natural generalizations of the characterizations mentioned above.

In §1, we introduce the direct generalizations of *aperiodicity* and *compressibility* to cocycles that come from viewing ρ as endowing each E-class with a notion of relative size. We also introduce the generalization of *smoothness* to cocycles that comes from the Glimm-Effros dichotomy. We note that, unfortunately, even when E is smooth, there are Borel cocycles on E admitting neither a compression nor an invariant Borel probability measure. In order to bypass this obstacle, we introduce the *quotient* of ρ by a finite subequivalence relation of E. Generalizing the observation of Dougherty-Jackson-Kechris, we show that the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is equivalent to the existence of a Borel subequivalence relation of E on which ρ is aperiodic and smooth. We also note that, at least when ρ is smooth, the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is the sole obstacle to the existence of a ρ-invariant Borel probability measure.

In §2, we introduce *Borel coboundaries*, a natural class of particularly simple Borel cocycles containing the constant cocycles. We note that, unfortunately, there are Borel coboundaries admitting neither an injective Borel compression of the quotient by a finite Borel subequivalence relation of E nor an invariant Borel probability measure. In order to bypass this new obstacle, we then drop the assumption of injectivity, and combine the Becker-Kechris generalization of Nadkarni's theorem, the Dougherty-Jackson-Kechris characterization of the existence of Borel compressions, and an approximation lemma to generalize Nadkarni's theorem to Borel coboundaries.

Theorem 1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is a Borel coboundary. Then exactly one of the following holds:*

1. *There is a finite-to-one Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
2. *There is a ρ-invariant Borel probability measure.*

In §3, we no longer restrict our attention to Borel coboundaries. Unfortunately, the direct generalization of Theorem 1 to Borel cocycles remains open. In order to bypass this final obstacle, we consider the weakening of the notion of a compression of the quotient of ρ by a finite subequivalence relation F of E obtained by only taking the quotient in the range, which we refer to as a *compression* of ρ over F. By augmenting the main argument of [Mil08a] with an additional approximation lemma, we generalize Nadkarni's theorem to Borel cocycles.

Theorem 2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *There is a finite-to-one Borel compression of ρ over a finite Borel subequivalence relation of E.*
2. *There is a ρ-invariant Borel probability measure.*

1. Smooth cocycles

One can think of a cocycle $\rho\colon E \to (0, \infty)$ as assigning a notion of relative size to each E-class C, with the *ρ-size* of a point $y \in C$ relative to a point $z \in C$ being $\rho(y, z)$. More generally, the *ρ-size* of a set $Y \subseteq C$ relative to z is given by $|Y|^\rho_z = \sum_{y \in Y} \rho(y, z)$. We say that

Y is *ρ-infinite* if this quantity is infinite. As the definition of cocycle ensures that $|Y|^\rho_{z'} = |Y|^\rho_z \rho(z, z')$ for all $z' \in C$, it follows that the notion of being ρ-infinite does not depend on the choice of $z \in C$. It also follows that the *ρ-size* of Y relative to a non-empty set $Z \subseteq C$, given by $|Y|^\rho_Z = |Y|^\rho_z / |Z|^\rho_z$, does not depend on the choice of $z \in C$.

We say that a cocycle $\rho \colon E \to (0, \infty)$ is *aperiodic* if every E-class is ρ-infinite. Note that the aperiodicity of ρ trivially yields that of E. Conversely, when ρ is bounded, the aperiodicity of E yields that of ρ.

We say that a function $\phi \colon X \to X$ is a *compression* of ρ if the graph of ϕ is contained in E, $|\phi^{-1}(x)|^\rho_x \leq 1$ for all $x \in X$, and the set $\{x \in X \mid |\phi^{-1}(x)|^\rho_x < 1\}$ is E-complete. Note that, when ρ is the constant cocycle, a function $\phi \colon X \to X$ is a compression of E if and only if it is a compression of ρ.

Proposition 1.1. *Suppose that X is a standard Borel space and E is an aperiodic smooth countable Borel equivalence relation on X. Then there is an aperiodic Borel cocycle $\rho \colon E \to (0, \infty)$ that does not admit a compression.*

Proof. Fix a strictly decreasing sequence $(r_n)_{n \in \mathbb{N}}$ of positive real numbers for which $\sum_{n \in \mathbb{N}} r_n = \infty$. As E is both aperiodic and smooth, the Lusin-Novikov uniformization theorem yields a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel transversals of E. For each $x \in X$, let $n(x)$ denote the unique natural number for which $x \in B_{n(x)}$, and define $\rho \colon E \to (0, \infty)$ by setting $\rho(x, y) = r_{n(x)} / r_{n(y)}$ whenever $x \mathrel{E} y$.

The fact that $\sum_{n \in \mathbb{N}} r_n = \infty$ ensures that ρ is aperiodic. To see that there is no compression of ρ, note that if $\phi \colon X \to X$ is a function such that the graph of ϕ is contained in E and $|\phi^{-1}(x)|^\rho_x \leq 1$ for all $x \in X$, then a straightforward induction on $n(x)$, using the fact that $(r_n)_{n \in \mathbb{N}}$ is strictly decreasing, shows that $\phi(x) = x$ for all $x \in X$. ⊠

A *digraph* on X is an irreflexive set $G \subseteq X \times X$. Given such a digraph, we say that a set $Y \subseteq X$ is *G-independent* if $G \cap (Y \times Y) = \emptyset$. A *$Y$-coloring* of G is a function $c \colon X \to Y$ with the property that $c^{-1}(y)$ is G-independent for all $y \in Y$.

The *vertical sections* of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y \mid (x, y) \in R\}$, where $x \in X$. When G is Borel, it follows from [KST99, Proposition 4.5] that there is a Borel $\mathbb{N}$-coloring of G if and only if X is the union of countably-many Borel sets $B \subseteq X$ for which the vertical sections of $G \cap (B \times B)$ are finite.

We say that a Borel measure μ on X is *E-ergodic* if every E-invariant Borel set is μ-conull or μ-null. Given a Borel cocycle $\rho \colon E \to \Gamma$ and a set $Z \subseteq \Gamma$, let G^ρ_Z denote the digraph on X with respect to which distinct points x and y are related if and only if they are E-equivalent and $\rho(x,y) \in Z$. The Glimm-Effros dichotomy for countable Borel equivalence relations (see [Wei84]) ensures that E is smooth if and only if there is no atomless E-ergodic E-invariant σ-finite Borel measure. In [Mil08b], this was generalized to show that if $\rho\colon E \to (0,\infty)$ is a Borel cocycle, then there is an open neighborhood $U \subseteq (0,\infty)$ of 1 for which there is a Borel $\mathbb{N}$-coloring of G^ρ_U if and only if there is no atomless E-ergodic ρ-invariant σ-finite Borel measure. Consequently, we say that a Borel cocycle $\rho\colon E \to (0,\infty)$ is *smooth* if it satisfies these equivalent conditions. Note that the smoothness of E trivially yields that of ρ. Conversely, when ρ is bounded, the smoothness of ρ ensures that X is the union of countably-many Borel sets whose intersection with each E-class is finite, thus E is smooth.

We say that a set $Y \subseteq X$ is *ρ-lacunary* if it is G^ρ_U-independent for some open neighborhood $U \subseteq (0,\infty)$ of 1.

Proposition 1.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a Polish group, and $\rho\colon E \to \Gamma$ is a Borel cocycle. If there is an open neighborhood $U \subseteq \Gamma$ of 1_Γ for which there is a Borel $\mathbb{N}$-coloring of G^ρ_U, then there is a Borel $\mathbb{N}$-coloring of G^ρ_K for all compact sets $K \subseteq \Gamma$.*

Proof. Given a digraph G on X, we say that a set $Y \subseteq X$ is a *G-clique* if all pairs of distinct points of Y are G-related. It is sufficient to show that if a set $Y \subseteq X$ does not contain an infinite G^ρ_U-clique, then the vertical sections of $G^\rho_K \cap (X \times Y)$ are finite. Towards this end, fix a non-empty open set $V \subseteq \Gamma$ with the property that $V^{-1}V \subseteq U$, as well as a finite sequence $(\gamma_i)_{i<n}$ of elements of Γ for which $K \subseteq \bigcup_{i<n} \gamma_i V$, and note that if $x \in X$, then $(G^\rho_K)_x \subseteq \bigcup_{i<n} (G^\rho_{\gamma_i V})_x$, so we need only show that each $(G^\rho_{\gamma_i V})_x$ is a G^ρ_U-clique. But if $i < n$ and $y, z \in (G^\rho_{\gamma_i V})_x$, then $\rho(y,z) = \rho(y,x)\rho(x,z) \in (\gamma_i V)^{-1}\gamma_i V = V^{-1}V \subseteq U$. ⊠

The following fact ensures that a Borel cocycle $\rho\colon E \to (0,\infty)$ is smooth if and only if there is an E-complete ρ-lacunary Borel set.

Proposition 1.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, Γ is a locally compact Polish group, $\rho\colon E \to \Gamma$ is a Borel cocycle, and $U \subseteq \Gamma$ is a pre-compact open neighborhood of 1_Γ. Then there is a Borel $\mathbb{N}$-coloring of G^ρ_U if and only if there is an E-complete G^ρ_U-independent Borel set.*

Proof. If $c\colon X \to \mathbb{N}$ is a Borel $\mathbb{N}$-coloring of G^ρ_U, then set $A_n = c^{-1}(n)$ and $B_n = A_n \setminus \bigcup_{m<n} [A_m]_E$ for all $n \in \mathbb{N}$. As the Lusin-Novikov uniformization theorem ensures that the latter sets are Borel, it follows that their union is an E-complete G^ρ_U-independent Borel set.

Conversely, suppose that $B \subseteq X$ is an E-complete G^ρ_U-independent Borel set. The Lusin-Novikov uniformization theorem then yields Borel functions $\phi_n \colon B \to X$ such that $E \cap (B \times X) = \bigcup_{n \in \mathbb{N}} \text{graph}(\phi_n)$, from which it follows that there are such functions satisfying the additional constraint that the sets $K_n = \rho(\text{graph}(\phi_n))$ are pre-compact. As Proposition 1.2 yields Borel $\mathbb{N}$-colorings of $G^\rho_{K_n U K_n^{-1}} \cap (B \times B)$, and the Lusin-Novikov uniformization theorem ensures that ϕ_n sends $G^\rho_{K_n U K_n^{-1}}$-independent Borel sets to G^ρ_U-independent Borel sets, there are Borel $\mathbb{N}$-colorings of $G^\rho_U \cap (\phi_n(B) \times \phi_n(B))$, and therefore of G^ρ_U. ⊠

Remark 1.4. Propositions 1.2 and 1.3 easily imply that a Borel cocycle $\rho \colon E \to (0, \infty)$ is smooth if and only if X is the union of countably-many ρ-lacunary Borel sets.

We say that a function $\phi \colon X \to X$ is *strictly ρ-increasing* if its graph is contained in E and $|\phi^{-1}(x)|^\rho_x < 1$ for all $x \in X$.

Proposition 1.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a smooth Borel cocycle. Then there is an E-invariant Borel set $B \subseteq X$ for which $E \restriction {\sim}B$ is smooth and there is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism.*

Proof. Fix a partition $(B_n)_{n \in \mathbb{N}}$ of X into ρ-lacunary Borel sets. For each $x \in X$, let $n(x)$ be the unique natural number for which $x \in B_{n(x)}$. Let $\preceq$ be the partial order on X with respect to which $x \preceq y$ if and only if $x \mathrel{E} y$, $n(x) = n(y)$, and $\rho(x, y) \leq 1$, and let B be the set of $x \in X$ such that for all $n \in \mathbb{N}$, either $B_n \cap [x]_E = \emptyset$ or $\preceq \restriction (B_n \cap [x]_E)$ is isomorphic to the usual ordering of $\mathbb{Z}$. Then $E \restriction {\sim}B$ is smooth, and the $(\preceq \restriction B)$-successor function is a strictly $(\rho \restriction (E \restriction B))$-increasing Borel automorphism. ⊠

Given a cocycle $\rho \colon E \to (0, \infty)$ and a finite subequivalence relation F of E, define $\rho/F \colon E/F \to (0, \infty)$ by $(\rho/F)([x]_F, [y]_F) = |[x]_F|^\rho_{[y]_F}$. The Lusin-Novikov uniformization theorem ensures that if F is Borel, then X/F is standard Borel, so that E/F is a countable Borel equivalence relation on a standard Borel space. Moreover, if ρ is Borel, then ρ/F is a Borel cocycle on E/F. The Lusin-Novikov uniformization theorem also implies that, when ρ is the constant cocycle, a Borel compression of ρ/F gives rise to a Borel compression of ρ. In spite of Proposition 1.1, such quotients allow us to generalize the fact that aperiodic smooth countable Borel equivalence relations admit Borel compressions.

Proposition 1.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is an* *aperiodic smooth Borel cocycle. Then there is a finite Borel subequivalence relation F of E for which there is a strictly (ρ/F)-increasing Borel injection.*

Proof. By Proposition 1.5, we can assume that E is smooth. As the aperiodicity of ρ yields that of E, there is a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel transversals of E. For each $x \in X$, let $n(x)$ be the unique natural number with $x \in B_{n(x)}$, set $n_i(x) = i$ for all $i < 2$, recursively define $n_{i+2}(x)$ to be the least natural number such that the ρ-size of the set $\{y \in [x]_E \mid n_{i+1}(x) \leq n(y) < n_{i+2}(x)\}$ relative to the set $\{y \in [x]_E \mid n_i(x) \leq n(y) < n_{i+1}(x)\}$ is strictly greater than one for all $i \in \mathbb{N}$, and let $i(x)$ be the unique natural number with the property that $n_{i(x)}(x) \leq n(x) < n_{i(x)+1}(x)$. Let F be the subequivalence relation of E with respect to which two E-equivalent points are F-equivalent if and only if $i(x) = i(y)$. Then the function $\phi \colon X/F \to X/F$, given by $\phi([x]_F) = \{y \in [x]_E \mid i(y) = i(x) + 1\}$, is a strictly (ρ/F)-increasing Borel injection. ⊠

The following fact yields an equivalent form of ρ-invariance that will prove useful when considering Borel injections.

Proposition 1.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then $\mu(T(B)) = \int_B \rho(T(x), x) \; d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T \colon B \to X$ whose graphs are contained in E.*

Proof. Fix a countable group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E, recursively define $B_n = \{x \in B \setminus \bigcup_{m<n} B_m \mid T(x) = \gamma_n \cdot x\}$ for all $n \in \mathbb{N}$, and note that

$$\begin{aligned} \mu(T(B)) &= \sum_{n \in \mathbb{N}} \mu(\gamma_n(B_n)) \\ &= \sum_{n \in \mathbb{N}} \int_{B_n} \rho(\gamma_n \cdot x, x) \; d\mu(x) \\ &= \int_B \rho(T(x), x) \; d\mu(x) \end{aligned}$$

by ρ-invariance. ⊠

The following fact yields an equivalent form of ρ-invariance that will prove useful when considering Borel functions.

Proposition 1.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure. Then $\mu(\phi^{-1}(B)) =$*

$\int_B |\phi^{-1}(x)|^\rho_x \, d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel functions $\phi \colon X \to X$ whose graphs are contained in E.

Proof. By the Lusin-Novikov uniformization theorem, there are Borel sets $B_n \subseteq B$ and Borel injections $T_n \colon B_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $\mathrm{graph}(\phi^{-1}) \cap (B \times X)$. Then

$$\int_B |\phi^{-1}(x)|^\rho_x \, d\mu(x) = \sum_{n \in \mathbb{N}} \int_{B_n} \rho(T_n(x), x) \, d\mu(x) = \mu(\phi^{-1}(B))$$

by Proposition 1.7. ⊠

Much as before, we say that a function $\phi \colon X \to X$ is a *compression* of ρ over a finite subequivalence relation F of E if the graph of ϕ is contained in E, $|\phi^{-1}([x]_F)|^\rho_{[x]_F} \leq 1$ for all $x \in X$, and the set $\{x \in X \mid |\phi^{-1}([x]_F)|^\rho_{[x]_F} < 1\}$ is E-complete. The Lusin-Novikov uniformization theorem ensures that every Borel compression of the quotient of ρ by a finite Borel subequivalence relation F of E gives rise to a Borel compression of ρ over F. It also implies that, when ρ is the constant cocycle, a Borel compression of ρ over a finite Borel subequivalence relation of E gives rise to a Borel compression of ρ.

Proposition 1.9. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and there is a Borel compression $\phi \colon X \to X$ of ρ over a finite Borel subequivalence relation F of E. Then there is no ρ-invariant Borel probability measure.*

Proof. By the Lusin-Novikov uniformization theorem, there exist a Borel transversal $B \subseteq X$ of F, Borel sets $B_n \subseteq B$, and Borel injections $T_n \colon B_n \to X$ for which $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $F \cap (B \times X)$. If μ is a ρ-invariant Borel measure, then Proposition 1.7 ensures that

$$\begin{aligned} \mu(X) &= \sum_{n \in \mathbb{N}} \mu(T_n(B_n)) \\ &= \sum_{n \in \mathbb{N}} \int_{B_n} \rho(T_n(x), x) \, d\mu(x) \\ &= \int_B |[x]_F|^\rho_x \, d\mu(x), \end{aligned}$$

whereas Propositions 1.7 and 1.8 imply that

$$\begin{aligned}\mu(X) &= \int |\phi^{-1}(x)|_x^\rho \, d\mu(x) \\ &= \sum_{n\in\mathbb{N}} \int_{T_n(B_n)} |\phi^{-1}(x)|_x^\rho \, d\mu(x) \\ &= \sum_{n\in\mathbb{N}} \int_{B_n} |(\phi^{-1}\circ T_n)(x)|_{T_n(x)}^\rho \, d((T_n^{-1})_*(\mu))(x) \\ &= \sum_{n\in\mathbb{N}} \int_{B_n} |(\phi^{-1}\circ T_n)(x)|_x^\rho \, d\mu(x) \\ &= \int_B |\phi^{-1}([x]_F)|_x^\rho \, d\mu(x).\end{aligned}$$

As the set $A = \{x \in B \mid |\phi^{-1}([x]_F)|_x^\rho < |[x]_F|_x^\rho\}$ is E-complete, it follows that if $\mu(X) > 0$, then $\mu(A) > 0$. As $|\phi^{-1}([x]_F)|_x^\rho \leq |[x]_F|_x^\rho$ for all $x \in B$, it follows that if $\mu(A) > 0$, then $\mu(X) = \infty$. ⊠

We next note the useful fact that smoothness is invariant under quotients by finite Borel subequivalence relations of E.

Proposition 1.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0,\infty)$ is a Borel cocycle, and F is a finite Borel subequivalence relation of E. Then ρ is smooth if and only if ρ/F is smooth.*

Proof. By partitioning X into countably-many F-invariant Borel sets, we can assume that there is a real number $r > 1$ with $|[x]_F|_x^\rho \leq r$ for all $x \in X$. As $[Y]_F/F$ is $G^{\rho/F}_{(1/r,r)}$-independent for all $G^\rho_{(1/r^2,r^2)}$-independent sets $Y \subseteq X$, the smoothness of ρ yields that of ρ/F. As every F-invariant set $Y \subseteq X$ for which Y/F is $G^{\rho/F}_{(1/r^2,r^2)}$-independent is itself $(G^\rho_{(1/r,r)} \setminus F)$-independent, the smoothness of ρ/F yields that of ρ. ⊠

Generalizing the Dougherty-Jackson-Kechris observation that there is a Borel compression of E if and only if there is an aperiodic smooth Borel subequivalence relation of E, we have the following.

Proposition 1.11. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0,\infty)$ is a Borel cocycle. Then the following are equivalent:*

1. *There is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
2. *There is a Borel subequivalence relation of E on which ρ is aperiodic and smooth.*

(3) *There exist an E-invariant Borel set $B \subseteq X$ and a Borel subequivalence relation F of E such that $F \restriction {\sim}B$ is smooth, $\rho \restriction (F \restriction {\sim}B)$ is aperiodic, and there is a strictly $(\rho \restriction (F \restriction B))$-increasing Borel automorphism.*

Proof. To see (1) $\implies$ (2), observe that by Proposition 1.10, we can assume that there is an injective Borel compression $\phi\colon X \to X$ of ρ. Set $A = \{x \in X \mid |\phi^{-1}(x)|_x^\rho < 1\}$, and let F be the orbit equivalence relation generated by ϕ. As the sets $A_r = \{x \in X \mid |\phi^{-1}(x)|_x^\rho < r\}$ are $(\rho \restriction F)$-lacunary for all $r < 1$, it follows that $\rho \restriction (F \restriction A)$ is smooth, thus $\rho \restriction (F \restriction [A]_F)$ is aperiodic and smooth. By the Lusin-Novikov uniformization theorem, there is a Borel extension $\psi\colon X \to [A]_F$ of the identity function on $[A]_F$ whose graph is contained in E, in which case the restriction of ρ to the pullback of $F \restriction [A]_F$ through ψ is aperiodic and smooth.

To see (2) $\implies$ (3), note that if condition (2) holds, then Proposition 1.5 immediately yields the weakening of condition (3) in which the set B need not be E-invariant. To see that this weakening yields condition (3) itself, note that if $B' \subseteq X$ is a Borel set and F' is a smooth Borel subequivalence relation of $E \restriction B'$ for which $\rho \restriction F'$ is aperiodic, then the Lusin-Novikov uniformization theorem yields a Borel extension $\pi\colon [B']_E \to B'$ of the identity function on B' whose graph is contained in E, the subequivalence relation F'' of $E \restriction [B']_E$ given by $x \mathrel{F''} y \iff \pi(x) \mathrel{F'} \pi(y)$ is smooth, and $\rho \restriction F''$ is aperiodic.

It only remains to note that Proposition 1.6 yields (3) $\implies$ (1). ⊠

We close this section by noting that, at least when ρ is smooth, the existence of an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E is the sole obstacle to the existence of a ρ-invariant Borel probability measure.

Proposition 1.12. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho\colon E \to (0, \infty)$ is a smooth Borel cocycle. Then exactly one of the following holds:*

(1) *There is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2) *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 1.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, note first that if ρ is aperiodic, then Proposition 1.6 yields a finite Borel subequivalence relation F of E for which there is a strictly (ρ/F)-increasing Borel injection. And if there is a ρ-finite equivalence class C of E, then the

Borel probability measure μ on X, given by $\mu(B) = |B \cap C|^{\rho}_{C}$ for all Borel sets $B \subseteq X$, is ρ-invariant. ⊠

2. Coboundaries

We say that a Borel cocycle $\rho \colon E \to (0, \infty)$ is a *Borel coboundary* if there is a Borel function $f \colon X \to (0, \infty)$ such that $\rho(x, y) = f(x)/f(y)$ for all $(x, y) \in E$. The following observation shows that, even for Borel coboundaries, the equivalent conditions of Proposition 1.11 do not characterize the non-existence of an invariant Borel probability measure.

Proposition 2.1. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X admitting an invariant Borel probability measure. Then there is a Borel coboundary $\rho \colon E \to (0, \infty)$ with the property that there is neither an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E nor a ρ-invariant Borel probability measure.*

Proof. Set $B_0 = X$ and let $\iota_0 \colon B_0 \to B_0$ be the identity function. Recursively apply [KM04, Proposition 7.4] to obtain Borel sets $B_{n+1} \subseteq \iota_n(B_n)$ and Borel involutions $\iota_{n+1} \colon \iota_n(B_n) \to \iota_n(B_n)$ such that the graph of ι_{n+1} is contained in E and the sets B_{n+1} and $\iota_{n+1}(B_{n+1})$ partition $\iota_n(B_n)$ for all $n \in \mathbb{N}$. For each $x \in X$, let $n(x)$ be the maximal natural number for which $x \in B_{n(x)}$, and set $f(x) = 2^{n(x)}$. Define $\rho \colon E \to (0, \infty)$ by setting $\rho(x, y) = f(x)/f(y)$ for all $(x, y) \in E$.

To see that there is no ρ-invariant Borel probability measure, note that if μ is a ρ-invariant Borel measure, then the fact that $\iota_{n+1}(B_{n+2})$ and $(\iota_{n+1} \circ \iota_{n+2})(B_{n+2})$ partition B_{n+1} for all $n \in \mathbb{N}$ ensures that

$$\mu(B_{n+1}) = \int_{B_{n+2}} \rho(\iota_{n+1}(x), x) + \rho((\iota_{n+1} \circ \iota_{n+2})(x), x) \; d\mu(x) = \mu(B_{n+2})$$

for all $n \in \mathbb{N}$, thus $\mu(X) \in \{0, \infty\}$.

Suppose, towards a contradiction, that there is an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. Then Proposition 1.11 yields an E-invariant Borel set $A \subseteq X$ and a Borel subequivalence relation F of E such that $F \upharpoonright {\sim}A$ is smooth, $\rho \upharpoonright (F \upharpoonright {\sim}A)$ is aperiodic, and there is a strictly $(\rho \upharpoonright (F \upharpoonright A))$-increasing Borel automorphism $\phi \colon A \to A$. Fix an E-invariant Borel probability measure μ. As $\iota_n(B_{n+1})$ and $(\iota_n \circ \iota_{n+1})(B_{n+1})$ partition B_n for all $n \in \mathbb{N}$, it follows that $\mu(B_n) = 2\mu(B_{n+1})$ for all $n \in \mathbb{N}$. As the aperiodicity of $\rho \upharpoonright (F \upharpoonright {\sim}A)$ yields that of $F \upharpoonright {\sim}A$, Propositions 1.6 and 1.9 imply that A is μ-conull, thus so too is $A \cap \bigcup_{n \in \mathbb{N}} B_{n+1}$. As the definition of ρ ensures that $\phi(A \cap \bigcup_{n \in \mathbb{N}} B_{n+1}) \subseteq A \cap \bigcup_{n \in \mathbb{N}} B_{n+2}$, and the latter set has μ-measure $1/2$, this contradicts E-invariance. ⊠

The following fact yields an equivalent of ρ-invariance that will prove useful when dealing with finite Borel subequivalence relations.

Proposition 2.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and μ is a ρ-invariant Borel measure on X. Then $\mu(B) = \int |B \cap [x]_F|^\rho_{[x]_F} \ d\mu(x)$ for all Borel sets $B \subseteq X$ and finite Borel subequivalence relations F of E.*

Proof. Fix a Borel transversal $A \subseteq X$ of F, Borel sets $A_n \subseteq A$, and Borel injections $T_n \colon A_n \to X$ with the property that $(\mathrm{graph}(T_n))_{n \in \mathbb{N}}$ partitions $F \cap (A \times X)$, and observe that

$$\begin{aligned}
\int |B \cap [x]_F|^\rho_{[x]_F} \ d\mu(x) &= \sum_{n \in \mathbb{N}} \int_{T_n(A_n)} |B \cap [x]_F|^\rho_{[x]_F} \ d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} |B \cap [x]_F|^\rho_{[x]_F} \ d((T_n^{-1})_* \mu)(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n} |B \cap [x]_F|^\rho_{[x]_F} \rho(T_n(x), x) \ d\mu(x) \\
&= \int_A |B \cap [x]_F|^\rho_x \ d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \int_{A_n \cap T_n^{-1}(B)} \rho(T_n(x), x) \ d\mu(x) \\
&= \sum_{n \in \mathbb{N}} \mu(T_n(A_n) \cap B) \\
&= \mu(B)
\end{aligned}$$

by Proposition 1.7. ⊠

Given a Borel set $R \subseteq X \times X$ with countable vertical sections and a Borel function $\rho \colon R \to (0, \infty)$, we say that a Borel measure μ on X is *ρ-invariant* if $\mu(T(B)) = \int_B \rho(T(x), x) \ d\mu(x)$ for all Borel sets $B \subseteq X$ and Borel injections $T \colon B \to X$ whose graphs are contained in R^{-1}. The *composition* of sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $R \circ S = \{(x, z) \in X \times Z \mid \exists y \in Y \ x \ R \ y \ S \ z\}$. The Lusin-Novikov uniformization theorem ensures that if R and S are Borel sets with countable vertical sections, then so too is their composition. The following fact will prove useful in verifying ρ-invariance.

Proposition 2.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $R, S \subseteq E$ are Borel, and $\rho\colon E \to (0,\infty)$ is a Borel cocycle. Then every $(\rho \restriction (R \cup S))$-invariant Borel measure μ is $(\rho \restriction (R \circ S))$-invariant.*

Proof. Note first that if $B \subseteq X$ is a Borel set, $T_S\colon B \to X$ is a Borel injection whose graph is contained in S^{-1}, and $T_R\colon T_S(B) \to X$ is a Borel injection whose graph is contained in R^{-1}, then

$$\begin{aligned}
\mu((T_R \circ T_S)(B)) &= \int_{T_S(B)} \rho(T_R(x), x)\ d\mu(x) \\
&= \int_B \rho((T_R \circ T_S)(x), T_S(x))\ d((T_S^{-1})_*\mu)(x) \\
&= \int_B \rho((T_R \circ T_S)(x), x)\ d\mu(x).
\end{aligned}$$

As the Lusin-Novikov uniformization theorem ensures that every Borel injection whose graph is contained in $(R \circ S)^{-1}$ can be decomposed into a disjoint union of countably-many Borel injections of the form $T_R \circ T_S$ as above, the proposition follows. ⊠

We say that Borel cocycles $\rho\colon E \to (0,\infty)$ and $\sigma\colon E \to (0,\infty)$ are *Borel cohomologous* if their ratio is a Borel coboundary. We say that a Borel function $f\colon X \to (0,\infty)$ *witnesses* that ρ and σ are Borel cohomologous if $f(x)/f(y) = \sigma(x,y)/\rho(x,y)$ for all $(x,y) \in E$.

Proposition 2.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $f\colon X \to (0,\infty)$ is a Borel function witnessing that Borel cocycles $\rho, \sigma\colon E \to (0,\infty)$ are Borel cohomologous, and μ is a ρ-invariant Borel measure. Then the Borel measure given by $\nu(B) = \int_B f\ d\mu$ is σ-invariant.*

Proof. Simply observe that if $B \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in E, then

$$\begin{aligned}\nu(T(B)) &= \int_{T(B)} f \; d\mu \\ &= \int_B f \circ T \; d((T^{-1})_* \mu) \\ &= \int_B (f \circ T)(x) \rho(T(x), x) \; d\mu(x) \\ &= \int_B f(x) \sigma(T(x), x) \; d\mu(x) \\ &= \int_B \sigma(T(x), x) \; d\nu(x)\end{aligned}$$

by ρ-invariance. ⊠

We say that a Borel set $B \subseteq X$ has *ρ-density* at least ϵ if there is a finite Borel subequivalence relation F of E such that $|B \cap [x]_F|^\rho_{[x]_F} \geq \epsilon$ for all $x \in X$. We say that a Borel set $B \subseteq X$ has *positive ρ-density* if there exists $\epsilon > 0$ for which B has ρ-density at least ϵ.

Proposition 2.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and $B \subseteq X$ is a Borel set with positive ρ-density. Then every $(\rho \restriction (E \restriction B))$-invariant finite Borel measure μ extends to a ρ-invariant finite Borel measure.*

Proof. Fix $\epsilon > 0$ for which B has ρ-density at least ϵ, as well as a finite Borel subequivalence relation F of E such that $|B \cap [x]_F|^\rho_{[x]_F} \geq \epsilon$ for all $x \in X$, and let $\overline{\mu}$ be the Borel measure on X given by

$$\overline{\mu}(A) = \int |A \cap [x]_F|^\rho_{B \cap [x]_F} \; d\mu(x)$$

for all Borel sets $A \subseteq X$.

As $\overline{\mu}(X) \leq \mu(B)/\epsilon$, it follows that $\overline{\mu}$ is finite, and Proposition 2.2 ensures that $\mu = \overline{\mu} \restriction B$.

Lemma 2.6. *Suppose that $f \colon X \to [0, \infty)$ is a Borel function. Then $\int f \; d\overline{\mu} = \int \sum_{y \in [x]_F} f(y) |\{y\}|^\rho_{B \cap [x]_F} \; d\mu(x)$.*

Proof. It is sufficient to check the special case that f is the characteristic function of a Borel set, which is a direct consequence of the definition of $\overline{\mu}$. ⊠

Lemma 2.7. *The measure $\overline{\mu}$ is $(\rho \restriction F)$-invariant.*

Proof. Simply observe that if $A \subseteq X$ is a Borel set and $T \colon X \to X$ is a Borel automorphism whose graph is contained in F, then

$$\begin{aligned}\int_A \rho(T(x), x) \ d\overline{\mu}(x) &= \int \sum_{y \in A \cap [x]_F} \rho(T(y), y) |\{y\}|^\rho_{B \cap [x]_F} \ d\mu(x) \\ &= \int |T(A \cap [x]_F)|^\rho_{B \cap [x]_F} \ d\mu(x) \\ &= \overline{\mu}(T(A))\end{aligned}$$

by Lemma 2.6. ⊠

As $E = F \circ (E \cap (B \times B)) \circ F$, two applications of Proposition 2.3 ensure that $\overline{\mu}$ is ρ-invariant. ⊠

The primary argument of this section will hinge on the following approximation lemma.

Proposition 2.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then for all Borel sets $A \subseteq X$ and positive real numbers $r < 1$, there exist an E-invariant Borel set $B \subseteq X$, a Borel set $C \subseteq B$, and a finite Borel subequivalence relation F of $E \restriction C$ such that $\rho \restriction (E \restriction {\sim}B)$ is smooth, $r < |A \cap [x]_F|^\rho_{[x]_F \setminus A} < 1$ for all $x \in C$, and $A \cap [x]_E \subseteq C$ or $[x]_E \setminus A \subseteq C$ for all $x \in B$.*

Proof. By [KM04, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $r < |A \cap S|^\rho_{S \setminus A} < 1$. Set $D = A \setminus \bigcup \mathcal{S}$ and $D' = ({\sim}A) \setminus \bigcup \mathcal{S}$.

Lemma 2.9. *Suppose that $(x, x') \in E$. Then there exists a real number $s > 1$ with the property that x has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in D or x' has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in D'.*

Proof. Fix $n, n' \in \mathbb{N}$ such that $(n/n')\rho(x, x')$ lies strictly between r and 1, and fix $s > 1$ sufficiently small that $(n/n')\rho(x, x')$ lies strictly between rs^2 and $1/s^2$. Suppose, towards a contradiction, that there are sets $S \subseteq D$ and $S' \subseteq D'$ of $G^\rho_{(1/s,s)}$-neighbors of x and x' of cardinalities n and n'. Then $n/s < |S|^\rho_x < ns$ and $n'\rho(x', x)/s < |S'|^\rho_x < n'\rho(x', x)s$, so the ρ-size of S relative to S' lies strictly between $(n/n')\rho(x, x')/s^2$ and $(n/n')\rho(x, x')s^2$. As these bounds lie strictly between r and 1, this contradicts the maximality of $\mathcal{S}$. ⊠

Lemma 2.9 ensures that $[D]_E \cap [D']_E$ is contained in the E-saturation of the union of the sets of the form $\{x \in D \mid |D \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$ and $\{x \in D' \mid |D' \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$, so $\rho \restriction (E \restriction ([D]_E \cap [D']_E))$ is smooth. Set $B = {\sim}([D]_E \cap [D']_E)$ and $C = B \cap \bigcup \mathcal{S}$, and let F be the equivalence relation on C whose classes are the subsets of C in $\mathcal{S}$. ⊠

We say that a Borel set $B \subseteq X$ has *σ-positive ρ-density* if X is the union of countably-many E-invariant Borel sets $A_n \subseteq X$ for which $A_n \cap B$ has positive $(\rho \restriction (E \restriction A_n))$-density.

Theorem 2.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, and $A \subseteq X$ is an E-complete Borel set. Then X is the union of an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has σ-positive $(\rho \restriction (E \restriction C))$-density, and an E-invariant Borel set $D \subseteq X$ for which there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$.*

Proof. Fix a positive real number $r < 1$. We will show that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, as well as countably-many E-invariant Borel sets $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, there are increasing sequences of finite Borel subequivalence relations F_n of E and E-complete F_n-invariant Borel sets $A_n \subseteq X$ with the property that $r < |A_n \cap [x]_{F_{n+1}}|^{\rho}_{(A_{n+1} \setminus A_n) \cap [x]_{F_{n+1}}} < 1$ for all $n \in \mathbb{N}$ and $x \in A_n$.

We begin by setting $A_0 = A$ and letting F_0 be equality. Suppose now that $n \in \mathbb{N}$ and we have already found A_n and F_n. By applying Proposition 2.8 to A_n/F_n, and throwing out an E-invariant Borel set $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, we obtain a finite Borel subequivalence relation $F_{n+1} \supseteq F_n$ of E and an F_{n+1}-invariant Borel set $A_{n+1} \subseteq X$ such that $r < |A_n \cap [x]_{F_{n+1}}|^{\rho}_{[x]_{F_{n+1}} \setminus A_n} < 1$ for all $x \in A_{n+1}$, and $A_n \cap [x]_E \subseteq A_{n+1}$ or $[x]_E \setminus A_n \subseteq A_{n+1}$ for all $x \in X$. By throwing out an E-invariant Borel set $C \subseteq X$ for which $A \cap C$ has positive $(\rho \restriction (E \restriction C))$-density, we can assume that $A_n \subseteq A_{n+1}$, completing the recursive construction.

Set $B_n = A_n \setminus \bigcup_{m<n} A_m$ and define $\phi_n \colon B_n/F_n \to B_{n+1}/F_{n+1}$ by setting $\phi_n(B_n \cap [x]_{F_n}) = B_{n+1} \cap [x]_{F_{n+1}}$ for all $n \in \mathbb{N}$ and $x \in B_n$. Then the union of $\bigcup_{n \in \mathbb{N}} \phi_n$ and the identity function on ${\sim}\bigcup_{n \in \mathbb{N}} A_n$ is a Borel compression of the quotient of ρ by the union of $\bigcup_{n \in \mathbb{N}} F_n \restriction B_n$ and equality. ⊠

As a corollary, we obtain the desired characterization.

Theorem 2.11. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel coboundary. Then exactly one of the following holds:*

(1) *There is a finite-to-one Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E.*
(2) *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 1.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, fix a bounded open neighborhood $U \subseteq (0, \infty)$ of 1. As ρ is a Borel coboundary, the Lusin-Novikov uniformization theorem implies that there is an E-complete Borel set $A \subseteq X$ for which $\rho(E \restriction A) \subseteq U$. By Theorem 2.10, after throwing out E-invariant Borel sets $B \subseteq X$ and $D \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth and there is a finite-to-one Borel compression of the quotient of $\rho \restriction (E \restriction D)$ by a finite Borel subequivalence relation of $E \restriction D$, we can assume that A has σ-positive ρ-density.

If there is a $(\rho \restriction (E \restriction A))$-invariant Borel probability measure μ, then by passing to an $(E \restriction A)$-invariant μ-positive Borel set, we can assume that A has positive ρ-density, in which case Proposition 2.5 yields a ρ-invariant Borel probability measure.

If there is no $(\rho \restriction (E \restriction A))$-invariant Borel probability measure, then Proposition 2.4 ensures that there is no $(E \restriction A)$-invariant Borel probability measure, in which case the Becker-Kechris generalization of Nadkarni's theorem and the Dougherty-Jackson-Kechris characterization of the existence of a Borel compression yield an aperiodic smooth Borel subequivalence relation F of $E \restriction A$. Then $\rho \restriction F$ is smooth, and the fact that $\rho \restriction (E \restriction A)$ is bounded ensures that $\rho \restriction F$ is also aperiodic. Fix a Borel extension $\phi \colon X \to A$ of the identity function on A whose graph is contained in E, and observe that ρ is aperiodic and smooth on the pullback of F through ϕ. Proposition 1.6 therefore yields an injective Borel compression of the quotient of ρ by a finite Borel subequivalence relation of E. ⊠

3. The general case

Here we generalize Nadkarni's theorem to Borel cocycles. As in §2, our primary argument will hinge on a pair of approximation lemmas. Given a finite set $S \subseteq X$ for which $S \times S \subseteq E$, let μ_S^ρ be the Borel probability measure on X given by $\mu_S^\rho(B) = |B \cap S|_S^\rho$.

Proposition 3.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel cocycle, $f \colon X \to [0, \infty)$ is Borel, $\delta > 0$, and $\epsilon > \sup_{(x,y) \in E} f(x) - f(y)$. Then there exist an E-invariant Borel set $B \subseteq X$ and a finite Borel subequivalence relation F of $E \restriction B$ for which $\rho \restriction (E \restriction {\sim}B)$ is smooth and $\delta\epsilon > \sup_{(x,y) \in E \restriction B} \int f \, d\mu_{[x]_F}^\rho - \int f \, d\mu_{[y]_F}^\rho$.*

Proof. We can clearly assume that $\delta < 1$, and since one can repeatedly apply the corresponding special case of the proposition over the corresponding quotients, we can also assume that $\delta > 2/3$. For each $x \in X$, let $\overline{f}([x]_E)$ be the average of $\inf f([x]_E)$ and $\sup f([x]_E)$. By [KM04, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $\epsilon(\delta - 1/2) > |\int f \ d\mu^\rho_S - \overline{f}([S]_E)|$. Set $C = \{x \in {\sim}\bigcup \mathcal{S} \mid f(x) < \overline{f}([x]_E)\}$ and $D = \{x \in {\sim}\bigcup \mathcal{S} \mid f(x) > \overline{f}([x]_E)\}$.

Lemma 3.2. *Suppose that $(x, y) \in E$. Then there exists a real number $r > 1$ such that x has only finitely-many $G^\rho_{(1/r,r)}$-neighbors in C or y has only finitely-many $G^\rho_{(1/r,r)}$-neighbors in D.*

Proof. As $\delta > 2/3$, a trivial calculation reveals that $-\epsilon(\delta - 1/2)$ is strictly below the average of $-\epsilon/2$ and $\epsilon(\delta - 1/2)$, and that the average of $-\epsilon(\delta - 1/2)$ and $\epsilon/2$ is strictly below $\epsilon(\delta - 1/2)$. In particular, by choosing $m, n \in \mathbb{N}$ for which the ratios $s = m/(m + n\rho(y, x))$ and $t = n\rho(y, x)/(m+n\rho(y, x))$ are sufficiently close to $1/2$, we can therefore ensure that the sums $s(\overline{f}([x]_E) - \epsilon/2) + t(\overline{f}([x]_E) + \epsilon(\delta - 1/2))$ and $s(\overline{f}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{f}([x]_E) + \epsilon/2)$ both lie strictly between $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$. Fix $r > 1$ such that they lie strictly between $(\overline{f}([x]_E) - \epsilon(\delta - 1/2))r^2$ and $(\overline{f}([x]_E) + \epsilon(\delta - 1/2))/r^2$.

Suppose, towards a contradiction, that there exist sets $S \subseteq C$ and $T \subseteq D$ of $G^\rho_{(1/r,r)}$-neighbors of x and y of cardinalities m and n. Then $m/r < |S|^\rho_x < mr$ and $n\rho(y, x)/r < |T|^\rho_x < n\rho(y, x)r$, from which a trivial calculation reveals that $s/r^2 < |S|^\rho_x/|S \cup T|^\rho_x < sr^2$ and $t/r^2 < |T|^\rho_x/|S \cup T|^\rho_x < tr^2$. As $\int f \ d\mu^\rho_S$ lies between $\overline{f}([x]_E) - \epsilon/2$ and $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$, and $\int f \ d\mu^\rho_T$ lies between $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon/2$, it follows that $\int f \ d\mu^\rho_{S\cup T}$ lies between $(s(\overline{f}([x]_E) - \epsilon/2) + t(\overline{f}([x]_E) + \epsilon(\delta - 1/2)))/r^2$ and $(s(\overline{f}([x]_E) - \epsilon(\delta - 1/2)) + t(\overline{f}([x]_E) + \epsilon/2))r^2$, so strictly between $\overline{f}([x]_E) - \epsilon(\delta - 1/2)$ and $\overline{f}([x]_E) + \epsilon(\delta - 1/2)$, contradicting the maximality of $\mathcal{S}$. ⊠

Lemma 3.2 ensures that $[C]_E \cap [D]_E$ is contained in the E-saturation of the union of the sets of the form $\{x \in C \mid |C \cap (G^\rho_{(1/r,r)})_x| < \aleph_0\}$ and $\{x \in D \mid |D \cap (G^\rho_{(1/r,r)})_x| < \aleph_0\}$, so $\rho \upharpoonright (E \upharpoonright ([C]_E \cap [D]_E))$ is smooth. Set $B = {\sim}([C]_E \cap [D]_E)$, and let F be the equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$ together with the singletons contained in $B \setminus \bigcup \mathcal{S}$. ⊠

Proposition 3.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $\rho \colon E \to (0, \infty)$ is a Borel* *cocycle, $f, g \colon X \to [0, \infty)$ are Borel, and $r > 1$. Then there exist an E-invariant Borel set $B \subseteq X$, a Borel set $C \subseteq B$, and a finite Borel subequivalence relation F of $E \restriction B$ such that $\rho \restriction (E \restriction {\sim}B)$ is smooth and $\int_C f \ d\mu^\rho_{[x]_F} \le \int_{B \setminus C} g \ d\mu^\rho_{[x]_F} \le r \int_C f \ d\mu^\rho_{[x]_F}$ for all $x \in B$.*

Proof. As the proposition holds trivially on $f^{-1}(0) \cup g^{-1}(0)$, we can assume that $f, g \colon X \to (0, \infty)$. By [KM04, Lemma 7.3], there is a maximal Borel set $\mathcal{S}$ of pairwise disjoint non-empty finite sets $S \subseteq X$ with $S \times S \subseteq E$ and $1 < \int_{S \setminus T} g \ d\mu^\rho_S \ / \int_T f \ d\mu^\rho_S < r$ for some $T \subseteq S$.

Set $D_{U,V} = (f^{-1}(U) \cap g^{-1}(V)) \setminus \bigcup \mathcal{S}$ for all $U, V \subseteq (0, \infty)$.

Lemma 3.4. *For all $x \in X$, there exists $s > 1$ such that x has only finitely-many $G^\rho_{(1/s,s)}$-neighbors in $D_{(f(x)/s, f(x)s),(g(x)/s, g(x)s)}$.*

Proof. Fix $m, n \in \mathbb{N}$ for which $1 < (g(x)/f(x))(n/m) < r$, as well as $s > 1$ sufficiently large that $s^6 < (g(x)/f(x))(n/m) < r/s^6$. Suppose, towards a contradiction, that there is a set $S \subseteq D_{(f(x)/s, f(x)s),(g(x)/s, g(x)s)}$ of $G^\rho_{(1/s,s)}$-neighbors of x of cardinality $k = m + n$, and fix $T \subseteq S$ of cardinality m. Then $f(x)\mu^\rho_S(T)/s < \int_T f \ d\mu^\rho_S < f(x)\mu^\rho_S(T)s$ and $(m/k)/s^2 < \mu^\rho_S(T) < (m/k)s^2$, so $f(x)(m/k)/s^3 < \int_T f \ d\mu^\rho_S < f(x)(m/k)s^3$. And $g(x)\mu^\rho_S(S \setminus T)/s < \int_{S \setminus T} g \ d\mu^\rho_S < g(x)\mu^\rho_S(S \setminus T)s$ and $(n/k)/s^2 < \mu^\rho_S(S \setminus T) < (n/k)s^2$, so $g(x)(n/k)/s^3 < \int_{S \setminus T} g \ d\mu^\rho_S < g(x)(n/k)s^3$. It follows that $\int_{S \setminus T} g \ d\mu^\rho_S \ / \int_T f \ d\mu^\rho_S$ lies strictly between $(g(x)/f(x))(n/m)/s^6$ and $(g(x)/f(x))(n/m)s^6$, and therefore strictly between 1 and r, contradicting the maximality of $\mathcal{S}$. ⊠

As Lemma 3.2 ensures that ${\sim}\bigcup \mathcal{S}$ is contained in the union of the sets of the form $\{x \in D_{U,V} \mid |D_{U,V} \cap (G^\rho_{(1/s,s)})_x| < \aleph_0\}$, it follows that $\rho \restriction (E \restriction [{\sim}\bigcup \mathcal{S}]_E)$ is smooth. Set $B = {\sim}[{\sim}\bigcup \mathcal{S}]_E$, let F be the Borel equivalence relation on B whose classes are the subsets of B in $\mathcal{S}$, and appeal to the Lusin-Novikov uniformization theorem to obtain a Borel set $C \subseteq B$ with the property that $1 < \int_{B \setminus C} g \ d\mu^\rho_{[x]_F} \ / \int_C f \ d\mu^\rho_{[x]_F} < r$ for all $x \in B$. ⊠

We are now ready to establish our primary result.

Theorem 3.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $\rho \colon E \to (0, \infty)$ is a Borel cocycle. Then exactly one of the following holds:*

1. *There is a finite-to-one Borel compression of ρ over a finite Borel subequivalence relation of E.*
2. *There is a ρ-invariant Borel probability measure.*

Proof. Proposition 1.9 ensures that conditions (1) and (2) are mutually exclusive. To see that at least one of them holds, fix a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E, and define $\rho_\gamma \colon X \to (0, \infty)$ by $\rho_\gamma(x) = \rho(\gamma \cdot x, x)$ for all $\gamma \in \Gamma$.

By standard change of topology results (see, for example, [Kec95, §13]), there exist a Polish topology on $[0, \infty)$ and a zero-dimensional Polish topology on X, compatible with the underlying Borel structures of $[0, \infty)$ and X, with respect to which every interval with rational end-points is clopen, Γ acts by homeomorphisms, and each ρ_γ is continuous. Fix a compatible complete metric on X, as well as a countable algebra $\mathcal{U}$ of clopen subsets of X, closed under multiplication by elements of Γ, and containing a basis for X as well as the pullback of every interval with rational endpoints under every ρ_γ.

We say that a function $f \colon X \to [0, \infty)$ is *$\mathcal{U}$-simple* if it is a finite linear combination of characteristic functions of sets in $\mathcal{U}$. Note that for all $\epsilon > 0$, $\gamma \in \Gamma$, and $Y \subseteq X$ on which ρ_γ is bounded, there is such a function with the further property that $|f(y) - \rho_\gamma(y)| \leq \epsilon$ for all $y \in Y$.

Fix a sequence $(\epsilon_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, as well as an increasing sequence $(\mathcal{U}_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathcal{U}$ whose union is $\mathcal{U}$.

By recursively applying Propositions 3.1 and 3.3 to functions of the form $[x]_F \mapsto \mu^\rho_{[x]_F}(A)$ and $[x]_F \mapsto \mu^\rho_{[x]_F}(B) - \mu^\rho_{[x]_F}(A)$, and throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which $\rho \restriction (E \restriction B)$ is smooth, we obtain increasing sequences of finite algebras $\mathcal{A}_n \supseteq \mathcal{U}_n$ of Borel subsets of X and finite Borel subequivalence relations F_n of E such that:

1. $\forall n \in \mathbb{N} \forall A \in \mathcal{A}_n \forall (x, y) \in E \ \mu^\rho_{[x]_{F_{n+1}}}(A) - \mu^\rho_{[y]_{F_{n+1}}}(A) \leq \epsilon_n$.
2. $\forall n \in \mathbb{N} \forall A, B \in \mathcal{A}_n \ (\forall x \in X \ \mu^\rho_{[x]_{F_n}}(A) \leq \mu^\rho_{[x]_{F_n}}(B) \implies$
   $\exists C \in \mathcal{A}_{n+1} \forall x \in X \ 0 \leq \mu^\rho_{[x]_{F_{n+1}}}(B \setminus C) - \mu^\rho_{[x]_{F_{n+1}}}(A) \leq \epsilon_n)$.

Set $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ and $F = \bigcup_{n \in \mathbb{N}} F_n$. Condition (1) ensures that we obtain finitely-additive probability measures μ_x on $\mathcal{U}$ by setting $\mu_x(U) = \lim_{n \to \infty} \mu^\rho_{[x]_{F_n}}(U)$ for all $U \in \mathcal{U}$ and $x \in X$.

Lemma 3.6. *Suppose that $(U_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{U}$ whose union is in $\mathcal{U}$ and $B = \{x \in X \mid \sum_{n \in \mathbb{N}} \mu_x(U_n) < \mu_x(\bigcup_{n \in \mathbb{N}} U_n)\}$. Then there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$.*

Proof. As $\mu_x(\bigcup_{m \geq n} U_m) - \sum_{m \geq n} \mu_x(U_m)$ is independent of n, it follows that for all $x \in B$, there exist $\delta > 0$ and $n \in \mathbb{N}$ with the property that $\delta + 2 \sum_{m \geq n} \mu_x(U_m) \leq \mu_x(\bigcup_{m \geq n} U_m)$. So by partitioning B into countably-many E-invariant Borel sets and passing to terminal segments of $(U_n)_{n \in \mathbb{N}}$ on each set, we can assume that $B = \{x \in X \mid \delta + 2 \sum_{n \in \mathbb{N}} \mu_x(U_n) \leq \mu_x(\bigcup_{n \in \mathbb{N}} U_n)\}$ for some $\delta > 0$. Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers whose sum is at most δ.

Sublemma 3.7. *There are pairwise disjoint sets $A_n \subseteq \bigcup_{m>n} U_m$ in $\mathcal{A}$ with the property that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $\forall x \in B \ 0 \leq \mu^{\rho}_{[x]_{F_k}}(A_n) - \mu^{\rho}_{[x]_{F_k}}(U_n) \leq \delta_n$.*

Proof. Suppose that $n \in \mathbb{N}$ and we have already found $(A_m)_{m<n}$. Note that if $x \in B$, then

$$\mu_x(U_n) + \sum_{m \geq n} \delta_m \leq \mu_x\left(\bigcup_{m \in \mathbb{N}} U_m\right) - \left(\mu_x(U_n) + \sum_{m<n} 2\mu_x(U_m) + \delta_m\right)$$
$$\leq \mu_x\left(\bigcup_{m>n} U_m\right) - \sum_{m<n} \mu_x(U_m) + \delta_m,$$

so $\forall x \in B \ \mu^{\rho}_{[x]_{F_k}}(U_n) \leq \mu^{\rho}_{[x]_{F_k}}(\bigcup_{m>n} U_m \setminus \bigcup_{m<n} A_m)$ for sufficiently large $k \in \mathbb{N}$, by condition (1). It then follows from condition (2) that there exists $A_n \subseteq \bigcup_{m>n} U_m \setminus \bigcup_{m<n} A_m$ in $\mathcal{A}$ with the property that $\forall x \in B \ 0 \leq \mu^{\rho}_{[x]_{F_k}}(A_n) - \mu^{\rho}_{[x]_{F_k}}(U_n) \leq \delta_n$ for sufficiently large $k \in \mathbb{N}$. ⊠

Fix $k_n \in \mathbb{N}$ with the property that $\mu^{\rho}_{[x]_{F_{k_n}}}(U_n) \leq \mu^{\rho}_{[x]_{F_{k_n}}}(A_n)$ for all $n \in \mathbb{N}$ and $x \in B$, as well as Borel functions $\phi_n \colon B \cap U_n \to A_n$ whose graphs are contained in F_{k_n} for all $n \in \mathbb{N}$. Then the union of $\bigcup_{n \in \mathbb{N}} \phi_n$ and the identity function on $B \setminus \bigcup_{n \in \mathbb{N}} U_n$ is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over the union of $\bigcup_{n \in \mathbb{N}} F_{k_n} \restriction (A_n \cap B)$ and equality on B. ⊠

Lemma 3.6 ensures that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that for all $\delta > 0$ and $U \in \mathcal{U}$, there is a partition $(U_n)_{n \in \mathbb{N}}$ of U into sets in $\mathcal{U}$ of diameter at most δ such that $\mu_x(U) = \sum_{n \in \mathbb{N}} \mu_x(U_n)$ for all $x \in X$.

Lemma 3.8. *Each μ_x is a measure on $\mathcal{U}$.*

Proof. Suppose, towards a contradiction, that there are pairwise disjoint sets $U_n \in \mathcal{U}$ with $\bigcup_{n \in \mathbb{N}} U_n \in \mathcal{U}$ but $\mu_x(\bigcup_{n \in \mathbb{N}} U_n) > \sum_{n \in \mathbb{N}} \mu_x(U_n)$, for some $x \in X$. Fix a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers converging to zero, and recursively construct a sequence $(V_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ of sets in $\mathcal{U}$, beginning with $V_\emptyset = \bigcup_{n \in \mathbb{N}} U_n$, such that $(V_{t \frown (n)})_{n \in \mathbb{N}}$ is a partition of V_t into sets of diameter at most $\delta_{|t|}$ with the property that

$\mu_x(V_t) = \sum_{n \in \mathbb{N}} \mu_x(V_{t \frown (n)})$, for all $t \in \mathbb{N}^{<\mathbb{N}}$. Set $r = \sum_{n \in \mathbb{N}} \mu_x(U_n)$, and recursively construct a sequence $(i_n)_{n \in \mathbb{N}}$ of natural numbers with the property that $\sum_{t \in T_n} \mu_x(V_t) > r$, where $T_n = \prod_{m<n} i_m$, for all $n \in \mathbb{N}$. Set $V_n = \bigcup_{t \in T_n} V_t$ for all $n \in \mathbb{N}$. As $(U_n)_{n \in \mathbb{N}}$ covers the compact set $K = \bigcap_{n \in \mathbb{N}} V_n$, so too does $(U_m)_{m<n}$, for some $n \in \mathbb{N}$. Set $U = \bigcup_{m<n} U_m$, and let T be the tree of all $t \in \bigcup_{m \in \mathbb{N}} T_m$ for which $V_t \nsubseteq U$. Note that T is necessarily well-founded, since any branch b through T would give rise to a singleton $\bigcap_{n \in \mathbb{N}} V_{t \restriction n}$ contained in $K \setminus U$. König's Lemma therefore yields $m \in \mathbb{N}$ with $T \subseteq \bigcup_{\ell<m} T_\ell$, in which case $V_m \subseteq U$, contradicting the fact that $\mu_x(V_m) > \mu_x(U)$. ⊠

As a consequence, Carathéodory's Theorem ensures that there is a unique extension of each μ_x to a Borel probability measure $\overline{\mu}_x$ on X.

Lemma 3.9. *Suppose that $\gamma \in \Gamma$, $U \in \mathcal{U}$, ρ_γ is bounded on U, and $B = \{x \in X \mid \overline{\mu}_x(\gamma(U)) \neq \int_U \rho_\gamma \, d\overline{\mu}_x\}$. Then there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$.*

Proof. By the symmetry of our argument, it is enough to establish the analogous lemma for the set $B = \{x \in X \mid \overline{\mu}_x(\gamma(U)) < \int_U \rho_\gamma \, d\overline{\mu}_x\}$. By partitioning B into countably-many E-invariant Borel sets, we can assume that $B = \{x \in X \mid \delta + \overline{\mu}_x(\gamma(U)) < \int_U \rho_\gamma \, d\overline{\mu}_x\}$ for some $\delta > 0$.

Sublemma 3.10. *For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ with the property that $|\int_U \rho_\gamma \, d\overline{\mu}_x - \int_U \rho_\gamma \, d\mu^\rho_{[x]_{F_n}}| \leq \epsilon$ for all $x \in X$.*

Proof. Fix a $\mathcal{U}$-simple function $f \colon X \to [0, \infty)$ with the property that $|f(x) - \rho_\gamma(x)| \leq \epsilon/3$ for all $x \in U$. By condition (1), there exists $n \in \mathbb{N}$ such that $|\int_U f \, d\overline{\mu}_x - \int_U f \, d\mu^\rho_{[x]_{F_n}}| \leq \epsilon/3$ for all $x \in X$. But then

$$\begin{aligned} \left| \int_U \rho_\gamma \, d\overline{\mu}_x - \int_U \rho_\gamma \, d\mu^\rho_{[x]_{F_n}} \right| &\leq \left| \int_U \rho_\gamma \, d\overline{\mu}_x - \int_U f \, d\overline{\mu}_x \right| + \\ &\quad \left| \int_U f \, d\overline{\mu}_x - \int_U f \, d\mu^\rho_{[x]_{F_n}} \right| + \\ &\quad \left| \int_U f \, d\mu^\rho_{[x]_{F_n}} - \int_U \rho_\gamma \, d\mu^\rho_{[x]_{F_n}} \right| \\ &\leq \epsilon \end{aligned}$$

for all $x \in X$. ⊠

Condition (1) and Sublemma 3.10 ensure that there exists $n \in \mathbb{N}$ such that $\mu^\rho_{[x]_{F_n}}(\gamma(U)) < \int_U \rho_\gamma \, d\mu^\rho_{[x]_{F_n}}$ for all $x \in B$. As the former quantity is $|\gamma(U) \cap [x]_{F_n}|^\rho_x / |[x]_{F_n}|^\rho_x$ and the latter is $|\gamma(U \cap [x]_{F_n})|^\rho_x / |[x]_{F_n}|^\rho_x$, it follows that $|\gamma(U) \cap [x]_{F_n}|^\rho_x < |\gamma(U \cap [x]_{F_n})|^\rho_x$ for all $x \in B$, so any function from $B \cap \gamma(U)$ to $B \cap \gamma(U)$, sending $\gamma(U) \cap [x]_{F_n}$ to $\gamma(U \cap [x]_{F_n})$ for all $x \in B \cap \gamma(U)$, is a compression of $\rho \restriction (E \restriction (B \cap \gamma(U)))$ over the equivalence relation $(\gamma \times \gamma)(F_n) \restriction (B \cap \gamma(U))$. The Lusin-Novikov uniformization theorem yields a Borel such function, and every Borel such function trivially extends to a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$. ⊠

Lemma 3.9 ensures that, after throwing out countably-many E-invariant Borel sets $B \subseteq X$ for which there is a finite-to-one Borel compression of $\rho \restriction (E \restriction B)$ over a finite Borel subequivalence relation of $E \restriction B$, we can assume that $\overline{\mu}_x(\gamma(U)) = \int_U \rho_\gamma \, d\overline{\mu}_x$ for all $\gamma \in \Gamma$, $U \in \mathcal{U}$ on which ρ_γ is bounded, and $x \in X$. As our choice of topologies ensures that every open set $U \subseteq X$ is a disjoint union of sets in $\mathcal{U}$ on which ρ_γ is bounded, we obtain the same conclusion even when $U \subseteq X$ is an arbitrary open set. As every Borel probability measure on a Polish space is regular (see, for example, [Kec95, Theorem 17.10]), we obtain the same conclusion even when $U \subseteq X$ is an arbitrary Borel set. And since every Borel automorphism $T \colon X \to X$ whose graph is contained in E is a disjoint union of restrictions of automorphisms in Γ to Borel subsets, it follows that each $\overline{\mu}_x$ is ρ-invariant. ⊠

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