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MEASURABLE CHROMATIC NUMBERS

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Abstract. We show that if $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then the globally Baire and universally measurable chromatic numbers of the graph of any Borel function on a Polish space are equal and at most three. In particular, this holds for the graph of the unilateral shift on $[\mathbb{N}]^{\mathbb{N}}$, although its Borel chromatic number is $\aleph_0$. We also show that if $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then the universally measurable chromatic number of every treeing of a measure amenable equivalence relation is at most three. In particular, this holds for "the" minimum analytic graph $\mathcal{G}_0$ with uncountable Borel (and Baire measurable) chromatic number. In contrast, we show that for all $\kappa \in \{2, 3, \ldots, \aleph_0, \mathfrak{c}\}$, there is a treeing of E_0 with Borel and Baire measurable chromatic number κ. Finally, we use a Glimm-Effros style dichotomy theorem to show that every basis for a non-empty initial segment of the class of graphs of Borel functions of Borel chromatic number at least three contains a copy of $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$.

§1. Introduction. A *directed graph* on X is an irreflexive set $\mathcal{G} \subseteq X \times X$. A *coloring* of $\mathcal{G}$ is a map $c : X \to Y$ such that $c(x_1) \neq c(x_2)$, for all $(x_1, x_2) \in \mathcal{G}$. For a set Γ of subsets of X, the Γ-*measurable chromatic number* of $\mathcal{G}$ is given by

$$\chi_\Gamma(\mathcal{G}) = \min\{|c(X)| : c \text{ is a } \Gamma\text{-measurable coloring of } \mathcal{G}\},$$

where c ranges over all functions from X to Polish spaces. When X is Polish and μ is a measure on X (by which we shall always mean a measure defined on the Borel subsets of X), we use $\chi_B(\mathcal{G})$, $\chi_{BP}(\mathcal{G})$, and $\chi_\mu(\mathcal{G})$ to denote the Borel, Baire, and μ-measurable chromatic numbers of $\mathcal{G}$, respectively. The first of these was studied extensively by Kechris-Solecki-Todorcevic [10]. Here we examine various questions which arise from their work.

In §2, we study chromatic numbers of directed graphs of the form

$$\mathcal{G}_f = \{(x, f(x)) : x \in X \text{ and } x \neq f(x)\},$$

where X is Polish and $f : X \to X$ is Borel. Kechris-Solecki-Todorcevic [10] have shown that $\chi_B(\mathcal{G}_f) \in \{1, 2, 3, \aleph_0\}$. We give a simple new proof of this theorem, which yields also a characterization of the circumstances under which $\chi_B(\mathcal{G}_f) = \aleph_0$. Using this characterization, we obtain the following:

Theorem A. *Suppose that X is a Polish space, μ is a probability measure on X, and $f : X \to X$ is Borel. Then $\chi_{BP}(\mathcal{G}_f) \leq 3$ and $\chi_\mu(\mathcal{G}_f) \leq 3$.*

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A set $B \subseteq X$ is *globally Baire* if for every Polish space Y and every continuous function $\pi : Y \to X$, the set $\pi^{-1}(B)$ is Baire measurable. We denote the family of such sets by GB. A set $B \subseteq X$ is *universally measurable* if it is μ-measurable, for every probability measure μ on X. We denote the family of such sets by UM. Let $\mathfrak{c}$ denote the cardinality of the continuum. We write $\mathtt{add}(\mathtt{meager}) = \mathfrak{c}$ to indicate that for every Polish space X, the union of strictly fewer than $\mathfrak{c}$-many meager subsets of X is meager. We write $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$ to indicate that for every probability measure μ on a Polish space X, the union of strictly fewer than $\mathfrak{c}$-many μ-null subsets of X is μ-null. Martin-Solovay [13] have shown that Martin's Axiom implies $\mathtt{add}(\mathtt{meager}) = \mathtt{add}(\mathtt{null}) = \mathfrak{c}$, and [1] easily implies that if $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then $\mathtt{add}(\mathtt{meager}) = \mathfrak{c}$. Using Theorem A, we obtain:

Theorem B ($\mathtt{add}(\mathtt{null}) = \mathfrak{c}$). *Suppose that X is Polish and $f : X \to X$ is Borel. Then* $(\chi_B(\mathcal{G}_f), \chi_{GB}(\mathcal{G}_f), \chi_{UM}(\mathcal{G}_f)) \in \{(1,1,1),(2,2,2),(3,3,3),(\aleph_0,3,3)\}$.

A countable equivalence relation E on X is *hyperfinite* if there are finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$. We say that E is *μ-hyperfinite* if there is a μ-conull Borel set $C \subseteq X$ such that $E|C$ is hyperfinite, and E is *measure amenable* if it is μ-hyperfinite, for every probability measure μ on X. The reader is directed to [6] for a thorough treatment of these notions.

A *graph* on X is an irreflexive symmetric subset of $X \times X$. The *symmetrization* of $\mathcal{G}$ is given by $\mathcal{G}^{\pm 1} = \mathcal{G} \cup \mathcal{G}^{-1}$, where $\mathcal{G}^{-1} = \{(y,x) \in X \times X : (x,y) \in \mathcal{G}\}$. A *(directed) graphing* of E is a Borel (directed) graph $\mathcal{G}$ such that the connected components of $\mathcal{G}^{\pm 1}$ are exactly the equivalence classes of E. A *(directed) forest* is a (directed) graph $\mathcal{T}$ such that $\mathcal{T}^{\pm 1}$ is acyclic, and a *(directed) treeing* of E is a (directed) graphing of E which is a (directed) forest.

We say that a function $f : X \to X$ is *aperiodic* if $x \neq f^n(x)$, for all $n \geq 1$ and $x \in X$. The *tail equivalence relation* associated with f is given by

$$xE_t(f)y \Leftrightarrow \exists m,n \in \mathbb{N}\ (f^m(x) = f^n(y)).$$

Theorem 8.2 of [2] ensures that if f is an aperiodic countable-to-one Borel function on a Polish space, then $E_t(f)$ is necessarily hyperfinite, thus $\mathcal{G}_f$ is a directed treeing of a hyperfinite equivalence relation.

In §3, we consider chromatic numbers of treeings of hyperfinite equivalence relations. Let E_0 denote the hyperfinite equivalence relation on $2^{\mathbb{N}}$ given by

$$xE_0y \Leftrightarrow \exists n \in \mathbb{N}\, \forall m \geq n\ (x(m) = y(m)).$$

Kechris-Solecki-Todorcevic [10] have described a treeing $\mathcal{G}_0$ of E_0 with uncountable Baire measurable chromatic number. In contrast, we show the following:

Theorem C. *Suppose that $\mathcal{T}$ is a directed treeing of a μ-hyperfinite equivalence relation on a Polish space. Then $\chi_\mu(\mathcal{T}) \leq 3$.*

As it should cause no confusion, we use the term *Lebesgue measure* to refer to both the usual Lebesgue measure on $\mathbb{R}$ and the $(1/2,1/2)$ product measure on $2^{\mathbb{N}}$. Kechris-Solecki-Todorcevic [10] have suggested that the Lebesgue measurable chromatic number of $\mathcal{G}_0$ is $\mathfrak{c}$. Using Theorem C, we show that this assertion becomes correct when $\mathfrak{c}$ is replaced with 3. In §6.C of [10], it is noted that an analytic graph has countable Borel chromatic number if and only if it has countable globally Baire chromatic number, and it is suggested that the analogous fact holds for universally measurable chromatic number. Under $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, however, we see that $\mathcal{G}_0$ is a counterexample to this claim. In fact, Theorem 6.6 of [10] then implies that under $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, every analytic forest with uncountable Borel chromatic number has a Borel subgraph with universally measurable chromatic number three and uncountable globally Baire chromatic number.

We explore also the extent to which the Baire measurable analog of Theorem C fails, and in the process obtain a characterization of the circumstances under which a given countable Borel equivalence relation E admits a treeing with a given Borel chromatic number. Recall that a *transversal* of E is a set which intersects every E-class in exactly one point, E is *smooth* if it admits a Borel transversal, and E is *treeable* if it admits a treeing.

Theorem D. *Suppose that X is a Polish space and E is a non-smooth treeable countable Borel equivalence relation on X. Then for each $\kappa \in \{2, 3, \ldots, \aleph_0, \mathfrak{c}\}$, there is a treeing $\mathcal{T}$ of E such that $\chi_B(\mathcal{T}) = \chi_{GB}(\mathcal{T}) = \kappa$. Moreover, if $\kappa \geq 3$ and $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then there is such a treeing for which $\chi_{UM}(\mathcal{T}) = 3$.*

Theorem D gives an alternate solution to Problem 3.3 of [10], which asks if there is a Borel forest with Borel chromatic number strictly between 3 and $\aleph_0$. This was originally answered by Laczkovich. His solution, which appears as an appendix in [10], yields graphs with Lebesgue measurable chromatic number strictly greater than three, however, so Theorem C implies that their induced equivalence relations are not measure amenable.

Finally, we turn our attention to a basis problem. A *homomorphism* from a directed graph $\mathcal{G}$ on X to a directed graph $\mathcal{H}$ on Y is a function $\pi : X \to Y$ such that $(\pi(x_1), \pi(x_2)) \in \mathcal{H}$, for all $(x_1, x_2) \in \mathcal{G}$. We write $\mathcal{G} \preceq_B \mathcal{H}$ to indicate the existence of a Borel homomorphism from $\mathcal{G}$ to $\mathcal{H}$. A *$\preceq_B$-basis* for a class $\mathcal{A}$ of directed graphs is a class $\mathcal{B} \subseteq \mathcal{A}$ such that $\forall \mathcal{G} \in \mathcal{A} \, \exists \mathcal{H} \in \mathcal{B} \; (\mathcal{H} \preceq_B \mathcal{G})$.

Kechris-Solecki-Todorcevic [10] have shown that their graph $\mathcal{G}_0$ forms a one-element $\preceq_B$-basis for the class of analytic graphs of uncountable Borel chromatic number. One of the outstanding open questions of [10] is whether there is such a $\preceq_B$-basis for the class of graphs of the form $\mathcal{G}_f^{\pm 1}$, where f is a Borel function on a Polish space and $\chi_B(\mathcal{G}_f) \geq \aleph_0$. While this question remains open, we investigate the analogous question for directed graphs in which $\aleph_0$ is replaced with 3.

In §4, we use an idea of Eigen-Hajian-Weiss [3] to prove an anti-basis theorem for a weakening of Borel homomorphism on the class of graphs of the form $\mathcal{G}_f$ with Borel chromatic number at least three, which gives the following:

Theorem E. *Suppose that $\mathcal{B}$ is a $\preceq_B$-basis for the class of directed graphs of the form $\mathcal{G}_f$ for which $\chi_B(\mathcal{G}_f) \geq 3$. Then $|\mathcal{B}| \geq \mathfrak{c}$.*

In §5, we prove a Glimm-Effros style dichotomy theorem which yields a basis for a strengthening of the quasi-order described in §4. By combining this basis theorem with the results of §4, we obtain the following:

Theorem F. *Suppose that f is a Borel function on a Polish space, $\chi_B(\mathcal{G}_f) \geq 3$, and $\mathcal{B}$ is a $\preceq_B$-basis for the class of directed graphs of the form $\mathcal{G}_g$ for which $\chi_B(\mathcal{G}_g) \geq 3$ and $\mathcal{G}_g \preceq_B \mathcal{G}_f$. Then there is an embedding of $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ into $(\mathcal{B}, \preceq_B)$.*

§2. Graphs induced by functions. We begin this section with a characterization of the circumstances under which $\mathcal{G}_f$ has finite Borel chromatic number:

THEOREM 2.1. *Suppose that X is a Polish space and $f : X \to X$ is a fixed-point free Borel function. Then the following are equivalent:*

1. *The Borel chromatic number of $\mathcal{G}_f$ is at most three.*
2. *The Borel chromatic number of $\mathcal{G}_f$ is finite.*
3. *There is a Borel set $B \subseteq X$ with the property that for all $x \in X$, there exist $m, n \in \mathbb{N}$ such that $f^m(x) \in B$ and $f^n(x) \notin B$.*

PROOF. To see (2) ⇒ (3), fix a Borel coloring $c : X \to \{1, \dots, n\}$ of $\mathcal{G}_f$, and define $i : X \to \{1, \dots, n\}$ by

$$i(x) = \min\{1 \leq m \leq n : \forall j \in \mathbb{N}\, \exists k \geq j\ (c(f^k(x)) = m)\}.$$

Then $x E_t(f) y \Rightarrow i(x) = i(y)$, so the set $B = \{x \in X : c(x) = i(x)\}$ is as desired.

To see (3) ⇒ (1), let $\mathbb{1}_B$ denote the characteristic function of B, set

$$j(x) = \min\{m \in \mathbb{N} : \mathbb{1}_B(x) \neq \mathbb{1}_B(f^m(x))\},$$

and define $c : X \to \{0, 1, 2\}$ by

$$c(x) = \begin{cases} \mathbb{1}_B(x) & \text{if } j(x) \text{ is odd,} \\ 2 & \text{if } j(x) \text{ is even.} \end{cases}$$

To see that c is a coloring of $\mathcal{G}_f$, it is enough to check that $c(x) \neq c(f(x))$, for all $x \in X$. If $j(x) > 1$, then $j(f(x)) = j(x) - 1$, so exactly one of $c(x), c(f(x))$ is 2, thus $c(x) \neq c(f(x))$. If $j(x) = 1$ and $j(f(x))$ is even, then $c(x) = \mathbb{1}_B(x) \neq 2 = c(f(x))$. If $j(x) = 1$ and $j(f(x))$ is odd, then $c(x) = \mathbb{1}_B(x) \neq \mathbb{1}_B(f(x)) = c(f(x))$. As (1) ⇒ (2) is trivial, this completes the proof of the theorem. ☼

We say that a function $f : X \to X$ is *periodic* if for all $x \in X$, there exist natural numbers $m < n$ such that $f^m(x) = f^n(x)$.

PROPOSITION 2.2. *Suppose that X is a Polish space and $f : X \to X$ is a periodic Borel function. Then $\chi_B(\mathcal{G}_f) \leq 3$.*

PROOF. Set $A = \{x \in X : \exists n \geq 1\ (x = f^n(x))\}$ and fix a Borel transversal B of $E_t(f)|A$. For each $x \in X$, let $i(x)$ be the least natural number such that $f^{i(x)}(x) \in B$, and define $c : X \to \{0, 1, 2\}$ by

$$c(x) = \begin{cases} i(x) \pmod 2 & \text{if } x \notin B, \\ 2 & \text{if } x \in B. \end{cases}$$

It is clear that c is a coloring of $\mathcal{G}_f$. ☼

We can now give the optimal upper bounds on $\chi_{BP}(\mathcal{G}_f)$ and $\chi_\mu(\mathcal{G}_f)$:

THEOREM 2.3. *Suppose that X is a Polish space, μ is a probability measure on X, and $f : X \to X$ is Borel. Then $\chi_{BP}(\mathcal{G}_f) \leq 3$ and $\chi_\mu(\mathcal{G}_f) \leq 3$.*

PROOF. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a map $\pi : X \to Y$ such that $x_1 E x_2 \Leftrightarrow \pi(x_1) F \pi(x_2)$, for all $x_1, x_2 \in X$. We write $E \leq_B F$ to indicate the existence of a Borel reduction of E to F. We say that E is *smooth* if there is a Polish space X such that $E \leq_B \Delta(X)$, where $\Delta(X) = \{(x, x) : x \in X\}$. The Lusin-Novikov uniformization theorem (see, for example, Theorem 18.10 of [7]) ensures that this definition agrees with the one given earlier for countable Borel equivalence relations. Note also that if E is smooth and $F \leq_B E$, then F is smooth. The *E-saturation* of a set $B \subseteq X$ is given by $[B]_E = \{x \in X : \exists y \in B\ (xEy)\}$, and we say that B is *E-invariant* if it is equal to its E-saturation. In what follows, we will freely use the fact that the tail equivalence relation induced by a Borel function is smooth if and only if it admits a Borel transversal, which follows from Theorem 5.10 of [16].

By Corollary 8.2 of [2], there is an increasing sequence of smooth Borel equivalence relations F_n whose union is $E_t(f)$. For each $n \in \mathbb{N}$, define

$$A_n = \{x \in X : \exists i \in \mathbb{N}\, \forall j \geq i\ (f^i(x) F_n f^j(x))\}.$$

Lemma 2.4. *The equivalence relation $E_t(f)|A_n$ is smooth.*

Proof. Define $i : A_n \to \mathbb{N}$ by $i(x) = \min\{m \in \mathbb{N} : \forall j \geq m\ (f^j(x) F_n f^m(x))\}$, and observe that the map $\pi(x) = f^{i(x)}(x)$ is a reduction of $E_t(f)|A_n$ to F_n. ☼

By Proposition 2.2, we can assume that f is aperiodic.

Lemma 2.5. *The Borel chromatic number of $\mathcal{G}_f|A_n$ is at most two.*

Proof. By Lemma 2.4, there is a Borel transversal B of $E_t(f)|A_n$. Define $j : A_n \to \mathbb{N}$ by

$$j(x) = \min\{m \in \mathbb{N} : \exists k \in \mathbb{N}\, \exists y \in B\ (f^m(x) = f^k(y))\}.$$

As "$\exists y \in B$" can just as well be replaced with "$\exists! y \in B$," a straightforward induction shows that j is Borel. Define $k : A_n \to \mathbb{N}$ by

$$k(x) = m \Leftrightarrow \exists y \in B\ (f^{j(x)}(x) = f^m(y)),$$

noting that graph(k) is analytic, thus k is Borel. As $j(x) + k(x)$ is simply the distance from x to B in the graph metric associated with $\mathcal{G}_f^{\pm 1}$, it follows that the function $c(x) = j(x) + k(x) \pmod 2$ is a coloring of $\mathcal{G}_f|A_n$. ☼

Lemma 2.5 allows us to assume that $A_n = \emptyset$, for all $n \in \mathbb{N}$, which in turn allows us to define functions $i_n : X \to \mathbb{N}$ by

$$i_n(x) = \max\{m \in \mathbb{N} : x F_n f(x) F_n \cdots F_n f^m(x)\}.$$

Set $B_{\leq n} = \{x \in X : x F_n f(x)\}$, $B_{<n} = \bigcup_{m<n} B_{\leq m}$, and $B_n = B_{\leq n} \setminus B_{<n}$. Note that for all $x \in X$ and $n \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that $f^i(x) \notin B_{<n}$. For each $\alpha \in 2^{\leq \mathbb{N}}$, define $C_\alpha = \bigcup_{\alpha(n)=1} B_n$.

Lemma 2.6. *There is a comeager $E_t(f)$-invariant Borel set $C \subseteq X$ such that $\chi_B(\mathcal{G}_f|C) \leq 3$.*

Proof. For all $x \in X$ and $s \in 2^{<\mathbb{N}}$, there exist $t \supseteq s$ and $i, j \in \mathbb{N}$ such that $f^i(x) \in C_t$ and $f^j(x) \in B_{<|t|} \setminus C_t$, thus

$$\forall x \in X\, \forall^* \alpha \in 2^{\mathbb{N}}\, \exists i, j \in \mathbb{N}\ (f^i(x) \in C_\alpha \text{ and } f^j(x) \notin C_\alpha),$$

where "$\forall^* \alpha \in 2^{\mathbb{N}}\ \phi(\alpha)$" indicates that the set $\{\alpha \in 2^{\mathbb{N}} : \phi(\alpha)\}$ is comeager. The Kuratowski-Ulam Theorem ensures that for comeagerly many $\alpha \in 2^{\mathbb{N}}$, the set

$$C = \{x \in X : \forall n \in \mathbb{N}\, \exists i, j \in \mathbb{N}\ (f^{i+n}(x) \in C_\alpha \text{ and } f^{j+n}(x) \notin C_\alpha)\}$$

is comeager, and Theorem 2.1 then ensures that $\chi_B(\mathcal{G}_f|C) \leq 3$. ☼

LEMMA 2.7. *There is a μ-conull $E_t(f)$-invariant Borel set $C \subseteq X$ such that $\chi_B(\mathcal{G}_f|C) \leq 3$.*

PROOF. For all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $m > n$ sufficiently large that

$$\mu(\{x \in X : \exists i \in \mathbb{N}\ (f^i(x) \in B_{<m} \setminus B_{<n})\}) \geq 1 - \epsilon.$$

It follows that for all $\epsilon > 0$ and $s \in 2^{<\mathbb{N}}$, there exists $t \supseteq s$ such that

$$\mu(\{x \in X : \exists i, j \in \mathbb{N}\ (f^i(x) \in C_t \text{ and } f^j(x) \in B_{<|t|} \setminus C_t)\}) \geq 1 - \epsilon.$$

We can therefore recursively find $\alpha \in 2^{\mathbb{N}}$ such that the set

$$C = \{x \in X : \forall n \in \mathbb{N}\, \exists i, j \in \mathbb{N}\ (f^{i+n}(x) \in C_\alpha \text{ and } f^{j+n}(x) \notin C_\alpha)\}$$

is μ-conull, and Theorem 2.1 then ensures that $\chi_B(\mathcal{G}_f|C) \leq 3$. ☼

The desired result clearly follows from Lemmas 2.6 and 2.7. ☼

Next, we give the optimal upper bounds on the globally Baire and universally measurable chromatic numbers of $\mathcal{G}_f$, under appropriate hypotheses:

THEOREM 2.8. *Suppose that X is a Polish space and $f : X \to X$ is Borel.*
1. *If $\mathtt{add}(\mathtt{meager}) = \mathfrak{c}$, then $\chi_{GB}(\mathcal{G}_f) \leq 3$.*
2. *If $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then $\chi_{UM}(\mathcal{G}_f) \leq 3$.*

PROOF. We prove (2) and leave the nearly identical proof of (1) to the reader. Fix an enumeration $\langle \mu_\alpha \rangle_{\alpha < \mathfrak{c}}$ of the probability measures on X. We will recursively construct μ_α-conull, $E_t(f)$-invariant Borel sets $B_\alpha \subseteq X$, Borel colorings $c_\alpha : B_\alpha \to \{0, 1, 2\}$ of $\mathcal{G}_f|B_\alpha$, and pairwise disjoint $E_t(f)$-invariant analytic sets $A_\alpha \subseteq B_\alpha$, for $\alpha < \mathfrak{c}$, such that each of the sets $C_\alpha = \bigcup_{\beta \leq \alpha} A_\beta$ is μ_α-conull. Granting that this has been accomplished strictly below α, Lemma 2.7 ensures that there is a μ_α-conull, $E_t(f)$-invariant Borel set $B_\alpha \subseteq X$ and a Borel coloring $c_\alpha : B_\alpha \to \{0, 1, 2\}$ of $\mathcal{G}_f|B_\alpha$. As $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, the set $A = \bigcup_{\beta < \alpha} A_\beta$ is μ_α-measurable, thus there is a Borel set $B \supseteq A$ such that $\mu_\alpha(B \setminus A) = 0$. It follows that the set $A_\alpha = B_\alpha \cap [X \setminus B]_{E_t(f)}$ is analytic and

$$\mu_\alpha(A_\alpha) = \mu_\alpha([X \setminus B]_{E_t(f)}) \geq \mu_\alpha(X \setminus B) = 1 - \mu_\alpha(A),$$

thus $\mu_\alpha(C_\alpha) = \mu_\alpha(A) + \mu_\alpha(A_\alpha) = 1$.

Fix a coloring $c : X \to \{0, 1, 2\}$ of $\mathcal{G}_f$ with $c|A_\alpha = c_\alpha|A_\alpha$, for all $\alpha < \mathfrak{c}$. To see that $c^{-1}(\{i\})$ is μ_α-measurable, simply observe that $c^{-1}(\{i\})$ agrees with $\bigcup_{\beta \leq \alpha} c_\beta^{-1}(\{i\}) \cap A_\beta$ off of a μ_α-null set, and our assumption that $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$ ensures that the latter set is universally measurable. ☼

We close this section by giving all possible values of the Borel, globally Baire, and universally measurable chromatic numbers of $\mathcal{G}_f$:

THEOREM 2.9 ($\mathtt{add}(\mathtt{null}) = \mathfrak{c}$). *Suppose that X is Polish and $f : X \to X$ is Borel. Then $(\chi_B(\mathcal{G}_f), \chi_{GB}(\mathcal{G}_f), \chi_{UM}(\mathcal{G}_f)) \in \{(1, 1, 1), (2, 2, 2), (3, 3, 3), (\aleph_0, 3, 3)\}$.*

PROOF. By Corollary 3.11 of [12], the Borel chromatic number of $\mathcal{G}_f$ is at least three if and only if there is a continuous map $\pi : 2^{\mathbb{N}} \to X$ such that

$$\forall (x, y) \in E_0\ (\pi(x) E_t(f) \pi(y) \text{ and } d_{\mathcal{G}_0}(x, y) \equiv d_{\mathcal{G}_f}(\pi(x), \pi(y)) \pmod 2),$$

where $d_{\mathcal{G}_0}, d_{\mathcal{G}_f}$ are the graph metrics associated with $\mathcal{G}_0, \mathcal{G}_f^{\pm 1}$. Note that the composition of a two coloring of $\mathcal{G}_f$ with such a map is a two coloring of $\mathcal{G}_0$.

Proposition 6.2 of [10] ensures that $\chi_{BP}(\mathcal{G}_0) = \mathfrak{c}$, and the upcoming Theorem 3.3 ensures that $\chi_\mu(\mathcal{G}_0) = 3$, where μ denotes Lebesgue measure. It follows that $\chi_B(\mathcal{G}_f) \geq 3 \Leftrightarrow \chi_{GB}(\mathcal{G}_f) \geq 3 \Leftrightarrow \chi_{UM}(\mathcal{G}_f) \geq 3$, thus the desired result is a consequence of Theorems 2.1 and 2.8, along with the fact that $\chi_B(\mathcal{G}_f) \leq \aleph_0$, which itself follows from Proposition 4.5 of [10]. ☼

Remark 2.10. The four possibilities in the conclusion of Theorem 2.9 are realized by the directed graphs associated with the identity function, the odometer on $2^{\mathbb{N}}$, the shift on $2^{\mathbb{Z}}$, and the shift on $[\mathbb{N}]^{\mathbb{N}}$ (see [10]).

§3. Treeings of hyperfinite equivalence relations. We begin this section with the following extension of Theorem 2.3:

THEOREM 3.1. *Suppose that X is a Polish space, μ is a probability measure on X, E is a μ-hyperfinite equivalence relation on X, and $\mathcal{T}$ is a treeing of E. Then there is a μ-conull, E-invariant Borel set $C \subseteq X$ such that $\chi_B(\mathcal{T}|C) \leq 3$.*

PROOF. Let A denote the set of $x \in X$ for which there is an infinite injective $\mathcal{T}$-path through $[x]_E$. As A is analytic and E-invariant, it follows that there are E-invariant Borel sets $B \subseteq A$ and $C \subseteq X \setminus A$ such that $\mu(B \cup C) = 1$. By Theorem 2.1 of [14], the equivalence relation $E|C$ is smooth and therefore admits a Borel transversal, thus $\chi_B(\mathcal{T}|C) \leq 2$.

By the proof of Lemma 3.19 of [6], after throwing away a μ-null, E-invariant Borel set if necessary, there are E-invariant Borel sets B_1, B_2 which partition B, a Borel function $f : B_1 \to B_1$ such that $\mathcal{T}|B_1 = \mathcal{G}_f^{\pm 1}$, and a Borel graph $\mathcal{L} \subseteq \mathcal{T}|B_2$ such that each equivalence class of $E|B_2$ contains exactly one non-trivial connected component of $\mathcal{L}$, which is a tree of vertex degree two. By Proposition 4.6 of [10], there is a Borel three coloring of $\mathcal{L}$, and this easily gives rise to a Borel three coloring of $\mathcal{T}|B_2$. Lemma 2.7 ensures the existence of an E-invariant Borel set $B_1' \subseteq B_1$ such that $\mu(B_1 \setminus B_1') = 0$ and $\chi_B(\mathcal{T}|B_1') \leq 3$, and it follows that the set $C = B_1' \cup B_2$ is as desired. ☼

As in §2, we obtain the following corollary:

THEOREM 3.2 ($\mathtt{add}(\mathtt{null}) = \mathfrak{c}$). *Suppose that $\mathcal{T}$ is a treeing of a measure amenable equivalence relation on a Polish space. Then $\chi_{UM}(\mathcal{T}) = \min(3, \chi_B(\mathcal{T}))$.*

Next, let us recall the graph $\mathcal{G}_0$ from [10]. We say that a sequence $\langle s_n \rangle \in \prod_{n \in \mathbb{N}} 2^n$ is *dense* if for all $s \in 2^{<\mathbb{N}}$, there exists $n \in \mathbb{N}$ such that $s \subseteq s_n$. Given such a sequence, recursively define T_n on 2^n by setting $T_0 = \emptyset$ and

$$T_{n+1} = \{(si, tj) : ((s,t) \in T_n \text{ and } i = j) \text{ or } (s = t = s_n \text{ and } i \neq j)\}.$$

The *instance of* $\mathcal{G}_0$ associated with $\langle s_n \rangle$ is the graph on $2^{\mathbb{N}}$ given by

$$\mathcal{G}_0^{\langle s_n \rangle} = \bigcup_{n \in \mathbb{N}} \{(sx, tx) : (s,t) \in T_n \text{ and } x \in 2^{\mathbb{N}}\}.$$

It is straightforward to check that every instance of $\mathcal{G}_0$ is a treeing of E_0, and Proposition 6.2 of [10] ensures that every instance of $\mathcal{G}_0$ has Baire measurable chromatic number $\mathfrak{c}$. In particular, it follows that every instance of $\mathcal{G}_0$ is a counterexample to the Baire category analog of Theorem 3.1. The proof given above breaks down because the Baire category analog of Lemma 3.19 of [6] is also false. The reader is directed to [5] for a characterization of the circumstances under which the Borel analog of Lemma 3.19 of [6] holds, and to [9] for another application of the failure of the Baire category analog of Lemma 3.19 of [6].

In the special case of an instance of $\mathcal{G}_0$ and Lebesgue measure, the set A from the proof of Theorem 3.1 is conull, and it is not difficult to see that there are instances of $\mathcal{G}_0$ for which the corresponding set B_1 is necessarily null, as well as instances of $\mathcal{G}_0$ for which the corresponding set B_2 is necessarily null.

An *embedding* of $\mathcal{G} \subseteq X \times X$ into $\mathcal{H} \subseteq Y \times Y$ is an injection $\pi : X \to Y$ such that $(x_1, x_2) \in \mathcal{G} \Leftrightarrow (\pi(x_1), \pi(x_2)) \in \mathcal{H}$, for all $x_1, x_2 \in X$. This is a stronger notion than that which appears in [10]. Theorem 15 of [11] implies that every instance of $\mathcal{G}_0$ continuously embeds into every other instance of $\mathcal{G}_0$. However, since any Borel isomorphism between instances of $\mathcal{G}_0$ is necessarily Lebesgue measure-preserving, it follows from the previous paragraph that there are instances of $\mathcal{G}_0$ which are not Borel isomorphic. Nevertheless, we will follow Kechris-Solecki-Todorcevic [10] in using $\mathcal{G}_0$ to denote instances of $\mathcal{G}_0$.

THEOREM 3.3. *The Lebesgue measurable chromatic number of $\mathcal{G}_0$ is three, and if $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then so too is its universally measurable chromatic number.*

PROOF. By Theorems 3.1 and 3.2, it is enough to show that $\chi_\mu(\mathcal{G}_0) \neq 2$, where μ denotes Lebesgue measure. Suppose, towards a contradiction, that there is a μ-measurable set $B \subseteq 2^{\mathbb{N}}$ such that every pair in $\mathcal{G}_0$ consists of a point of B and a point of $X \setminus B$. It is clear that $d_{\mathcal{G}_0}(s0x, s1x)$ is odd, for all $s \in 2^{<\mathbb{N}}$ and $x \in 2^{\mathbb{N}}$, thus the map $six \mapsto s(1-i)x$ sends $B \cap \mathcal{N}_s$ to $(X \setminus B) \cap \mathcal{N}_s$. It then follows that B is of density 1/2 within every basic clopen set, which contradicts the analog of the Lebesgue density theorem in $2^{\mathbb{N}}$, which itself can be seen as a corollary of either (1) the proof of the usual Lebesgue density theorem (see, for example, Theorem 3.20 of [17]), or (2) the analog of the Lebesgue density theorem for Polish ultrametric spaces (see, for example, Lemma 2.5 of [15]). ☼

Next, we see that the Baire measurable analog of Theorem 3.1 fails in the worst possible way. We begin by defining treeings $\mathcal{T}_\kappa$ of E_0, for $3 \leq \kappa \leq \aleph_0$. We write $u \perp v$ to indicate the existence of $i < j < \kappa$ such that $1^i 0 \subseteq u$ and $1^j 0 \subseteq v$. Fix sequences $u_n, v_n \in 2^{n+1}$ such that:

1. $\forall n \in \mathbb{N}\ (u_n(n) \neq v_n(n))$.
2. $\forall n \geq 1\ (u_n \perp v_n)$.
3. $\forall u, v \in 2^{<\mathbb{N}}\ (u \perp v \Rightarrow \exists n \in \mathbb{N}\ (u \subseteq u_n \text{ and } v \subseteq v_n))$.

Recursively define T_n on 2^n by setting $T_0 = \emptyset$ and

$$T_{n+1} = \{(ui, vj) : ((u, v) \in T_n \text{ and } i = j) \text{ or } (\{ui, vj\} = \{u_n, v_n\})\},$$

and define $\mathcal{T}_\kappa$ on $2^{\mathbb{N}}$ by

$$\mathcal{T}_\kappa = \bigcup_{n \in \mathbb{N}} \{(ux, vx) : (u, v) \in T_n \text{ and } x \in 2^{\mathbb{N}}\}.$$

Condition (1) ensures that $\mathcal{T}_\kappa$ is a treeing of E_0.

PROPOSITION 3.4. $\chi_B(\mathcal{T}_\kappa) = \chi_{BP}(\mathcal{T}_\kappa) = \kappa$.

PROOF. To see that $\chi_B(\mathcal{T}_\kappa) \leq \kappa$, define $c : 2^{\mathbb{N}} \to \kappa$ by

$$c(x) = \begin{cases} i & \text{if } i < \kappa \text{ and } 1^i 0 \subseteq x, \\ 1 & \text{if } 1^\kappa \subseteq x. \end{cases}$$

Conditions (1) and (2) ensure that c is a coloring of $\mathcal{T}_\kappa$.

We say that a set $B \subseteq 2^{\mathbb{N}}$ is *$\mathcal{T}_\kappa$-discrete* if $\mathcal{T}_\kappa \cap (B \times B) = \emptyset$. To see that $\chi_{BP}(\mathcal{T}_\kappa) \geq \kappa$, it is enough to show that if $B \subseteq X$ is $\mathcal{T}_\kappa$-discrete and Baire measurable, then there is at most one $i < \kappa$ such that B is non-meager in $\mathcal{N}_{1^i 0}$. Suppose, towards a contradiction, that there exist $i < j < \kappa$ such that B is non-meager in both $\mathcal{N}_{1^i 0}$ and $\mathcal{N}_{1^j 0}$, and find $u \supseteq 1^i 0$ and $v \supseteq 1^j 0$ such that B is comeager in both $\mathcal{N}_u$ and $\mathcal{N}_v$. It follows from condition (3) that there exists $n \in \mathbb{N}$ such that $u \subseteq u_n$ and $v \subseteq v_n$. Fix $x \in 2^{\mathbb{N}}$ such that $u_n x, v_n x \in B$, and observe that $(u_n x, v_n x) \in \mathcal{T}_\kappa$, the desired contradiction. ☼

As noted earlier, this gives an alternate solution to Problem 3.3 of [10], which asks if there is a Borel forest whose Borel chromatic number lies strictly between 3 and $\aleph_0$. However, the following question remains open:

Question 3.5. Is there a locally finite Borel forest whose Borel chromatic number lies strictly between 3 and $\aleph_0$?

A negative answer to this question would imply that every analytic subgraph of $\mathcal{G}_0$ has Baire measurable chromatic number 1, 2, 3, $\aleph_0$, or $\mathfrak{c}$. This is a simple consequence of the following observation:

PROPOSITION 3.6. *Suppose that $\mathcal{T}$ is an analytic subgraph of $\mathcal{G}_0$ with countable Baire measurable chromatic number. Then there is a comeager E_0-invariant Borel set $C \subseteq 2^{\mathbb{N}}$ such that $\mathcal{T}|C$ is locally finite.*

PROOF. It is sufficient to show that the set $A = \{x \in 2^{\mathbb{N}} : |\mathcal{T}_x| = \aleph_0\}$ is meager. Suppose, towards a contradiction, that there exists $s \in 2^{<\mathbb{N}}$ such that A is comeager in $\mathcal{N}_s$. We will show that no non-meager Borel subset of $\mathcal{N}_s$ is $\mathcal{T}$-discrete, which implies that $\chi_{BP}(\mathcal{T}) > \aleph_0$, the desired contradiction. Towards this end, suppose that $B \subseteq \mathcal{N}_s$ is a non-meager Borel set, and fix $t \supseteq s$ with B comeager in $\mathcal{N}_t$, as well as $n \geq |t|$ such that $t \subseteq s_n$ and the set $\{x \in 2^{\mathbb{N}} : (s_n 0x, s_n 1x) \in \mathcal{T}\}$ is non-meager. Then there exists $x \in 2^{\mathbb{N}}$ such that $s_n 0x, s_n 1x \in B$ and $(s_n 0x, s_n 1x) \in \mathcal{T}$, thus B is not $\mathcal{T}$-discrete. ☼

Next, we establish the analog of Theorem 3.3 for our new treeings:

THEOREM 3.7. *The Lebesgue measurable chromatic number of $\mathcal{T}_\kappa$ is three, and if $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then so too is its universally measurable chromatic number.*

PROOF. By Theorems 3.1 and 3.2, it is enough to show that $\chi_\mu(\mathcal{T}_\kappa) \neq 2$, where μ denotes Lebesgue measure. Let d_n denote the graph metric of T_n.

LEMMA 3.8. *There are infinitely many $n \in \mathbb{N}$ such that $d_n(u_n|n, v_n|n)$ is even.*

PROOF. The fact that T_2 is a tree easily implies that there are distinct sequences $u \in \{00, 10\}$ and $v \in \{10, 11\}$ such that $d_2(u, v)$ is even. Suppose, towards a contradiction, that there exists $n \geq 3$ such that $d_m(u_m|m, v_m|m)$ is odd, for all $m \geq n$. A simple induction then shows that if $m \geq n$ and $u', v' \in 2^m$ extend $u0^{n-2}, v0^{n-2}$, then $d_m(u', v')$ is also even. In particular, for no $m \in \mathbb{N}$

can it be the case that $u0^{n-2} \subseteq u_m$ and $v0^{n-2} \subseteq v_m$, since $d_{m+1}(u_m, v_m) = 1$ is odd, and this contradicts condition (3) in the definition of $\mathcal{T}_\kappa$. ☼

Note that if $d_n(u_n|n, v_n|n)$ is even, then $d_{n+1}(s0, s1)$ is odd, for all $s \in 2^n$. Lemma 3.8 therefore implies that there are infinitely many $n \in \mathbb{N}$ such that $d_{n+1}((x|n)0, (x|n)1)$ is odd, for all $x \in 2^{\mathbb{N}}$. Suppose, towards a contradiction, that there is a μ-measurable set $B \subseteq 2^{\mathbb{N}}$ such that every pair in $\mathcal{T}_\kappa$ consists of a point of B and a point of $X \setminus B$. Then $\mu(B) > 0$, so there is a density point x of B. Fix $n \in \mathbb{N}$ sufficiently large that $\mu(B \cap \mathcal{N}_{x|m})/\mu(\mathcal{N}_{x|m}) > 1/2$, for all $m \geq n$, as well as $m \geq n$ such that $d_{m+1}((x|m)0, (x|m)1)$ is odd. It then follows that the map $(x|m)iy \mapsto (x|m)(1-i)y$ sends $B \cap \mathcal{N}_{x|m}$ to $(X \setminus B) \cap \mathcal{N}_{x|m}$, thus B has density $1/2$ within $\mathcal{N}_{x|m}$, the desired contradiction. ☼

Next, we characterize the circumstances under which a treeable countable Borel equivalence relation admits a treeing of a given Borel chromatic number. Clearly every treeing of a smooth countable Borel equivalence relation has Borel chromatic number at most two. This is the only obstacle:

Theorem 3.9. *Suppose that X is a Polish space, E is a non-smooth treeable countable Borel equivalence relation on X, and $\kappa \in \{2, 3, \ldots, \aleph_0, \mathfrak{c}\}$. Then there is a treeing $\mathcal{T}$ of E such that $\chi_B(\mathcal{T}) = \chi_{GB}(\mathcal{T}) = \kappa$. Moreover, if $\kappa \geq 3$ and* $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, *then there is such a treeing for which $\chi_{UM}(\mathcal{T}) = 3$.*

Proof. We begin with the following special case of the theorem:

Lemma 3.10. *There is a treeing of E whose Borel chromatic number is two.*

Proof. We can clearly assume that every equivalence class of E is infinite. By Proposition 7.4 of [8], there is a fixed-point free Borel involution $i : X \to X$ whose graph is contained in E. Fix a Borel linear ordering $<$ of X, and put $B = \{x \in X : x < i(x)\}$. By Proposition 3.3 of [6], there is a treeing $\mathcal{T}_B$ of $E|B$. For each $e \in \mathcal{T}_B$, let $x_0(e) < x_1(e)$ be the two points connected by e, and set

$$\mathcal{T} = \mathrm{graph}(i) \cup \{(x_0(e), i(x_1(e))) : e \in \mathcal{T}_B\}^{\pm 1}.$$

It is clear that $\mathcal{T}$ is a treeing of E, and since the characteristic function of B is a coloring of $\mathcal{T}$, the lemma follows. ☼

Now we handle the case $\kappa \geq 3$. By Theorem 1.1 of [4], there is a continuous injective reduction $\pi : 2^{\mathbb{N}} \to X$ of E_0 to E. Set $B = \pi(2^{\mathbb{N}})$ and $\mathcal{T} = \{(\pi(x), \pi(y)) : (x, y) \in \mathcal{T}_\kappa\}$. By the Lusin-Novikov uniformization theorem, there is a Borel function $f : [B]_E \setminus B \to B$ with $\mathrm{graph}(f) \subseteq E$. Proposition 3.4 implies that $\mathcal{T}' = \mathcal{T} \cup \mathcal{G}_f^{\pm 1}$ is a treeing of $E|[B]_E$ with Borel and globally Baire measurable chromatic number κ. By Lemma 3.10, there is a treeing $\mathcal{T}''$ of $E|(X \setminus [B]_E)$ with Borel chromatic number two. Then $\mathcal{T}' \cup \mathcal{T}''$ is a treeing of E with Borel and globally Baire measurable chromatic number κ. Moreover, if $\kappa \geq 3$ and $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then Theorem 3.7 ensures that $\chi_{UM}(\mathcal{T}) = 3$. ☼

Remark 3.11. If $E|C$ is non-smooth for every comeager Borel set $C \subseteq X$, then it is possible to ensure also that $\chi_{BP}(\mathcal{T}) = \kappa$.

By adding a copy of the complete graph on λ vertices to the forest whose existence is ensured by Theorem 3.9, we obtain the following:

THEOREM 3.12. *Suppose that X is a Polish space and E is a non-smooth countable Borel equivalence relation on X. Then for each $\kappa \in \{2, 3, \ldots, \aleph_0, \mathfrak{c}\}$ and $2 \leq \lambda \leq \min(\aleph_0, \kappa)$, there is a graphing $\mathcal{G}$ of E such that $\chi(\mathcal{G}) = \lambda$ and $\chi_B(\mathcal{G}) = \chi_{GB}(\mathcal{G}) = \kappa$. Moreover, if $\kappa \geq 3$ and $\mathtt{add}(\mathtt{null}) = \mathfrak{c}$, then there is such a graphing for which $\chi_{UM}(\mathcal{G}) = \max(\lambda, 3)$.*

§4. The inexistence of small bases. Given a Borel function $f : X \to X$, let $E_0(f)$ denote the equivalence relation on X given by

$$xE_0(f)y \Leftrightarrow \exists n \in \mathbb{N}\ (f^n(x) = f^n(y)).$$

We use f_0 to denote the injection of $X/E_0(f)$ into $X/E_0(f)$ induced by f. If f is injective, then $E_0(f)$ is trivial, in which case we use f and f_0 interchangeably. The *distance* between equivalence classes $[x]_{E_0(f)}$ and $[y]_{E_0(f)}$ is given by

$$d_f(x, y) = \begin{cases} \min\{|m - n| : f^m(x) = f^n(y)\} & \text{if } xE_t(f)y, \\ \infty & \text{otherwise.} \end{cases}$$

The *distance set* associated with a set $B \subseteq X$ is given by

$$\Delta_f(B) = \{d_f(x, y) : x, y \in B \text{ and } xE_t(f)y\}.$$

We say that B is *evenly spaced* if $\Delta_f(B) \subseteq 2\mathbb{N}$, and we say that B is *two spaced* if it is both evenly spaced and equal to its *even saturation*, which is given by $[B]_f^{\text{even}} = \bigcup_{i,j \in \mathbb{N}} f^{-2i}(f^{2j}(B))$. We say that B is an *f-complete section* if it intersects every $E_t(f)$-class.

PROPOSITION 4.1. *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel function. Then the following are equivalent:*

1. *The Borel chromatic number of $\mathcal{G}_f$ is at most two.*
2. *There is a two-spaced Borel f-complete section.*
3. *There is an evenly-spaced analytic f-complete section.*

PROOF. The proofs of (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are straightforward.

LEMMA 4.2. *Every evenly-spaced analytic set is contained in a two-spaced Borel set.*

PROOF. Suppose that $A_0 \subseteq X$ is an evenly-spaced analytic set. As the property of being evenly spaced is coanalytic on analytic, it follows from the first reflection theorem (see, for example, Theorem 35.10 of [7]) that given an evenly-spaced analytic set $A_n \subseteq X$, there is an evenly-spaced Borel set $B_n \supseteq A_n$. Let $A_{n+1} = [B_n]_f^{\text{even}}$. Then $\bigcup_{n \in \mathbb{N}} B_n$ is the desired two-spaced Borel set. ☼

It is clear that (3) $\Rightarrow$ (2) is a consequence of Lemma 4.2. ☼

The *f_0-diameter* of a set $B \subseteq X$ is given by $\operatorname{diam}_f(B) = \sup \Delta_f(B)$. A *partial transversal* of f_0 is a set of f_0-diameter zero, and a *transversal* of f_0 is an f-complete section of f_0-diameter zero. We say that f_0 is *smooth* if it admits a Borel transversal, in which case Proposition 4.1 implies that if f is aperiodic, then $\chi_B(\mathcal{G}_f) \leq 2$.

PROPOSITION 4.3. *Suppose that X is a Polish space, $f : X \to X$ is an aperiodic Borel function, and X can be covered by countably many analytic sets of finite f_0-diameter. Then f_0 is smooth.*

Proof. We note first the following pair of lemmas:

Lemma 4.4. *Every analytic set $A \subseteq X$ is contained in an $E_0(f)$-invariant Borel set $B \subseteq X$ such that $\Delta_f(A) = \Delta_f(B)$ and $[B]_{E_t(f)}$ is Borel.*

Proof. Set $\Delta = \Delta_f(A)$ and $A_0 = A$. Given an analytic set $A_n \subseteq X$ such that $\Delta_f(A_n) \subseteq \Delta$, observe that the property of having one's difference set contained in Δ is coanalytic on analytic, thus the first reflection theorem ensures that there is a Borel set $A_{n+1} \supseteq f(A_n)$ such that $\Delta_f(A_{n+1}) \subseteq \Delta$. Set $B = \bigcup_{n\in\mathbb{N}} f^{-n}(A_n)$. It is clear that $\Delta_f(B) = \Delta$ and both B and the set $[B]_{E_t(f)} = \bigcup_{m,n\in\mathbb{N}} f^{-m}(A_n)$ are Borel. To see that B is $E_0(f)$-invariant, suppose that $x \in B$ and $xE_0(f)y$, fix $n \in \mathbb{N}$ sufficiently large that $f^n(x) = f^n(y)$, fix $m \geq n$ sufficiently large that $x \in f^{-m}(A_m)$, and observe that $y \in f^{-m}(A_m) \subseteq B$. ☼

Lemma 4.5. *For each $k \in \mathbb{N}$, every analytic set of f_0-diameter k is contained in the union of $k+1$ Borel partial transversals.*

Proof. By induction on k. The case $k = 0$ follows from the definition of partial transversal. Suppose now that we have established the lemma strictly below k, and $A \subseteq X$ is an analytic set of f_0-diameter k. By Lemma 4.4, there is an $E_0(f)$-invariant Borel set $B \supseteq A$ of f_0-diameter k. Then the set $C = \bigcup_{n\geq 1} f^{-n}(B)$ has f_0-diameter $k-1$, and is therefore contained in the union of k Borel partial transversals. As the set $B \setminus C$ is a partial transversal of f_0, it follows that A is contained in the union of $k+1$ Borel partial transversals. ☼

As X can be covered with countably many analytic sets of finite f_0-diameter, it follows from Lemma 4.5 that X can be covered with countably many Borel partial transversals, thus Lemma 4.4 ensures that X can be covered with Borel partial transversals $B_0, B_1, \ldots \subseteq X$ whose $E_t(f)$-saturations are Borel. Then the set $\bigcup_{n\in\mathbb{N}} B_n \setminus \bigcup_{m<n} [B_m]_{E_t(f)}$ is a transversal of f_0, thus f_0 is smooth. ☼

Suppose now that $f : X \to X$ and $g : Y \to Y$ are aperiodic Borel functions. A *lifting* of a function $\pi : X/E_0(f) \to Y/E_0(g)$ is a map $\tilde{\pi} : X \to Y$ such that $\tilde{\pi}(x) \in \pi([x]_{E_0(f)})$, for all $x \in X$. We say that π is *Borel* if it admits a Borel lifting. For $\epsilon > 0$, an *ϵ-Lipschitz homomorphism* from f_0 to g_0 is a map $\pi : X/E_0(f) \to Y/E_0(g)$ such that $\epsilon d_f(x_1, x_2) \leq d_g(x_1', x_2') \leq (1/\epsilon) d_f(x_1, x_2)$, for all $x_1 E_t(f) x_2$ and $x_i' \in \pi([x_i]_{E_0(f)})$. We write $f_0 \preceq_L g_0$ to indicate the existence of a Borel ϵ-Lipschitz homomorphism from f_0 to g_0, for some $\epsilon > 0$. We say that f_0, g_0 are *orthogonal*, or $f_0 \perp g_0$, if the only aperiodic Borel functions h such that $h_0 \preceq_L f_0$ and $h_0 \preceq_L g_0$ are those for which h_0 is smooth.

Given a set $S \subseteq \mathbb{N}$ and $\epsilon > 0$, let $[S]_\epsilon = \{n \in \mathbb{N} : \exists m \in S\ (\epsilon m \leq n \leq (1/\epsilon)m)\}$. We say that $S, T \subseteq \mathbb{N}$ are *orthogonal*, or $S \perp T$, if $|[S]_\epsilon \cap [T]_\epsilon| < \aleph_0$, for all $\epsilon > 0$.

Proposition 4.6. *Suppose that X_i is a Polish space, $f_i : X_i \to X_i$ is an aperiodic Borel function, $A_i \subseteq X_i$ is an analytic f_i-complete section, and $\Delta_{f_1}(A_1) \perp \Delta_{f_2}(A_2)$. Then $(f_1)_0 \perp (f_2)_0$.*

Proof. Suppose that Y is a Polish space, $g : Y \to Y$ is an aperiodic Borel function, $\epsilon > 0$, and $\pi_i : Y \to X_i$ is a Borel lifting of an ϵ-Lipschitz homomorphism from g_0 to $(f_i)_0$. Set $A_i^k = [f_i^k(A_i)]_{E_0(f_i)}$ and $B_i^k = \pi_i^{-1}(A_i^k)$. Then

$$\begin{aligned}\Delta_g(B_1^{k_1} \cap B_2^{k_2}) &\subseteq \Delta_g(B_1^{k_1}) \cap \Delta_g(B_2^{k_2}) \\ &\subseteq [\Delta_{f_1}(A_1^{k_1})]_\epsilon \cap [\Delta_{f_2}(A_2^{k_2})]_\epsilon \\ &= [\Delta_{f_1}(A_1)]_\epsilon \cap [\Delta_{f_2}(A_2)]_\epsilon,\end{aligned}$$

thus $B_1^{k_1} \cap B_2^{k_2}$ has finite g_0-diameter. As each of these sets are analytic and their union is Y, Proposition 4.3 implies that g_0 is smooth, so $(f_1)_0 \perp (f_2)_0$. ☼

The *odometer* is the isometry of $2^{\mathbb{N}}$ given by

$$\sigma(x) = \begin{cases} 0^n 1y & \text{if } x = 1^n 0y, \\ 0^\infty & \text{if } x = 1^\infty. \end{cases}$$

Although the function $x \mapsto x(0)$ is a two coloring of $\mathcal{G}_\sigma$, we can obtain automorphisms whose graphs do not admit Borel two colorings by building towers over the odometer. The maps we consider will be indexed by sequences $\alpha \in \mathbb{N}^{\mathbb{N}}$ with the property that $\alpha(n) > \sum_{i<n} \alpha(i)$, for all $n \in \mathbb{N}$. We use Ω to denote the set of such sequences. For each $\alpha \in \Omega$, define $T_\alpha : 2^{\mathbb{N}} \to \mathbb{N}$ by

$$T_\alpha(x) = \begin{cases} \alpha(n) - \sum_{i<n} \alpha(i) & \text{if } 1^n 0 \subseteq x, \\ 1 & \text{if } x = 1^\infty. \end{cases}$$

Let $X_\alpha = \{(x, i) \in 2^{\mathbb{N}} \times \mathbb{N} : i < T_\alpha(x)\}$, and define $\sigma_\alpha : X_\alpha \to X_\alpha$ by

$$\sigma_\alpha(x, i) = \begin{cases} (\sigma(x), 0) & \text{if } i = T_\alpha(x) - 1, \\ (x, i+1) & \text{otherwise.} \end{cases}$$

Proposition 4.7. *Suppose that $\alpha \in \Omega$, $n \in \mathbb{N}$, $s \in 2^n$, and $x \in 2^{\mathbb{N}}$. Then*

$$\sigma_\alpha^{\sum_{i<n} \alpha(i)s(i)}(0^n x, 0) = (sx, 0).$$

Proof. By induction on n. The case $n = 0$ is a triviality, so suppose that we have shown the proposition up to n and we are given $s \in 2^{n+1}$. If $s(n) = 0$, then

$$\sigma_\alpha^{\sum_{i<n+1} \alpha(i)s(i)}(0^{n+1}x, 0) = \sigma_\alpha^{\sum_{i<n} \alpha(i)s(i)}(0^n 0x, 0) = (sx, 0),$$

by the induction hypothesis. If $s(n) = 1$, then

$$\begin{aligned}\sigma_\alpha^{\sum_{i<n+1} \alpha(i)s(i)}(0^{n+1}x, 0) &= \sigma_\alpha^{\sum_{i<n} \alpha(i)s(i)} \circ \sigma_\alpha^{\alpha(n) - \sum_{i<n} \alpha(i)} \circ \\ &\quad\ \sigma_\alpha^{\sum_{i<n} \alpha(i)}(0^n 0x, 0) \\ &= \sigma_\alpha^{\sum_{i<n} \alpha(i)s(i)} \circ \sigma_\alpha^{\alpha(n) - \sum_{i<n} \alpha(i)}(1^n 0x, 0) \\ &= \sigma_\alpha^{\sum_{i<n} \alpha(i)s(i)}(0^n 1x, 0) \\ &= (sx, 0),\end{aligned}$$

by two applications of the induction hypothesis and the definition of σ_α. ☼

As Proposition 4.6 of [10] ensures that $\chi_B(\mathcal{G}_{\sigma_\alpha}) \in \{2, 3\}$, the following fact (along with Proposition 4.1) gives its exact value:

Proposition 4.8. *Suppose that $\alpha \in \Omega$. Then the following are equivalent:*

1. *The automorphism σ_α admits a two-spaced Borel complete section.*

2. *There exists $n \in \mathbb{N}$ such that $\alpha(m)$ is even, for all $m \geq n$.*

PROOF. To see (2) $\Rightarrow$ (1), fix $n \in \mathbb{N}$ sufficiently large that $\alpha(m)$ is even, for all $m \geq n$. Proposition 4.7 then ensures that the set $B = \{(0^n x, 0) : x \in 2^{\mathbb{N}}\}$ is an evenly-spaced Borel complete section, and it follows that $[B]^{\mathrm{even}}_{\sigma_\alpha}$ is a two-spaced Borel complete section.

To see $\neg(2) \Rightarrow \neg(1)$, suppose that $A \subseteq X_\alpha$ is a Borel complete section, and fix $i \in \mathbb{Z}$ and $s \in 2^{<\mathbb{N}}$ such that the set $B = \{x \in 2^{\mathbb{N}} : (x, 0) \in \sigma^i_\alpha(A)\}$ is comeager in $\mathcal{N}_s$. The failure of (2) ensures the existence of $n \geq |s|$ such that $\alpha(n)$ is odd. Fix $x \in 2^{\mathbb{N}}$ such that $s0^{n-|s|}0x, s0^{n-|s|}1x \in B$. Then $(s0^{n-|s|}0x, 0), (s0^{n-|s|}1x, 0) \in \sigma^i_\alpha(A)$, and since Proposition 4.7 ensures that $d_{\sigma_\alpha}((s0^{n-|s|}0x, 0), (s0^{n-|s|}1x, 0)) = \alpha(n)$, it follows that $\sigma^i_\alpha(A)$ is not evenly spaced, thus A is not evenly spaced. ☼

Proposition 4.8 implies that the automorphisms indexed by elements of the set $\Omega_{\mathrm{odd}} = \{\alpha \in \Omega : \forall n \in \mathbb{N}\ (\alpha(n) \equiv 1 \pmod 2)\}$ do not admit two-spaced Borel complete sections, so their graphs have Borel chromatic number three.

The *IP-set* associated with a sequence $\alpha \in \mathbb{N}^{\leq \mathbb{N}}$ is the set $\mathrm{IP}(\alpha)$ of natural numbers of the form $\sum_{i \in S} \alpha(i)$, where $S \subseteq \mathrm{dom}(\alpha)$ is finite. The *distance set* associated with α is given by $\Delta(\alpha) = \{|i - j| : i, j \in \mathrm{IP}(\alpha)\}$.

PROPOSITION 4.9. *Suppose that $\alpha \in \Omega$. Then σ_α admits a Borel complete section $B_\alpha \subseteq X_\alpha$ such that $\Delta_f(B_\alpha) = \Delta(\alpha)$.*

PROOF. Let $B = \{(x, 0) \in X_\alpha : x \text{ is not eventually constant}\}$. Proposition 4.7 ensures that $\Delta_{\sigma_\alpha}(B) = \Delta(\alpha)$, so the set $B_\alpha = B \cup \{(0^\infty, 0)\}$ is as desired. ☼

We say that sequences $\alpha, \beta \in \Omega$ are *orthogonal* if $\Delta(\alpha) \perp \Delta(\beta)$.

PROPOSITION 4.10. *Suppose that $\alpha \in \Omega$. Then there is a pairwise orthogonal family of $\mathfrak{c}$ many subsequences of α.*

PROOF. Fix an injection $\phi : 2^{<\mathbb{N}} \to \mathbb{N}$ such that $s \subseteq t \Rightarrow \phi(s) \leq \phi(t)$, for all $s, t \in 2^{<\mathbb{N}}$. Then $|s| \leq \phi(s)$, for all $s \in 2^{<\mathbb{N}}$. Fix $k_n \in \mathbb{N}$ such that $\alpha(k_n) > (n+1)(n^2+1)\alpha(k_{n-1})$, for all $n \geq 1$. For each $x \in 2^{\mathbb{N}}$, set $\alpha_x(n) = \alpha(k_{\phi(x|n)})$.

LEMMA 4.11. *Suppose that $x(0) \dots x(n) \neq y(0) \dots y(n)$ and $n \geq 1/\epsilon$. Then*

$$[\Delta(\alpha_x)]_\epsilon \cap [\Delta(\alpha_y)]_\epsilon \subseteq [\Delta(\alpha_x|n)]_\epsilon \cap [\Delta(\alpha_y|n)]_\epsilon.$$

PROOF. Given $\delta \in [\Delta(\alpha_x)]_\epsilon \cap [\Delta(\alpha_y)]_\epsilon$, fix $m \in \mathbb{N}$ least for which there exist $i_x \geq j_x$ in $\mathrm{IP}(\alpha_x(0) \dots \alpha_x(m))$ and $i_y \geq j_y$ in $\mathrm{IP}(\alpha_y(0) \dots \alpha_y(m))$ such that

$$\epsilon(i_x - j_x), \epsilon(i_y - j_y) \leq \delta \leq (1/\epsilon)(i_x - j_x), (1/\epsilon)(i_y - j_y).$$

Suppose, towards a contradiction, that $m \geq n$. By reversing the roles of x and y if necessary, we can assume that $i_y \notin \mathrm{IP}(\alpha_y|m)$, $j_y \in \mathrm{IP}(\alpha_y|m)$, and if $i_x \notin \mathrm{IP}(\alpha_x|m)$, then $\phi(x|m) < \phi(y|m)$. Then $i_x, j_y \leq (m+1)\alpha(k_{\phi(y|m)-1})$, so

$$\begin{aligned}
\alpha(k_{\phi(y|m)}) &\leq (i_y - j_y) + j_y \\
&\leq (1/\epsilon^2)(i_x - j_x) + j_y \\
&\leq m^2(m+1)\alpha(k_{\phi(y|m)-1}) + (m+1)\alpha(k_{\phi(y|m)-1}) \\
&= (m+1)(m^2+1)\alpha(k_{\phi(y|m)-1}) \\
&\leq (\phi(y|m)+1)(\phi(y|m)^2+1)\alpha(k_{\phi(y|m)-1}),
\end{aligned}$$

which contradicts our choice of $k_{\phi(y|m)}$. ☼

By Lemma 4.11, if $x \neq y$, $\epsilon > 0$, and $n \in \mathbb{N}$ is sufficiently large, then

$$[\Delta(\alpha_x)]_\epsilon \cap [\Delta(\alpha_y)]_\epsilon \subseteq [\Delta(\alpha_x|n)]_\epsilon \cap [\Delta(\alpha_y|n)]_\epsilon.$$

As the latter set is finite, it follows that $\langle \alpha_x \rangle_{x \in 2^\mathbb{N}}$ is pairwise orthogonal. ☼

An *embedding* of f_0 into g_0 is an injection $\pi : X/E_0(f) \to Y/E_0(g)$ with $\pi \circ f_0 = g_0 \circ \pi$. We write $f_0 \sqsubseteq g_0$ if there is a Borel embedding of f_0 into g_0.

Proposition 4.12. *If $\alpha, \beta \in \Omega$ and α is a subsequence of β, then $\sigma_\alpha \sqsubseteq \sigma_\beta$.*

Proof. Set $X = \{x \in 2^\mathbb{N} : x \text{ is not eventually constant}\}$. It is clearly sufficient to produce a Borel embedding of $\sigma_\alpha|(X_\alpha \cap (X \times \mathbb{N}))$ into σ_β. Towards this end, fix a strictly increasing sequence of natural numbers k_i such that $\alpha(i) = \beta(k_i)$, and set $\ell_0 = k_0$ and $\ell_{i+1} = k_{i+1} - k_i - 1$. Then $k_i = \ell_0 + 1 + \cdots + \ell_{i-1} + 1 + \ell_i$. Define $\pi : X_\alpha \to X_\beta$ by

$$\pi(x, i) = \sigma_\beta^i(0^{\ell_0} x(0) 0^{\ell_1} x(1) \ldots, 0).$$

Lemma 4.13. $\forall x \in X$ $(\pi \circ \sigma_\alpha^{T_\alpha(x)}(x, 0) = \sigma_\beta^{T_\alpha(x)} \circ \pi(x, 0))$.

Proof. Simply note that if $x = 1^n 0 y$, then Proposition 4.7 ensures that

$$\begin{aligned}
\sigma_\beta^{T_\alpha(x)} \circ \pi(x, 0) &= \sigma_\beta^{\alpha(n) - \sum_{i<n} \alpha(i)}(0^{\ell_0} 1 \ldots 0^{\ell_{n-1}} 1 0^{\ell_n} 0 z, 0) \\
&= \sigma_\beta^{\beta(k_n) - \sum_{i<n} \beta(k_i)}(0^{\ell_0} 1 \ldots 0^{\ell_{n-1}} 1 0^{\ell_n} 0 z, 0) \\
&= (0^{\ell_0} 0 \ldots 0^{\ell_{n-1}} 0 0^{\ell_n} 1 z, 0) \\
&= \pi(0^n 1 y, 0) \\
&= \pi \circ \sigma_\alpha^{T_\alpha(x)}(x, 0),
\end{aligned}$$

for an appropriately chosen $z \in 2^\mathbb{N}$. ☼

Lemma 4.13 clearly implies that $\pi|(X_\alpha \cap (X \times \mathbb{N}))$ is as desired. ☼

A *squashed basis* for a class $\mathcal{A}$ of Borel functions on Polish spaces is a class $\mathcal{B} \subseteq \mathcal{A}$ such that $\forall f \in \mathcal{A}\, \exists g \in \mathcal{B}$ $(g_0 \preceq_L f_0)$.

Theorem 4.14. *Suppose that $\alpha \in \Omega_{\mathrm{odd}}$ and $\mathcal{B}$ is a squashed basis for the class of Borel functions f which do not admit two-spaced Borel complete sections and for which $f_0 \preceq_L (\sigma_\alpha)_0$. Then there is a pairwise orthogonal subset of $\mathcal{B}$ of cardinality $\mathfrak{c}$.*

Proof. By Proposition 4.10, there is a pairwise orthogonal sequence $\langle \alpha_x \rangle_{x \in 2^\mathbb{N}}$ of subsequences of α. By Proposition 4.8, none of the functions σ_{α_x} admit two-spaced Borel complete sections, and by Proposition 4.12, each of the functions σ_{α_x} Borel embeds into σ_α. By Proposition 4.9, there are Borel σ_{α_x}-complete sections B_x such that $\Delta_{\sigma_{\alpha_x}}(B_x) = \Delta(\alpha_x)$. Then $\Delta_{\sigma_{\alpha_x}}(B_x) \perp \Delta_{\sigma_{\alpha_y}}(B_y)$, for all $x \neq y$, so the sequence $\langle \sigma_{\alpha_x} \rangle_{x \in 2^\mathbb{N}}$ is pairwise orthogonal, by Proposition 4.6. For each $x \in 2^\mathbb{N}$, fix $f_x \in \mathcal{B}$ such that $(f_x)_0 \preceq_L \sigma_{\alpha_x}$, and observe that the sequence $\langle (f_x)_0 \rangle_{x \in 2^\mathbb{N}}$ is pairwise orthogonal. ☼

§5. A basis theorem. In order to considerably strengthen Theorem 4.14, we give next a Glimm-Effros style characterization of the circumstances under which an aperiodic Borel function admits a two-spaced Borel complete section:

THEOREM 5.1. *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel function. Then exactly one of the following holds:*

1. *The function f admits a two-spaced Borel complete section.*
2. *There exists $\alpha \in \Omega_{\text{odd}}$ such that $\sigma_\alpha \sqsubseteq f_0$.*

PROOF. Proposition 4.8 easily implies that (1) and (2) are mutually exclusive, so it only remains to show $\neg(1) \Rightarrow (2)$. We will prove that if (1) fails, then there is a sequence $\alpha \in \Omega_{\text{odd}}$ and a continuous injection $\pi_\infty : 2^{\mathbb{N}} \to X$ such that:

(a) $\forall n \in \mathbb{N} \, \forall x \in 2^{\mathbb{N}} \; (f^{\alpha(n)} \circ \pi_\infty(0^n 0x) E_0(f) \pi_\infty(0^n 1x))$.
(b) $\forall x, y \in 2^{\mathbb{N}} \; ((x, y) \notin E_0 \Rightarrow (\pi_\infty(x), \pi_\infty(y)) \notin E_t(f))$.

The map $(x, i) \mapsto f^i \circ \pi_\infty(x)$ then induces the desired embedding, off of the set of points (x, i) for which x is eventually constant.

Define $x <_{f_0} y \Leftrightarrow \exists i, j \in \mathbb{N} \; (j < i \text{ and } f^i(x) = f^j(y))$, and for each $n \in \mathbb{N}$, let $<_0$ denote the *reverse lexicographic order* on 2^n given by $s <_0 t \Leftrightarrow (s \neq t \text{ and } s(i) < t(i))$, where $i < n$ is largest such that $s(i) \neq t(i)$. Set

$$\mathcal{F}_n = \{\pi \in X^{2^n} : \forall s, t \in 2^n \; (s <_0 t \Rightarrow \pi(s) <_{f_0} \pi(t))\}.$$

For each $\pi \in \mathcal{F}_{n+1}$, we use π_0, π_1 to denote the elements of $\mathcal{F}_n$ given by $\pi_i(s) = \pi(si)$. For each $\mathcal{A} \subseteq \mathcal{F}_n$ and $s \in 2^n$, define $\mathcal{A}(s) = \{\pi(s) : \pi \in \mathcal{A}\}$. Fix a compatible Polish metric d on X. For $k_0 \not\equiv k_1 \pmod 2$, a (k_0, k_1)-*extension* of $\mathcal{A} \subseteq \mathcal{F}_n$ is a set $\mathcal{A}' \subseteq \mathcal{F}_{n+1}$ such that:

- $\forall \pi \in \mathcal{A}' \; (\pi_0, \pi_1 \in \mathcal{A} \text{ and } f^{k_0} \circ \pi_0(0^n) = f^{k_1} \circ \pi_1(0^n))$.
- $\forall i, j \leq n \, \forall s, t \in 2^n \; (f^i(\mathcal{A}'(s0)) \cap f^j(\mathcal{A}'(t1)) = \emptyset)$.
- $\forall i \leq n \, \forall s \in 2^{n+1} \; (\operatorname{diam}(f^i(\mathcal{A}'(s))) \leq 1/n)$.

We use $\mathcal{I}_f$ to denote the σ-ideal generated by the evenly-spaced Borel sets, and we use $\mathcal{I}_n$ to denote the σ-ideal of sets $\mathcal{A} \subseteq \mathcal{F}_n$ such that $\mathcal{A}(0^n) \in \mathcal{I}_f$.

LEMMA 5.2. *Suppose that $\mathcal{A} \subseteq \mathcal{F}_n$ is an $\mathcal{I}_n$-positive analytic set and $k \in \mathbb{N}$. Then there exist $k_0, k_1 \in \mathbb{N}$ such that $k_0 - k_1 > \max(k, n)$ and $\mathcal{A}$ admits an $\mathcal{I}_{n+1}$-positive analytic (k_0, k_1)-extension.*

PROOF. Fix $\ell \in \mathbb{N}$ such that the set $\mathcal{A}_\ell = \{\pi \in \mathcal{A} : \pi(1^n) <_{f_0} f^\ell \circ \pi(0^n)\}$ is not in $\mathcal{I}_n$, define $S \subseteq \mathbb{N} \times \mathbb{N}$ by

$$S = \{(k_0, k_1) \in \mathbb{N} \times \mathbb{N} : k_0 \not\equiv k_1 \pmod 2 \text{ and } k_0 - k_1 > \max(k, \ell + n)\},$$

and for each $(k_0, k_1) \in S$, define $\mathcal{A}_{(k_0,k_1)} \subseteq \mathcal{F}_{n+1}$ by

$$\mathcal{A}_{(k_0,k_1)} = \{\pi \in \mathcal{F}_{n+1} : \pi_0, \pi_1 \in \mathcal{A}_\ell \text{ and } f^{k_0} \circ \pi_0(0^n) = f^{k_1} \circ \pi_1(0^n)\}.$$

SUBLEMMA 5.3. *There exists $(k_0, k_1) \in S$ such that $\mathcal{A}_{(k_0,k_1)} \notin \mathcal{I}_{n+1}$.*

PROOF. Suppose, towards a contradiction, that $\mathcal{A}_{(k_0,k_1)} \in \mathcal{I}_{n+1}$, for all pairs $(k_0, k_1) \in S$. Fix a Borel set $B \in \mathcal{I}_f$ with $\bigcup_{(k_0,k_1) \in S} \mathcal{A}_{(k_0,k_1)}(0^{n+1}) \subseteq B$, and set $\mathcal{A}' = \{\pi \in \mathcal{A}_\ell : \pi(0^n) \notin B\}$. Then $\Delta_f(\mathcal{A}'(0^n)) \subseteq \{0, \ldots, \max(k, \ell + n)\} \cup 2\mathbb{N}$, so Lemma 4.4 ensures that there is an $E_0(f)$-invariant Borel set $B' \supseteq \mathcal{A}'(0^n)$ with $\Delta_f(B') \subseteq \{0, \ldots, \max(k, \ell + n)\} \cup 2\mathbb{N}$. Then the set $A = B' \setminus \bigcup_{i \in \mathbb{N}} f^{-(2i+1)}(B')$ is evenly spaced, so $[A]_{E_t(f)} \in \mathcal{I}_f$, thus $\mathcal{A}' \in \mathcal{I}_n$, the desired contradiction. ⊣

Fix a pair $(k_0, k_1) \in S$ such that $\mathcal{A}_{(k_0,k_1)} \notin \mathcal{I}_{n+1}$, and note that if $\pi \in \mathcal{A}_{(k_0,k_1)}$, then $d_f(\pi_0(1^n), \pi_1(0^n)) > n$, so $f^i \circ \pi(s0) \neq f^j \circ \pi(t1)$, for all $i, j \leq n$ and $s, t \in 2^n$. Fix a countable open basis $U_0, U_1, \dots$ for X consisting of sets of diameter $\leq 1/n$, and let $\mathcal{F}$ denote the family of all functions $\phi : \{0, \dots, n\} \times 2^{n+1} \to \mathbb{N}$ such that $U_{\phi(i,s0)} \cap U_{\phi(j,t1)} = \emptyset$, for all $i, j \leq n$ and $s, t \in 2^n$. Then for all $\pi \in \mathcal{A}_{(k_0,k_1)}$, there exists $\phi \in \mathcal{F}$ such that π is in the set $\mathcal{A}_\phi = \{\pi \in \mathcal{A}_{(k_0,k_1)} : \forall i \leq n \, \forall s \in 2^{n+1} \; (f^i \circ \pi(s) \in U_{\phi(i,s)})\}$. Fix $\phi \in \mathcal{F}$ such that $\mathcal{A}_\phi \notin \mathcal{I}_{n+1}$, and observe that $\mathcal{A}_\phi$ is as desired.

A *Souslin scheme* is a sequence $\langle C_t \rangle_{t \in \mathbb{N}^{<\mathbb{N}}}$ of closed subsets of X such that $\mathrm{diam}(C_t) \leq 1/|t|$, for all $t \in \mathbb{N}^{<\mathbb{N}}$. Associated with such a scheme are the sets $A_t = \bigcup_{x \supseteq t} \bigcap_{n \in \mathbb{N}} C_{x|n}$, for $t \in \mathbb{N}^{<\mathbb{N}}$. It is well known that the analytic sets are those of the form $A_\emptyset$, for some Souslin scheme (see, for example, §14 of [7]).

We will next construct analytic sets $\mathcal{A}_n \subseteq \mathcal{F}_n$ which serve as approximations to the desired embedding. We will simultaneously find natural numbers $k_0^n > k_1^n$, Souslin schemes $\langle C_t^s \rangle_{t \in \mathbb{N}^{<\mathbb{N}}}$ for $\mathcal{A}_n(s)$, and finite sequences of natural numbers $t_n^s \subsetneq t_{n+1}^s \subsetneq \cdots$, for $n \in \mathbb{N}$ and $s \in 2^n$, such that:

- $k_0^n - k_1^n > \sum_{i<n} k_0^i - k_1^i$.
- $\mathcal{A}_{n+1}$ is an $\mathcal{I}_{n+1}$-positive (k_0^n, k_1^n)-extension of $\mathcal{A}_n$.
- $\forall s \in 2^n \, \forall i \leq n \; (\mathcal{A}_n(s) \subseteq A^{s|i}_{t_n^{s|i}})$.

We begin by setting $\mathcal{A}_0 = \mathcal{F}_0$, which is not in $\mathcal{I}_0$ by Lemma 4.2. We fix also a Souslin scheme $\langle C_t^\emptyset \rangle_{t \in \mathbb{N}^{<\mathbb{N}}}$ for $\mathcal{A}_0(\emptyset)$, and we set $t_0^\emptyset = \emptyset$.

Suppose that we have found $\mathcal{A}_i$, $\langle C_t^s \rangle_{t \in \mathbb{N}^{<\mathbb{N}}}$, and $t_i^s \subsetneq \cdots \subsetneq t_n^s$, for $i \leq n$ and $s \in 2^i$, as well as $k_0^i, k_1^i \in \mathbb{N}$, for $i < n$. By Lemma 5.2, there exist $k_0^n, k_1^n \in \mathbb{N}$ such that $k_0^n - k_1^n > \sum_{i<n} k_0^i - k_1^i$, as well as an $\mathcal{I}_{n+1}$-positive (k_0^n, k_1^n)-extension $\mathcal{A}$ of $\mathcal{A}_n$. Fix $t_{n+1}^s \supsetneq t_n^s$, for $i \leq n$ and $s \in 2^i$, such that the set

$$\mathcal{A}_{n+1} = \{\pi \in \mathcal{A} : \forall s \in 2^{n+1} \, \forall i \leq n \; (\pi(s) \in A^{s|i}_{t_{n+1}^{s|i}})\}$$

is not in $\mathcal{I}_{n+1}$. For each $s \in 2^{n+1}$, fix Souslin schemes $\langle C_t^s \rangle_{t \in \mathbb{N}^{<\mathbb{N}}}$ for $\mathcal{A}_{n+1}(s)$, and set $t_{n+1}^s = \emptyset$. This completes the recursive construction.

Observe now that for each $x \in 2^\mathbb{N}$, the closed sets

$$C^{x|0}_{t_0^{x|0}}, C^{x|0}_{t_1^{x|0}} \cap C^{x|1}_{t_1^{x|1}}, \dots, C^{x|0}_{t_n^{x|0}} \cap C^{x|1}_{t_n^{x|1}} \cap \cdots \cap C^{x|n}_{t_n^{x|n}}, \dots$$

are decreasing and of vanishing diameter, thus the map

$$\pi_\infty(x) = \text{the unique element of } \bigcap_{n \in \mathbb{N}} \bigcap_{i \leq n} C^{x|i}_{t_n^{x|i}}$$

is a continuous injection. Noting that $\mathrm{diam}(\mathcal{A}_i(x|i)) \to 0$ as $i \to \infty$ and

$$\{\pi_\infty(x)\} = \bigcap_{i \in \mathbb{N}} \bigcap_{n \geq i} C^{x|i}_{t_n^{x|i}} \subseteq \bigcap_{i \in \mathbb{N}} \mathcal{A}_i(x|i),$$

it follows that

$$\pi_\infty(x) = \text{the unique element of } \bigcap_{i \in \mathbb{N}} \mathcal{A}_i(x|i).$$

Define $\alpha \in \Omega_{\mathrm{odd}}$ by $\alpha(n) = k_0^n - k_1^n$. To see (a), it is enough to show:

LEMMA 5.4. *If $n \in \mathbb{N}$ and $x \in 2^{\mathbb{N}}$, then $f^{k_0^n} \circ \pi_\infty(0^n 0x) = f^{k_1^n} \circ \pi_\infty(0^n 1x)$.*

PROOF. Fix $i \geq k_1^n - n$ and $\pi \in \mathcal{A}_{n+1+i}$ such that $\pi(0^n 0(x|i)) = \pi_\infty(0^n 0x)$, and observe that

$$\begin{aligned} f^{k_0^n} \circ \pi_\infty(0^n 0x) &= f^{k_0^n} \circ \pi(0^n 0(x|i)) \\ &= f^{k_1^n} \circ \pi(0^n 1(x|i)) \\ &\in f^{k_1^n}(\mathcal{A}_{n+1+i}(0^n 1(x|i))). \end{aligned}$$

Then $f^{k_1^n} \circ \pi_\infty(0^n 1x) \in f^{k_1^n}(\mathcal{A}_{n+1+i}(0^n 1(x|i)))$ and the diameter of the latter set is at most $1/(n+i)$, so $d(f^{k_0^n} \circ \pi_\infty(0^n 0x), f^{k_1^n} \circ \pi_\infty(0^n 1x)) < 1/(n+i)$. Letting $i \to \infty$, it follows that $f^{k_0^n} \circ \pi_\infty(0^n 0x) = f^{k_1^n} \circ \pi_\infty(0^n 1x)$. ☼

To see (b), it is enough to show that

$$\forall x, y \in 2^{\mathbb{N}}\ (x(n) \neq y(n) \Rightarrow \forall i, j \leq n\ (f^i \circ \pi_\infty(x) \neq f^j \circ \pi_\infty(y))),$$

which follows from the fact that if $x(n) \neq y(n)$, then $f^i(\mathcal{A}_{n+1}(x(0) \dots x(n))) \cap f^j(\mathcal{A}_{n+1}(y(0) \dots y(n))) = \emptyset$. ☼

This leads to the following fact regarding pairwise orthogonality:

THEOREM 5.5. *Suppose that X is a Polish space and $f : X \to X$ is an aperiodic Borel function which does not admit a two-spaced Borel complete section. Then there is a sequence $\langle B_x \rangle_{x \in 2^{\mathbb{N}}}$ of $E_t(f)$-invariant Borel subsets of X such that none of the restrictions $f|B_x$ admit two-spaced Borel complete sections and the sequence $\langle (f|B_x)_0 \rangle_{x \in 2^{\mathbb{N}}}$ is pairwise orthogonal.*

PROOF. By Theorem 5.1, there exists $\alpha \in \Omega_{\mathrm{odd}}$ such that $\sigma_\alpha \sqsubseteq f_0$. By the proof of Theorem 4.14, there is a sequence $\langle \alpha_x \rangle_{x \in 2^{\mathbb{N}}}$ of subsequences of α such that the corresponding sequence $\langle \sigma_{\alpha_x} \rangle_{x \in 2^{\mathbb{N}}}$ is pairwise orthogonal. For each $x \in 2^{\mathbb{N}}$, fix a Borel lifting $\pi_x : X_{\alpha_x} \to X$ of an embedding of σ_{α_x} into f_0. Then the image of each $E_t(\sigma_{\alpha_x})$-class under π_x is countable, so the Lusin-Novikov uniformization theorem ensures that the set $B_x = [\pi_x(X_{\alpha_x})]_{E_t(f)}$ is Borel, and it easily follows that the sequence $\langle B_x \rangle_{x \in 2^{\mathbb{N}}}$ is as desired. ☼

We can give now the promised strengthening of Theorem 4.14:

THEOREM 5.6. *Suppose that X is a Polish space, $f : X \to X$ is an aperiodic Borel function which does not admit a two-spaced Borel complete section, and $\mathcal{B}$ is a squashed basis for the class of Borel functions g which do not admit two-spaced Borel complete sections and for which $g_0 \preceq_L f_0$. Then there is an embedding of $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ into $(\mathcal{B}, \preceq_L)$.*

PROOF. Put $f_\emptyset = f$. Given $f_s : X_s \to X_s$, for some $s \in \mathbb{R}^{<\mathbb{N}}$, Theorem 5.5 ensures that there is a pairwise orthogonal sequence $\langle f_{sr} : X_{sr} \to X_{sr} \rangle_{r \in \mathbb{R}}$ of Borel functions in $\mathcal{B}$ which embed into f_s. The map $s \mapsto f_s$ is as desired. ☼

It is clear that the proof of Theorem 5.6 adapts to give the following:

THEOREM 5.7. *Suppose that f is a Borel function on a Polish space, $\chi_B(\mathcal{G}_f) \geq 3$, and $\mathcal{B}$ is a $\preceq_B$-basis for the class of directed graphs of the form $\mathcal{G}_g$ for which $\chi_B(\mathcal{G}_g) \geq 3$ and $\mathcal{G}_g \preceq_B \mathcal{G}_f$. Then there is an embedding of $(\mathbb{R}^{<\mathbb{N}}, \supseteq)$ into $(\mathcal{B}, \preceq_B)$.*

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