Contemporary Mathematics

A Classical Proof of the Kanovei-Zapletal Canonization

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Abstract. We give a classical proof of the Kanovei-Zapletal canonization of Borel equivalence relations on Polish spaces [**5, 6**].

1. Introduction

A *canonization theorem* for a class $\mathcal{M}$ of structures is a result asserting that for some small subclass $\mathcal{N} \subseteq \mathcal{M}$, some class $\mathcal{X}$ of large sets, and every structure $M \in \mathcal{M}$, there exist $N \in \mathcal{N}$ and $X \in \mathcal{X}$ with the property that $M \restriction X = N \restriction X$.

Here we consider such theorems in the context of descriptive set theory. A well-known example is the following straightforward corollary of Mycielski's theorem on meager subsets of the plane (see §8 of [**7**]):

Theorem 1 (Galvin). *Suppose that X is a perfect Polish space and E is a Baire measurable equivalence relation on X. Then there is a perfect set $B \subseteq X$ with the property that $E \restriction B \in \{\Delta(B), B \times B\}$, where $\Delta(B) = \{(x,x) \mid x \in B\}$.*

It is natural to ask whether there are analogous theorems if we consider even larger sets. One must of course be careful here, as in the presence of the axiom of choice, perfect subsets of Polish spaces are as large as they come. Fortunately, work in descriptive set theory over the past two decades has provided us with a natural successor of the continuum among the definable cardinals.

Suppose that X is a standard Borel space. A Borel equivalence relation F on X is *smooth* if there is a Borel map $\varphi\colon X \to 2^\omega$ such that

$$\forall x_0, x_1 \in X\ (x_0 F x_1 \iff \varphi(x_0) = \varphi(x_1)).$$

Suppose that $B \subseteq X$ is a Borel set. We say that B is *F-smooth* if $F \restriction B$ is smooth. Otherwise, we say that B is *F-non-smooth.*

Suppose now that X is a Polish space and F is a Borel equivalence relation on X which is not smooth. Theorem 1 implies that every perfect subset of X contains an F-smooth perfect set, and the Harrington-Kechris-Louveau dichotomy theorem [**4**] implies that every F-non-smooth Borel subset of X contains an F-non-smooth

2010 *Mathematics Subject Classification.* Primary 03E15; Secondary 03E05.
Key words and phrases. Canonization, Glimm-Effros dichotomy.
I would like to thank the organizers of the 2009 Boise Extravaganza in Set Theory for supporting my visit to Boise and encouraging me to submit this paper.

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perfect set. In particular, it follows that the condition of being F-non-smooth is strictly stronger than the condition of containing a perfect set.

One natural attempt at strengthening Theorem 1 is to fix a Borel equivalence relation F on X which is not smooth, and to augment the conclusion of the theorem by asking that the set B is F-non-smooth. Unfortunately, this version of the result cannot possibly hold in the special case that $E = F$, as the equivalence relations $\Delta(B)$ and $B \times B$ are themselves smooth. The Kanovei-Zapletal canonization theorem [**5, 6**] asserts that this is the only counterexample to the stronger result:

Theorem 2 (Kanovei-Zapletal). *Suppose that X is a Polish space, E and F are Borel equivalence relations on X, and F is not smooth. Then there is an F-non-smooth Borel set $B \subseteq X$ with the property that $E \restriction B \in \{\Delta(B), F \restriction B, B \times B\}$.*

The original proof of this result used effective descriptive set theory and forcing. The goal of this note is to point out that, at least in the special case when $X = 2^\omega$ and F is the equivalence relation given by

$$xE_0y \iff \exists m \in \omega \forall n \in \omega \setminus m \ (x(n) = y(n)),$$

it can also be seen as a corollary of two essentially well-known facts using purely classical methods, and that the full theorem can then be obtained via an application of the Harrington-Kechris-Louveau dichotomy theorem [**4**].

2. Preliminaries

Suppose that X and Y are sets. A *homomorphism* from a set $R \subseteq X \times X$ to a set $S \subseteq Y \times Y$ is a function $\varphi \colon X \to Y$ such that

$$\forall x_0, x_1 \in X \ ((x_0, x_1) \in R \implies (\varphi(x_0), \varphi(x_1)) \in S).$$

A *homomorphism* from a sequence $(R_i)_{i \in I}$ of subsets of $X \times X$ to a sequence $(S_i)_{i \in I}$ of subsets of $Y \times Y$ is a function $\varphi \colon X \to Y$ which is a homomorphism from R_i to S_i for all $i \in I$.

The following Mycielski-style fact is implicit in many arguments involving Borel equivalence relations and Baire category:

Proposition 3. *Suppose that $R \subseteq 2^\omega \times 2^\omega$ is meager. Then there is a continuous homomorphism from $(\Delta(2^\omega)^c, E_0^c, E_0)$ to $(\Delta(2^\omega)^c, R^c, E_0)$.*

Proof. Fix a decreasing sequence of dense open sets $U_n \subseteq \Delta(2^\omega)^c$ with the property that $R \cap \bigcap_{n \in \omega} U_n = \emptyset$. We will recursively construct natural numbers $k_n \in \omega$ and functions $u_n \colon 2^n \to 2^{k_n}$ such that:

(1) $\forall n \in \omega \ (k_n < k_{n+1})$.
(2) $\forall i \in 2 \forall n \in \omega \forall s \in 2^n \ (u_n(s) \sqsubseteq u_{n+1}(s ^\frown i))$.
(3) $\forall i \in 2 \forall n \in \omega \forall s, t \in 2^n \ (\mathcal{N}_{u_{n+1}(s ^\frown i)} \times \mathcal{N}_{u_{n+1}(t ^\frown (1-i))} \subseteq U_n)$.
(4) $\forall i, j \in 2 \forall n \in \omega \forall s, t \in 2^n$
$(i = j \iff u_{n+1}(s ^\frown i) \restriction [k_n, k_{n+1}) = u_n(t ^\frown j) \restriction [k_n, k_{n+1}))$.

We begin by setting $k_0 = 0$ and $u_0(\emptyset) = \emptyset$. Suppose now that we have found $k_n \in \omega$ and $u_n \colon 2^n \to 2^{k_n}$. By a straightforward recursion of finite length, we can find $k \in \omega$ and distinct sequences $v_0, v_1 \in 2^k$ such that $\mathcal{N}_{u_n(s) ^\frown v_i} \times \mathcal{N}_{u_n(t) ^\frown v_{1-i}} \subseteq U_n$ for all $i \in 2$ and $s, t \in 2^n$. Set $k_{n+1} = k_n + k$ and $u_{n+1}(s ^\frown i) = u_n(s) ^\frown v_i$ for $i \in 2$ and $s \in 2^n$. This completes the recursive construction.

Conditions (1) and (2) ensure that we obtain a continuous function $\pi\colon 2^\omega \to 2^\omega$ by setting $\pi(x) = \lim_{n\to\omega} u_n(x \restriction n)$, condition (3) implies that π is a homomorphism from $(\Delta(2^\omega)^c, E_0^c)$ to $(\Delta(2^\omega)^c, R^c)$, and condition (4) ensures that π is a homomorphism from E_0 to E_0. ⊠

The following fact is a straightforward generalization of the Glimm-Effros dichotomy theorem [**1, 3**]:

THEOREM 4. *Suppose that X is a Polish space, $E \subseteq F$ are countable Borel equivalence relations on X, and E is not smooth. Then there is a continuous homomorphism from $(\Delta(2^\omega)^c, E_0^c, E_0)$ to $(\Delta(X)^c, F^c, E)$.*

PROOF. The *orbit equivalence relation* associated with a group G of permutations of X is given by

$$x_0 E_G^X x_1 \iff \exists g \in G\ (g \cdot x_0 = x_1).$$

In [**2**], Feldman-Moore established that every countable Borel equivalence relation on a Polish space is the orbit equivalence relation associated with a countable group of Borel automorphisms (see also Theorem 1.3 of [**8**]). This trivially implies that there are countable groups $G \leq H$ of Borel automorphisms of X such that $E = E_G^X$ and $F = E_H^X$. Fix an increasing sequence of finite symmetric sets $H_n \subseteq H$ containing 1_H such that $H = \bigcup_{n\in\omega} H_n$. By standard change of topology results (see §13 of [**7**]), we can assume that X is a zero-dimensional Polish metric space and H is a group of homeomorphisms of X.

We will recursively construct clopen sets $U_n \subseteq X$ and homeomorphisms $g_n \in G$ such that for all $n \in \omega$, the following conditions hold:

(1) U_n is E-non-smooth.
(2) $U_{n+1} \subseteq U_n \cap g_n^{-1}(U_n)$.
(3) $\forall s \in 2^{n+1}\ (\mathrm{diam}(g_s(U_{n+1})) \leq 1/(n+1))$, where $g_s = \prod_{i\in|s|} g_i^{s(i)}$.
(4) $\forall h \in H_n \forall s,t \in 2^n\ (hg_s(U_{n+1}) \cap g_t g_n(U_{n+1}) = \emptyset)$.

We begin by setting $U_0 = X$.

Suppose now that $n \in \omega$ and we have already found $U_n \subseteq X$ and $g_m \in G$ for all $m \in n$. For each $g \in G$, define $V_g \subseteq X$ by

$$V_g = \{x \in U_n \cap g^{-1}(U_n) \mid \forall h \in H_n \forall s,t \in 2^n\ (hg_s \cdot x \neq g_t g \cdot x)\}.$$

Set $C = U_n \setminus \bigcup_{g\in G} V_g$, and observe that if $(x,y) \in E \restriction C$, then there exists $g \in G$ such that $g \cdot x = y$, so the fact that $x \notin V_g$ ensures the existence of $h \in H_n$ and $s,t \in 2^n$ such that $hg_s \cdot x = g_t g \cdot x = g_t \cdot y$, thus $y = g_t^{-1} hg_s \cdot x$. As there are only finitely many possible values of $g_t^{-1} hg_s$, it follows that C intersects each equivalence class of E in a finite set, and is therefore E-smooth. As $U_n = C \cup \bigcup_{g\in G} V_g$ and U_n is E-non-smooth, there exists $g_n \in G$ such that V_{g_n} is E-non-smooth.

Our topological assumptions ensure that V_{g_n} is the union of countably many clopen sets $U \subseteq X$ which satisfy the following conditions:

(a) $\forall s \in 2^{n+1}\ (\mathrm{diam}(g_s(U)) \leq 1/(n+1))$.
(b) $\forall h \in H_n \forall s,t \in 2^n\ (hg_s(U) \cap g_t g_n(U) = \emptyset)$.

Let U_{n+1} denote any such E-non-smooth set. This completes the construction.

Conditions (2) and (3) ensure that for all $x \in 2^\omega$, the clopen sets of the form $g_{x\restriction n}(U_n)$, for $n \in \omega$, are decreasing and of vanishing diameter. We therefore obtain a continuous function $\pi\colon 2^\omega \to X$ by setting

$$\pi(x) = \text{the unique element of } \bigcap_{n\in\omega} g_{x\restriction n}(U_n).$$

Lemma 5. *If $k \in \omega$, $s \in 2^k$, and $x \in 2^\omega$, then $\pi(s \frown x) = g_s \cdot \pi(0^k \frown x)$.*

Proof of lemma. Simply observe that

$$\begin{aligned}\{\pi(s\frown x)\} &= \bigcap_{n\in\omega} g_{s\frown(x\restriction n)}(U_{k+n}) \\ &= \bigcap_{n\in\omega} g_s g_{0^k\frown(x\restriction n)}(U_{k+n}) \\ &= g_s\left(\bigcap_{n\in\omega} g_{0^k\frown(x\restriction n)}(U_{k+n})\right) \\ &= g_s(\{\pi(0^k\frown x)\}),\end{aligned}$$

thus $\pi(s\frown x) = g_s \cdot \pi(0^k\frown x)$. ⊠

Lemma 5 implies that π is a homomorphism from E_0 to E.

Lemma 6. *If $n \in \omega$, $x, y \in 2^\omega$, and $x(n) \neq y(n)$, then $\pi(y) \notin H_n \cdot \pi(x)$.*

Proof of lemma. By reversing the roles of x and y if necessary, we can assume that $x(n) = 0$ and $y(n) = 1$. Suppose, towards a contradiction, that there exists $h \in H_n$ with $\pi(y) = h \cdot \pi(x)$. Set $s = x \restriction n$ and $t = y \restriction n$. Lemma 5 ensures that the points $x' = g_s^{-1} \cdot \pi(x) = g_s^{-1}h^{-1} \cdot \pi(y)$ and $y' = g_n^{-1}g_t^{-1} \cdot \pi(y)$ are both in U_{n+1}, thus $\pi(y) \in hg_s(U_{n+1}) \cap g_t g_n(U_{n+1})$, which contradicts condition (4). ⊠

Lemma 6 ensures that π is a homomorphism from $(\Delta(2^\omega)^c, E_0^c)$ to $(\Delta(X)^c, F^c)$, which completes the proof of the theorem. ⊠

3. Canonization

A *reduction* of a set $R \subseteq X \times X$ to a set $S \subseteq Y \times Y$ is a homomorphism from (R^c, R) to (S^c, S). An *embedding* is an injective reduction.

Theorem 7 (Kanovei-Zapletal). *Suppose that E is a Borel equivalence relation on 2^ω. Then there is an E_0-non-smooth Borel set $B \subseteq 2^\omega$ with the property that $E \restriction B \in \{\Delta(B), E_0 \restriction B, B \times B\}$.*

Proof. If there exists $x \in 2^\omega$ such that $[x]_E$ is non-meager, then the set $B = [x]_E$ is as desired, since $E \restriction B = B \times B$. Otherwise, the Kuratowski-Ulam theorem (see §8 of [**7**]) implies that E is meager, so Proposition 3 ensures the existence of a continuous homomorphism $\varphi\colon 2^\omega \to 2^\omega$ from $(\Delta(2^\omega)^c, E_0^c, E_0)$ to $(\Delta(2^\omega)^c, (E \cup E_0)^c, E_0)$. Set $F = (\varphi \times \varphi)^{-1}(E)$, noting that $F \subseteq E_0$.

If F is smooth, then there is a Borel transversal $A \subseteq 2^\omega$ of F, in which case A is E_0-non-smooth and $F \restriction A = \Delta(A)$, so the set $B = \varphi(A)$ is as desired, since $E \restriction B = \Delta(B)$. If F is not smooth, then Theorem 4 gives a continuous homomorphism $\psi\colon 2^\omega \to 2^\omega$ from $(\Delta(2^\omega)^c, E_0^c, E_0)$ to $(\Delta(X)^c, E_0^c, F)$, so the set $B = \varphi \circ \psi(2^\omega)$ is as desired, since $E \restriction B = E_0 \restriction B$. ⊠

Theorem 8 (Kanovei-Zapletal). *Suppose that X is a Polish space, E and F are Borel equivalence relations on X, and F is not smooth. Then there is an F-non-smooth Borel set $B \subseteq X$ with the property that $E \restriction B \in \{\Delta(B), F \restriction B, B \times B\}$.*

PROOF. The Harrington-Kechris-Louveau dichotomy theorem [**4**] yields a continuous embedding $\pi\colon 2^\omega \to X$ of E_0 into F. Set $E' = (\pi \times \pi)^{-1}(E)$ and $F' = (\pi \times \pi)^{-1}(F) = E_0$. By Theorem 7, there is an F'-non-smooth set $B' \subseteq 2^\omega$ such that $E' \restriction B' \in \{\Delta(B'), F' \restriction B', B' \times B'\}$. The set $B = \pi(B')$ is as desired. $\boxtimes$

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