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A BOUND ON MEASURABLE CHROMATIC NUMBERS OF LOCALLY FINITE BOREL GRAPHS

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Abstract. We show that the Baire measurable chromatic number of every locally finite Borel graph on a non-empty Polish space is strictly less than twice its ordinary chromatic number, provided this ordinary chromatic number is finite. In the special case that the connectedness equivalence relation is hyperfinite, we obtain the analogous result for the μ-measurable chromatic number.

Introduction

A *graph* on a set X is an irreflexive, symmetric set $G \subseteq X \times X$. Such a graph is *locally finite* if every point has only finitely many G-neighbors. A (κ-)*coloring* of such a graph is a function $c\colon X \to \kappa$ with the property that $\forall (x, y) \in G \; c(x) \neq c(y)$. The *chromatic number* of such a graph, or $\chi(G)$, is the least cardinal κ for which there is such a κ-coloring. Note that any locally finite graph may be colored with countably many colors. In this paper, we consider measurable analogs of these notions, a subject of increasing interest over the last few years due to its connections with descriptive set-theoretic dichotomies and dynamical properties of group actions.

A subset of a topological space is *Borel* if it is in the σ-algebra generated by the underlying topology, and a function between topological spaces is *Borel* if pre-images of open sets are Borel. A *Polish space* is a separable topological space which admits a compatible complete metric. While it is hardly standard terminology, we use the term *Polish cardinal* to refer to a cardinal equipped with a Polish topology. Thus the Polish cardinals are exactly those in the set $\{0, 1, \ldots, \aleph_0, 2^{\aleph_0}\}$, with the two infinite cardinals supporting various topologies.

When X is a Polish space, the *Borel chromatic number* of G, or $\chi_B(G)$, is the least Polish cardinal κ for which there is a Borel κ-coloring of G. The *Baire measurable chromatic number* of G, or $\chi_{BM}(G)$, is the least Polish cardinal κ for which there is a Baire measurable κ-coloring of G. And given a Borel probability measure μ on X, the *μ-measurable chromatic number* of G, or $\chi_\mu(G)$, is the least Polish cardinal κ for which there is a μ-measurable κ-coloring of G. Mirroring the situation for ordinary chromatic numbers, [5, Proposition 4.5] implies that $\chi_B(G) \leq \aleph_0$ whenever G is a locally finite Borel graph. In this paper, we study what further bounds may be gleaned when $\chi(G)$ is finite.

We say that an equivalence relation is *countable* if all of its equivalence classes are countable, and *finite* if all of its equivalence classes are finite. We say that a Borel

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equivalence relation E on a Polish space X is *hyperfinite* if there is an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of finite Borel equivalence relations on X whose union is E.

Ruling out a strong connection between ordinary and measurable chromatic numbers, [1, Corollary 0.8] yields a locally finite Borel graph G and a Borel probability measure μ on a Polish space for which $\chi(G) = 2$ and $\chi_\mu(G) = \aleph_0$. However, the equivalence relation E_G generated by G is quite complicated, in the sense that it is not hyperfinite. In §1, we show that this is no accident.

Theorem A. *Suppose that X is a non-empty Polish space, G is a locally finite Borel graph on X for which $\chi(G) < \aleph_0$ and E_G is hyperfinite, and μ is a Borel probability measure on X. Then there is a μ-conull E_G-invariant Borel set $C \subseteq X$ such that $\chi_B(G \restriction C) \leq 2\chi(G) - 1$, thus $\chi_\mu(G) \leq 2\chi(G) - 1$.*

It is natural to ask whether the analogous result holds for Baire category. As [3, Theorem 6.2] implies that every countable Borel equivalence relation is hyperfinite on a comeager invariant Borel set, such an analog would necessarily imply its generalization in which the assumption that E_G is hyperfinite is removed, thereby ruling out any analog of [1, Corollary 0.8] for Baire category. Perhaps it is then a surprise that, after establishing a technical preliminary result concerning Borel chromatic numbers in §2, we do indeed establish such an analog in §3.

Theorem B. *Suppose that X is a non-empty Polish space and G is a locally finite Borel graph on X for which $\chi(G) < \aleph_0$. Then there is a comeager E_G-invariant Borel set $C \subseteq X$ such that $\chi_B(G \restriction C) \leq 2\chi(G) - 1$, thus $\chi_{BM}(G) \leq 2\chi(G) - 1$.*

In §4, we show that these results imply their generalizations to analytic graphs.

1. Measurable chromatic numbers

In this section, we obtain our bound on μ-measurable chromatic numbers in terms of ordinary chromatic numbers.

Before getting to our primary result, we first note a well-known assumption under which the Borel and ordinary chromatic numbers agree. A *reduction* of an equivalence relation E on X to an equivalence relation F on Y is a function $\pi \colon X \to Y$ with the property that $x_1 \mathrel{E} x_2 \iff \pi(x_1) \mathrel{F} \pi(x_2)$ for all $x_1, x_2 \in X$, and a Borel equivalence relation is *smooth* if it is Borel reducible to equality on $2^{\mathbb{N}}$.

The Lusin-Novikov uniformization theorem for Borel subsets of the plane with countable vertical sections (see, for example, [4, Theorem 18.10]) easily implies that every finite Borel equivalence relation is smooth. We will only need the special case of the following fact for finite Borel equivalence relations, whose natural proof is somewhat simpler, in particular avoiding the need for the uniformization theorem for Borel subsets of the plane with compact vertical sections (see, for example, [4, Theorem 28.8]).

Proposition 1. *Suppose that X is a Polish space and G is a locally countable Borel graph on X for which E_G is smooth. Then $\chi(G) = \chi_B(G)$.*

Proof. Fix a Borel reduction $\pi \colon X \to 2^{\mathbb{N}}$ of E_G to equality, and appeal to the uniformization theorem for Borel subsets of the plane with countable vertical sections to see that $\pi(X)$ is Borel, and to obtain Borel functions $\pi_n \colon \pi(X) \to X$ such that $\operatorname{graph}(\pi^{-1}) = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(\pi_n)$.

As the function $c\colon X \to \mathbb{N}$ given by

$$c(x) = \min\{n \in \mathbb{N} \mid x = (\pi_n \circ \pi)(x)\}$$

is a Borel coloring of G, it follows that $\chi_B(G) \leq \aleph_0$. In particular, if $\chi(G)$ is infinite, then $\aleph_0 \leq \chi(G) \leq \chi_B(G) \leq \aleph_0$, from which it follows that $\chi(G) = \chi_B(G) = \aleph_0$. We can therefore assume that $\chi(G)$ is finite.

Given a point $y \in \pi(X)$, we say that a function $d\colon \mathbb{N} \to \chi(G)$ *codes* a coloring of G on $\pi^{-1}(y)$ if the following conditions hold:

(1) $\forall m, n \in \mathbb{N}\ (\pi_m(y) = \pi_n(y) \implies d(m) = d(n))$.
(2) $\forall m, n \in \mathbb{N}\ (\pi_m(y)\ G\ \pi_n(y) \implies d(m) \neq d(n))$.

Observe now that the set

$$R = \{(y, d) \in \pi(X) \times \chi(G)^{\mathbb{N}} \mid d \text{ codes a coloring of } G \text{ on } \pi^{-1}(y)\}$$

is Borel and has compact vertical sections, so the uniformization theorem for Borel subsets of the plane with compact vertical sections yields a Borel uniformization $e\colon \pi(X) \to \chi(G)^{\mathbb{N}}$ of R. Then the function $f(x) = e(\pi(x))(c(x))$ is a Borel $\chi(G)$-coloring of G. □

For each $n \in \mathbb{N}$, a *G-path* of length n is a sequence $(x_i)_{i \leq n}$ with $\forall i < n\ x_i\ G\ x_{i+1}$. The *graph metric* induced by G on a connected component of G is given by

$$d_G(x, y) = \min\{n \in \mathbb{N} \mid \text{there is a } G\text{-path from } x \text{ to } y \text{ of length } n\}.$$

A *G-ray* is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $\forall n \in \mathbb{N}\ x_n\ G\ x_{n+1}$. A *$G$-barrier* for a point x is a set $Y \subseteq X$ with the property that every injective G-ray emanating from x intersects Y.

Theorem 2. *Suppose that X is a non-empty Polish space, G is a locally finite Borel graph on X for which $\chi(G) < \aleph_0$ and E_G is hyperfinite, and μ is a Borel probability measure on X. Then there is a μ-conull E_G-invariant Borel set $C \subseteq X$ such that $\chi_B(G \restriction C) \leq 2\chi(G) - 1$, thus $\chi_\mu(G) \leq 2\chi(G) - 1$.*

Proof. We can assume that $\chi(G) \geq 2$. Fix real numbers $\epsilon_n > 0$ with $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, as well as an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on X whose union is E_G. As G is locally finite, it follows that if $k \in \mathbb{N}$ and $x \in X$, then $d_G([x]_{E_k}, [x]_{E_G} \setminus [x]_{E_\ell}) \geq 5$ for all sufficiently large $\ell \in \mathbb{N}$. In particular, this ensures that if $k \in \mathbb{N}$ and $\epsilon > 0$, then $\mu(\{x \in X \mid d_G([x]_{E_k}, [x]_{E_G} \setminus [x]_{E_\ell}) \leq 4\}) \leq \epsilon$ for all sufficiently large $\ell \in \mathbb{N}$. This implies that one can recursively construct $k_n \in \mathbb{N}$ such that $\mu(\{x \in X \mid d_G([x]_{E_{k_n}}, [x]_{E_G} \setminus [x]_{E_{k_{n+1}}}) \leq 4\}) \leq \epsilon_n$ for all $n \in \mathbb{N}$.

Define $C_n = \{x \in X \mid \forall m \geq n\ d_G([x]_{E_{k_m}}, [x]_{E_G} \setminus [x]_{E_{k_{m+1}}}) \geq 5\}$, as well as $A_n = \{x \in X \mid 2 \leq d_G(x, [x]_{E_G} \setminus [x]_{E_{k_{n+1}}}) \leq 3\} \cap C_{n+1}$. The latter definition ensures that if $x \in A_n$, then every G-neighbor of x is $E_{k_{n+1}}$-related to x. In particular, it follows that every connected component of $G \restriction A_n$ is contained in an equivalence class of $E_{k_{n+1}}$, and is therefore finite, so Proposition 1 yields a Borel $\chi(G)$-coloring c_n of $G \restriction A_n$. Set $B_n = \{x \in A_n \mid c_n(x) > 0\}$, $B = \bigcup_{n \in \mathbb{N}} B_n$, and $C = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} C_m$.

Note that if $m < n$ and $x \in A_m$, then $x \in C_n$, from which it follows that $d_G(x, [x]_{E_G} \setminus [x]_{E_{k_{n+1}}}) \geq 5$. In particular, it follows that if y is a G-neighbor of x, then $d_G(y, [y]_{E_G} \setminus [y]_{E_{k_{n+1}}}) \geq 4$, hence $y \notin A_n$. This implies that no point in B_m

is G-related to a point in B_n, for distinct $m, n \in \mathbb{N}$, hence $\bigcup_{n \in \mathbb{N}} c_n \restriction B_n$ is a Borel $(\chi(G) - 1)$-coloring of $G \restriction B$.

As $\mu({\sim}C_n) \leq \sum_{m \geq n} \epsilon_m$, it follows that $\mu(\bigcup_{m \geq n} C_m) = 1$, thus $\mu(C) = 1$. The E_{k_n}-invariance of C_n ensures that C is E_G-invariant.

Observe that if $x \in C$, then there exists $n \in \mathbb{N}$ for which $x \in C_n$, in which case any G-path from x to $[x]_{E_G} \setminus [x]_{E_{k_{n+1}}}$ necessarily passes through two G-related points of A_n, and therefore through a point of B_n. In particular, it follows that B is a G-barrier for C. König's Lemma therefore ensures that every connected component of $G \restriction (C \setminus B)$ is finite, so Proposition 1 yields a Borel $\chi(G)$-coloring of $G \restriction (C \setminus B)$. Finally, amalgamating the $(\chi(G) - 1)$-coloring of $G \restriction B$ and the $\chi(G)$-coloring of $G \restriction (C \setminus B)$ yields a Borel $(2\chi(G) - 1)$-coloring of $G \restriction C$, and amalgamating this with a $\chi(G)$-coloring of $G \restriction {\sim}C$ yields a μ-measurable $(2\chi(G) - 1)$-coloring of G. □

The hypothesis that G is locally finite is essential: The graph G on $2^{\mathbb{N}}$ relating two elements if they differ in exactly one coordinate satisfies $\chi(G) = 2$ (since one can fix a function $\phi\colon 2^{\mathbb{N}}/E_G \to 2^{\mathbb{N}}$ choosing a point out of each E_G-class, and color by the parity of $|\mathrm{supp}(c) \bigtriangleup \mathrm{supp}(\phi([c]_{E_G}))|$, where $\mathrm{supp}(c) = \{n \in \mathbb{N} \mid c(n) = 1\}$), but $\chi_{BM}(G) = \chi_\mu(G) = 2^{\aleph_0}$ when μ is the $(1/2, 1/2)$-product measure.

2. Intersection graphs

In this section, we obtain bounds on Borel chromatic numbers of very specific sorts of graphs.

The *intersection graph* on a family $\mathcal{X}$ of subsets of a set X consists of all pairs of distinct sets in $\mathcal{X}$ with non-empty intersection. When $\mathcal{X}$ is the collection of finite subsets of a Polish space X, one obtains a Borel structure on $\mathcal{X}$ via its identification with the family of sequences in $X^{<\mathbb{N}}$ (which is itself a Polish space, for example, by [4, Proposition 3.3]) which are strictly increasing with respect to some fixed Borel linear ordering of X (which exists, for example, by [4, Theorem 15.6]). Given in addition an equivalence relation E on X, we use $[X]^{<\aleph_0}_E$ to denote the family of all finite subsets of X which are contained in a single E-class, with the Borel structure it inherits as a Borel subset of the space of finite subsets of X.

Proposition 3 (Kechris-Miller). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a Borel $\aleph_0$-coloring of the intersection graph on $[X]^{<\aleph_0}_E$.*

Proof. Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a base for X. By the uniformization theorem for Borel subsets of the plane with countable sections, there is a Borel function associating with each finite set $S \subseteq X$ an enumeration $(x^S_i)_{i < |S|}$ of S, in addition to Borel functions $f_n\colon X \to X$ with the property that $E = \bigcup_{n \in \mathbb{N}} \mathrm{graph}(f_n)$.

Define $c\colon [X]^{<\aleph_0}_E \to \mathbb{N}^{<\mathbb{N}}$ by letting $c(S)$ be the lexicographically least sequence $(k^S_i)_{i < |S|}$ of natural numbers such that the sets $U_{k^S_i}$ are pairwise disjoint and $x^S_i \in U_{k^S_i}$ for all $i < |S|$. Define $d\colon [X]^{<\aleph_0}_E \to \mathbb{N}^{<(\mathbb{N} \times \mathbb{N})}$ by letting $d(S)$ be the lexicographically least sequence $(k^S_{i,j})_{i,j < |S|}$ such that $x^S_i = f_{k^S_{i,j}}(x^S_j)$ for all $i, j < |S|$.

It remains to show that $c \times d$ is a coloring of the intersection graph on $[X]^{<\aleph_0}_E$. Suppose, towards a contradiction, that S and T are neighbors with the property that $(c \times d)(S) = (c \times d)(T)$. Set $n = |S| = |T|$ and fix $j, k < n$ such that $x^S_j = x^T_k$. As the sets of the form $V_i = U_{k_i^S} = U_{k_i^T}$ are pairwise disjoint, it follows that $j = k$. But then $x_i^S = f_{k_{i,j}^S}(x_j^S) = f_{k_{i,j}^T}(x_j^T) = x_i^T$ for all $i < n$, thus $S = T$, a contradiction. □

We next turn our attention to a somewhat more general collection of graphs. Let $([X]_E^{<\aleph_0})_E^{<\mathbb{N}}$ denote the family of all finite sequences of sets in $[X]_E^{<\aleph_0}$ which are contained in the same E-class.

Proposition 4. *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then there is a Borel $\aleph_0$-coloring of the graph on $([X]_E^{<\aleph_0})_E^{<\mathbb{N}}$ consisting of all pairs of distinct non-empty sequences whose zeroth entries have non-empty intersection.*

Proof. We use the following general lemma. Recall that if $R \subseteq X \times X$ and $S \subseteq Y \times Y$, a map $\phi\colon X \to Y$ is a *homomorphism* from R to S if $x_0 \mathrel{R} x_1 \implies \phi(x_0) \mathrel{S} \phi(x_1)$. We denote by $\Delta(X)$ the *diagonal* of X, namely $\{(x, x) \mid x \in X\} \subseteq X \times X$.

Lemma 5. *Suppose that X and Y are Polish spaces, and G and H are graphs on X and Y respectively. If $\chi_B(H)$ is countable and there is a countable-to-one Borel homomorphism from G to $H \cup \Delta(Y)$, then $\chi_B(G)$ is countable.*

Proof. Fix a countable-to-one Borel homomorphism $\phi\colon X \to Y$ from G to $H \cup \Delta(Y)$ and a Borel coloring $c\colon Y \to \mathbb{N}$ of H. Using the uniformization theorem for Borel subsets of the plane with countable vertical sections we may fix Borel functions $f_n\colon Y \to X$ such that $\phi^{-1}(y) = \{f_n(y) \mid n \in \mathbb{N}\}$. Define a Borel function $d\colon X \to \mathbb{N}$ by $d(x) = \min\{n \in \mathbb{N} \mid x = (f_n \circ \phi)(x)\}$. Finally, $(c \circ \phi) \times d$ is our desired countable coloring of G. □

The proposition then follows by observing that projection onto the zeroth coordinate is a countable-to-one homomorphism from the graph in question to the union of the diagonal and the intersection graph. □

3. Baire measurable chromatic numbers

In this section, we obtain our bound on Baire measurable chromatic numbers in terms of ordinary chromatic numbers.

Given $r \in \mathbb{R}$ and $Y \subseteq X$, the *closed d_G-ball* of radius r around Y is given by $\mathcal{B}_{d_G}(Y, r) = \{x \in X \mid \exists y \in Y \ d_G(x, y) \leq r\}$.

Theorem 6. *Suppose that X is a non-empty Polish space and G is a locally finite Borel graph on X for which $\chi(G) < \aleph_0$. Then there is a comeager E_G-invariant Borel set $C \subseteq X$ such that $\chi_B(G \restriction C) \leq 2\chi(G) - 1$, thus $\chi_{BM}(G) \leq 2\chi(G) - 1$.*

Proof. We can assume that $\chi(G) \geq 2$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ goes *through* a set $Y \subseteq X$ if each point x_n of the sequence is in Y. We will recursively construct a sequence $(B_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of Borel subsets of X satisfying the following:

(1) No point of any B_s is G-related to a point of any $B_{s^\frown(n)} \setminus B_s$.
(2) Every connected component of every $G \restriction B_s$ is a finite set on which the chromatic number of G is at most $\chi(G) - 1$.
(3) For all $s \in \mathbb{N}^{<\mathbb{N}}$ and $x \in X$, some $B_{s^\frown(n)}$ is a G-barrier for x.
(4) There is no injective G-ray through $\mathcal{B}_{d_G}(B_s, 2)$.

We begin by setting $B_\emptyset = \emptyset$.

Suppose now that $s \in \mathbb{N}^{<\mathbb{N}}$ and B_s has already been defined. Let $\mathcal{X}_s$ denote the set of all triples $(x, S, T) \in X \times [X]^{<\aleph_0}_{E_G} \times [X]^{<\aleph_0}_{E_G}$, where $S, T \subseteq [x]_{E_G}$, S is a G-barrier for x, no point of S is G-related to a point of B_s, $\chi(G \restriction S) \leq \chi(G) - 1$, and there is no G-path from $\mathcal{B}_{d_G}(S, 2)$ to $\sim T$ through $\mathcal{B}_{d_G}(B_s \cup S, 2)$. Let $\mathcal{G}_s$ denote the graph on $\mathcal{X}_s$ in which two distinct triples (x, S, T) and (x', S', T') are related if T and T' have non-empty intersection. By Proposition 4, there is a Borel coloring $c_s \colon \mathcal{X}_s \to \mathbb{N}$ of $\mathcal{G}_s$. The uniformization theorem for Borel subsets of the plane with countable vertical sections ensures that for all $n \in \mathbb{N}$, the set

$$B_{s^\frown(n)} = B_s \cup \bigcup \{S \in [X]^{<\aleph_0}_{E_G} \mid \exists x \in [S]_{E_G} \exists T \subseteq [S]_{E_G}\ c_s(x, S, T) = n\}$$

is Borel.

The definition of $\mathcal{X}_s$ ensures that no point of B_s is G-related to a point of any $B_{s^\frown(n)} \setminus B_s$, and the definition of $\mathcal{G}_s$ implies that if $c_s(x, S, T) = c_s(x', S', T')$ but $(x, S, T) \neq (x', S', T')$, then $T \cap T' = \emptyset$. The definition of $\mathcal{X}_s$ then ensures that $S \cap S' = \emptyset$ and no point of S is G-related to a point of S', and therefore that every connected component of every $G \restriction B_{s^\frown(n)}$ is a finite set on which the chromatic number of G is at most $\chi(G) - 1$.

To see that for all $x \in X$, some $B_{s^\frown(n)}$ is a G-barrier for x, we observe first that there is a finite G-barrier $R \subseteq [x]_{E_G} \setminus \mathcal{B}_{d_G}(B_s, 2)$ for x. Towards this end, note that if $x \notin \mathcal{B}_{d_G}(B_s, 2)$, then the set $R = \{x\}$ is as desired. Otherwise, the definition of B_s ensures that there are no infinite G-rays through the set Y of points y for which there is a G-path from x to y through $\mathcal{B}_{d_G}(B_s, 2)$, so König's lemma ensures that Y is finite, thus we can take R to be the set of G-neighbors of points of Y which are not themselves in Y.

Fix a coloring $c_R \colon \mathcal{B}_{d_G}(R, 1) \to \chi(G)$ of the restriction of G to $\mathcal{B}_{d_G}(R, 1)$, and define $S = \{y \in \mathcal{B}_{d_G}(R, 1) \mid c(y) > 0\}$, noting that S is a G-barrier for x, no point of S is G-related to a point of B_s, and $\chi(G \restriction S) \leq \chi(G) - 1$. One more application of the inexistence of injective G-rays through $\mathcal{B}_{d_G}(B_s, 2)$ and König's Lemma then yields a finite set $T \subseteq [x]_{E_G}$ for which there is no G-path from $\mathcal{B}_{d_G}(S, 2)$ to $\sim T$ through $\mathcal{B}_{d_G}(B_s \cup S, 2)$. It follows that $(x, S, T) \in \mathcal{X}_s$, in which case $B_{s^\frown(n)}$ is a G-barrier for x, where $n = c_s(x, S, T)$.

To see that there is no injective G-ray through any $\mathcal{B}_{d_G}(B_{s^\frown(n)}, 2)$, note that if $(x_i)_{i \in \mathbb{N}}$ is such a G-ray, then the inexistence of injective G-rays through $\mathcal{B}_{d_G}(B_s, 2)$ ensures that $d_G(x_i, B_{s^\frown(n)} \setminus B_s) \leq 2$ for some $i \in \mathbb{N}$. Fix $(x, S, T) \in \mathcal{X}_s$ such that $c_s(x, S, T) = n$ and $d_G(x_i, S) \leq 2$. We will show that $x_i \in T$ for all $i \in \mathbb{N}$, contradicting the fact that T is finite. Suppose, towards a contradiction, that there is a G-path from x_i to $\sim T$ through $\mathcal{B}_{d_G}(B_{s^\frown(n)}, 2)$. As the definition of $\mathcal{X}_s$ ensures that there is no such path through $\mathcal{B}_{d_G}(B_s \cup S, 2)$, it follows that there exists $(x', S', T') \in \mathcal{X}_s$ such that $(x, S, T) \neq (x', S', T')$, $c(x', S', T') = n$, and $\mathcal{B}_{d_G}(S', 2) \cap T \neq \emptyset$. As the definition of $\mathcal{X}_s$ implies that $\mathcal{B}_{d_G}(S', 2) \subseteq T'$, this contradicts the definition of $\mathcal{G}_s$.

This completes the description of the recursive construction. For each $p \in \mathbb{N}^\mathbb{N}$, define $B_p = \bigcup_{n \in \mathbb{N}} B_{p \restriction n}$ and $C_p = \{x \in X \mid \forall y \in [x]_{E_G}\ B_p \text{ is a } G\text{-barrier for } y\}$. König's lemma and the uniformization theorem for Borel subsets of the plane with countable vertical sections imply that the set of pairs $(p, x) \in \mathbb{N}^\mathbb{N} \times X$ for which $x \in C_p$ is Borel.

Lemma 7. *For comeagerly many $p \in \mathbb{N}^\mathbb{N}$, the set C_p is comeager.*

Proof. The uniformization theorem for Borel subsets of the plane with countable vertical sections yields Borel functions $f_n\colon X \to X$ with $E_G = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(f_n)$. Given $x \in X$, condition (3) ensures that for all $n \in \mathbb{N}$, every $s \in \mathbb{N}^{<\mathbb{N}}$ has an extension $t \in \mathbb{N}^{<\mathbb{N}}$ such that B_t is a G-barrier for $f_n(x)$. It follows that the set of $p \in \mathbb{N}^\mathbb{N}$ such that B_p is a G-barrier for $f_n(x)$ is dense and open, thus the set of $p \in \mathbb{N}^\mathbb{N}$ such that B_p is a G-barrier for all $y \in [x]_{E_G}$ is comeager. As the set of $(p, x) \in \mathbb{N}^\mathbb{N} \times X$ for which $x \in C_p$ has the Baire property, the Kuratowski-Ulam quantifier exchange theorem for comeager subsets of the plane (see, for example, [4, Theorem 8.41]) gives the desired result. □

Fix a sequence $p \in \mathbb{N}^\mathbb{N}$ for which C_p is comeager, and set $B = B_p$ and $C = C_p$. Conditions (1) and (2), along with Proposition 1, yield a Borel $(\chi(G) - 1)$-coloring of $G \restriction B$, and the fact that B is a G-barrier for every point of C, together with Proposition 1, gives a Borel $\chi(G)$-coloring of $G \restriction (C \setminus B)$. As in the proof of Theorem 2, we may then amalgamate the colorings to get a Borel $(2\chi(G) - 1)$-coloring of $G \restriction C$, and then amalgamate the result with a $\chi(G)$-coloring of $G \restriction {\sim}C$ to get a Baire measurable $(2\chi(G) - 1)$-coloring of G. □

4. Analytic graphs

In this final section, we show that our earlier results generalize to analytic graphs. The *horizontal sections* of a set $R \subseteq X \times Y$ are given by $R^y = \{x \in X \mid x \mathrel{R} y\}$, and the *vertical sections* of a set $R \subseteq X \times Y$ are given by $R_x = \{y \in Y \mid x \mathrel{R} y\}$. A property P of subsets of a Polish space Y is $\mathbf{\Pi}^1_1$*-on-*$\mathbf{\Sigma}^1_1$ if whenever X is a Polish space and $R \subseteq X \times Y$ is analytic, the set $\{x \in X \mid R_x \text{ satisfies } P\}$ is co-analytic. The *first reflection theorem* ensures that every analytic set satisfying such a property P is contained in a Borel set satisfying P (see, for example, [4, Theorem 35.10]). This will be our primary tool in the arguments to come.

The generalizations of Propositions 3 and 4 to analytic equivalence relations are consequences of the following well-known fact.

Proposition 8. *Suppose that X is a Polish space and E is a countable analytic equivalence relation on X. Then there is a countable Borel equivalence relation F on X such that $E \subseteq F$.*

Proof. By a result of Mazurkiewicz-Sierpiński, the property of being countable is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$ (see, for example, [4, Theorem 29.19]), thus so too is the property (of subsets of $X \times X$) that every horizontal and vertical section is countable. The first reflection theorem therefore yields a Borel set $R \subseteq X \times X$, all of whose horizontal and vertical sections are countable, such that $E \subseteq R$.

Define $S = \{(x, y) \in X \times X \mid x \mathrel{R} y \text{ or } y \mathrel{R} x\}$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $f_n\colon X \to X$ such that $S = \bigcup_{n\in\mathbb{N}} \operatorname{graph}(f_n)$. For each sequence $s \in \mathbb{N}^{<\mathbb{N}}$, let f_s denote the composition of the functions of the form $f_{s(i)}$, for $i < |s|$. As graphs of Borel functions are themselves Borel (see, for example, [4, Proposition 12.4]), it follows that the equivalence relation $F = \bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} \operatorname{graph}(f_s)$ is as desired. □

Similarly, the generalization of Theorem 6 to analytic graphs is a consequence of the following fact, along with the observation that if $G \subseteq H$ are graphs, then every coloring of H is a coloring of G, and every E_H-invariant set is E_G-invariant.

Proposition 9. *Suppose that X is a Polish space and G is a locally finite analytic graph on X. Then there is a locally finite Borel graph H on X, with $\chi(G) = \chi(H)$, such that $G \subseteq H$.*

Proof. A *directed graph* on X is an irreflexive set $H \subseteq X \times X$. The notions of coloring and chromatic number extend to directed graphs in the obvious way. Note that, by the axiom of choice, if $n \in \mathbb{N}$ is a natural number, then there is an n-coloring of G if and only if for every finite set $Y \subseteq X$, there is an n-coloring of $G \restriction Y$ (see [2]). In particular, it follows that the property of being a directed graph with chromatic number at most n is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$. As the property of having finite horizontal and vertical sections is also $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$, it follows that there is a Borel directed graph K on X, with the same chromatic number as G, as well as with finite horizontal and vertical sections, such that $G \subseteq K$. Then the graph $H = \{(x, y) \in X \times X \mid x \mathrel{K} y \text{ or } y \mathrel{K} x\}$ is as desired. □

To see that our use of the axiom of choice was unnecessary, note that the proof of Theorem 6 actually yields a comeager E_G-invariant Borel set $C \subseteq X$ such that $\chi_B(G \restriction C) \leq \sup\{\chi(G \restriction Y) \mid Y \text{ is a finite subset of } X\}$. But even without the axiom of choice, the idea behind the proof of Proposition 9 gives a locally finite Borel graph $H \supseteq G$ on X such that the quantities $\sup\{\chi(G \restriction Y) \mid Y \text{ is a finite subset of } X\}$ and $\sup\{\chi(H \restriction Y) \mid Y \text{ is a finite subset of } X\}$ agree.

It remains to discuss the generalization of Theorem 2. Before getting to this, however, we first note the following.

Proposition 10. *Suppose that X is a Polish space and E is a finite analytic equivalence relation on X. Then there is a finite Borel equivalence relation F on X such that $E \subseteq F$.*

Proof. As the property of being finite is $\mathbf{\Pi}^1_1$-on-$\mathbf{\Sigma}^1_1$, so too is the property (of subsets of $X \times X$) that every horizontal and vertical section of the transitive closure of the symmetrization of the set in question is finite. The first reflection theorem therefore yields a Borel set $R \subseteq X \times X$, with the latter property, such that $E \subseteq R$.

Define $S = \{(x, y) \in X \times X \mid x \mathrel{R} y \text{ or } y \mathrel{R} x\}$. By the uniformization theorem for Borel subsets of the plane with countable vertical sections, there are Borel functions $f_n : X \to X$ such that $S = \bigcup_{n \in \mathbb{N}} \operatorname{graph}(f_n)$. For each sequence $s \in \mathbb{N}^{<\mathbb{N}}$, let f_s denote the composition of the functions of the form $f_{s(i)}$, for $i < |s|$. As graphs of Borel functions are themselves Borel, it follows that the relation $F = \bigcup_{s \in \mathbb{N}^{<\mathbb{N}}} \operatorname{graph}(f_s)$ is as desired. □

An analytic equivalence relation E on X is *hyperfinite* if there is an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite analytic equivalence relations on X whose union is E. The generalization of Theorem 2 to analytic graphs is a consequence of Proposition 9, the following fact, and the observation that if $G \subseteq H$ are graphs, then every coloring of H is a coloring of G, and every E_H-invariant set is E_G-invariant.

Proposition 11. *Suppose that X is a Polish space and E is a hyperfinite analytic equivalence relation on X. Then there is a hyperfinite Borel equivalence relation F on X such that $E \subseteq F$.*

Proof. Fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite analytic equivalence relations on X whose union is E. By Proposition 10, there are finite Borel equivalence relations F_n on X such that $E_n \subseteq F_n$. Then we obtain an increasing sequence of finite Borel equivalence relations by setting $F'_n = \bigcap_{m \geq n} F_m$. As $E_n \subseteq F'_n$, it follows that the equivalence relation $F = \bigcup_{n \in \mathbb{N}} F'_n$ is as desired. □

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